On Subordination Principles for Generalized Shannon Sampling Series

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Abstract:

This paper provides some subordination equalities and their applications for the generalized Shannon sampling series.

1. Introduction

For the uniformly continuous and bounded functions $f \in C(\mathbb{R})$ the generalized Shannon sampling series (see [3] and references cited there) are given by ($t \in \mathbb{R}$; $W > 0$)

$$(S_W f)(t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{W}) s(Wt - k), \tag{1}$$

where the condition for the operator $S_W : C(\mathbb{R}) \to C(\mathbb{R})$ to be well-defined is that for the kernel function $s = s(t)$ we assume

$$\sum_{k=-\infty}^{\infty} |s(u - k)| < \infty \quad (u \in \mathbb{R}).$$

Let be given an even window function $\lambda \in C_{[-1,1]}$, $\lambda(0) = 1$, $\lambda(u) = 0$ ($|u| \geqslant 1$,) then in our approach the kernel function will be defined by the equality

$$s(t) := s_\lambda(t) := \int_0^1 \lambda(u) \cos(\pi t u)\, du. \tag{2}$$

Many window functions have been used in applications (see, e.g. [1], [2], [4], [8]), in Signal Analysis in particular. Next window functions are important for our subordination equalities.

1) $\lambda_{(r)}(u) = 1 - u^r$, $r \geq 1$ defines the Zygmund (or Riesz) kernel, denoted by $z_r = z_r(t)$, which special case $r = 1$, the Fejér (or Bartlett, see [8]) kernel $s_F(t) = \frac{1}{2}\text{sinc}^2 \frac{t}{2}$, is well-known; the special case $r = 2$ is called also as the Welch [8] kernel;

2) $\lambda_j(u) := \cos \pi(j + 1/2)u$, $j = 0, 1, 2, \ldots$ defines the Rogosinski-type kernel (see [5]) in the form

$$r_j(t) := \frac{(-1)^j}{\pi} \frac{(j + 1/2) \cos \pi t}{(j + 1/2)^2 - t^2}; \tag{3}$$

3) $\lambda_H(u) := \cos^2 \frac{\pi u}{2} = \frac{1}{2}(1 + \cos \pi u)$ defines the Hann kernel (see [6])

$$s_H(t) := \frac{1}{2} \frac{\text{sinc} t}{1 - t^2}. \tag{4}$$

Concerning some direct (Jackson-type) approximation theorems we present certain subordination equalities, which show that the sampling operators, like Rogosinski, Zygmund, and Hann, are in some sense basic.

2. Subordination equalities

Subordination equalities state some relations between two sampling operators.

2.1 Subordination by the Rogosinski-type sampling series

Let consider the Rogosinski-type sampling operators $R_{W,j}$ defined by the kernel functions r_j in (3). These kernel functions are deduced by the window functions $\lambda_j(u) := \cos \pi(j + 1/2)u$, ($j \in N$) and as a family of functions it forms an orthogonal system on $[0, 1]$. Therefore, we may represent a quite arbitrary window function λ by its Fourier series. But the Fourier representation allows us to prove for a given kernel function s the sampling series

$$s(t) = 2 \sum_{j=0}^{\infty} s(j + 1/2)\; r_j(t).$$

In following B^p_σ stands for the Bernstein class, it consists of those bounded functions $f \in L^p(\mathbb{R})$ ($1 \leqslant p \leqslant \infty$), which can be extended to an entire function $f(z)$ ($z \in \mathbb{C}$) of exponential type σ. For $s \in B^1_\pi$ the sampling series above is absolutely convergence and by (1) we get formally the equalities

$$S_W f = 2 \sum_{j=0}^{\infty} s(j + 1/2) R_{W,j} f,$$

$$f - S_W f = 2 \sum_{j=0}^{\infty} s(j + 1/2)(f - R_{W,j} f),$$

calling as the subordination equalities, since the approximation properties of the general sampling operators (1) can be described via the approximation properties of the Rogosinski-type sampling operators $R_{W,j} : C(\mathbb{R}) \to C(\mathbb{R})$. We have proved that [5]

