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TITRE DE LA THÈSE :

Études de petites valeurs propres du Laplacien de Witten

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Chapitre 1

Introduction

1.1 Positionnement du problème

1.1.1 Introduction

Dans cette thèse, nous nous intéressons à l'étude des valeurs propres exponentiellement petites d'un opérateur de Schrödinger semi-classique, le Laplacien de Witten. C'est-à-dire que nous travaillons avec un petit paramètre réel positif h issu de l'analyse semi-classique et que nous cherchons à connaître précisément le comportement asymptotique de valeurs propres de la forme $A(h)e^{-\frac{C}{h}}$ (C est un paramètre réel strictement positif) de cet opérateur lorsque h tend vers 0^+. Nous voulons plus exactement déterminer le préfacteur $A(h)$.

Cet opérateur fut initialement introduit par E. Witten dans [Wit] - en déformant le Laplacien de Hodge avec une fonction de Morse f - pour démontrer les inégalités de Morse de façon analytique. Dans ce cadre, un calcul aussi précis des petites valeurs propres de l'opérateur n'est cependant pas nécessaire (regarder également [Hen][CyFrKiSi][Hel2][HeSj4]).

Cette étude spectrale est intimement liée, comme nous le verrons dans la Partie 1.1.3, à l'étude de processus stochastiques de diffusion (réversibles) en Physique statistique, dans le cadre de la théorie des grandes déviations de ces processus stochastiques.

L'interprétation physique de ces valeurs propres exponentiellement petites correspond à l'inverse de temps de sortie moyens d'états métastables pour une particule soumise à un champ de gradient aléatoirement perturbé (c.f. Partie 1.1.3). Le petit paramètre h correspond quant à lui à la température du système dans lequel évolue cette particule.

Ces temps de sortie moyens sont donc exponentiellement longs si la température est petite, et d'autant plus courts que celle-ci est importante.

L'étude de ce type de comportements remonte au moins à S.A. Arrhenius qui énonça en 1889 la loi d'Arrhenius, permettant de décrire, en cinétique chimique, la variation de la vitesse d'une réaction chimique en fonction de la température.

Cette loi explique substantiellement que le coefficient k de vitesse de la réaction chimique s'écrit $k = Ae^{-\frac{E_a}{cT}}$, où c est une constante positive et E_a l'énergie d'activation d'Arrhenius, dépendant de la réaction étudiée. Cette énergie d'activation est la quantité d'énergie nécessaire pour initier le processus chimique (c.f. Figure 1.1 ci-dessous).

On peut même s'amuser à remonter juqu'à L. Pasteur qui, sans s'intéresser à des calculs de temps exponentiellement longs, mit en évidence la présence d'êtres vivants dans l'air grâce à des expériences faisant intervenir un flacon avec "col de cygne" (c.f. Figure 1.2). Il remarqua en effet que l'air ambiant provoquait l'apparition d'être vivants sur une goutte placée à l'entrée du col de cygne alors que dans un très grand nombre de cas, il n'en apparaissait pas sur les substances placées au fond du flacon.

Les bactéries, plus lourdes que l'air, sont en effet piégées dans le creux du col de cygne et ont besoin d'un afflux d'air important pour pouvoir franchir le col les séparant du flacon... On pourrait ainsi certainement mettre en évidence que le temps de sortie moyen d'une particule piégèe dans ce puits est exponentiellement long, et d'autant plus long que le col à franchir est haut.

Une étude via les processus stochastiques et la théorie des grandes déviations en particulier a été largement développée à la fin des années soixante-dix - début des années quatre-vingt pour ce type de problèmes par différentes équipes, notamment D.W. Stroock et S.R. Varadhan (c.f. par exemple [StVa]) ou M.I. Freidlin et A.D. Wentzell (c.f. [FrWe]).

Dans cette dernière référence, [FrWe], les auteurs démontrent en particulier que ces temps de sorties sont exponentiellement petits (i.e. de la forme $\mathcal{O}(e^{-\frac{C}{h}})$) en donnant des équivalents logarithmiques, et ce, dans un cadre tout à fait général, non réduit aux processus de diffusion réversibles.

Parallèlement, ce type de comportements exponentiels a également été étudié en vue des algorithmes de recuit simulé (c.f. [HoKuSt]).

Une étude plus fine de ces temps de sorties - ou encore de ces valeurs propres - visant à déterminer précisément le préfacteur $A(h)$ de la valeur propre $A(h)e^{-\frac{C}{h}}$ a été menée plus récemment dans les travaux [BoEcGaKl] et [BoGaKl], dans le cadre de processus de diffusion réversibles dans $\mathbb{R}^n$. Il

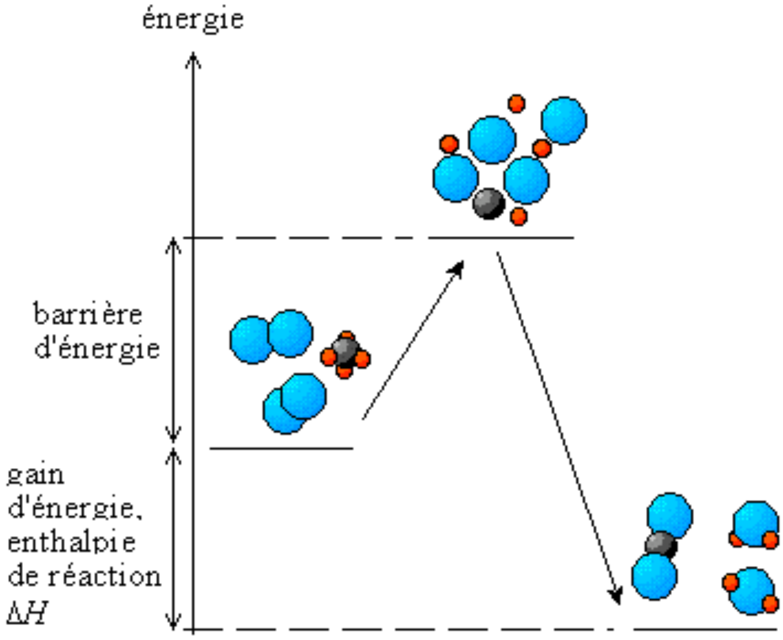


Figure 1.1 – Variation de l'énergie au cours de la réaction chimique, barrière énergétique et enthalpie de réaction (exemple de la combustion du méthane dans le dioxygène). *Source : Wikipédia.*

