Linear Signal Reconstruction from Jittered Sampling

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Abstract:

This paper presents an accurate and simple method to evaluate the performance of AD/DA converters affected by clock jitter, which is based on the analysis of the mean square error (MSE) between the reconstructed signal and the original one. Using an approximation of the linear minimum MSE (LMMSE) filter as reconstruction technique, we derive analytic expressions of the MSE. Through asymptotic analysis, we evaluate the performance of digital signal reconstruction as a function of the clock jitter, number of quantization bits, signal bandwidth and sampling rate.

1. Introduction

A significant problem in Analog Digital Conversion (ADC) of wide-band signals is clock jitter and its impact on the quality of signal reconstruction. Indeed, even small amounts of jitter can measurably degrade the performance of analog to digital and digital to analog converters.

Clock jitter is typically detrimental because the analog to digital process relies upon a sample clock to indicate when a sample or snapshot of the analog signal is taken. The sample clock must be evenly spaced in time; any deviation will result in a distortion of the digitization process. If one had a perfect ADC and a perfect DAC and used the same clock to drive both units, then jitter would not have any impact on the reconstructed signal. In a real world system, however, a digitized signal travels through multiple processors, usually it is stored on a disk or piece of tape for a while, and then goes through more processing before being converted back to analog. Thus, during reconstruction, the clock pulses used to sample the signal are replaced with newer ones with their own subtle variations. Jitter may have different probability distributions which may have different effects on the quality of the reconstructed signal.

While several results are available in the literature on jittered sampling [4, 5] as well as on experimental measurements and instruments performance [1, 3, 6, 7], an analytical methodology for the performance study of the AD/DA conversion is still missing.

In this paper we fill this gap and propose a method for evaluating the performance of AD/DC converters affected by jitter, which is based on the analysis of the mean square error (MSE) between the reconstructed signal and the original one [7].

As reconstruction technique, we consider linear filtering methods, which typically have low complexity and are used in a wide variety of fields. If jitter were known exactly, the linear minimum MSE (LMMSE) reconstruction technique would be optimal, since it minimizes the MSE of the reconstructed signal. In practice this is not the case, hence we apply a reconstruction filter with the same structure of the LMMSE filter, where we let the jitter vanish. Then, we apply asymptotic analysis to derive analytical expressions of the MSE on the quality of the reconstructed signal. We then show that our asymptotic expressions provide an excellent approximation of the MSE even for small values of the system parameters, with the advantage of greatly reducing the computation complexity. We apply our method to study the performance of the AD/DA conversion system as a function of the clock jitter, number of quantization bits, signal bandwidth and sampling rate.

2. System model

Throughout the paper we use the following notations. Column vectors are denoted by bold lowercase letters and matrices are denoted by bold upper case letters. The (k, q)-th entry of the generic matrix $\mathbf{Z}$ is denoted by $(\mathbf{Z})_{k,q}$. The $n \times n$ identity matrix is denoted by $\mathbf{I}_n$, while $\mathbf{I}$ is the generic identity matrix. $(\cdot)^{\mathrm{T}}$ is the transpose operator, while $(\cdot)^{\dagger}$ is the conjugate transpose operator. We denote by $f_x(z)$ the probability density function (pdf) of the generic random variable x, and by $\mathbb{E}[\cdot]$ the average operator.

2.1 Signal sampling and reconstruction

We consider an analog signal $s(t)$ sampled at constant rate $f_s = 1/T_s$ over the finite interval $[0, MT_s)$. T_s is the sample spacing. When observed over a finite interval, $s(t)$ admits an infinite Fourier series expansion. Let N' denote the largest index of the non-negligible Fourier coefficients, then N'/T_s can be considered as the approximate one-sided bandwidth of the signal. We therefore represent the signal by using a truncated Fourier series with $N = 2N' +$

