



# Uniform stabilization in weighted Sobolev spaces for the KdV equation posed on the half-line

Ademir F. Pazoto <sup>\*</sup>      Lionel Rosier <sup>†</sup>

February 3, 2010

## Abstract

Studied here is the large-time behavior of solutions of the Korteweg-de Vries equation posed on the right half-line under the effect of a localized damping. Assuming as in [20] that the damping is active on a set  $(a_0, +\infty)$  with  $a_0 > 0$ , we establish the exponential decay of the solutions in the weighted spaces  $L^2((x+1)^m dx)$  for  $m \in \mathbb{N}^*$  and  $L^2(e^{2bx} dx)$  for  $b > 0$  by a Lyapunov approach. The decay of the spatial derivatives of the solution is also derived.

**MSC:** Primary: 93D15, 35Q53; Secondary: 93B05.

**Key words.** Exponential Decay, Korteweg-de Vries equation, Stabilization.

## 1 Introduction

The Korteweg-de Vries (KdV) equation was first derived as a model for the propagation of small amplitude long water waves along a channel [9, 16, 17]. It has been intensively studied from various aspects for both mathematics and physics since the 1960s when solitons were discovered through solving the KdV equation, and the inverse scattering method, a so-called nonlinear Fourier transform, was invented to seek solitons [14, 22]. It is now well known that the KdV equation is not only a good model for water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance weak nonlinear and dispersive effects.

The initial boundary value problems (IBVP) arise naturally in modeling small-amplitude long waves in a channel with a wavemaker mounted at one end [1, 2, 3, 29]. Such mathematical formulations have received considerable attention in the past, and a satisfactory theory of global well-posedness is available for initial and boundary conditions satisfying physically relevant smoothness and consistency assumptions (see e.g. [1, 4, 6, 7, 11, 12, 13] and the references therein).

The analysis of the long-time behavior of IBVP on the quarter-plane for KdV has also received considerable attention over recent years, and a review of some of the results related to the issues we address here can be found in [5, 7, 19]. For stabilization and controllability issues on the half line, we refer the reader to [20] and [27, 28], respectively.

In this work, we are concerned with the asymptotic behavior of the solutions of the IBVP for the KdV equation posed on the positive half line under the presence of a localized damping represented by the function  $a$ ; that is,

$$(1) \quad \begin{cases} u_t + u_x + u_{xxx} + uu_x + a(x)u = 0, & x, t \in \mathbb{R}^+, \\ u(0, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x > 0. \end{cases}$$

<sup>\*</sup>Instituto de Matemática, Universidade Federal do Rio de Janeiro, P.O. Box 68530, CEP 21945-970, Rio de Janeiro, RJ, Brasil (ademir@im.ufrj.br)

<sup>†</sup>Institut Elie Cartan, UMR 7502 UHP/CNRS/INRIA, B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France (rosier@iecn.u-nancy.fr)

Assuming  $a(x) \geq 0$  a.e. and that  $u(\cdot, t) \in H^3(\mathbb{R}^+)$ , it follows from a simple computation that

$$(2) \quad \frac{dE}{dt} = - \int_0^\infty a(x)|u(x, t)|^2 dx - \frac{1}{2}|u_x(0, t)|^2$$

where

$$(3) \quad E(t) = \frac{1}{2} \int_0^\infty |u(x, t)|^2 dx$$

is the total energy associated with (1). Then, we see that the term  $a(x)u$  plays the role of a feedback damping mechanism and, consequently, it is natural to wonder whether the solutions of (1) tend to zero as  $t \rightarrow \infty$  and under what rate they decay. When  $a(x) > a_0 > 0$  almost everywhere in  $\mathbb{R}^+$ , it is very simple to prove that  $E(t)$  converges to zero as  $t$  tends to infinity. The problem of stabilization when the damping is effective only in a subset of the domain is much more subtle. The following result was obtained in [20].

**Theorem 1.1** *Assume that the function  $a = a(x)$  satisfies the following property*

$$(4) \quad a \in L^\infty(\mathbb{R}^+), a \geq 0 \text{ a.e. in } \mathbb{R}^+ \text{ and } a(x) \geq a_0 > 0 \text{ a.e. in } (x_0, +\infty)$$

for some numbers  $a_0, x_0 > 0$ . Then for all  $R > 0$  there exist two numbers  $C > 0$  and  $\nu > 0$  such that for all  $u_0 \in L^2(\mathbb{R}^+)$  with  $\|u_0\|_{L^2(\mathbb{R}^+)} \leq R$ , the solution  $u$  of (1) satisfies

$$(5) \quad \|u(t)\|_{L^2(\mathbb{R}^+)} \leq Ce^{-\nu t} \|u_0\|_{L^2(\mathbb{R}^+)}.$$

Actually, Theorem 1.1 was proved in [20] under the additional hypothesis that

$$(6) \quad a(x) \geq a_0 \text{ a.e. in } (0, \delta)$$

for some  $\delta > 0$ , but (6) may be dropped by replacing the unique continuation property [20, Lemma 2.4] by [30, Theorem 1.6]. The exponential decay of  $E(t)$  is obtained following the methods in [23, 25, 26] which combine multiplier techniques and compactness arguments to reduce the problem to some unique continuation property for weak solutions of KdV.

Along this work we assume that the real-valued function  $a = a(x)$  satisfies the condition (4) for some given positive numbers  $a_0, x_0$ . In this paper we investigate the stability properties of (1) in the weighted spaces introduced by Kato in [15]. More precisely, for  $b > 0$  and  $m \in \mathbb{N}$ , we prove that the solution  $u$  exponentially decays to 0 in  $L_b^2$  and  $L_{(x+1)^m dx}^2$  (if  $u(0)$  belongs to one of these spaces), where

$$L_b^2 = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \int_0^\infty |u(x)|^2 e^{2bx} dx < \infty\},$$

$$L_{(x+1)^m dx}^2 = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \int_0^\infty |u(x)|^2 (x+1)^m dx < \infty\}.$$

The following weighted Sobolev spaces

$$H_b^s = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \partial_x^i u \in L_b^2 \text{ for } 0 \leq i \leq s; u(0) = 0 \text{ if } s \geq 1\}$$

and

$$H_{(x+1)^m dx}^s = \{u : \mathbb{R}^+ \rightarrow \mathbb{R}; \partial_x^i u \in L_{(x+1)^{m-i} dx}^2 \text{ for } 0 \leq i \leq s; u(0) = 0 \text{ if } s \geq 1\},$$

endowed with their usual inner products, will be used thereafter. Note that  $H_b^0 = L_b^2$  and that  $H_{(x+1)^m dx}^0 = L_{(x+1)^m dx}^2$ .

The exponential decay in  $L^2_{(x+1)^m dx}$  is obtained by constructing a convenient Lyapunov function (which actually decreases strictly on the sequence of times  $\{kT\}_{k \geq 0}$ ) by induction on  $m$ . For  $u_0 \in L^2_{(x+1)^m dx}$ , we also prove the following estimate

$$(7) \quad \|u(t)\|_{H^1_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^m dx}}$$

in two situations: (i)  $m = 1$  and  $\|u_0\|_{L^2_{(x+1)^m dx}}$  is arbitrarily large; (ii)  $m \geq 2$  and  $\|u_0\|_{L^2_{(x+1)^m dx}}$  is small enough. In the situation (ii), we first establish a similar estimate for the linearized system and next apply the contraction mapping principle in a space of functions fulfilling the exponential decay. Note that (7) combines the (global) Kato smoothing effect to the exponential decay.

The exponential decay in  $L^2_b$  is established for any initial data  $u_0 \in L^2_b$  under the additional assumption that  $4b^3 + b < a_0$ . Next, we can derive estimates of the form

$$\|u(t)\|_{H^s_b} \leq C \frac{e^{-\mu t}}{t^{s/2}} \|u_0\|_{L^2_b}$$

for any  $s \geq 1$ , revealing that  $u(t)$  decays exponentially to 0 in strong norms.

It would be interesting to see if such results are still true when the function  $a$  has a smaller support. It seems reasonable to conjecture that similar positive results can be derived when the support of  $a$  contains a set of the form  $\cup_{k \geq 1} [ka_0, ka_0 + b_0]$  where  $0 < b_0 < a_0$ , while a negative result probably holds when the support of  $a$  is a finite interval, as the  $L^2$  norm of a soliton-like initial data may not be sufficiently dissipated over time. Such issues will be discussed elsewhere.

The plan of this paper is as follows. Section 2 is devoted to global well-posedness results in the weighted spaces  $L^2_b$  and  $L^2_{(x+1)^2 dx}$ . In section 3, we prove the exponential decay in  $L^2_{(x+1)^m dx}$  and  $L^2_b$ , and establish the exponential decay of the derivatives as well.

## 2 Global well-posedness

### 2.1 Global well-posedness in $L^2_b$

Fix any  $b > 0$ . To begin with, we apply the classical semigroup theory to the linearized system

$$(8) \quad \begin{cases} u_t + u_x + u_{xxx} + a(x)u = 0, & x, t \in \mathbb{R}^+, \\ u(0, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), & x > 0. \end{cases}$$

Let us consider the operator

$$A : D(A) \subset L^2_b \rightarrow L^2_b$$

with domain

$$D(A) = \{u \in L^2_b; \partial_x^i u \in L^2_b \text{ for } 1 \leq i \leq 3 \text{ and } u(0) = 0\}$$

defined by

$$Au = -u_{xxx} - u_x - a(x)u.$$

Then, the following result holds.

**Lemma 2.1** *The operator  $A$  defined above generates a continuous semigroup of operators  $(S(t))_{t \geq 0}$  in  $L^2_b$ .*

**Proof.** We first introduce the new variable  $v = e^{bx}u$  and consider the following (IBVP)

$$(9) \quad \begin{cases} v_t + (\partial_x - b)v + (\partial_x - b)^3v + a(x)v = 0, & x, t \in \mathbb{R}^+, \\ v(0, t) = 0, & t > 0, \\ v(x, 0) = v_0(x) = e^{bx}u_0(x), & x > 0. \end{cases}$$

Clearly, the operator  $B : D(B) \subset L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$  with domain

$$D(B) = \{u \in H^3(\mathbb{R}^+); u(0) = 0\}$$

defined by

$$Bv = -(\partial_x - b)v - (\partial_x - b)^3v - a(x)v$$

is densely defined and closed. So, we are done if we prove that for some real number  $\lambda$  the operator  $B - \lambda$  and its adjoint  $B^* - \lambda$  are both dissipative in  $L^2(\mathbb{R}^+)$ . It is readily seen that  $B^* : D(B^*) \subset L^2(\mathbb{R}^+) \rightarrow L^2(\mathbb{R}^+)$  is given by  $B^*v = (\partial_x + b)v + (\partial_x + b)^3v - a(x)v$  with domain

$$D(B^*) = \{v \in H^3(\mathbb{R}^+); v(0) = v'(0) = 0\}.$$

Pick any  $v \in D(B)$ . After some integration by parts, we obtain that

$$(Bv, v)_{L^2} = -\frac{1}{2}v_x^2(0) - 3b \int_0^\infty v_x^2 dx + (b + b^3) \int_0^\infty v^2 dx - \int_0^\infty a(x)v^2 dx,$$

that is,

$$([B - (b^3 + b)]v, v)_{L^2} \leq 0.$$

Analogously, we deduce that for any  $v \in D(B^*)$

$$(v, [B^* - (b^3 + b)]v)_{L^2} \leq 0$$

which completes the proof. ■

The following linear estimates will be needed.