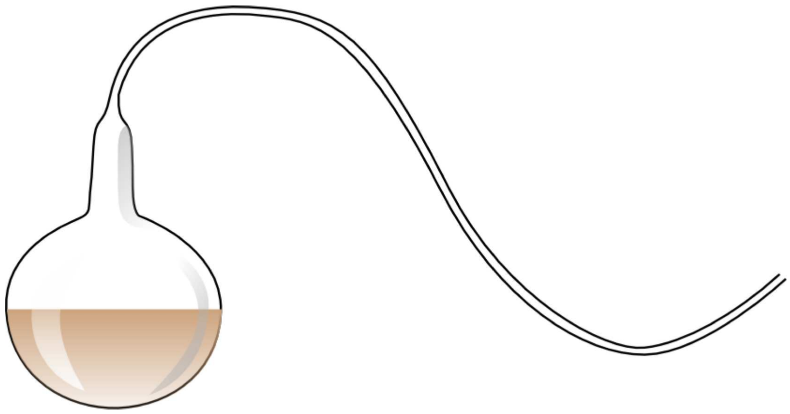


Figure 1.2 – Flacon avec "col de cygne" de Pasteur. *Source : Wikipédia.*

y est notamment montré que $A(h)$ est de la forme $Dh\left(1+\mathcal{O}(h^{\frac{1}{2}}|\ln h|)\right)$, où D est une constante strictement positive.

Ces derniers résultats ont été améliorés par les travaux [HeKlNi] et [HeNi] (et le Chapitre 4), faits du point de vue de l'étude du Laplacien de Witten, dans le cas général de variétés riemanniennes avec ou sans bord. Il y est montré, sous des hypothèses de régularité cependant plus fortes, que le préfacteur $A(h)$ admet un développement asymptotique complet, le terme $\mathcal{O}(h^{\frac{1}{2}}|\ln h|)$ étant en particulier remplacé par $\mathcal{O}(h)$.

1.1.2 Complexe de Witten avec ou sans (conditions au) bord

Nous allons, dans ce premier chapitre, parler de Laplaciens de Witten sur des variétés avec ou sans bord. Ces variétés seront le plus simple possible : riemanniennes, de classe $\mathcal{C}^\infty$, compactes, orientées, connexes et de dimension finie $n \in \mathbb{N}^*$. Pour ne pas nous encombrer de trop nombreuses notations, notons une telle variété $\overline{\Omega} = \Omega \cup \partial\Omega$. Ainsi, lorsque nous supposerons cette variété sans bord, nous prendrons tout simplement $\partial\Omega = \emptyset$ et $\Omega = \overline{\Omega}$, Ω étant compacte. Par contre, lorsque nous considérerons une variété à bord, $\partial\Omega$ sera naturellement le bord en question et Ω son intérieur.

Sur cette variété $\overline{\Omega}$, notons respectivement d et d^* l'opérateur de différentiation extérieure et son adjoint par rapport au produit scalaire naturel L^2 hérité de la structure riemannienne. La restriction de d à l'espace des p-formes différentielles ($p \in \{0,\dots,n\}$), $d^{(p)}$, est à valeurs dans l'espace des $p+1$-formes. $d^{*,(p)}$ envoie quant à lui, par dualité, des $p+1$-formes sur des p-formes. Le Laplacien de Hodge est l'opérateur suivant :

$$\Delta_H := dd^* + d^*d = (d+d^*)^2\,.$$

Sa restriction à l'espace des p-formes, $\Delta_H^{(p)}$, est évidemment à valeur dans le même espace. De plus, sur une variété sans bord, cet opérateur est autoadjoint positif, à spectre discret, sur l'espace $\Lambda L^2(\Omega)$ des sections $L^2(\Omega)$ du fibré des formes différentielles, une fois muni du domaine $\Lambda H^2(\Omega)$ des sections H^2.

Dans ce cas, un résultat fondamental de la théorie de Hodge dit que $\mathcal{H}^p(\Omega) := \ker d^{(p)}/\operatorname{Ran} d^{(p-1)}$, le p-ième groupe de cohomologie de Rham de la variété Ω, est le noyau de $\Delta_H^{(p)}$. Sa dimension est en outre finie, égale à $b_p(M)$, le p-ième nombre de Betti de la variété Ω. Ces nombres sont des invariants topologiques, indépendants de la structure métrique considérée sur Ω.

Remarque 1.1.1. *Lorsque nous nous plaçons dans $\mathbb{R}^n$ avec sa métrique usuelle, $\Delta_H^{(0)}$ est tout simplement l'opposé de Δ, le Laplacien classique.*

Le Laplacien de Witten, initialement introduit par E. Witten pour démontrer les inégalités de Morse de façon analytique (c.f. [Wit]), est une déformation du Laplacien de Hodge par une fonction de Morse f et un paramètre strictement positif h. Ces inégalités donnent une relation entre les nombres de Betti d'une variété et le nombre de points critiques d'une fonction de Morse sur cette variété.

Rappelons qu'une fonction de Morse est une fonction $\mathcal{C}^\infty$ n'ayant qu'un nombre fini de points critiques, tous non dégénérés. Un tel point (critique et non dégénéré) est d'ailleurs isolé par conséquence directe du Lemme de Morse. Travaillant ici sur des variétés compactes, la finitude de l'ensemble de ces points est donc une hypothèse superflue.

Etant donnés une fonction de Morse f et $h > 0$, $d_{f,h}$ et $d^*_{f,h}$ sont les opérateurs différentiels déformés suivants,

$$d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} \quad \text{et} \quad d_{f,h} = e^{\frac{f}{h}}(hd^*)e^{-\frac{f}{h}} \, ,$$

et le Laplacien de Witten est alors défini comme l'était le Laplacien de Hodge :

$$\Delta_{f,h} := d_{f,h}d^*_{f,h} + d^*_{f,h}d_{f,h} = (d_{f,h} + d^*_{f,h})^2 \, .$$

Notant ∇ le gradient et $\mathcal{L}$ la dérivée de Lie, le Laplacien de Witten a également l'expression suivante :

$$\Delta_{f,h} \;=\; h^2\Delta_H + |\nabla f|^2 + h\left(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}\right) \, . \tag{1.1.1}$$

En tant qu'opérateur agissant sur les 0-formes (i.e. sur les fonctions) dans $\mathbb{R}^n$ muni de sa métrique usuelle, le Laplacien de Witten s'écrit :

$$\Delta^{(0)}_{f,h} = -h^2\Delta + |\nabla f|^2 - h\Delta f \, . \tag{1.1.2}$$

Remarque 1.1.2. *Un point remarquable de la démonstration des inégalités de Morse par E. Witten est le suivant : le p-ième groupe de cohomologie associé à $d_{f,h}$, à savoir $\mathcal{H}^p_{f,h}(\Omega) := \ker d^{(p)}_{f,h}/\mathrm{Rand}^{(p-1)}_{f,h}$, est isomorphe à $\mathcal{H}^p(\Omega)$. Sa dimension est donc $b_p(\Omega)$; elle est par conséquent indépendante de la fonction f et du paramètre h.*

Laplaciens de Witten autoadjoints

Dans le cas d'une variété compacte sans bord, le Laplacien de Witten $\Delta_{f,h}$ de domaine $D(\Delta_{f,h}) = \{\omega \in \Lambda H^2(\Omega)\}$ est, comme le Laplacien de Hodge, autoadjoint et positif.