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1 complex harmonics as

$$s(t)=\frac{1}{\sqrt{N}}\sum_{\ell=-N'}^{N'}a_\ell\exp\left(\mathrm{j}2\pi\ell\frac{t}{MT_s}\right),\qquad(1)$$

$0\le t<MT_s$. The vector $\mathbf{a}=[a_{-N'},\dots,a_0,\dots,a_{N'}]^{\mathrm{T}}$ represents the complex discrete spectrum of the signal.
Observe that the signal representation given in (1) includes sine waves of any fractional frequency $f_0=f_sN'/M$ (when $a_\ell=0$ for $-N'<\ell<N'$ and $a_{-N'}=a^*_{N'}$), which are frequently used as reference signal for calibration of ADC [1, 2]. We note that when the signal $s(t)$ is observed in the frequency domain through its M samples, the spectral resolution is given by $\Delta f=1/(MT_s)$. Therefore, considering the expression in (1), the signal bandwidth is given by $B=\frac{N\Delta f}{2}=\frac{N}{2MT_s}$. By defining the parameter

$$\beta=\frac{M}{N}\qquad(2)$$

as the *oversampling factor* of the signal $s(t)$ with respect to the Nyquist rate, we can also write:

$$B=\frac{f_s/2}{\beta}\qquad(3)$$

In this work, we consider that sampling locations suffer from jitter, i.e., the m-th sampling location is given by

$$t_m=mT_s+d_m,\qquad(4)$$

$m=0,\dots,M-1$, where d_m is the associated independent random jitter whose distribution is denoted by $f_d(z)$. Typically, we have $|d_m|\ll T_s$.
Let the signal samples be $\mathbf{s}=[s_0,\dots,s_{M-1}]^{\mathrm{T}}$ where $s_m=s(t_m)$, $0\le m\le M-1$. Using (1), the set of signal samples can be written as

$$\mathbf{s}=\mathbf{V}^\dagger\mathbf{a}$$

where $\mathbf{V}$ is an $N\times M$ random Vandermonde matrix defined as

$$(\mathbf{V})_{\ell,m}=\frac{1}{\sqrt{N}}\exp\left(-\mathrm{j}2\pi\ell\frac{t_m}{MT_s}\right)\qquad(5)$$

$\ell=-N',\dots,N'$, and $m=0,\dots,N-1$. Note that $\mathbf{V}$ accounts for the jitter in the AD/DA conversion process, and that the parameter β defined in (2) also represents the *aspect ratio* of matrix $\mathbf{V}$.
Furthermore, in addition to jittered sampling, we assume that signal samples are affected by some additive noise and are therefore given by

$$\mathbf{y}=\mathbf{s}+\mathbf{n}$$

where $\mathbf{n}$ is a vector of M noise samples, modeled as zero mean i.i.d. random variables. In practice, the dominant additive noise error is due to the n-bit quantization process [10].
In order to reconstruct the signal we consider a reconstruction technique that provides an estimate $\hat{\mathbf{a}}$ of the discrete spectrum $\mathbf{a}$. The reconstruction $\hat{s}(t)$ of $s(t)$ obtained from $\hat{\mathbf{a}}$ is given by

$$\hat{s}(t)=\frac{1}{\sqrt{N}}\sum_{\ell=-N'}^{N'}\hat{a}_\ell\exp\left(\mathrm{j}2\pi\ell\frac{t}{MT_s}\right)$$

2.2 Reconstruction error

We consider as performance metric of the AD/DA conversion process the mean square error (MSE) associated to the estimate. The MSE, evaluated in the observation interval $[0,MT_s)$, can be equivalently computed in both time and frequency domains as:

$$\mathrm{MSE}=\mathbb{E}\left[\int_0^{MT_s}|s(t)-\hat{s}(t)|^2\,\mathrm{d}t\right]=\frac{\mathbb{E}\left[\|\mathbf{a}-\hat{\mathbf{a}}\|^2\right]}{N}$$

More specifically, we consider the MSE relative to the signal average power, i.e.,

$$J=\frac{\mathrm{MSE}}{\sigma_a^2}$$

which can be thought of as a noise to signal ratio and will be plotted using a dB scale in our results.
Among the possible techniques that can be applied to reconstruct the original signal, we focus on linear filters that provide an estimate of $\mathbf{a}$ through the linear operation $\hat{\mathbf{a}}=\mathbf{B}\mathbf{y}$ where $\mathbf{B}$ is an $N\times M$ matrix.