**Lemma 2.2** *Let  $u_0 \in L_b^2$  and  $u = S(\cdot)u_0$ . Then, for any  $T > 0$*

$$(10) \quad \frac{1}{2} \int_0^\infty |u(x, T)|^2 dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x)|u|^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0$$

$$(11) \quad \begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 e^{2bx} dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt \\ & - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt + \int_0^T \int_0^\infty a(x)|u|^2 e^{2bx} dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0. \end{aligned}$$

As a consequence,

$$(12) \quad \|u\|_{L^\infty(0, T; L_b^2)} + \|u_x\|_{L^2(0, T; L_b^2)} \leq C \|u_0\|_{L_b^2},$$

where  $C = C(T)$  is a positive constant.

**Proof.** Pick any  $u_0 \in D(A)$ . Multiplying the equation in (1) by  $u$  and integrating over  $(0, +\infty) \times (0, T)$ , we obtain (10). Then, the identity may be extended to any initial state  $u_0 \in L_b^2$  by a density argument. To derive (11) we first multiply the equation by  $(e^{2bx} - 1)u$  and integrate by parts over  $(0, +\infty) \times (0, T)$  to deduce that

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u(x, T)|^2 (e^{2bx} - 1) dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 (e^{2bx} - 1) dx + \\ & + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt + \\ & + \int_0^T \int_0^\infty a(x) |u|^2 (e^{2bx} - 1) dx dt = 0. \end{aligned}$$

Adding the above equality and (10) hand to hand, we obtain (11) using the same density argument. Then, Gronwall inequality, (4) and (11) imply that

$$\|u\|_{L^\infty(0, T; L_b^2)} \leq C \|u_0\|_{L_b^2},$$

with  $C = C(T) > 0$ . This estimate together with (11) gives us

$$\|u_x\|_{L^2(0, T; L_b^2)} \leq C \|u_0\|_{L_b^2},$$

where  $C = C(T)$  is a positive constant. ■

The global well-posedness result reads as follows:

**Theorem 2.3** *For any  $u_0 \in L_b^2$  and any  $T > 0$ , there exists a unique solution  $u \in C([0, T]; L_b^2) \cap L^2(0, T; H_b^1)$  of (1).*

**Proof.** By computations similar to those performed in the proof of Lemma 2.2, we obtain that for any  $f \in C^1([0, T]; L_b^2)$  and any  $u_0 \in D(A)$ , the solution  $u$  of the system

$$\begin{cases} u_t + u_x + u_{xxx} + a(x)u = f, & x \in \mathbb{R}^+, t \in (0, T), \\ u(0, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^+, \end{cases}$$

fulfills

$$(13) \quad \sup_{0 \leq t \leq T} \|u(t)\|_{L_b^2} + \left( \int_0^T \int_0^\infty |u_x|^2 e^{2bx} dx dt \right)^{\frac{1}{2}} \leq C \left( \|u_0\|_{L_b^2} + \int_0^T \|f\|_{L_b^2} dt \right)$$

for some constant  $C = C(T)$  nondecreasing in  $T$ . A density argument yields that  $u \in C([0, T]; L_b^2)$  when  $f \in L^1(0, T; L_b^2)$  and  $u_0 \in L_b^2$ .

Let  $u_0 \in L_b^2$  be given. To prove the existence of a solution of (1) we introduce the map  $\Gamma$  defined by

$$(\Gamma u)(t) = S(t)u_0 + \int_0^t S(t-s)N(u(s)) ds$$

where  $N(u) = -uu_x$ , and the space

$$F = C([0, T]; L_b^2) \cap L^2(0, T; H_b^1)$$

endowed with its natural norm. We shall prove that  $\Gamma$  has a fixed-point in some ball  $B_R(0)$  of  $F$ . We need the following

CLAIM 1. If  $u \in H_b^1$  then

$$\|u^2 e^{2bx}\|_{L^\infty(\mathbb{R}^+)} \leq (2 + 2b) \|u\|_{L_b^2} \|u\|_{H_b^1}.$$

From Cauchy-Schwarz inequality, we get for any  $\bar{x} \in \mathbb{R}^+$

$$\begin{aligned} u^2(\bar{x})e^{2b\bar{x}} &= \int_0^{\bar{x}} [u^2 e^{2bx}]_x dx = \int_0^{\bar{x}} [2uu_x e^{2bx} + 2bu^2 e^{2bx}] dx \\ &\leq 2\left(\int_0^\infty u^2 e^{2bx} dx\right)^{\frac{1}{2}} \left(\int_0^\infty u_x^2 e^{2bx} dx\right)^{\frac{1}{2}} + 2b \int_0^\infty u^2 e^{2bx} dx \leq (2 + 2b) \|u\|_{L_b^2} \|u\|_{H_b^1} \end{aligned}$$

which guarantees that Claim 1 holds.

CLAIM 2. There exists a constant  $K > 0$  such that for  $0 < T \leq 1$

$$\|\Gamma(u) - \Gamma(v)\|_F \leq KT^{\frac{1}{4}} (\|u\|_F + \|v\|_F) \|u - v\|_F, \quad \forall u, v \in F.$$

According to the previous analysis,

$$\|\Gamma(u) - \Gamma(v)\|_F \leq C \|uu_x - vv_x\|_{L^1(0,T;L_b^2)}.$$

So, applying triangular inequality and Hölder inequality, we have

$$(14) \quad \begin{aligned} \|\Gamma(u) - \Gamma(v)\|_F &\leq C \{ \|u - v\|_{L^2(0,T;L^\infty(0,\infty))} \|u\|_{L^2(0,T;H_b^1)} + \\ &\quad + \|v\|_{L^2(0,T;L^\infty(0,\infty))} \|u - v\|_{L^2(0,T;H_b^1)} \}. \end{aligned}$$

Now, by Claim 1, we have

$$(15) \quad \|u\|_{L^2(0,T;L^\infty(0,\infty))} \leq CT^{\frac{1}{4}} \|u\|_{L^\infty(0,T;L_b^2)}^{\frac{1}{2}} \|u\|_{L^2(0,T;H_b^1)}^{\frac{1}{2}}.$$

Then, combining (14) and (15), we deduce that

$$(16) \quad \|\Gamma(u) - \Gamma(v)\|_F \leq CT^{\frac{1}{4}} \{ \|u\|_F + \|v\|_F \} \|u - v\|_F.$$

Let  $T > 0$ ,  $R > 0$  be numbers whose values will be specified later, and let  $u \in B_R(0) \subset F$  be given. Then, by Claim 2 and Lemma 2.2,  $\Gamma u \in F$  and

$$\|\Gamma u\|_F \leq C (\|u_0\|_{L_b^2} + T^{\frac{1}{4}} \|u\|_F^2).$$

Consequently, for  $R = 2C\|u_0\|_{L_b^2}$  and  $T > 0$  small enough,  $\Gamma$  maps  $B_R(0)$  into itself. Moreover, we infer from (16) that this mapping contracts if  $T$  is small enough. Then, by the contraction mapping theorem, there exists a unique solution  $u \in B_R(0) \subset F$  to the problem (1) for  $T$  small enough.

In order to prove that this solution is global, we need some a priori estimates. So, we proceed as in the proof of Lemma 2.2 to obtain for the solution  $u$  of (1)

$$(17) \quad \frac{1}{2} \int_0^\infty |u(x, T)|^2 dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 dx + \int_0^T \int_0^\infty a(x) |u|^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0, t) dt = 0$$

and

$$(18) \quad \begin{aligned} &\frac{1}{2} \int_0^\infty |u(x, T)|^2 e^{2bx} dx - \frac{1}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^T u_x^2(0, t) dt \\ &\quad + 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt - (4b^3 + b) \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \\ &\quad + \int_0^T \int_0^\infty a(x) |u|^2 e^{2bx} dx dt - \frac{2b}{3} \int_0^T \int_0^\infty u^3 e^{2bx} dx dt = 0. \end{aligned}$$

First, observe that

$$\left| \int_0^\infty u^2 e^{2bx} dx \right| = \left| -\frac{1}{b} \int_0^\infty uu_x e^{2bx} dx \right| \leq \frac{1}{b} \left( \int_0^\infty u^2 e^{2bx} dx \right)^{\frac{1}{2}} \left( \int_0^\infty u_x^2 e^{2bx} dx \right)^{\frac{1}{2}},$$

therefore,

$$\int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{b^2} \int_0^\infty u_x^2 e^{2bx} dx.$$

Combined to Claim 1, this yields

$$\|u(x)e^{bx}\|_{L^\infty(\mathbb{R}^+)} \leq C\|u_x\|_{L_b^2}.$$

On the other hand, it follows from (17) that

$$\|u(t)\|_{L^2(\mathbb{R}^+)} \leq \|u_0\|_{L^2(\mathbb{R}^+)},$$

hence

$$\begin{aligned} \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt &\leq \int_0^T \|ue^{bx}\|_{L^\infty(\mathbb{R}^+)} \left( \int_0^\infty |u|^2 e^{bx} dx \right) dt \\ &\leq C \int_0^T \|u_x\|_{L_b^2} \|u\|_{L_b^2} \|u\|_{L^2} dt \\ &\leq \delta \|u_x\|_{L^2(0,T;L_b^2)}^2 + C_\delta \|u\|_{L^2(0,T;L_b^2)}^2, \end{aligned}$$

where  $\delta > 0$  is arbitrarily chosen and  $C = C(b, \delta, \|u_0\|_{L^2(\mathbb{R}^+)})$  is a positive constant. Combining this inequality (with  $\delta < 9/2$ ) to (18) results in

$$\|u(T)\|_{L_b^2}^2 \leq \|u_0\|_{L_b^2}^2 + C \int_0^T \|u\|_{L_b^2}^2 dt$$

where  $C = C(b, \|u_0\|_{L^2(\mathbb{R}^+)})$  does not depend on  $T$ . It follows from Gronwall lemma that

$$\|u(T)\|_{L_b^2}^2 \leq \|u_0\|_{L_b^2}^2 e^{CT}$$

for all  $T > 0$ , which gives the global well-posedness. ■

## 2.2 Global well-posedness in $L^2_{(x+1)^2 dx}$

**Definition 2.4** For  $u_0 \in L^2_{(x+1)^2 dx}$  and  $T > 0$ , we denote by a mild solution of (1) any function  $u \in C([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$  which solves (1), and such that for some  $b > 0$  and some sequence  $\{u_{n,0}\} \subset L_b^2$  we have

$$\begin{aligned} u_{n,0} &\rightarrow u_0 \text{ strongly in } L^2_{(x+1)^2 dx}, \\ u_n &\rightarrow u \text{ weakly* in } L^\infty(0, T; L^2_{(x+1)^2 dx}), \\ u_n &\rightarrow u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}), \end{aligned}$$

$u_n$  denoting the solution of (1) emanating from  $u_{n,0}$  at  $t = 0$ .

**Theorem 2.5** For any  $u_0 \in L^2_{(x+1)^2 dx}$  and any  $T > 0$ , there exists a unique mild solution  $u \in C([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$  of (1).