Il est d'ailleurs toujours autoadjoint et positif sur $\mathbb{R}^n$, mais alors le domaine n'a en général pas de formulation globale simple. Néanmoins, ce domaine est également $\Lambda H^2(\Omega)$ si l'on suppose en outre des conditions de décroissance à l'infini pour la fonction de Morse f.

Dans le cas d'une variété compacte à bord, une réalisation autoadjointe du Laplacien de Witten fait évidemment intervenir des conditions supplémentaires, au bord. Les domaines correspondant aux réalisations autoadjointes du Laplacien de Witten avec conditions de Neumann ($\Delta^N_{f,h}$) et de Dirichlet ($\Delta^D_{f,h}$) sont les suivants (c.f. [ChLi], [HeNi] et le Chapitre 4) :

$$D(\Delta^N_{f,h}) = \{\omega \in \Lambda H^2(\Omega)\,,\ \mathbf{n}\omega = 0\,,\ \mathbf{n}d_{f,h}\omega = 0\}$$

et

$$D(\Delta^D_{f,h}) = \left\{\omega \in \Lambda H^2(\Omega)\,,\ \mathbf{t}\omega = 0\,,\ \mathbf{t}d^*_{f,h}\omega = 0\right\}\,,$$

où $\mathbf{t}$ et $\mathbf{n}$ désignent respectivement les composantes tangentielles et normales, au bord, d'une forme différentielle.

Ces conditions au bord ne sont d'ailleurs pas tout à fait les conditions naturelles de Neumann ou de Dirichlet, mais des conditions déformées de Neumann ou de Dirichlet, correspondant aux déformations $d_{f,h}$ et $d^*_{f,h}$ des opérateurs d et d^*.

Les conditions naturelles de Neumann et de Dirichlet pour une forme différentielle ω sont en effet les suivantes :

$$\textit{Conditions de Neumann :}\quad \mathbf{n}\omega = 0 \quad \text{et} \quad \mathbf{n}d\omega = 0\,,$$
$$\textit{Conditions de Dirichlet :}\quad \mathbf{t}\omega = 0 \quad \text{et} \quad \mathbf{t}d^*\omega = 0\,.$$

Ces dernières conditions correspondent respectivement aux groupes de cohomologies absolue et relative de la variété à bord $\overline{\Omega}$ (c.f. [Gil][Mas]).

Lorsque ω est une fonction, cela correspond bien aux conditions usuelles de Neumann et de Dirichlet, les conditions énoncées ci-dessus étant équivalentes aux conditions $\frac{\partial}{\partial n}\omega = 0$ sur le bord pour le cas Neumann et $\omega = 0$ sur le bord pour le cas Dirichlet...

Complexe de Witten

Un autre point structurel fondamental utilisé par E. Witten pour démontrer les inégalités de Morse est la structure de complexe avec carrés autoadjoints de dimension finie (c.f. [Bis] pour le cas de carrés non autoadjoints).

En effet, désignant par $\overline{\Delta^{(p)}_{f,h}}$ l'une des réalisations autoadjointes du Laplacien de Witten mentionnées précédemment, $\overline{\Delta^{(p)}_{f,h}}$ satisfait, pour u dans $D(\overline{\Delta^{(p)}_{f,h}})$:

$$\left\{\begin{array}{rcl}\overline{\Delta^{(p+1)}_{f,h}}d^{(p)}_{f,h}u & = & d^{(p)}_{f,h}\overline{\Delta^{(p)}_{f,h}}u\\ \overline{\Delta^{(p-1)}_{f,h}}d^{(p-1),*}_{f,h}u & = & d^{(p-1),*}_{f,h}\overline{\Delta^{(p)}_{f,h}}u\,.\end{array}\right.$$

Ainsi, si u est un vecteur propre de $\overline{\Delta^{(p)}_{f,h}}$ associé à une valeur propre λ, $d^{(p)}_{f,h}u$ (resp. $d^{(p),*}_{f,h}u$) est, à condition bien sûr d'être non nul, un vecteur propre de $\overline{\Delta^{(p+1)}_{f,h}}$ (resp. de $\overline{\Delta^{(p-1)}_{f,h}}$) associé à la même valeur propre.

Par ailleurs, pour $I(h) = [0, h^c[$ (pour un certain $c > 1$) et h assez petit, l'espace $F^{(p)} = \mathrm{Ran}1_{I(h)}(\overline{\Delta^{(p)}_{f,h}})$ est de dimension finie, la dimension $m_p(f)$ dépendant fortement du nombre de points critiques de la fonction f. Considérons alors la restriction de la dérivée extérieure déformée et de son adjoint à $F^{(p)}$:

$$\beta^{(p)}_{f,h} = d_{f,h}\big|_{F^{(p)}} = d^{(p)}_{f,h}\big|_{F^{(p)}} \quad \text{et} \quad \beta^{(p-1)}_{f,h} = d^{*}_{f,h}\big|_{F^{(p)}} = d^{*,(p-1)}_{f,h}\big|_{F^{(p)}}\,.$$

La structure de complexe de dimension finie est la suivante :

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F^{(0)} & \xrightarrow{\beta^{(0)}_{f,h}} & F^{(1)} & \xrightarrow{\beta^{(1)}_{f,h}} & \cdots & \xrightarrow{\beta^{(n-1)}_{f,h}} & F^{(n)} & \longrightarrow & 0 \\ 0 & \longleftarrow & F^{(0)} & \xleftarrow{\beta^{(0),*}_{f,h}} & F^{(1)} & \xleftarrow{\beta^{(1),*}_{f,h}} & \cdots & \xleftarrow{\beta^{(n-1),*}_{f,h}} & F^{(n)} & \longleftarrow & 0 \end{array}\,. \tag{1.1.3}$$

Cette structure est également très importante pour étudier les petites valeurs propres de Laplaciens de Witten, que ce soit dans les cas sans bord (c.f. [HeKlNi]), ou avec conditions de Dirichlet (c.f. [HeNi]) ou de Neumann (c.f. Chapitre 4).