$$\|R_{W,j}\| = \frac{4}{\pi} \sum_{\ell=0}^{2j} \frac{1}{2\ell + 1} = \frac{2}{\pi} \log(j + 1) + O(1),$$

thus the subordination equalities are valid, when

$$\sum_{j=0}^{\infty} |s(j+1/2)| \log(j+1) < \infty.$$

Similar subordination equalities can be deduced for some interpolating sampling series, i.e. for which the equation $(\tilde{S}_W f)(\frac{k}{W}) = f(\frac{k}{W}) \quad (k \in \mathbb{Z})$ is valid. In [7] we have proved that the interpolating sampling operators will be defined by (1) using the kernel $\tilde{s}(t) := 2s(2t)$, where the kernel s is generated by (2) with a window function λ for which $\lambda(u) + \lambda(1-u) = 1 \quad (u \in [0,1])$.
Let the operator $S_W^\alpha : C(\mathbb{R}) \to C(\mathbb{R})$ be defined by the kernel $s_\alpha := \alpha\, s(\alpha \cdot) \in B^1_{\alpha\pi} \quad (0 < \alpha \le 2)$, where $s \in B^1_\pi$, and the modified Hann operator $H^\alpha_{W,j}$ is defined by the kernel

$$s^\alpha_{H,j}(t) := \frac{\alpha}{2}\, \frac{(2j+1)^2}{(2j+1)^2 - (\alpha t)^2}\, \mathrm{sinc}(\alpha t). \qquad (5)$$

Then here we have (see [7], Th. 2.3 and 2.4)

$$S^\alpha_W f = 4 \sum_{j=0}^{\infty} s(2j+1) H^\alpha_{W,j} f,$$

$$f - S^\alpha_W f = 4 \sum_{j=0}^{\infty} s(2j+1)(f - H^\alpha_{W,j} f).$$

2.2 Subordination by the Rogosinski-type sampling series: 2D case

The two-dimensional generalized sampling series has the form

$$(S_W f)(x,y) := \sum_{k,l=-\infty}^{\infty} f(\frac{k}{W}, \frac{l}{W}) s(Wx - k, Wy - l),$$

in particular, the multiplicative Rogosinski-type sampling series we define as

$$(R_{W;i,j} f)(x,y) := \sum_{k,l=-\infty}^{\infty} f(\frac{k}{W}, \frac{l}{W}) r_i(Wx - k) r_j(Wy - l),$$

where the Rogosinski-type kernel r_j is defined by (3). Here our subordination equalities read as

$$S_W f = 4 \sum_{i,j=0}^{\infty} s(i+1/2, j+1/2) R_{W;i,j} f,$$

$$f - S_W f = 4 \sum_{i,j=0}^{\infty} s(i+1/2, j+1/2)(f - R_{W;i,j} f),$$

provided

$$\sum_{i,j=1}^{\infty} |s(i+1/2, j+1/2)| \log i \log j < \infty.$$

By given subordination equalities we see that the non-multiplicative sampling series may be studied by the multiplicative Rogosinski-type sampling series.

2.3 Subordination by the Zygmund sampling series

The Zygmund sampling operator Z^r_W will be defined by the window function $\lambda_{(r)}(u) = 1 - u^r, \ r \ge 1$. Let us consider the kernel s in (2), for which the corresponding window function has the power series representation

$$\lambda(u) = 1 - \sum_{j=r}^{\infty} c_j u^j.$$

Then the formal subordination equalities are in the shape

$$S_W f = \sum_{j=r}^{\infty} c_j Z^j_W f,$$

$$f - S_W f = \sum_{j=r}^{\infty} c_j (f - Z^j_W f).$$

Several other subordination equalities and their applications will be presented.

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