**Proof.** We prove the existence and the uniqueness in two steps.

STEP 1. EXISTENCE

Since the embedding  $L_b^2 \subset L_{(x+1)^2 dx}^2$  is dense, for any given  $u_0 \in L_{(x+1)^2 dx}^2$  we may construct a sequence  $\{u_{n,0}\} \subset L_b^2$  such that  $u_{n,0} \rightarrow u_0$  in  $L_{(x+1)^2 dx}^2$  as  $n \rightarrow \infty$ . For each  $n$ , let  $u_n$  denote the solution of (1) emanating from  $u_{n,0}$  at  $t = 0$ , which is given by Theorem 2.3. Then  $u_n \in C([0, T]; L_b^2) \cap L^2(0, T; H_b^1)$  and it solves

$$(19) \quad u_{n,t} + u_{n,x} + u_{n,xxx} + u_n u_{n,x} + a(x)u_n = 0,$$

$$(20) \quad u_n(0, t) = 0$$

$$(21) \quad u_n(x, 0) = u_{n,0}(x).$$

Multiplying (19) by  $(x+1)^2 u_n$  and integrating by parts, we obtain

$$(22) \quad \begin{aligned} & \frac{1}{2} \int_0^\infty (x+1)^2 |u_n(x, T)|^2 dx + 3 \int_0^T \int_0^\infty (x+1) |u_{n,x}|^2 dx dt + \frac{1}{2} \int_0^T |u_{n,x}(0, t)|^2 dt \\ & - \int_0^T \int_0^\infty (x+1) |u_n|^2 dx dt - \frac{2}{3} \int_0^T \int_0^\infty (x+1) u_n^3 dx dt + \int_0^T \int_0^\infty (x+1)^2 u_n^2 a(x) dx \\ & = \frac{1}{2} \int_0^\infty (x+1)^2 |u_{n,0}(x)|^2 dx. \end{aligned}$$

Scaling in (19) by  $u_n$  gives

$$\begin{aligned} & \frac{1}{2} \int_0^\infty |u_n(x, T)|^2 dx + \frac{1}{2} \int_0^T |u_{n,x}(0, t)|^2 dt + \int_0^T \int_0^\infty a(x) |u_n(x, t)|^2 dx dt \\ & = \frac{1}{2} \int_0^\infty |u_{n,0}(x)|^2 dx, \end{aligned}$$

hence

$$(23) \quad \|u_n\|_{L^2(\mathbb{R}^+)} \leq \|u_{n,0}\|_{L^2(\mathbb{R}^+)} \leq C$$

where  $C = C(\|u_0\|_{L^2(\mathbb{R}^+)})$ . It follows that

$$(24) \quad \begin{aligned} \frac{2}{3} \int_0^\infty (x+1) |u_n|^3 dx & \leq \frac{2\sqrt{2}}{3} \|u_{n,x}\|_{L^2(\mathbb{R}^+)}^{\frac{1}{2}} \|u_n\|_{L^2(\mathbb{R}^+)}^{\frac{3}{2}} \|(x+1)u_n\|_{L^2(\mathbb{R}^+)} \\ & \leq \int_0^\infty (x+1) |u_{n,x}|^2 dx + C \int_0^\infty (x+1)^2 |u_n|^2 dx \end{aligned}$$

which, combined to (22), gives

$$\begin{aligned} & \frac{1}{2} \int_0^\infty (x+1)^2 |u_n(x, T)|^2 dx + 2 \int_0^T \int_0^\infty (x+1) |u_{n,x}|^2 dx dt + \frac{1}{2} \int_0^T |u_{n,x}(0, t)|^2 dt \\ & \leq \frac{1}{2} \int_0^\infty (x+1)^2 |u_{n,0}(x)|^2 dx + C \int_0^T \int_0^\infty (x+1)^2 |u_n(x, t)|^2 dx dt. \end{aligned}$$

An application of Gronwall's lemma yields

$$\begin{aligned} \|u_n\|_{L^\infty(0, T; L_{(x+1)^2 dx}^2)} & \leq C(T, \|u_{n,0}\|_{L_{(x+1)^2 dx}^2}), \\ \|u_{n,x}\|_{L^2(0, T; H_{(x+1)^2 dx}^1)} & \leq C(T, \|u_{n,0}\|_{L_{(x+1)^2 dx}^2}), \\ \|u_{n,x}(0, \cdot)\|_{L^2(0, T)} & \leq C(T, \|u_{n,0}\|_{L_{(x+1)^2 dx}^2}). \end{aligned}$$

Therefore, there exists a subsequence of  $\{u_n\}$ , still denoted by  $\{u_n\}$ , such that

$$\begin{cases} u_n \rightharpoonup u \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}), \\ u_n \rightharpoonup u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}), \\ u_{n,x}(0, \cdot) \rightharpoonup u_x(0, \cdot) \text{ weakly in } L^2(0, T). \end{cases}$$

Note that, for all  $L > 0$ ,  $\{u_n\}$  is bounded in  $L^2(0, T; H^1(0, L)) \cap H^1(0, T; H^{-2}(0, L))$ , hence by Aubin's lemma, we have (after extracting a subsequence if needed)

$$u_n \rightarrow u \text{ strongly in } L^2(0, T; L^2(0, L)) \text{ for all } L > 0.$$

This gives that  $u_n u_{n,x} \rightarrow u u_x$  in the sense of distributions, hence the limit  $u \in L^\infty(0, T; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$  is a solution of (1). Let us check that  $u \in C([0, T]; L^2_{(x+1)^2 dx})$ . Since  $u \in C([0, T]; H^{-2}(\mathbb{R}^+)) \cap L^\infty(0, T; L^2_{(x+1)^2 dx})$ , we have that  $u \in C_w([0, T]; L^2_{(x+1)^2 dx})$  (see e.g. [21]), where  $C_w([0, T]; L^2_{(x+1)^2 dx})$  denotes the space of sequentially weakly continuous functions from  $[0, T]$  into  $L^2_{(x+1)^2 dx}$ .

We claim that  $u \in L^3(0, T; L^3(\mathbb{R}^+))$ . Indeed, from Moser estimate (see [31])

$$(25) \quad \|u\|_{L^\infty(\mathbb{R}^+)} \leq \sqrt{2} \|u_x\|_{L^2(\mathbb{R}^+)}^{\frac{1}{2}} \|u\|_{L^2(\mathbb{R}^+)}^{\frac{1}{2}}$$

and Young inequality we get

$$(26) \quad \int_0^\infty |u|^3 dx \leq \|u\|_{L^\infty} \|u\|_{L^2}^2 \leq \sqrt{2} \|u_x\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{5}{2}} \leq \varepsilon \|u_x\|_{L^2}^2 + c_\varepsilon \|u\|_{L^2}^{\frac{10}{3}}$$

where  $\varepsilon > 0$  is arbitrarily chosen and  $c_\varepsilon$  denotes some positive constant. Since  $u \in C_w([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$ , it follows that  $u \in L^3(0, T; L^3(\mathbb{R}^+))$ . On the other hand,  $u(0, t) = 0$  for  $t \in (0, T)$  and  $u_x(0, \cdot) \in L^2(0, T)$ . Scaling in (1) by  $(x+1)^2 u$  yields for all  $t_1, t_2 \in (0, T)$

$$(27) \quad \begin{aligned} & \frac{1}{2} \int_0^\infty (x+1)^2 |u(x, t_2)|^2 dx - \frac{1}{2} \int_0^\infty (x+1)^2 |u(x, t_1)|^2 dx \\ &= -3 \int_{t_1}^{t_2} \int_0^\infty (x+1) |u_x|^2 dx dt - \frac{1}{2} \int_{t_1}^{t_2} |u_x(0, t)|^2 dt + \int_{t_1}^{t_2} \int_0^\infty (x+1) |u|^2 dx dt \\ &+ \frac{2}{3} \int_{t_1}^{t_2} \int_0^\infty (x+1) u^3 dx dt - \int_{t_1}^{t_2} \int_0^\infty (x+1)^2 a(x) |u|^2 dx dt. \end{aligned}$$

Therefore  $\lim_{t_1 \rightarrow t_2} \left| \|u(t_2)\|_{L^2_{(x+1)^2 dx}}^2 - \|u(t_1)\|_{L^2_{(x+1)^2 dx}}^2 \right| = 0$ . Combined to the fact that  $u \in C_w([0, T]; L^2_{(x+1)^2 dx})$  this yields  $u \in C([0, T], L^2_{(x+1)^2 dx})$ .

STEP 2. UNIQUENESS

Here,  $C$  will denote a universal constant which may vary from line to line. Pick  $u_0 \in L^2_{(x+1)^2 dx}$ , and let  $u, v \in C([0, T]; L^2_{(x+1)^2 dx}) \cap L^2(0, T; H^1_{(x+1)^2 dx})$  be two mild solutions of (1). Pick two sequences  $\{u_{n,0}\}, \{v_{n,0}\}$  in  $L^2_b$  for some  $b > 0$  such that

$$(28) \quad u_{n,0} \rightarrow u_0 \text{ strongly in } L^2_{(x+1)^2 dx},$$

$$(29) \quad u_n \rightarrow u \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}),$$

$$(30) \quad u_n \rightarrow u \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx})$$

and also

$$(31) \quad v_{n,0} \rightarrow u_0 \text{ strongly in } L^2_{(x+1)^2 dx},$$

$$(32) \quad v_n \rightarrow v \text{ weakly } * \text{ in } L^\infty(0, T; L^2_{(x+1)^2 dx}),$$

$$(33) \quad v_n \rightarrow v \text{ weakly in } L^2(0, T; H^1_{(x+1)^2 dx}).$$

We shall prove that  $w = u - v$  vanishes on  $\mathbb{R}^+ \times [0, T]$  by providing some estimate for  $w_n = u_n - v_n$ . Note first that  $w_n$  solves the system

$$(34) \quad w_{n,t} + w_{n,x} + w_{n,xxx} + aw_n = f_n = v_n v_{n,x} - u_n u_{n,x},$$

$$(35) \quad w_n(0, t) = 0,$$

$$(36) \quad w_n(x, 0) = w_{n,0}(x) = u_{n,0}(x) - v_{n,0}(x).$$

Scaling in (34) by  $(x+1)w_n$  yields

$$\begin{aligned} & \frac{1}{2} \int_0^\infty (x+1)|w_n(x, t)|^2 dx + \frac{3}{2} \int_0^t \int_0^\infty |w_{n,x}|^2 dx d\tau - \frac{1}{2} \int_0^t \int_0^\infty |w_n|^2 dx d\tau \\ & \leq \frac{1}{2} \int_0^\infty (x+1)|w_{n,0}|^2 dx + \int_0^t \left( \int_0^\infty (x+1)|w_n|^2 dx \right)^{\frac{1}{2}} \left( \int_0^\infty (x+1)|f_n|^2 dx \right)^{\frac{1}{2}} d\tau \\ & \leq \frac{1}{2} \int_0^\infty (x+1)|w_{n,0}|^2 dx + \frac{1}{4} \sup_{0 < \tau < t} \int_0^\infty (x+1)|w_n(x, \tau)|^2 dx \\ & \quad + \left[ \int_0^T \left( \int_0^\infty (x+1)|f_n|^2 dx \right)^{\frac{1}{2}} d\tau \right]^2. \end{aligned}$$