Si $b_p(f)$ désigne le p-ième nombre de Betti (pour p dans $\{0, \dots, n\}$) du complexe $\beta^{(p)}_{f,h}$, alors les polynômes à coefficients entiers positifs,

$$M(X) = \sum_{p=0}^{n} m_p(f)X^p \quad \text{et} \quad B(X) = \sum_{p=0}^{n} b_p(f)X^p$$

satisfont, pour un polynôme $Q(X)$ à coefficients entiers positifs,

$$M(X) - B(X) = (1 + X)Q(X)\,. \tag{1.1.4}$$

Dans le cas sans bord, pour tout p, le nombre $m_p(f)$ est exactement le nombre de points critiques d'indice p de f.

Une idée pour le comprendre consiste à regarder la formulation 1.1.1 du Laplacien de Witten, $\Delta_{f,h} = h^2\Delta_H + |\nabla f|^2 + h\left(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}\right)$, lorsque nous sommes dans $\mathbb{R}^n$ avec sa métrique usuelle. Le symbole principal de cet opérateur est donné par $\xi^2 + |\nabla f(x)|^2$, dont la valeur 0 est atteinte lorsque $\nabla f(x) = 0$ (et $\xi = 0$). Les vecteurs propres associés à la valeur propre 0 - ou au moins aux valeurs propres de taille $\mathcal{O}(h^c)$ - se concentrent donc autour des points critiques de la fonction f lorsque $h \to 0^+$. Une étude plus poussée du symbole sous-principal nous montre que ces valeurs propres de $\Delta^{(p)}_{f,h}$ de taille $\mathcal{O}(h^c)$ sont exactement au nombre des points critiques de f d'indice p, où se concentrent les vecteurs propres associés. Cela laisse bien entendre une relation entre les points critiques de f et le noyau de $\Delta^{(p)}_{f,h}$, dont la dimension est étroitement liée aux nombres de Betti et indépendante de h en vertu de la Remarque 1.1.2 (c.f. [Hen][CyFrKiSi][Hel2][HeSj4]).

C'est le cœur de la démonstration de Witten des inégalités de Morse. Remarquons également que dans le cas d'une variété sans bord, (1.1.4) est bien équivalente aux inégalités fortes de Morse :

$$\left\{\begin{array}{ll} \sum_{p=0}^{k} & (-1)^{k-p}m_p(f) \geq \sum_{p=0}^{k}(-1)^{k-p}b_p(f)\ ,\ 0 \leq k < n \\ \\ \sum_{p=0}^{n} & (-1)^{p}m_p(f) = \sum_{p=0}^{n}(-1)^{p}b_p(f)\,. \end{array}\right.$$

Remarque 1.1.3. *Grâce à cette structure de complexe avec carrés autoadjoints pour le Laplacien de Witten,*

$$\Delta_{f,h} = (d_{f,h} + d^*_{f,h})^2\,,$$

il est équivalent de connaître les petites valeurs singulières de $d_{f,h}$ et de $d^*_{f,h}$ ou les petites valeurs propres de $\Delta^{(p)}_{f,h}$. En particulier, s'agissant de 0-formes, les valeurs singulières à considérer sont uniquement celles de $d^{(0)}_{f,h}$.

Nous n'avons malheureusement plus cette stucture si nous nous intéressons à des opérateurs de Fokker-Planck issus de la théorie cinétique comme ceux étudiés dans [BiLe][Leb][HerSjSt][HerHiSj][HerNi]. Dans ce cas, nous avons bien une structure de complexe de type "carrée" satisfaite par des déformations des opérateurs précédents,

$$\Delta^{(p),A}_{f,h} = (d^A_{f,h} + d^{A,*}_{f,h})^2\,;$$

cependant, l'opérateur $\Delta^{(p),A}_{f,h}$ n'est plus autoadjoint.

Petites valeurs propres de Laplaciens de Witten

Les études des complexes de Witten faites dans [Wit] et dans [ChLi], pour les cas à bord avec conditions de Dirichlet et de Neumann, utilisent le fait que la dimension de $\mathrm{Ran}1_{I(h)}(\overline{\Delta^{(p)}_{f,h}})$ (où $\overline{\Delta^{(p)}_{f,h}}$ est l'une des réalisations autoadjointes du Laplacien de Witten mentionnées précédemment) est, pour h assez petit, finie, indépendante de h et que les vecteurs propres engendrant ces espaces se concentrent près des points critiques de f.

Plus précisément, dans le cas sans bord étudié dans [Wit], cette dimension est le nombre de points critiques d'indice p de la fonction de Morse f et les vecteurs propres associés se concentrent autour de ces points critiques.

Les travaux de K. C. Chang et J. Liu dans [ChLi] étendent ce résultat aux cas à bord. Il convient alors de supposer quelques conditions supplémentaires sur la fonction f au bord et de généraliser quelque peu la notion de point critique au bord en fonction des conditions étudiées.

Cependant, ils ne s'intéressent en aucun cas à la taille effective de ces valeurs propres. Seul le fait qu'elles sont plus petites que h^c est nécessaire.

Comme les nombres de Betti sont des invariants topologiques de la variété, ils peuvent ainsi jongler avec les données géométriques ; cela ne modifie en rien les dimensions concernées. Autrement dit, ils peuvent se ramener à des hypothèses simplificatrices concernant la métrique.

Par contre, si nous voulons étudier précisément les petites valeurs propres de ces Laplaciens, comme cela est fait dans [HeKlNi], [HeNi] et au Chapitre 4, nous sommes obligés de travailler avec une métrique fixée, l'expression exacte des valeurs propres dépendant de la métrique. Nous devons donc mener une étude géométrique plus fine puisque la géométrie de la variété, inhérente à la métrique fixée, est imposée.

1.1.3 Lien avec l'approche probabiliste

En Physique statistique, dans le cadre de l'étude des processus de diffusion réversibles et de la théorie des grandes déviations de ces processus, les petites valeurs propres du Laplacien de Witten correspondent aux inverses de temps de sortie d'états métastables pour une particule soumise à un champ de gradient, les états métastables étant les minima locaux de la fonction de Morse associée au Laplacien de Witten.

Le champ de gradient correspond en fait à la fonction de Morse et pour sortir d'un état métastable, une particule doit franchir un col, i.e. un point critique d'indice 1 de cette fonction (c.f. [FrWe][BoEcGaKl][BoGaKl]) .

Expliquons un peu le lien entre Laplacien de Witten et processus stochastiques. Nous renvoyons par exemple le lecteur à [Ris] et [Ev] pour de plus amples détails. Les calculs exposés sont formels, c'est-à-dire que nous ne tenons pas compte d'hypothèses de régularité ou de ce genre.