3. Jittered AD/DA conversion with linear filtering

Let us assume $\|\mathbf{a}\|^2=\sigma_a^2N$ and $\mathbb{E}[\mathbf{n}\mathbf{n}^\dagger]=\sigma_n^2\mathbf{I}$, then we define the signal to noise ratio (SNR) in absence of jitter as

$$\gamma=\frac{\sigma_a^2}{\sigma_n^2}$$

Under the assumption that $\mathbb{E}[\mathbf{a}\mathbf{a}^\dagger]=\sigma_a^2\mathbf{I}$, the linear filter that provides the best performance in terms of MSE is the linear minimum mean square error (LMMSE) filter, which is given by

$$\mathbf{B}_{\mathrm{opt}}=\left(\mathbf{V}\mathbf{V}^\dagger+\frac{1}{\gamma}\mathbf{I}\right)^{-1}\mathbf{V}\qquad(6)$$

In [8], it has been shown that, by applying the LMMSE filter, we obtain:

$$J=\frac{1}{\sigma_a^2N}\mathbb{E}\left[\|\mathbf{a}-\hat{\mathbf{a}}\|^2\right]=\mathbb{E}\left[\mathrm{tr}\left\{\left(\gamma\mathbf{V}\mathbf{V}^\dagger+\mathbf{I}\right)^{-1}\right\}\right]$$

where $\mathrm{tr}\{\cdot\}$ is the normalized matrix trace operator and the average is over the randomness in $\mathbf{V}$.
Note, however, that the filter in (6) cannot be employed in practice, since the jitters d_m (hence the matrix $\mathbf{V}$) are unknown (see the definition of $\mathbf{V}$ in (5)). We therefore resort to an approximation of the optimum filter $\mathbf{B}_{\mathrm{opt}}$, based on the assumption that jitter has a zero mean.
In particular, we approximate $\mathbf{V}$ with the matrix $\mathbf{F}$ defined as,

$$\mathbf{F}=\mathbf{V}|_{d_m=0}$$

with the generic element of $\mathbf{F}$ given by, $(\mathbf{F})_{\ell,m}=\exp\left(-\mathrm{j}2\pi\ell\frac{m}{M}\right)/\sqrt{N}$, $\ell=-N',\dots,N'$, and $m=0,\dots,N-1$. We observe that $\mathbf{F}$ is such that: $\mathbf{F}\mathbf{F}^\dagger=\beta\mathbf{I}$ and it is related to the discrete Fourier transform matrix.
Substituting the approximation of $\mathbf{V}$ in (6), we obtain:

$$\mathbf{B}=\left(\beta+\frac{1}{\gamma}\right)^{-1}\mathbf{F}\qquad(7)$$

Notice that the filter in (7) is the LMMSE filter adapted to the linear model $\mathbf{y} = \mathbf{F}^\dagger\mathbf{a} + \mathbf{n}$. By letting $\omega = (\beta + 1/\gamma)^{-1}$, the noise to signal ratio J provided by the approximate filter (7) is given by

$$\begin{aligned} J &= \frac{1}{\sigma_a^2 N}\,\mathbb{E}\left[\|\mathbf{a} - \omega\mathbf{F}\mathbf{y}\|^2\right] \\ &= \mathsf{tr}\left\{\omega^2 \mathbb{E}_d\left[\mathbf{F}\mathbf{V}^\dagger\mathbf{V}\mathbf{F}^\dagger\right] - 2\omega\Re\{\mathbb{E}_d\left[\mathbf{F}\mathbf{V}^\dagger\right]\}\right\} \\ &\qquad +1 + \frac{\omega^2\beta}{\gamma} \end{aligned} \tag{8}$$

where the operator $\mathbb{E}_d[\cdot]$ averages over the random jitters $d_m, m = 0, \ldots, M-1$.