Since  $\|w_n(t)\|_{L^2(\mathbb{R}^+)} \leq \|w_n(t)\|_{L^2_{(x+1)dx}}$ , this yields for  $T < 1/10$

$$(37) \quad \begin{aligned} & \sup_{0 < t < T} \int_0^\infty (x+1)|w_n(x, t)|^2 dx + \int_0^T \int_0^\infty |w_{n,x}|^2 dx dt \\ & \leq C \left[ \int_0^\infty (x+1)|w_{n,0}(x)|^2 dx + \left( \int_0^T \left( \int_0^\infty (x+1)|f_n|^2 dx \right)^{\frac{1}{2}} d\tau \right)^2 \right]. \end{aligned}$$

It remains to estimate  $\int_0^T \left( \int_0^\infty (x+1)|f_n|^2 dx \right)^{\frac{1}{2}} dt$ . We split  $f_n$  into

$$f_n = (v_n - u_n)v_{n,x} + u_n(v_{n,x} - u_{n,x}) = f_n^1 + f_n^2.$$

We have that

$$\begin{aligned} \int_0^T \left( \int_0^\infty (x+1)|f_n^1|^2 dx \right)^{\frac{1}{2}} dt &= \int_0^T \left( \int_0^\infty (x+1)|w_n|^2 |v_{n,x}|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \int_0^T \|w_n\|_{L^\infty(\mathbb{R}^+)} \left( \int_0^\infty (x+1)|v_{n,x}|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \left( \int_0^T \|w_n\|_{L^\infty(\mathbb{R}^+)}^2 dt \right)^{\frac{1}{2}} \left( \int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

By Sobolev embedding, we have that

$$\begin{aligned} \left( \int_0^T \|w_n\|_{L^\infty(\mathbb{R}^+)}^2 dt \right)^{\frac{1}{2}} &\leq \left( \int_0^T \|w_n\|_{H^1(\mathbb{R}^+)}^2 dt \right)^{\frac{1}{2}} \\ &\leq \sqrt{T} \sup_{0 < t < T} \|w_n\|_{L^2(\mathbb{R}^+)} + \|w_{n,x}\|_{L^2(0, T; L^2(\mathbb{R}^+))}. \end{aligned}$$

Thus

$$(38) \quad \int_0^T \left( \int_0^\infty (x+1)|f_n^1|^2 dx \right)^{\frac{1}{2}} dt \leq \|v_{n,x}\|_{L^2(0,T;L^2_{(x+1)dx})} \left( \sqrt{T} \sup_{0 < t < T} \|w_n\|_{L^2(\mathbb{R}^+)} \right) + \|w_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))}$$

On the other hand, we have that

$$(39) \quad \begin{aligned} & \int_0^T \left( \int_0^\infty (x+1)|f_n^2|^2 dx \right)^{\frac{1}{2}} dt \\ &= \int_0^T \left( \int_0^\infty (x+1)|u_n|^2 |w_{n,x}|^2 dx \right)^{\frac{1}{2}} dt \\ &\leq \int_0^T \|(x+1)^{\frac{1}{2}} u_n\|_{L^\infty(\mathbb{R}^+)} \|w_{n,x}\|_{L^2(\mathbb{R}^+)} dt \\ &\leq C \int_0^T \left( \|(x+1)^{\frac{1}{2}} u_n\|_{L^2(\mathbb{R}^+)} + \|(x+1)^{\frac{1}{2}} u_{n,x}\|_{L^2(\mathbb{R}^+)} \right) \|w_{n,x}\|_{L^2(\mathbb{R}^+)} dt \\ &\leq C \left( \sqrt{T} \|(x+1)u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^+))} \right. \\ &\quad \left. + \|(x+1)^{\frac{1}{2}} u_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))} \right) \|w_{n,x}\|_{L^2(0,T;L^2(\mathbb{R}^+))}. \end{aligned}$$

Gathering together (37), (38) and (39), we conclude that for  $T < 1/10$

$$h_n(T) \leq K_n(T)h_n(T) + C\|w_{n,0}\|_{L^2_{(x+1)dx}}^2$$

where

$$(40) \quad h_n(t) := \sup_{0 < \tau < T} \int_0^\infty (x+1)|w_n(x,\tau)|^2 dx + \int_0^T \int_0^\infty |w_{n,x}|^2 dx dt$$

$$(41) \quad \begin{aligned} K_n(T) &\leq C \left( \int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt + T\|(x+1)u_n\|_{L^\infty(0,T;L^2(\mathbb{R}^+))}^2 \right. \\ &\quad \left. + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \right) \end{aligned}$$

and  $C$  denotes a universal constant. The following claim is needed.

CLAIM 3.

$$\lim_{T \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt = 0, \quad \lim_{T \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty (x+1)|v_{n,x}|^2 dx dt = 0.$$

Clearly, it is sufficient to prove the claim for the sequence  $\{u_n\}$  only. From (27) applied with  $u = u_n$  on  $[0, T]$ , we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^\infty (x+1)^2 |u_n(x,T)|^2 dx + 3 \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \\ & \leq \frac{1}{2} \int_0^\infty (x+1)^2 |u_{n,0}|^2 dx + \int_0^T \int_0^\infty (x+1)|u_n|^2 dx dt + \frac{2}{3} \int_0^T \int_0^\infty (x+1)|u_n|^3 dx dt. \end{aligned}$$

Combined to (23)-(24), this gives

$$(42) \quad \begin{aligned} & \|u_n(T)\|_{L^2_{(x+1)^2 dx}}^2 + \int_0^T \int_0^\infty (x+1)|u_{n,x}|^2 dx dt \\ & \leq \|u_{n,0}\|_{L^2_{(x+1)^2 dx}}^2 + C \int_0^T \|u_n\|_{L^2_{(x+1)^2 dx}}^2 dt. \end{aligned}$$

It follows from Gronwall lemma that

$$(43) \quad \|u_n(t)\|_{L^2_{(x+1)^2 dx}}^2 \leq \|u_{n,0}\|_{L^2_{(x+1)^2 dx}}^2 e^{Ct}$$

Using (43) in (42) and taking the limit sup as  $n \rightarrow \infty$  gives for a.e.  $T$

$$\|u(T)\|_{L^2_{(x+1)^2 dx}}^2 + \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dx dt \leq e^{CT} \|u_0\|_{L^2_{(x+1)^2 dx}}^2$$

As  $u$  is continuous from  $\mathbb{R}^+$  to  $L^2_{(x+1)^2 dx}$ , we infer that

$$\lim_{T \rightarrow 0} \limsup_{n \rightarrow \infty} \int_0^T \int_0^\infty |u_{n,x}|^2 dx dt = 0.$$

The claim is proved. Therefore, we have that for  $T > 0$  small enough and  $n$  large enough,  $K_n(T) < \frac{1}{2}$ , and hence

$$h_n(T) \leq 2C \|w_n(0)\|_{L^2_{(x+1) dx}}^2.$$

This yields

$$\|u - v\|_{L^\infty(0,T;L^2_{(x+1) dx})}^2 \leq \liminf_{n \rightarrow \infty} h_n(T) \leq 2C \liminf_{n \rightarrow \infty} \|w_n(0)\|_{L^2_{(x+1) dx}}^2 = 0$$

and  $u = v$  for  $0 < t < T$ . This proves the uniqueness for  $T$  small enough. The general case follows by a classical argument.  $\blacksquare$

**Remark 2.6** 1. If we assume only that  $u_0 \in L^2_{(x+1) dx}$ , then a proof similar to Step 1 gives the existence of a mild solution  $u \in C([0, T]; L^2_{(x+1) dx}) \cap L^2(0, T; H^1_{(x+1) dx})$  of (1). The uniqueness of such a solution is open. The existence and uniqueness of a solution issuing from  $u_0 \in L^2_{(x+1) dx}$  in a class of functions involving a Bourgain norm has been given in [13].

2. If  $u_0 \in L^2_{(x+1)^m dx}$  with  $m \geq 3$ , then  $u \in C([0, T]; L^2_{(x+1)^m dx}) \cap L^2(0, T; H^1_{(x+1)^m dx})$  for all  $T > 0$  (see below Theorem 3.1).

### 3 Asymptotic Behavior

#### 3.1 Decay in $L^2_{(x+1)^m dx}$

**Theorem 3.1** Assume that the function  $a = a(x)$  satisfies (4). Then, for all  $R > 0$  and  $m \geq 1$ , there exist numbers  $C > 0$  and  $\nu > 0$  such that

$$\|u(t)\|_{L^2_{(x+1)^m dx}} \leq C e^{-\nu t} \|u_0\|_{L^2_{(x+1)^m dx}}$$

for any solution given by Theorem 2.5, whenever  $\|u_0\|_{L^2_{(x+1)^m dx}} \leq R$ .

**Proof.** The proof will be done by induction in  $m$ . We set

$$(44) \quad V_0(u) = E(u) = \frac{1}{2} \int_0^\infty u^2 dx$$

and define the Lyapunov function  $V_m$  for  $m \geq 1$  in an inductive way

$$(45) \quad V_m(u) = \frac{1}{2} \int_0^\infty (x+1)^m u^2 dx + d_{m-1} V_{m-1}(u),$$

where  $d_{m-1} > 0$  is chosen sufficiently large (see below).

Suppose first that  $m = 1$  and put  $V = V_1$ . Multiplying the first equation in (1) by  $u$  and integrating by parts over  $\mathbb{R}^+ \times (0, T)$ , we obtain

$$(46) \quad \frac{1}{2} \int_0^\infty |u(x, T)|^2 dx = \frac{1}{2} \int_0^\infty |u_0(x)|^2 dx - \int_0^T \int_0^\infty a(x)|u|^2 dx dt - \frac{1}{2} \int_0^T u_x^2(0, t) dt.$$

Now, multiplying the equation by  $xu$ , we deduce that

$$(47) \quad \begin{aligned} & \frac{1}{2} \int_0^\infty x|u(x, T)|^2 dx - \frac{1}{2} \int_0^\infty x|u_0(x)|^2 dx + \frac{3}{2} \int_0^T \int_0^\infty u_x^2 dx dt \\ & - \frac{1}{2} \int_0^T \int_0^\infty u^2 dx dt - \frac{1}{3} \int_0^T \int_0^\infty u^3 dx dt + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt = 0. \end{aligned}$$

Combining (46) and (47) it follows that

$$(48) \quad \begin{aligned} & V(u) - V(u_0) + (d_0 + 1) \left( \frac{1}{2} \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x)|u|^2 dx dt \right) \\ & + \frac{3}{2} \int_0^T \int_0^\infty u_x^2 dx dt - \frac{1}{2} \int_0^T \int_0^\infty u^2 dx dt - \frac{1}{3} \int_0^T \int_0^\infty u^3 dx dt \\ & + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt = 0. \end{aligned}$$

The next step is devoted to estimate the nonlinear term in the left hand side of (48). To do that, we first assume that  $\|u_0\|_{L^2} \leq 1$ .