Nous nous plaçons de plus ici dans $\mathbb{R}^n$ et non dans une variété compacte. Cependant, les propriétés discutées dans les précédentes parties dans le cas d'une variété compacte sans bord restent valables dans $\mathbb{R}^n$ à condition de supposer certaines hypothèses sur le comportement Ĺ l'infini de la fonction f (on peut par exemple regarder [HeKlNi]).

Equations différentielles stochastiques

Prenons

$$b : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^n \quad \text{et} \quad B : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{M}_n(\mathbb{R})\,,$$

où $\mathbb{M}_n(\mathbb{R})$ désigne l'espace des matrices carrées de taille n à coefficients réels. Considérons alors l'équation différentielle stochastique (SDE),

$$dX = b(X,t)dt + B(X,t)dW\,, \tag{1.1.5}$$

où $X(\cdot)$ est un processus stochastique à valeurs dans $\mathbb{R}^n$ et $W(\cdot)$ est un mouvement brownien de dimension n.

Remarque 1.1.4. *On dit aussi que $W(\cdot)$ est un processus de Wiener de dimension n ou encore que $dW(\cdot)$ est un "bruit blanc" de dimension n.*

Si v_0 est une observable indépendante de la variable t, les formules de Itô mènent à la relation suivante :

$$v_0(X(t)) = v_0(X(0)) + \int_0^t \left(b.\partial_x v_0 + \frac{1}{2}\partial_x.(B^tB)\partial_x v_0(X(s))\right) ds + \int_0^t \partial_x v_0(X(s)).BdW\,. \tag{1.1.6}$$

Le lien entre les équations différentielles stochastiques et les semi-groupes de diffusion est obtenu après calcul de l'espérance conditionnelle,

$$v(x_0,t) = \mathbb{E}(v_0(X,t);\, X(0) = x_0)\,,$$

pour une observable v_0. Cela conduit au générateur de semi-groupe suivant :

$$L = -b.\partial_x - \frac{1}{2}\partial_x.(B^tB).\partial_x\,. \tag{1.1.7}$$

C'est-à-dire que l'on obtient :

$$v(x_0,t) = v_0(x_0) + \int_0^t (-Lv)(s)ds\,,$$

qui n'est rien d'autre que la forme intégrale de :

$$v(t) = e^{-tL}v_0 \text{ ou encore } \begin{cases} \partial_t v = -Lv = b.\partial_x v + \frac{1}{2}\partial_x.(B^tB)\partial_x v \\ v(t=0) = v_0\,. \end{cases} \tag{1.1.8}$$

Application aux processus de diffusion réversibles

Considérons le cas où le champ de transport est un gradient, $b(x) = -\partial_x V(x)$ (le processus de diffusion est alors dit réversible), et $B = \sqrt{2}\,Id$. La SDE (1.1.5) se lit alors

$$dX = -\partial_x V(X)dt + \sqrt{2}dW$$

et le générateur du semi-groupe correpondant (c.f. (1.1.7)) vaut :

$$L = \partial_x V(x).\partial_x - \Delta_x = (\partial_x V(x) - \partial_x).\partial_x\,.$$

Illustrons l'équation précédente, en dimension 1, pour trois particules soumises au même champ de gradient V.

La mesure de probabilité invariante est donnée par :

$$\mu_V = \frac{e^{-V(x)}}{\int_{\mathbb{R}^n} e^{-V(x)}dx}dx$$

(à condition que e^{-V} soit dans $L^1(\mathbb{R}^n, dx)$).

Supposons donc que $e^{-V} \in L^1(\mathbb{R}^n, dx)$, ou de façon équivalente, que $e^{-\frac{V}{2}} \in L^2(\mathbb{R}^n, dx)$ et écrivons :

$$\begin{aligned} e^{-\frac{V}{2}}Le^{\frac{V}{2}} &= e^{-\frac{V}{2}}(\partial_x V(x) - \partial_x).(e^{\frac{V}{2}}(\partial_x + \frac{\partial_x V(x)}{2})) \\ &= e^{-\frac{V}{2}}[e^{\frac{V}{2}}(\partial_x V(x).\partial_x + \frac{|\partial_x V(x)|^2}{2}) - \frac{\partial_x V(x)}{2}e^{\frac{V}{2}}(\partial_x + \frac{\partial_x V(x)}{2}) \\ &\qquad - e^{\frac{V}{2}}(\Delta_x + \frac{\partial_x V(x)}{2}.\partial_x + \frac{\Delta_x V(x)}{2})] \\ &= -\Delta_x + \frac{1}{4}|\partial_x V(x)|^2 - \frac{1}{2}\Delta_x V(x)\,. \end{aligned}$$

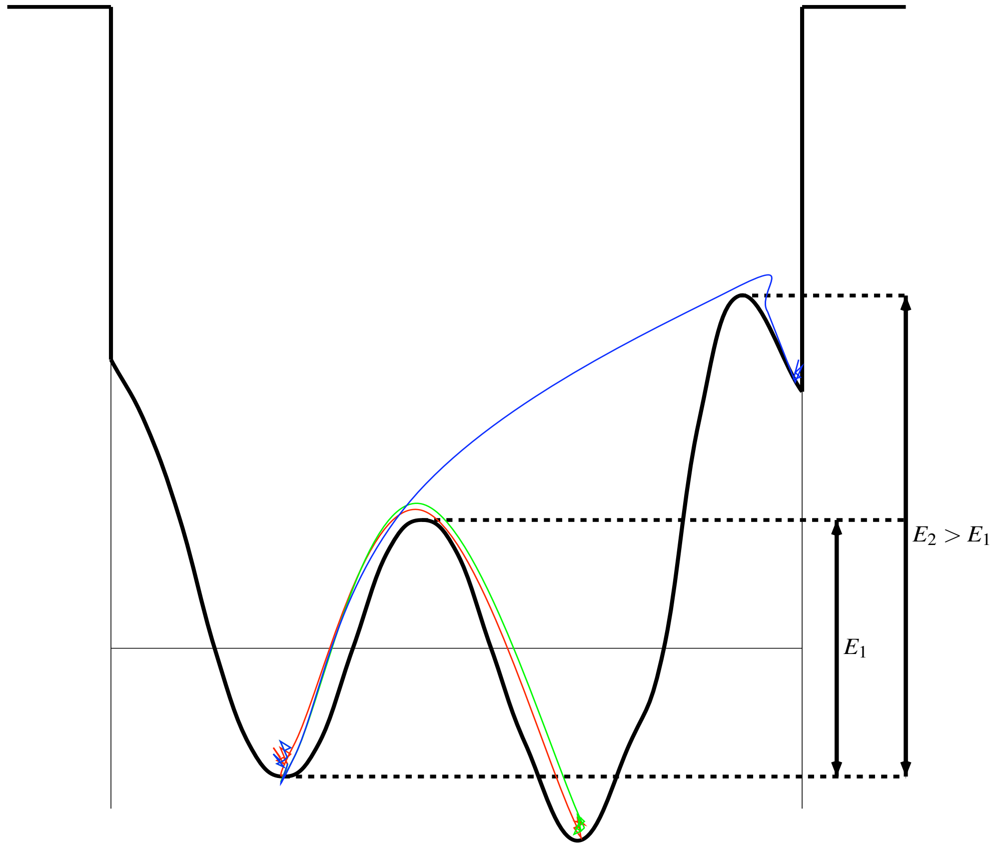


FIGURE 1.3 – Trois particules soumises au même champ de gradient (bien sûr, les mouvements sont en réalité beaucoup plus chaotiques).