Assuming the jitters to be independent [1] and with characteristic function $C_d(w) = \mathbb{E}_d[\exp(\mathrm{j}wz)]$, the first two terms in (8) are given by

$$\mathsf{tr}\,\mathbb{E}_d\left[\mathbf{F}\mathbf{V}^\dagger\right] = \frac{\beta}{N}\sum_{\ell=-N'}^{N'} C_d\left(\frac{2\pi\ell}{MT_s}\right)$$

$$\mathbb{E}_d\left[\mathbf{F}\mathbf{V}^\dagger\mathbf{V}\mathbf{F}^\dagger\right] = \beta + \beta\frac{(\beta-1)}{N}\sum_{\ell=-N'}^{N'}\left|C_d\left(\frac{2\pi\ell}{MT_s}\right)\right|^2$$

Hence, we can write:

$$\begin{aligned} J &= 1 + \omega^2\beta\left(1+\frac{1}{\gamma}\right) - 2\omega\frac{\beta}{N}\sum_{\ell=-N'}^{N'} C_d\left(\frac{2\pi\ell}{MT_s}\right) \\ &\qquad +\omega^2\beta\frac{(\beta-1)}{N}\sum_{\ell=-N'}^{N'}\left|C_d\left(\frac{2\pi\ell}{MT_s}\right)\right|^2 \end{aligned} \tag{9}$$

In order to reduce the complexity of the computation of the reconstruction error and provide simple but accurate analytical tools, in the next section we let the parameters N and M go to infinity, while the ratio $\beta = M/N$ is kept constant. We therefore derive an asymptotic expression of J, which we will show well approximates the expression in (9) even for small N and M.

4. Asymptotic analysis

When N and M grow to infinity while β is kept constant, we define the *asymptotic* noise to signal ratio J as:

$$J_\infty^{(\beta,\gamma)} = \lim_{\substack{N,M\to+\infty\\ \beta}} J$$

In [8], it has been shown that $J_\infty^{(\beta,\gamma)}$ provides an excellent approximation of MSE/σ_a^2 even for small values of N and M, with the advantage of greatly simplifying the computation.

In the limit $N, M \to \infty$ with constant β, we compute

$$\begin{aligned} \mu_1 &= \lim_{\substack{N,M\to+\infty\\ \beta}} \frac{1}{N}\sum_{\ell=-N'}^{N'} C_d\left(\frac{2\pi\ell}{MT_s}\right) \\ &= \int_{-1/2}^{1/2} C_d\left(4\pi Bx\right)\,\mathrm{d}x \end{aligned} \tag{10}$$

where, from (3), we used the fact that $1/\beta T_s = f_s/\beta = 2B$. Similarly, we define

$$\begin{aligned} \mu_2 &= \lim_{\substack{N,M\to+\infty\\ \beta}} \frac{1}{N}\sum_{\ell=-N'}^{N'} C_d\left(\frac{2\pi\ell}{MT_s}\right)^2 \\ &= \int_{-1/2}^{1/2} \left|C_d\left(4\pi Bx\right)\right|^2\,\mathrm{d}x \end{aligned} \tag{11}$$

By using (10) (11), and (9), the asymptotic expression of J is given by

$$J_\infty^{(\beta,\gamma)} = 1+\omega^2\beta(1+1/\gamma)-2\omega\beta\mu_1+\omega^2\beta(\beta-1)\mu_2 \tag{12}$$

It is worth mentioning that for large SNRs (i.e., in absence of measurement noise), $J_\infty^{(\beta,\gamma)}$ reduces to

$$J_\infty^{(\beta)} = \lim_{\gamma\to\infty} J_\infty^{(\beta,\gamma)} = 1+\frac{1}{\beta}-2\mu_1+\left(1-\frac{1}{\beta}\right)\mu_2 \tag{13}$$

Equation (13) provides us with a floor that represent the best quality of the reconstructed signal (minimum MSE) we can hope for.