By (26) we have that

$$\int_0^\infty |u|^3 dx \leq \varepsilon \|u_x\|_{L^2}^2 + c_\varepsilon \|u\|_{L^2}^{\frac{10}{3}}$$

for any  $\varepsilon > 0$  and some constant  $c_\varepsilon > 0$ . Thus, if  $\|u_0\|_{L^2} \leq 1$ , we have  $\|u\|_{L^2}^{\frac{10}{3}} \leq \|u\|_{L^2}^2$  and

$$(49) \quad \int_0^T \int_0^\infty |u|^3 dx dt \leq \varepsilon \int_0^T \int_0^\infty u_x^2 dx dt + c_\varepsilon \int_0^T \int_0^\infty u^2 dx dt.$$

Moreover, according to [20], there exists  $c_1 > 0$ , satisfying

$$(50) \quad \int_0^T \int_0^\infty u^2 dx dt \leq c_1 \left\{ \frac{1}{2} \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x)u^2 dx dt \right\}.$$

Now, combining (48)-(50) and taking  $\varepsilon < \frac{1}{2}$  and  $d_0 := 2c_1(\frac{1}{2} + \frac{c_\varepsilon}{3})$  we obtain

$$(51) \quad \begin{aligned} & V(u(T)) - V(u_0) + \frac{d_0 + 1}{2} \left( \frac{1}{2} \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x)|u|^2 dx dt \right) \\ & + \left( \frac{3}{2} - \frac{\varepsilon}{3} \right) \int_0^T \int_0^\infty u_x^2 dx dt + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt \leq 0 \end{aligned}$$

or

$$(52) \quad V(u(T)) - V(u_0) \leq -\tilde{c} \left\{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty (x+1)a(x)|u|^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt \right\}$$

where  $\tilde{c} > 0$ . We aim to prove the existence of a constant  $c > 0$  satisfying

$$(53) \quad V(u(T)) - V(u_0) \leq -cV(u_0)$$

Indeed, such an inequality gives at once the decay  $V(u(t)) \leq ce^{-\nu t}V(u_0)$ . To this end, we need to establish two claims.

CLAIM 4. There exists  $c > 0$  such that

$$\int_0^T V(u)dt \leq c \left\{ \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dxdt \right\}.$$

Since  $u_0 \in L^2_{(x+1)dx} \subset L^2$ , from (4) and (50) we get

$$\begin{aligned} \int_0^T V(u)dt &= \frac{1}{2} \int_0^T \int_0^\infty (x+1)u^2 dxdt + \frac{d_0}{2} \int_0^T \int_0^\infty u^2 dxdt \\ &\leq \frac{c_1 d_0}{2} \left\{ \frac{1}{2} \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt \right\} \\ &\quad + \frac{1}{2} \int_0^T \int_0^{x_0} (x+1)u^2 dxdt + \frac{1}{2} \int_0^T \int_{x_0}^\infty (x+1)u^2 dxdt \\ &\leq \frac{c_1 d_0}{2} \left\{ \frac{1}{2} \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty a(x)u^2 dxdt \right\} \\ &\quad + \frac{1}{2}(x_0+1) \int_0^T \int_0^{x_0} u^2 dxdt + \frac{1}{2} \int_0^T \int_{x_0}^\infty (x+1) \frac{a(x)}{a_0} u^2 dxdt \\ &\leq c \left\{ \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dxdt \right\}. \end{aligned}$$

CLAIM 5.

$$(54) \quad V(u_0) \leq C \left( \int_0^T u_x^2(0, t)dt + \int_0^T \int_0^\infty (x+1)a(x)u^2 dxdt + \int_0^T \int_0^\infty u_x^2 dxdt \right)$$

where  $C > 0$ .

Multiplying the first equation in (1) by  $(T-t)u$  and integrating by parts in  $(0, \infty) \times (0, T)$ , we obtain

$$(55) \quad \begin{aligned} \frac{T}{2} \int_0^\infty |u_0(x)|^2 dx = \\ \frac{1}{2} \int_0^T \int_0^\infty |u|^2 dxdt + \int_0^T \int_0^\infty (T-t)a(x)|u|^2 dxdt + \frac{1}{2} \int_0^T (T-t)u_x^2(0, t)dt, \end{aligned}$$

and therefore, using (50)

$$(56) \quad \int_0^\infty |u_0(x)|^2 dx \leq C \left( \int_0^T \int_0^\infty a(x)|u|^2 dxdt + \int_0^T u_x^2(0, t)dt \right).$$

Now, multiplying by  $(T-t)u$ , it follows that

$$\begin{aligned} -\frac{T}{2} \int_0^\infty x|u_0(x)|^2 dx + \frac{1}{2} \int_0^T \int_0^\infty x|u|^2 dxdt + \frac{3}{2} \int_0^T \int_0^\infty (T-t)u_x^2 dxdt \\ -\frac{1}{2} \int_0^T \int_0^\infty (T-t)u^2 dxdt + \int_0^T \int_0^\infty (T-t)xa(x)|u|^2 dxdt - \\ -\frac{1}{3} \int_0^T \int_0^\infty (T-t)u^3 dxdt = 0. \end{aligned}$$

The identity above and (49) allow us to conclude that

$$(57) \quad \begin{aligned} & \int_0^\infty x|u_0(x)|^2 dx \\ & \leq C \left\{ \int_0^T \int_0^\infty (x+1)|u|^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt + \int_0^T \int_0^\infty xa(x)|u|^2 dx dt + \right. \\ & \quad \left. + \int_0^T \int_0^\infty |u|^3 dx dt \right\} \leq C \left\{ \int_0^T V(u(t)) dt + \int_0^T \int_0^\infty xa(x)u^2 dx dt + \int_0^T \int_0^\infty u_x^2 dx dt \right\} \end{aligned}$$

for some  $C > 0$ . Claim 5 follows from Claim 4 and (56)-(57).  $\blacksquare$

The previous computations give us (53) (and the exponential decay) when  $\|u_0\|_{L^2} \leq 1$ . The general case is proved as follows. Let  $u_0 \in L^2_{(x+1)dx} \subset L^2$  be such that  $\|u_0\|_{L^2} \leq R$ . Since  $u \in C(\mathbb{R}^+; L^2(\mathbb{R}^+))$  and  $\|u(t)\|_{L^2} \leq \alpha e^{-\beta t} \|u_0\|_{L^2}$ , where  $\alpha = \alpha(R)$  and  $\beta = \beta(R)$  are positive constants,  $\|u(T)\|_{L^2} \leq 1$  if we pick  $T$  satisfying  $\alpha e^{-\beta T} R < 1$ . Then, it follows from (48)-(26) and (53) that for some constants  $\nu > 0$ ,  $c > 0$ ,  $C > 0$

$$V(u(t+T)) \leq ce^{-\nu t} V(u(T)) \leq c(T\|u_0\|_{L^2}^2 + T\|u_0\|_{L^2}^{\frac{10}{3}} + V(u_0))e^{-\nu t},$$

hence

$$V(u(t)) \leq Ce^{-\nu t} V(u_0),$$

where  $C = C(R)$ , which concludes the proof when  $m = 1$ .

**Induction Hypothesis:** There exist  $c > 0$  and  $\rho > 0$  such that if  $V_{m-1}(u_0) \leq \rho$ , we have

$$\begin{aligned} & V_m(u) - V_m(u_0) && (*)_m \\ & \leq -c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt \right\} \\ & V_m(u_0) && (**)_m \\ & \leq c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt \right\}. \end{aligned}$$

By (52)-(54), the induction hypothesis is true for  $m = 1$ . Pick now an index  $m \geq 2$  and assume that  $d_0, \dots, d_{m-2}$  have been constructed so that  $(*)_k - (**)_k$  are fulfilled for  $1 \leq k \leq m-1$ . We aim to prove that for a convenient choice of the constant  $d_{m-1}$  in (45), the properties  $(*)_m - (**)_m$  hold true.

Let us investigate first  $(*)_m$ . We multiply the first equation in (1) by  $(x+1)^m u$  to obtain

$$(58) \quad \begin{aligned} & V_m(u) - V_m(u_0) - d_{m-1}(V_{m-1}(u) - V_{m-1}(u_0)) \\ & - \frac{m(m-1)(m-2)}{2} \int_0^T \int_0^\infty (x+1)^{m-3} u^2 dx dt + \frac{1}{2} \int_0^T u_x^2(0,t) dt \\ & + \frac{3m}{2} \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt - \frac{m}{2} \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt \\ & - \frac{m}{3} \int_0^T \int_0^\infty (x+1)^{m-1} u^3 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt = 0. \end{aligned}$$

The next steps are devoted to estimate the terms in the above identity. First, combining (4) and (50) we infer the existence of a positive constant  $c > 0$  such that

$$\begin{aligned}
& \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt \\
&= \int_0^T \int_0^{x_0} (x+1)^{m-1} u^2 dx dt + \int_0^T \int_{x_0}^\infty (x+1)^{m-1} u^2 dx dt \\
(59) \quad &\leq (x_0+1)^{m-1} \int_0^T \int_0^\infty u^2 dx dt + \frac{1}{a_0} \int_0^T \int_0^\infty a(x)(x+1)^{m-1} u^2 dx dt \\
&\leq c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} a(x) u^2 dx dt \right\} \\
&\leq -c \{V_{m-1}(u) - V_{m-1}(u_0)\}
\end{aligned}$$

where we used  $(*)_{m-1}$ . In the same way

$$\begin{aligned}
(60) \quad & \int_0^T \int_0^\infty (x+1)^{m-3} u^2 dx dt \\
&\leq \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt \leq -c \{V_{m-1}(u) - V_{m-1}(u_0)\}
\end{aligned}$$

where  $c > 0$  is a positive constant. Moreover, assuming  $V_{m-1}(u_0) \leq \rho$  with  $\rho > 0$  small enough (so that by exponential decay of  $V_{m-1}(u(t))$  we have  $\int_0^\infty (x+1)^{m-1} |u(x,t)|^2 dx \leq 1$  for all  $t \geq 0$ ) and proceeding as in the case  $m = 1$ , we obtain the existence of  $\varepsilon > 0$  and  $c_\varepsilon > 0$  satisfying

$$\begin{aligned}
(61) \quad & \int_0^T \int_0^\infty (x+1)^{m-1} |u|^3 dx dt \\
&\leq \varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + c_\varepsilon \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt.
\end{aligned}$$

Indeed,

$$\begin{aligned}
(62) \quad & \int_0^\infty (x+1)^{m-1} |u|^3 dx \\
&\leq \|u\|_{L^\infty} \int_0^\infty (x+1)^{m-1} u^2 dx \leq \sqrt{2} \|u_x\|_{L^2}^{\frac{1}{2}} \|u\|_{L^2}^{\frac{1}{2}} \int_0^\infty (x+1)^{m-1} u^2 dx \\
&\leq \varepsilon \int_0^\infty (x+1)^{m-1} u_x^2 dx + c_\varepsilon \int_0^\infty u^2 dx + c_\varepsilon \left( \int_0^\infty (x+1)^{m-1} u^2 dx \right)^2.
\end{aligned}$$

Then, if we return to (58) and take  $\varepsilon < 9/2$  and  $d_{m-1} > 0$  large enough, from (59)-(61) it follows that

$$\begin{aligned}
(63) \quad & V_m(u) - V_m(u_0) \\
&\leq -c \left\{ \int_0^T u_x^2(0,t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty a(x)(x+1)^m u^2 dx dt \right\} \\
&\quad + \frac{d_{m-1}}{2} (V_{m-1}(u) - V_{m-1}(u_0)).
\end{aligned}$$

This yields  $(*)_m$ , by  $(*)_{m-1}$ . Let us now check  $(**)_m$ . It remains to estimate the terms in the right hand side of (63). We multiply the first equation in (1) by  $(T-t)(x+1)^m u$  to obtain

$$\begin{aligned} \frac{T}{2} \int_0^\infty (x+1)^m u_0^2 dx &= \frac{1}{2} \int_0^T \int_0^\infty (x+1)^m u^2 dx dt \\ &- \frac{m(m-1)(m-2)}{2} \int_0^T \int_0^\infty (T-t)(x+1)^{m-3} u^2 dx dt + \frac{1}{2} \int_0^T (T-t) u_x^2(0, t) dt \\ &+ \frac{3m}{2} \int_0^T \int_0^\infty (T-t)(x+1)^{m-1} u_x^2 dx dt - \frac{m}{2} \int_0^T \int_0^\infty (T-t)(x+1)^{m-1} u^2 dx dt \\ &- \frac{m}{3} \int_0^T \int_0^\infty (T-t)(x+1)^{m-1} u^3 dx dt + \int_0^T \int_0^\infty (T-t)(x+1)^m a(x) u^2 dx dt. \end{aligned}$$

Then, proceeding as above, we deduce that

$$\begin{aligned} &\int_0^T (x+1)^m u_0^2 dx \\ &\leq c \left\{ \int_0^T \int_0^\infty (x+1)^{m-1} u^2 dx dt + \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt \right. \\ &\quad \left. + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt \right\} \\ &\leq c \left\{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt + \int_0^T \int_0^\infty (x+1)^m a(x) u^2 dx dt \right\}. \end{aligned}$$

Combined to  $(**)_m$ , this yields  $(**)_m$ . This completes the construction of the sequence  $\{V_m\}_{m \geq 1}$  by induction.