Revenant à l'équation (1.1.2), cette dernière expression n'est rien d'autre que le Laplacien de Witten associé à $f = \frac{V}{2}$ et $h = 1$ agissant sur les fonctions. Ce qui revient encore à écrire :

$$e^{-\frac{V}{2}} L e^{\frac{V}{2}} = \Delta^{(0)}_{\frac{V}{2},1} \, .$$

Cas de problèmes à bord

L'intuition probabiliste, ainsi que les résultats obtenus dans [HeNi] et au Chapitre 4, montrent que les problèmes à bord avec conditions de Dirichlet et de Neumann correspondent respectivement à des potentiels $-\infty$ et $+\infty$ sur le bord (ou à l'extérieur).

Cela revient encore à dire que dans le cas avec conditions au bord de Dirichlet, une particule tend à sortir de la variété au bout d'un temps fini (car 0 n'est pas valeur propre de $\Delta^{D}_{f,h}$) tandis que dans le cas avec conditions au bord de Neumann, la particule y est piégée (car 0 est valeur propre de $\Delta^{N}_{f,h}$). Ceci est illustré par les deux figures suivantes, les Figures 1.4 et 1.5.

Ainsi, au flacon à col de cygne évoqué au début de cette introduction (sans liquide et à basse température) correspond un problème mixte Dirichlet-Neumann : des conditions de Neumann sur la paroi du flacon et des conditions de Dirichlet à l'entrée de celui-ci, là où l'air peut s'infiltrer. Les parois du flacon représentent en effet un potentiel infranchissable pour une bactérie située à l'intérieur de celui-ci tandis qu'une bactérie arrivant à l'entrée du flacon en sort définitivement.

1.2 Problème d'algèbre linéaire (Chapitre 2, [Lep1])

La démonstration finale de [HeKlNi] et de [HeNi] est donnée après réduction à un problème d'algèbre linéaire en dimension finie. La démarche qui est suivie dans ces articles pour résoudre ce problème d'algèbre linéaire est identique et repose sur des outils de théorie spectrale assez subtils.

L'idée était donc initialement d'extraire ce résultat pour l'écrire indépendamment de ces articles, comme cela a été partiellement fait dans [Nie1]. Ainsi, pour des démonstrations futures de problèmes similaires, et en particulier dans notre cas pour le problème avec conditions de Neumann, on pourrait s'y référer.

Avant d'énoncer le théorème et ses hypothèses, rappelons brièvement le cadre dans lequel nous nous plaçons.

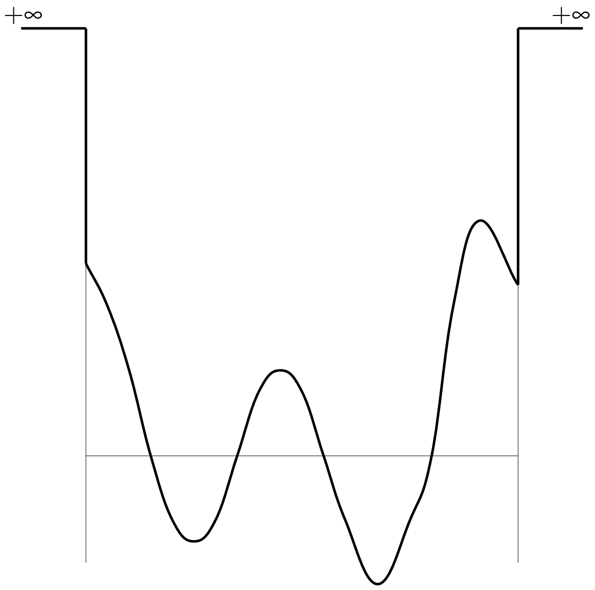


Figure 1.4 – Potentiel au bord, cas Neumann.

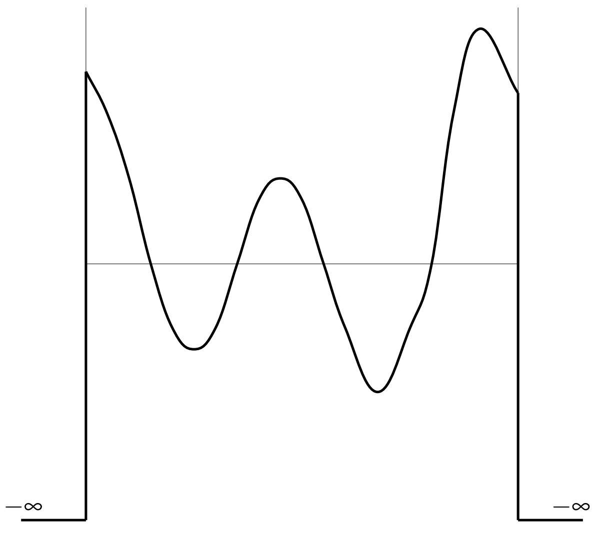


Figure 1.5 – Potentiel au bord, cas Dirichlet.

On veut déterminer les petites valeurs propres de réalisations autoadjointes de Laplaciens de Witten agissant sur les 0-formes. Une idée naturelle pourrait consister à chercher des vecteurs propres approchés pour ces Laplaciens.

Or, grâce à la structure du complexe de Witten avec carrés autoadjoints, $\Delta^{(0)}_{f,h} = d^{(0),*}_{f,h} d^{(0)}_{f,h}$ sur les 0-formes (et plus généralement $\Delta^{(p)}_{f,h} = (d^*_{f,h} + d_{f,h})^2$ sur les p-formes), il s'avère plus naturel de travailler avec les valeurs singulières de $d^{(0)}_{f,h}$, qui sont les racines carrées des valeurs propres de $\Delta^{(0)}_{f,h}$.