4.1 Example: uniform jitter distribution

Let us now assume the jitter to be uniformly distributed with pdf given by

$$f_d(z) = \begin{cases} \frac{1}{2d_{\max}} & -d_{\max} \le z \le d_{\max} \\ 0 & \text{elsewhere} \end{cases}$$

where $d_{\max}$ is the maximum jitter, independent of the sampling frequency f_s. In this case, the characteristic function of the jitter is given by $C_d(w) = \sin(d_{\max}w)/(d_{\max}w)$. Then,

$$\mu_1 = \frac{\mathrm{Si}(2\pi\eta_u)}{2\pi\eta_u}$$

and

$$\mu_2 = \frac{\cos^2(2\pi\eta_u) + 2\pi\eta_u\mathrm{Si}(4\pi\eta_u) - 1}{4\pi^2\eta_u^2}$$

where $\mathrm{Si}(\cdot)$ is the integral sine function and $\eta_u = d_{\max}B$ is a dimensionless parameter which relates maximum jitter and signal bandwidth.

5. Results

For the ease of representation, we assume that the dominant component of the additive noise is due to quantization, and we express the SNR in absence of jitter, γ, as a function of the number of quantization bits n of the ADC [9]:

$$(\gamma)_{\mathrm{dB}} = 6.02n + 1.76$$

Then, in the following plots we show the value of J as a function of γ or, equivalently, of the number of quantization bits n.

Figure 1 compares the value of J obtained through its asymptotic expression against the performance of a system with finite parameters values (i.e., the value of J computed using (9)). The results are derived for $\eta_u =$

$10^{-1}, 10^{-2}, 10^{-3}$, and $\beta = 10$. Solid lines refer to the asymptotic expression (12), while markers represent the values of J computed through (9), with $N' = 100$. We observe an excellent matching between our approximation of $J_\infty^{(\beta,\gamma)}$ and the results computed through (9), even for small values of N and M. We point out that this tight match can be observed for any $\beta > 1$ and $\eta_u \ll 1$.

We also notice that J shows a floor, whose expression is given by (13). This floor is due to the mismatch between the filter $\mathbf{F}$ employed in the reconstruction and the matrix $\mathbf{V}$ characterizing the sampling system.

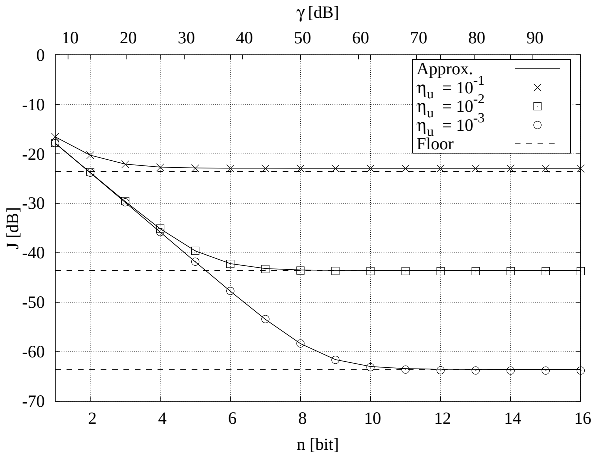

Figure 1: Comparison between the reconstruction error J derived through (9), the approximation of $J_\infty^{(\beta,\gamma)}$ and the floor $J_\infty^{(\beta)}$ in (13).