Let us now check the exponential decay of  $V_m$  for  $m \geq 2$ . It follows from  $(*)_m - (**)_m$  that

$$V_m(u) - V_m(u_0) \leq -c V_m(u_0)$$

where  $c > 0$ , which completes the proof when  $V_{m-1}(u_0) \leq \rho$ . The global result ( $V_{m-1}(u_0) \leq R$ ) is obtained as above for  $m = 1$ .  $\blacksquare$

**Corollary 3.2** *Let  $a = a(x)$  fulfilling (4) and  $a \in W^{2,\infty}(0, \infty)$ . Then for any  $R > 0$ , there exist positive constants  $c = c(R)$  and  $\mu = \mu(R)$  such that*

$$(64) \quad \|u_x(t)\|_{L^2(\mathbb{R}^+)} \leq c \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)dx}}$$

for all  $t > 0$  and all  $u_0 \in L^2_{(x+1)dx}$  satisfying  $\|u_0\|_{L^2_{(x+1)dx}} \leq R$ .

**Proof.** Pick any  $R > 0$  and any  $u_0 \in L^2_{(x+1)dx}$  with  $\|u_0\|_{L^2_{(x+1)dx}} \leq R$ . By Theorem 3.1 there are some constants  $C = C(R)$  and  $\nu = \nu(R)$  such that

$$(65) \quad \|u(t)\|_{L^2_{(x+1)dx}} \leq C e^{-\nu t} \|u_0\|_{L^2_{(x+1)dx}}.$$

Using the multiplier  $t(u^2 + 2u_{xx})$  we obtain after some integrations by parts that for all  $0 < t_1 < t_2$

$$\begin{aligned} &t_2 \int_0^\infty u_x^2(x, t_2) dx + \int_{t_1}^{t_2} t u_x^2(0, t) dt + 2 \int_{t_1}^{t_2} \int_0^\infty t a(x) u_x^2 dx dt + \int_{t_1}^{t_2} t u_{xx}^2(0, t) dt \\ &= -\frac{1}{3} \int_{t_1}^{t_2} \int_0^\infty u^3 dx dt + \frac{t_2}{3} \int_0^\infty u^3(x, t_2) dx + \int_{t_1}^{t_2} \int_0^\infty t u^3 a(x) dx dt \\ (66) \quad &+ \int_{t_1}^{t_2} \int_0^\infty u_x^2 dx dt + \int_{t_1}^{t_2} \int_0^\infty t a''(x) u^2 dx dt. \end{aligned}$$

1. Let us assume first that  $T > 1$ . Applying (66) on the time interval  $[T - 1, T]$ , we infer that

$$(67) \quad \int_0^\infty |u_x(x, T)|^2 dx \leq c \left( \int_{T-1}^T \int_0^\infty |u|^3 dx dt + \|u(T)\|_{L^3(\mathbb{R}^+)}^3 + \int_{T-1}^T \|u\|_{H^1(\mathbb{R}^+)}^2 dt \right).$$

To estimate the cubic terms in (67), we use (26) to obtain

$$(68) \quad \int_0^\infty |u_x(x, T)|^2 dx \leq \varepsilon \int_0^\infty |u_x(x, T)|^2 dx + c_\varepsilon \left( \|u(T)\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}} + \int_{T-1}^T (\|u\|_{H^1(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt \right).$$

Note that by (65)

$$\|u(T)\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}} \leq (C e^{-\nu T} \|u_0\|_{L^2_{(x+1)dx}})^{\frac{10}{3}} \leq C^{\frac{10}{3}} R^{\frac{4}{3}} e^{-\nu T} \|u_0\|_{L^2_{(x+1)dx}}^2.$$

It follows from (48), (26), and (65) that

$$(69) \quad \begin{aligned} & \int_{T-1}^T (\|u\|_{H^1(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt \\ & \leq C \left( V_1(u(T-1)) + \int_{T-1}^T (\|u\|_{L^2(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt \right) \\ & \leq C e^{-\nu T} \|u_0\|_{L^2_{(x+1)dx}}^2 \end{aligned}$$

where  $C = C(R, \nu)$ . (64) for  $T \geq 1$  follows from (68) and (69) by choosing  $\varepsilon < 1$  and  $\mu < \nu$ .

2. Assume now that  $T \leq 1$ . Estimating again the cubic terms in (66) (with  $[t_1, t_2] = [0, T]$ ) by using (26), we obtain

$$(70) \quad \begin{aligned} T \int_0^\infty u_x^2(x, T) dx & \leq \frac{T}{3} \left( \varepsilon \|u_x(T)\|_{L^2(\mathbb{R}^+)}^2 + C_\varepsilon \|u(T)\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}} \right) \\ & + C_\varepsilon \int_0^T (\|u\|_{H^1(\mathbb{R}^+)}^2 + \|u\|_{L^2(\mathbb{R}^+)}^{\frac{10}{3}}) dt. \end{aligned}$$

By (48), (26) and (65), we have that

$$(71) \quad \int_0^1 \int_0^\infty |u_x|^2 dx dt \leq C(R) \|u_0\|_{L^2_{(x+1)dx}}^2$$

which, combined to (70) with  $\varepsilon = 1$  and (65), gives

$$\|u_x(T)\|_{L^2(\mathbb{R}^+)}^2 \leq C(R) T^{-1} \|u_0\|_{L^2_{(x+1)dx}}^2$$

for all  $T < 1$ . This gives (64) for  $T < 1$ . ■

Corollary 3.2 may be extended (locally) to the weighted space  $L^2_{(x+1)^m dx}$  ( $m \geq 2$ ) in following the method of proof of [24, Theorem 1.1].

**Corollary 3.3** *Let  $a = a(x)$  fulfilling (4) and  $m \geq 2$ . Then there exist some constants  $\rho > 0$ ,  $C > 0$  and  $\mu > 0$  such that*

$$\|u(t)\|_{H^1_{(x+1)^m dx}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^m dx}}$$

for all  $t > 0$  and all  $u_0 \in L^2_{(x+1)^m dx}$  satisfying  $\|u_0\|_{L^2_{(x+1)^m dx}} \leq \rho$ .

**Proof.** We first prove estimates for the linearized problem

$$\begin{aligned} (72) \quad & u_t + u_x + u_{xxx} + au = 0 \\ (73) \quad & u(0, t) = 0 \\ (74) \quad & u(x, 0) = u_0(x) \end{aligned}$$

and next apply a perturbation argument to extend them to the nonlinear problem (1). Let us denote by  $W(t)u_0 = u(t)$  the solution of (72)-(74). By computations similar to those performed in the proof of Theorem 3.1, we have that

$$\|W(t)u_0\|_{L^2_{(x+1)^{m dx}}} \leq C_0 e^{-\nu t} \|u_0\|_{L^2_{(x+1)^{m dx}}}.$$

We need the

CLAIM 6. Let  $k \in \{0, \dots, 3\}$ . Then there exists a constant  $C_k > 0$  such that for any  $u_0 \in H^k_{(x+1)^{m dx}}$ ,

$$(75) \quad \|W(t)u_0\|_{H^k_{(x+1)^{m dx}}} \leq C_k e^{-\nu t} \|u_0\|_{H^k_{(x+1)^{m dx}}}.$$

Indeed, if  $u_0 \in H^3_{(x+1)^{m dx}}$ , then  $u_t(\cdot, 0) \in L^2_{(x+1)^{m-3 dx}}$ , and since  $v = u_t$  solves (72)-(73), we also have that

$$\|u_t(\cdot, t)\|_{L^2_{(x+1)^{m-3 dx}}} \leq C_0 e^{-\nu t} \|u_t(\cdot, 0)\|_{L^2_{(x+1)^{m-3 dx}}}.$$

Using (72), this gives

$$\|W(t)u_0\|_{H^3_{(x+1)^{m dx}}} \leq C_3 e^{-\nu t} \|u_0\|_{H^3_{(x+1)^{m dx}}}.$$

This proves (75) for  $k = 3$ . The fact that (75) is valid for  $k = 1, 2$  follows from a standard interpolation argument, for  $H^k_{(x+1)^{m dx}} = [H^0_{(x+1)^{m dx}}, H^3_{(x+1)^{m dx}}]_{\frac{k}{3}}$ .

**Lemma 3.4** *Pick any number  $\mu \in (0, \nu)$ . Then there exists some constant  $C = C(\mu) > 0$  such that for any  $u_0 \in L^2_{(x+1)^{m dx}}$*

$$(76) \quad \|W(t)u_0\|_{H^1_{(x+1)^{m dx}}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^{m dx}}}.$$

**Proof.** Let  $u_0 \in L^2_{(x+1)^{m dx}}$  and set  $u(t) = W(t)u_0$  for all  $t \geq 0$ . By scaling in (72) by  $(x+1)^m u$ , we see that for some constant  $C_K = C_K(T)$

$$\|u\|_{L^2(0,1; H^1_{(x+1)^{m dx}})} \leq C_K \|u_0\|_{L^2_{(x+1)^{m dx}}}.$$

This implies that  $u(t) \in H^1_{(x+1)^{m dx}}$  for a.e.  $t \in (0, 1)$  which, combined to (75), gives that  $u(t) \in H^1_{(x+1)^{m dx}}$  for all  $t > 0$ . Pick any  $T \in (0, 1]$ . Note that, by (75),

$$(77) \quad \|u(T)\|_{H^1_{(x+1)^{m dx}}} \leq C_1 e^{-\nu(T-t)} \|u(t)\|_{H^1_{(x+1)^{m dx}}}, \quad \forall t \in (0, T).$$