Avec les notations de la partie précédente concernant le complexe de Witten, $B(h) := \beta^{(0)}_{f,h} = d_{f,h}\big|_{F^{(0)}}$ envoie $F^{(0)}$, de dimension finie m_0, sur $F^{(1)}$, de dimension finie m_1, et $A_0(h) := \Delta^{(0)}_{f,h}\big|_{F^{(0)}}$ satisfait $A_0(h) = B^*(h)B(h)$.

Les hypothèses et le résultat du Chapitre 2 sont alors les suivants :

Hypothèse 1.2.1. *Il existe deux bases (de $F^{(0)}$ et de $F^{(1)}$ respectivement) dépendant de $(\varepsilon, h) \in (0, \varepsilon_0] \times (0, h_0]$ et un réel strictement positif α indépendant de $(\varepsilon\,, h) \in (0, \varepsilon_0] \times (0,\, h_0]$ tels que :*

$$\begin{aligned}\psi^{(0)}_k = \psi^{(0)}_k(\varepsilon, h)\ (k \in \{1, \dots, m_0\})\,, \left\langle \psi^{(0)}_k \mid \psi^{(0)}_{k'} \right\rangle = \delta_{kk'} + \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}}) \quad ,\\ \psi^{(1)}_j = \psi^{(1)}_j(\varepsilon, h)\ (j \in \{1, \dots, m_1\})\,, \left\langle \psi^{(1)}_j \mid \psi^{(1)}_{j'} \right\rangle = \delta_{jj'} + \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}}) \quad .\end{aligned}$$

Notons que les quasimodes ci-dessus dépendent de h mais aussi d'un autre paramètre ε. Ce paramètre provient de la construction des quasimodes faite dans [HeKlNi], [HeNi] et au Chapitre 4.

La notation $\mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})$ signifie que la majoration par $e^{-\frac{\alpha}{h}}$ dépend du paramètre ε.

Comme nous travaillons avec un $h > 0$ très petit, l'Hypothèse 1.2.1 signifie donc que nous nous donnons deux bases presque orthonormales, l'une pour $F^{(0)}$ et l'autre pour $F^{(1)}$.

Hypothèse 1.2.2. *Il existe aussi une application injective $j : \{1, \dots, m_0\} \to \{1, \dots, m_1\}$, une suite strictement décroissante de réels $(\alpha_k)_{k\in\{1,\dots,m_0\}}$, et $d > 0$ (indépendant de $(\varepsilon\,, h) \in (0, \varepsilon_0] \times (0,\, h_0]$) tels que :*

$$\begin{aligned}&\forall \varepsilon \in (0, \varepsilon_0],\ \exists\, C_\varepsilon > 1, \forall k \in \{1, \dots, m_0\}\,,\\ &\qquad \forall h \in (0, h_0], \quad C_\varepsilon^{-1} e^{-\frac{\alpha_k + d\varepsilon}{h}} \le \left| \left\langle \psi^{(1)}_{j(k)} \mid B(h)\psi^{(0)}_k \right\rangle \right| \le C_\varepsilon e^{-\frac{\alpha_k - d\varepsilon}{h}}\\ &\qquad \forall h \in (0, h_0], \forall j' \neq j(k), \quad \left| \left\langle \psi^{(1)}_{j'} \mid B(h)\psi^{(0)}_k \right\rangle \right| \le C_\varepsilon e^{-\frac{\alpha_k + \alpha}{h}}\,.\end{aligned}$$

Cette deuxième hypothèse, un peu technique, vient de l'appariement des minima locaux et des points selles d'indice 1 de la fonction de Morse f effectué dans [HeKlNi], [HeNi] et au Chapitre 4 (la façon adéquate d'apparier ces points a été établie dans les travaux probabilistes [FrWe][BoEcGaKl][BoGaKl]).

Elle peut en fait se résumer de la façon suivante : la matrice constituée de tous les produits scalaires estimés, $B'(h) = \left(\left\langle \psi_j^{(1)} \mid B(h)\psi_k^{(0)} \right\rangle\right)_{j,k}$ (avec $j \in \{1, \ldots, m_1\}$ et $k \in \{1, \ldots, m_0\}$) possède un coefficient par colonne exponentiellement grand par rapport à tous les autres, à la position $(j(k), k)$. De plus, ces gros coefficients sont eux-mêmes exponentiellement ordonnés par ordre croissant du numéro de colonne. Enfin, l'injectivité de l'application j assure qu'il y a au moins autant de lignes que de colonnes et qu'il ne peut y avoir deux de ces gros coefficients sur la même ligne. Par contre, deux coefficients d'une même ligne ne sont a priori pas comparables (c.f. l'exemple donné ci-dessous).

Enonçons maintenant le théorème du Chapitre 2.

Théorème 1.2.3. *Sous les Hypothèses 1.2.1 et 1.2.2, il existe des réels strictement positifs $h_0' \leq h_0$ et $\varepsilon_0' \leq \varepsilon_0$ tels que les valeurs propres $0 \leq \lambda_1(h) \leq \cdots \leq \lambda_{m_0}(h)$ de $A_0(h)$ satisfont :*

$$0 < \lambda_1(h) < \cdots < \lambda_{m_0}(h)\,,$$
$$\forall k \in \{1, \ldots, m_0\}\,, \quad \lambda_k(h) = \left|\left\langle \psi_{j(k)}^{(1)} \mid B(h)\psi_k^{(0)} \right\rangle\right|^2 (1 + \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}))\,,$$

où $\eta > 0$ est indépendant de $(\varepsilon, h) \in (0, \varepsilon_0'] \times (0, h_0']$.

Ce théorème nous dit ainsi que les valeurs propres de $A_0(h)$ sont, à une perturbation exponentiellement petite près, les gros coefficients des colonnes de la matrice $B'(h)$.

La démonstration de ce résultat effectuée dans [HeKlNi] et dans [HeNi] est assez compliquée car elle repose sur des arguments récursifs assez subtils utilisant des outils de théorie spectrale.

Dans le Chapitre 2, elle est en fait simplement ramenée à la combinaison du pivot de Gauss, appliqué à la matrice $B'(h)$, aux inégalités de Fan, inégalités qui étaient déjà utilisées dans [HeKlNi] et dans [HeNi].