Furthermore, in the case of unknown jitter, and, thus, of a floor in the behavior of J, there exists a number of quantization bits $n = n^*$ beyond which a further increase in the ADC precision does not provide a noticeable decrease in the reconstruction error J. The relation between η_u, β, and n^* is shown in Figure 2. Note that n^* is lightly affected by an increase of β, provided that $\beta > 1$, and a good compromise for choosing the oversampling rate is $\beta = 5$.

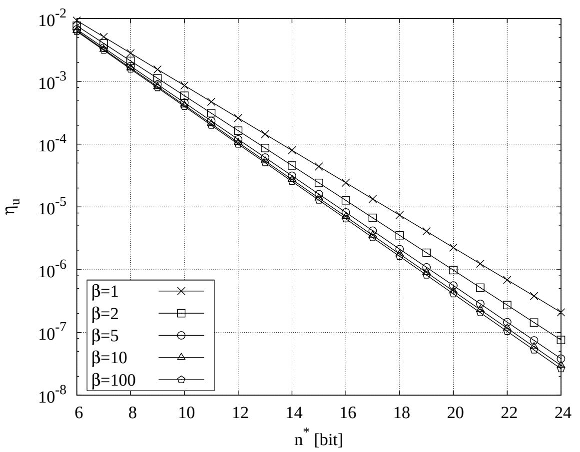

Figure 2: Minimum number of bits n^* required to reach the floor of $J_\infty^{(\beta,\gamma)}$ as a function of β and η_u.

6. Conclusions

We studied the performance of AD/DA converters, in presence of clock jitter and quantization errors. We considered that a linear filter approximating the LMMSE filter is used for signal reconstruction, and evaluated the system performance in terms of MSE. Through asymptotic analysis, we derived analytical expressions of the MSE which provide an accurate and simple method to evaluate the behavior of AD/DA converters as clock jitter, number of quantization bits, signal bandwidth and sampling rate vary. We showed that our asymptotic approach provides an excellent approximation of the MSE even for small values of the system parameters. Furthermore, we derived the MSE floor, which represents the best reconstruction quality level we can hope for and gives useful insights for the design of AD/DA converters.

References:

[1] *Project DYNAD, SMT4-CT98, Draft Standard Version 3.4*, Jul. 12, 2001.

[2] IEEE Standard for Terminology and Test Methods for Analog-to-Digital Converters, IEEE Std. 1241, 2000.

[3] P. Arpaia, P. Daponte, and S. Rapuano, "Characterization of digitizer timebase jitter by means of the Allan variance," *Computer Standards & Interfaces*, Vol. 25, pp. 15–22, 2003.

[4] B. Liu, and T. P. Stanley, "Error bounds for jittered sampling," *IEEE Transactions on Automatic Control*, Vol. 10, No. 4, pp. 449–454, Oct. 1965.

[5] J. Tourabaly, and A. Osseiran, "A jittered-sampling correction technique for ADCs," *IEEE International Workshop on Electronic Design, Test and Applications*, pp. 249–252, Los Alamitos, CA, USA, 2008.

[6] E. Rubiola, A. Del Casale, and A. De Marchi, "Noise induced time interval measurement biases," *46th IEEE Frequency Control Symposium*, pp. 265–269, May 1992.

[7] J. Verspecht, "Accurate spectral estimation based on measurements with a distorted-timebase digitizer," *IEEE Trans. on Instrumentation and Measurement* Vol. 43, pp. 210–215, Apr. 1994.

[8] A. Nordio, C.-F. Chiasserini, and E. Viterbo "Performance of linear field reconstruction techniques with noise and uncertain sensor locations," *IEEE Trans. on Signal Processing*, Vol. 56, No. 8, pp. 3535–3547, Aug. 2008.

[9] G. Gielen, "Analog building blocks for signal processing," ESAT-MICAS, Leuven, Belgium, 2006.

[10] S. C. Ergen, and P. Varaiya, "Effects of A-D conversion nonidealities on distributed sampling in dense sensor networks," IPSN '06, Nashville, Tennessee, Apr. 2006.