Integrating with respect to  $t$  in (77) yields

$$[C_1^{-1} \|u(T)\|_{H^1_{(x+1)^{m dx}}}]^2 \int_0^T e^{2\nu(T-t)} dt \leq \int_0^T \|u(t)\|_{H^1_{(x+1)^{m dx}}}^2 dt,$$

and hence

$$\begin{aligned} \|u(T)\|_{H^1_{(x+1)^{m dx}}} &\leq C_K C_1 \sqrt{\frac{2\nu}{e^{2\nu T} - 1}} \|u_0\|_{L^2_{(x+1)^{m dx}}} \\ &\leq \frac{C_K C_1}{\sqrt{T}} \|u_0\|_{L^2_{(x+1)^{m dx}}} \end{aligned}$$

for  $0 < T \leq 1$ . Therefore

$$(78) \quad \|u(t)\|_{H^1_{(x+1)^{m dx}}} \leq C_K C_1 e^\nu \frac{e^{-\nu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^{m dx}}} \quad \forall t \in (0, 1).$$

(76) follows from (78) and (75), since  $\mu < \nu$ . ■

Let us return to the proof of Corollary 3.3. Fix a number  $\mu \in (0, \nu)$ , where  $\nu$  is as in (75), and let us introduce the space

$$F = \{u \in C(\mathbb{R}^+; H^1_{(x+1)^{m dx}}); \|e^{\mu t} u(t)\|_{L^\infty(\mathbb{R}^+; H^1_{(x+1)^{m dx}})} < \infty\}$$

endowed with its natural norm. Note that (1) may be recast in the following integral form

$$(79) \quad u(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s)) ds$$

where  $N(u) = -uu_x$ . We first show that (79) has a solution in  $F$  provided that  $u_0 \in H^1_{(x+1)^{m dx}}$  with  $\|u_0\|_{H^1_{(x+1)^{m dx}}}$  small enough. Let  $u_0 \in H^1_{(x+1)^{m dx}}$  and  $u \in F$  with  $\|u_0\|_{H^1_{(x+1)^{m dx}}} \leq r_0$  and  $\|u\|_F \leq R$ ,  $r_0$  and  $R$  being chosen later. We introduce the map  $\Gamma$  defined by

$$(80) \quad (\Gamma u)(t) = W(t)u_0 + \int_0^t W(t-s)N(u(s)) ds \quad \forall t \geq 0.$$

We shall prove that  $\Gamma$  has a fixed point in the closed ball  $B_R(0) \subset F$  provided that  $r_0 > 0$  is small enough.

For the forcing problem

$$\begin{cases} u_t + u_x + u_{xxx} + au = f \\ u(0, t) = 0 \\ u(x, 0) = u_0(x) \end{cases}$$

we have the following estimate

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u(t)\|_{L^2_{(x+1)^{m dx}}}^2 + \int_0^T \int_0^\infty (x+1)^{m-1} u_x^2 dx dt \\ \leq C \left( \|u_0\|_{L^2_{(x+1)^{m dx}}}^2 + \|f\|_{L^1(0, T; L^2_{(x+1)^{m dx}})}^2 \right). \end{aligned}$$

Let us take  $f = N(u) = -uu_x$ . Observe that for all  $x > 0$

$$\begin{aligned} (x+1)u^2(x) &= \left| \int_0^\infty \frac{d}{dx} [(x+1)u^2(x)] dx \right| \\ &\leq C \left( \int_0^\infty (x+1)^m |u|^2 dx + \int_0^\infty (x+1)^{m-1} |u_x|^2 dx \right) \end{aligned}$$

whenever  $m \geq 2$ . It follows that for some constant  $K > 0$

$$\begin{aligned} \|uu_x\|_{L^2_{(x+1)^{m dx}}}^2 &\leq \|(x+1)u^2\|_{L^\infty(\mathbb{R}^+)} \int_0^\infty (x+1)^{m-1} |u_x|^2 dx \\ &\leq K \|u\|_{H^1_{(x+1)^{m dx}}}^4. \end{aligned}$$

Therefore, for any  $T > 0$ ,

$$\begin{aligned} \sup_{0 \leq t \leq T} \|(\Gamma u)(t)\|_{L^2_{(x+1)^{m dx}}}^2 + \int_0^T \int_0^\infty (x+1)^{m-1} |(\Gamma u)_x|^2 dx dt \\ \leq C \left( \|u_0\|_{L^2_{(x+1)^{m dx}}}^2 + \left( \int_0^T \|u(t)\|_{H^1_{(x+1)^{m dx}}}^2 dt \right)^2 \right) < \infty. \end{aligned}$$

Thus  $\Gamma u \in C(\mathbb{R}^+, L^2_{(x+1)^{m dx}}) \cap L^2_{loc}(\mathbb{R}^+; H^1_{(x+1)^{m dx}})$  with  $(\Gamma u)(0) = u_0$ . We claim that  $\Gamma u \in F$ . Indeed, by (75),

$$\|e^{\mu t} W(t)u_0\|_{H^1_{(x+1)^{m dx}}} \leq C_1 \|u_0\|_{H^1_{(x+1)^{m dx}}}$$

and for all  $t \geq 0$

$$\begin{aligned} \|e^{\mu t} \int_0^t W(t-s)N(u(s))ds\|_{H^1_{(x+1)^{m dx}}} &\leq C e^{\mu t} \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{t-s}} \|N(u(s))\|_{L^2_{(x+1)^{m dx}}} ds \\ &\leq C \int_0^t \frac{e^{\mu s}}{\sqrt{t-s}} K(e^{-\mu s} \|u\|_F)^2 ds \\ &\leq CK \|u\|_F^2 \int_0^t \frac{e^{-\mu(t-s)}}{\sqrt{s}} ds \\ &\leq CK(2 + \mu^{-1}) \|u\|_F^2 \end{aligned}$$

where we used Lemma 3.4. Pick  $R > 0$  such that  $CK(2 + \mu^{-1})R \leq \frac{1}{2}$ , and  $r_0$  such that  $C_1 r_0 = \frac{R}{2}$ . Then, for  $\|u_0\|_{H^1_{(x+1)^{m dx}}} \leq r_0$  and  $\|u\|_F \leq R$ , we obtain that

$$\|e^{\mu t} (\Gamma u)(t)\|_{H^1_{(x+1)^{m dx}}} \leq C_1 r_0 + CK(2 + \mu^{-1})R^2 \leq R, \quad t \geq 0.$$

Hence  $\Gamma$  maps the ball  $B_R(0) \subset F$  into itself. Similar computations show that  $\Gamma$  contracts. By the contraction mapping theorem,  $\Gamma$  has a unique fixed point  $u$  in  $B_R(0)$ . Thus  $\|u(t)\|_{H^1_{(x+1)^{m dx}}} \leq C e^{-\mu t} \|u_0\|_{H^1_{(x+1)^{m dx}}}$  provided that  $\|u_0\|_{H^1_{(x+1)^{m dx}}} \leq r_0$  with  $r_0$  small enough. Proceeding as in the proof of Lemma 3.4, we have that

$$\|u(t)\|_{H^1_{(x+1)^{m dx}}} \leq C \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L^2_{(x+1)^{m dx}}} \quad \text{for } 0 < t < 1,$$

provided that  $\|u_0\|_{L^2_{(x+1)^{m dx}}} \leq \rho_0$  with  $\rho_0 < 1$  small enough. The proof is complete with a decay rate  $\mu' < \mu$ .  $\blacksquare$

**Corollary 3.5** *Assume that  $a(x)$  satisfies (4) and that  $\partial_x^k a \in L^\infty(\mathbb{R}^+)$  for all  $k \geq 0$ . Pick any  $u_0 \in L^2_{(x+1)^{m dx}}$ . Then for all  $\varepsilon > 0$ , all  $T > \varepsilon$ , and all  $k \in \{1, \dots, m\}$ , there exists a constant  $C = C(\varepsilon, T, k) > 0$  such that*

$$(81) \quad \int_\varepsilon^\infty (x+1)^{m-k} |\partial_x^k u(x, t)|^2 dx \leq C \|u_0\|_{L^2_{(x+1)^{m dx}}}^2 \quad \forall t \in [\varepsilon, T].$$

**Proof.** The proof is very similar to the one in [18, Lemma 5.1] and so we only point out the small changes. First, it should be noticed that the presence in the KdV equation of the extra terms  $u_x$  and  $a(x)u$  does not cause any serious trouble. On the other hand, choosing a cut-off function in  $x$  of the form  $\eta(x) = \psi_0(x/\varepsilon)$  (instead of  $\eta(x) = \psi_0(x - x_0 + 2)$  as in [18]) where  $\psi_0 \in C^\infty(\mathbb{R}, [0, 1])$  satisfies  $\psi_0(x) = 0$  for  $x \leq 1/2$  and  $\psi_0(x) = 1$  for  $x \geq 1$ , allows to overcome the fact that  $u$  is a solution of (1) on the half-line only.  $\blacksquare$

### 3.2 Decay in $L^2_b$

This section is devoted to the exponential decay in  $L^2_b$ . Our result reads as follows:

**Theorem 3.6** Assume that the function  $a = a(x)$  satisfies (4) with  $4b^3 + b < a_0$ . Then, for all  $R > 0$ , there exist  $C > 0$  and  $\nu > 0$ , such that

$$\|u(t)\|_{L_b^2} \leq C e^{-\nu t} \|u_0\|_{L_b^2} \quad t \geq 0$$

for any solution  $u$  given by Theorem 2.3.

**Proof.** We introduce the Lyapunov function

$$(82) \quad V(u) = \frac{1}{2} \int_0^\infty u^2 e^{2bx} dx + c_b \int_0^\infty u^2 dx,$$

where  $c_b$  is a positive constant that will be chosen later. Then, adding (17) and (18) hand by hand we obtain

$$(83) \quad \begin{aligned} V(u) - V(u_0) &= (4b^3 + b) \int_0^T \int_{x_0}^\infty u^2 e^{2bx} dx dt + (4b^3 + b) \int_0^T \int_0^{x_0} u^2 e^{2bx} dx dt \\ &\quad - 3b \int_0^\infty \int_0^T u_x^2 e^{2bx} dx dt + \frac{2b}{3} \int_0^T \int_0^\infty u^3 e^{2bx} dx dt \\ &\quad - (c_b + \frac{1}{2}) \int_0^T u_x^2(0, t) dt - \int_0^T \int_0^\infty a(x) |u|^2 (e^{2bx} + 2c_b) dx dt, \end{aligned}$$

where  $x_0$  is the number introduced in (4). On the other hand, since  $L_b^2 \subset L_{(x+1)}^2$ ,  $\|u(t)\|_{L^2(0, \infty)}$  and  $\|u_x(t)\|_{L^2(0, \infty)}$  decays to zero exponentially. Consequently, from Moser estimate we deduce that  $\|u(t)\|_{L^\infty(0, \infty)} \rightarrow 0$ . We may assume that  $(2b/3)\|u(t)\|_{L^\infty} < \varepsilon = a_0 - (4b^3 + b)$  for all  $t \geq 0$ , by changing  $u_0$  into  $u(t_0)$  for  $t_0$  large enough. Therefore

$$(84) \quad \begin{aligned} &\frac{2b}{3} \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt \\ &\leq \frac{2b}{3} \int_0^T \|u(t)\|_{L^\infty(0, \infty)} \left( \int_0^\infty |u|^2 e^{2bx} dx \right) dt \leq \varepsilon \int_0^T \int_0^\infty u^2 e^{2bx} dx dt. \end{aligned}$$

So, returning to (83), the following holds

$$(85) \quad \begin{aligned} &V(u) - V(u_0) - (4b^3 + b + \varepsilon) \int_0^T \int_0^{x_0} u^2 e^{2bx} dx dt \\ &+ 3b \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt + (c_b + \frac{1}{2}) \int_0^T u_x^2(0, t) dt + 2c_b \int_0^T \int_0^\infty a(x) |u|^2 dx dt \leq 0. \end{aligned}$$

Moreover, according to [20] there exists  $C > 0$  satisfying

$$\begin{aligned} &\int_0^T \int_0^{x_0} u^2 e^{2bx} dx dt \\ &\leq e^{2bx_0} \int_0^T \int_0^{x_0} u^2 dx dt \leq C \left\{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x) u^2 dx dt \right\} \end{aligned}$$

since  $L_b^2 \subset L^2(\mathbb{R}^+)$ . Then, choosing  $c_b$  sufficiently large, the above estimate and (85) give us that

$$(86) \quad \begin{aligned} V(u) - V(u_0) &\leq -C \left\{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty a(x) u^2 dx dt \right. \\ &\quad \left. + \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt \right\} \leq -C V(u_0), \end{aligned}$$

which allows to conclude that  $V(u)$  decays exponentially. The last inequality is a consequence of the following results:

CLAIM 7. There exists a positive constant  $C > 0$ , such that

$$\int_0^T V(u(t))dt \leq C \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt.$$

First, observe that

$$\left| \int_0^\infty u^2 e^{2bx} dx \right| = \left| -\frac{1}{b} \int_0^\infty uu_x e^{2bx} dx \right| \leq \frac{1}{b} \left( \int_0^\infty u^2 e^{2bx} dx \right)^{\frac{1}{2}} \left( \int_0^\infty u_x^2 e^{2bx} dx \right)^{\frac{1}{2}},$$

therefore,

$$(87) \quad \int_0^\infty u^2 e^{2bx} dx \leq \frac{1}{b^2} \int_0^\infty u_x^2 e^{2bx} dx.$$

Then, from (4) and (87) we have

$$V(u(t)) \leq \left(\frac{1}{2} + c_b\right) \int_0^\infty u^2 e^{2bx} dx \leq \left(\frac{1}{2} + c_b\right) b^{-2} \int_0^\infty u_x^2 e^{2bx} dx$$

which gives us Claim 7.

CLAIM 8.

$$V(u_0) \leq C \left\{ \int_0^T u_x^2(0, t) dt + \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt + \int_0^T V(u(t)) dt \right\},$$

where  $C$  is a positive constant.

Multiplying the first equation in (1) by  $(T-t)ue^{2bx}$  and integrating by parts in  $(0, \infty) \times (0, T)$ , we obtain

$$(88) \quad \begin{aligned} & -\frac{T}{2} \int_0^\infty |u_0(x)|^2 e^{2bx} dx + \frac{1}{2} \int_0^T \int_0^\infty |u|^2 e^{2bx} dx dt + 3b \int_0^T \int_0^\infty (T-t) u_x^2 e^{2bx} dx dt \\ & + \frac{1}{2} \int_0^T (T-t) u_x^2(0, t) dt - (4b^3 + b) \int_0^T \int_0^\infty (T-t) u^2 e^{2bx} dx dt \\ & + \int_0^T \int_0^\infty (T-t) a(x) |u|^2 e^{2bx} dx dt - \frac{2b}{3} \int_0^T \int_0^\infty (T-t) u^3 e^{2bx} dx dt = 0 \end{aligned}$$

and therefore,

$$(89) \quad \begin{aligned} & \int_0^\infty |u_0(x)|^2 e^{2bx} dx \leq C \left( \int_0^T u_x^2(0, t) dt + \frac{1}{2} \int_0^T \int_0^\infty u^2 e^{2bx} dx dt \right. \\ & \left. + \int_0^T \int_0^\infty u_x^2 e^{2bx} dx dt + \int_0^T \int_0^\infty |u|^3 e^{2bx} dx dt \right). \end{aligned}$$

Then, combining (87) and (84), we derive Claim 8. (86) follows at once. This proves the exponential decay when  $\|u(t)\|_{L^\infty} \leq 3\varepsilon/(2b)$ . The general case is obtained as in Theorem 3.1  $\blacksquare$

**Corollary 3.7** *Assume that the function  $a = a(x)$  satisfies (4) with  $4b^3 + b < a_0$ . Then for any  $R > 0$ , there exist positive constants  $c = c(R)$  and  $\mu = \mu(R)$  such that*

$$(90) \quad \|u_x(t)\|_{L_b^2} \leq c \frac{e^{-\mu t}}{\sqrt{t}} \|u_0\|_{L_b^2}$$

for all  $t > 0$  and all  $u_0 \in L_b^2$  satisfying  $\|u_0\|_{L_b^2} \leq R$ .

**Corollary 3.8** Assume that the function  $a = a(x)$  satisfies (4) with  $4b^3 + b < a_0$ , and let  $s \geq 2$ . Then there exist some constants  $\rho > 0$ ,  $C > 0$  and  $\mu > 0$  such that

$$\|u(t)\|_{H_b^s} \leq C \frac{e^{-\mu t}}{t^{\frac{s}{2}}} \|u_0\|_{L_b^2}$$

for all  $t > 0$  and all  $u_0 \in L_b^2$  satisfying  $\|u_0\|_{L_b^2} \leq \rho$ .

The proof of Corollary 3.7 (resp. 3.8) is very similar to the proof of Corollary 3.2 (resp. 3.3), so it is omitted.

## Acknowledgments.

This work was achieved while the first author (AP) was visiting Université Paris-Sud with the support of the Cooperation Agreement Brazil-France and the second author (LR) was visiting IMPA and UFRJ. LR was partially supported by the “Agence Nationale de la Recherche” (ANR), Project CISIFS, Grant ANR-09-BLAN-0213-02.

## References

- [1] J. L. Bona and R. Winther, *The Korteweg-de Vries equation, posed in a quarter-plane*, SIAM J. Math. Anal. **14** (1983), 1056–1106.
- [2] J. L. Bona, W. G. Pritchard and L. R. Scott, *An evaluation of a model equation for water waves*, Philos. Trans. Royal Soc. London, Series A, **302** (1981), 457–510.
- [3] J. L. Bona and P. J. Bryant, *A mathematical model for long waves generated by wavemakers in non-linear dispersive systems*, Proc. Cambridge Philos. Soc., **73** (1973), 391–405.
- [4] J. L. Bona, S. M. Sun and B.-Y. Zhang, *A non-homogeneous boundary-value problem for the Korteweg-de Vries equation in a quarter plane*, Trans. American Math. Soc., **354** (2002), 427–490.
- [5] J. L. Bona, S. M. Sun and B.-Y. Zhang, *A forced oscillations of a damped Korteweg-de Vries equation in a quarter plane*, Comm. Cont. Math. **5** (2003), 369–400.
- [6] J. L. Bona, S. M. Sun and B.-Y. Zhang, *Boundary smoothing properties of the Korteweg-de Vries equation in a quarter plane and applications*, Dynamics Partial Differential Eq. **3** (2006), 1–70.
- [7] J. L. Bona, S. M. Sun, and B. Y. Zhang, *Nonhomogeneous problems for the Korteweg-de Vries and the Korteweg-de Vries-Burgers equations in a quarter plane*, Ann. Inst. H. Poincaré Anal. Non Linéaire **25** (2008), 1145–1185.
- [8] J. L. Bona and Jiahong Wu, *Temporal growth and eventual periodicity for dispersive wave equations in a quarter plane*, Discrete Contin. Dyn. Syst. **23** (2009), 1141–1168.
- [9] J. Boussinesq, *Essai sur la théorie des eaux courantes; Mémoires présentés par divers savants, à l’Acad. des Sci. Inst. Nat. France*, **23** (1877), 1C680.
- [10] E. Cerpa and E. Crépeau, *Rapid exponential stabilization for a linear Korteweg-de Vries equation*, Discrete Contin. Dyn. Syst. Ser. B, **11** (2009), no. 3, 655–668.

- [11] J. E. Colliander and C. E. Kenig, *The generalized Korteweg-de Vries equation on the half line*, Comm. Partial Diff. Eq., **27** (2002), 2187–2266.
- [12] A. V. Faminskii, *A mixed problem in a semistrip for the Korteweg-de Vries equation and its generalizations*, (Russian) Dinamika Sploshn Sredy, **51** (1988), 54–94.
- [13] A. V. Faminskii, *An initial boundary-value problem in a half- strip for the Korteweg-de Vries equation in fractional-order Sobolev spaces*. Comm. Partial Differential Equations **29** (2004), no. 11-12, 1653–1695.
- [14] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Lett., **19** (1967), 1095–1097.
- [15] T. Kato, *On the Cauchy problem for the (Generalized) Korteweg-de Vries Equation*, Stud. Appl. Math. Adv. Math. Suppl. Stud. **8** (1983), 93–128.
- [16] E. M. de Jager, *On the origin of the Korteweg-de Vries equation*, arXiv:math.HO/0602661.
- [17] D. J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Philos. Mag., **39** (1895), 422–443.
- [18] S. N. Kruzhkov, A. V. Faminskii *Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation*, Mat. Sb. **120** (1983) (162)(3):346–425. English transl. in Sb. Math. 1984, **48**(2):391–421.
- [19] J. A. Leach and D. J. Needham *The large-time development of the solution to an initial-value problem for the Korteweg-de Vries equation. I. Initial data has a discontinuous expansive step*, Nonlinearity **21** (2008), 2391–2408.
- [20] F. Linares and A. F. Pazoto, *Asymptotic behavior of the Korteweg-de Vries equation posed in a quarter plane*, J. Differential Equations **246** (2007), 1342–1353.
- [21] J.-L. Lions and E. Magenes, “Problèmes aux limites non homogènes et applications”, Tome 1, Dunod, Paris, 1968.
- [22] R. M. Miura, *The Korteweg-de Vries equation: A survey of results*, SIAM Rev., **18** (1976), 412–459.
- [23] A. Pazoto, *Unique continuation and decay for the Korteweg-de Vries equation with localized damping*, ESAIM Control Optim. Calc. Var. **11** (2005), 473–486.
- [24] A. Pazoto and L. Rosier, *Stabilization of a Boussinesq system of KdV-KdV type*, Systems & Control Lett. **57** (2008), 595–601.
- [25] G. Perla Menzala, C.F. Vasconcellos and E. Zuazua, *Stabilization of the Korteweg-de Vries equation with localized damping*, Quart. Appl. Math. **60** (2002), 111–129.
- [26] L. Rosier, *Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain*, ESAIM Control Optim. Calc. Var. **2** (1997), 33–55 (electronic).
- [27] L. Rosier, *Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line*, SIAM J. Control Optim. **39** (2000), 331–351.
- [28] L. Rosier, *A fundamental solution supported in a strip for a dispersive equation*, Computational and Applied Mathematics **21** (2002), 355–367.

- [29] L. Rosier, *Control of the surface of a fluid by a wavemaker*, ESAIM Control Optim. Calc. Var. **10** (2004), 346–380
- [30] L. Rosier and B.-Y. Zhang, *Global stabilization of the generalized Korteweg-de Vries equation posed on a finite domain*, SIAM J. Control Optim. **45** (2006), 927–956.
- [31] M. E. Taylor, “Partial Differential Equations III, Nonlinear Equations”, Series: Applied Mathematical Sciences 117, Springer-Verlag New York Inc., 1996.