Donnons une illustration de ce résultat dans un cas simple. Considérons $A_0(h)$ un endomorphisme de $\mathbb{R}^2$ et $B(h)$ une application linéaire de $\mathbb{R}^2$ dans $\mathbb{R}^3$ tels que $A_0(h) = B^*(h)B(h)$ et la matrice $B'(h)$ soit :

$$B'(h) = \begin{pmatrix} e^{-\frac{4}{h}} & e^{-\frac{1}{h}} \\ e^{-\frac{10}{h}} & e^{-\frac{2}{h}} \\ e^{-\frac{3}{h}} & e^{-\frac{2}{h}} \end{pmatrix}.$$

Le gros coefficient de la première colonne est $e^{-\frac{3}{h}}$ et celui de la deuxième colonne est $e^{-\frac{1}{h}}$. L'application injective j est donc l'application de $\{1,2\}$ dans $\{1,2,3\}$ envoyant 1 sur 3 et 2 sur 1 et le Théorème 1.2.3 nous assure alors l'existence d'un $\eta > 0$ tel que la plus petite valeur propre de $A_0(h)$ est $(e^{-\frac{3}{h}})^2(1+\mathcal{O}(e^{-\frac{\eta}{h}}))$ et l'autre $(e^{-\frac{1}{h}})^2(1+\mathcal{O}(e^{-\frac{\eta}{h}}))$. D'après la preuve du Chapitre 2, la constante η peut d'ailleurs ici être choisie égale à 1.

Un point de la preuve consiste à remarquer que $B'(h)$ est presque la matrice de $B(h)$ dans des bases orthonormales. Ainsi, les valeurs propres de $A_0(h)$ sont presque celles de la matrice $B'^*(h)B'(h)$. On peut donc s'amuser à vérifier que les valeurs propres de $B'^*(h)B'(h)$ sont bien les valeurs données ci-dessus.

Remarquons aussi que le Théorème 1.2.3 n'est pas spécifique au Laplacien de Witten agissant sur les 0-formes mais à la combinaison du caractère autoadjoint de cet opérateur à sa structure de complexe de dimension finie (c.f. Remarque 1.1.3). Ce résultat pourrait aussi s'appliquer au Laplacien de Witten agissant sur les p-formes, à condition d'avoir les Hypothèses 1.2.1 et 1.2.2. Mais il convient alors de prendre respectivement pour $B(h)$ les applications $\beta^{(p)}_{f,h}$ et $\beta^{(p+1),*}_{f,h}$.

1.3 Méthodes WKB pour des problèmes à bord (Chapitre 3, [Lep2])

Dans la partie précédente, nous expliquions comment obtenir les valeurs propres de Laplaciens de Witten autoadjoints (agissant sur les 0-formes), une fois obtenues certaines estimations de produits scalaires utilisant des 0-formes et des 1-formes bien choisies. Ces formes sont des vecteurs propres approchés pour $\Delta^{(0)}_{f,h}$ et $\Delta^{(1)}_{f,h}$.

Pour affirmer l'existence d'un développement asymptotique (pour h au voisinage de 0^+), et en particulier pour calculer précisément le premier terme de ce développement, on utilise, dans [HeKlNi], [HeNi] et au Chapitre 4, une approximation WKB de la 1-forme en question.

Ces 1-formes sont concentrées autour de certains points critiques (d'indice 1) comme cela a déjà été mentionné. Dans le cas sans bord [HeKlNi], les approximations WKB sont issues de l'analyse faite dans [HeSj4]. Cette analyse est d'ailleurs tout à fait générale et ne se limite pas qu'aux 1-formes. Dans [HeNi], traitant le cas à bord avec conditions de Dirichlet, il faut en plus certaines approximations WKB localisées près du bord. Or une simple astuce permet d'éviter une analyse complète au bord dans l'esprit de [HeSj4].

Cependant, ce genre d'astuce ne convient plus quand il s'agit de regarder le cas à bord avec conditions de Neumann. Une étude générale au bord est donc nécessaire, du moins concernant les 1-formes.

Dans le Chapitre 3, nous expliquons donc comment construire, au bord, des approximations WKB pour des p-formes, et ce pour les conditions de Neumann et de Dirichlet.

Donnons ici une version de ces résultats, néanmoins moins explicite que celle donnée au Chapitre 3 pour ne pas trop alourdir de notations et de définitions. Les coordonnées locales qui y sont introduites sont telles que le bord corresponde localement à $\{x_n = 0\}$ et l'intérieur à $\{x_n < 0\}$.

Théorème 1.3.1. ***Cas Neumann***
Soit U un point critique d'indice $p \in \{0, \dots, n-1\}$ de $f\big|_{\partial\Omega}$ en lequel la dérivée normale extérieure de f, $\frac{\partial f}{\partial n}(U)$, est strictement négative.
Il existe un système de coordonnées locales (x', x^n) centrées en U et, dans un voisinage de $x = 0$, u_p^{wkb} une solution $\mathcal{C}^\infty$ de :

$$\begin{aligned} \Delta_{f,h}^{(p)} u_p^{wkb} &= e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) && (1.3.1)\\ \mathbf{n} u_p^{wkb} &= 0 \ \ on\ \partial\Omega && (1.3.2)\\ \mathbf{n} d_{f,h} u_p^{wkb} &= 0 \ \ on\ \partial\Omega\,, && (1.3.3) \end{aligned}$$

où u_p^{wkb} est de la forme :

$$\begin{aligned} u_p^{wkb} &= a(x,h) e^{-\frac{\Phi}{h}}\,,\\ avec \quad a(x,h) &\sim \sum_k a^k(x) h^k \ \ et\ a^0(0) = dx^1 \wedge \cdots \wedge dx^p\,. \end{aligned}$$

Théorème 1.3.2. ***Cas Dirichlet***
Soit U un point critique d'indice $p \in \{1, \dots, n\}$ de $f\big|_{\partial\Omega}$ tel que $\frac{\partial f}{\partial n}(U) > 0$.
Il existe un système de coordonnées locales (x', x^n) centrées en U et, dans un voisinage de $x = 0$, u_p^{wkb} une solution $\mathcal{C}^\infty$ de :

$$\begin{aligned} \Delta_{f,h}^{(p)} u_p^{wkb} &= e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) && (1.3.4)\\ \mathbf{t} u_p^{wkb} &= 0 \ \ on\ \partial\Omega && (1.3.5)\\ \mathbf{t} d^*_{f,h} u_p^{wkb} &= 0 \ \ on\ \partial\Omega\,, && (1.3.6) \end{aligned}$$

où u_p^{wkb} est de la forme :

$$\begin{aligned} u_p^{wkb} &= a(x,h) e^{-\frac{\Phi}{h}}\,,\\ avec \quad a(x,h) &\sim \sum_k a^k(x) h^k \ \ et\ a^0(0) = dx^1 \wedge \cdots \wedge dx^{p-1} \wedge dx^n\,. \end{aligned}$$

La fonction Φ intervenant dans chacun des théorèmes est la distance d'Agmon au point U, c'est une distance dégénérée en U.

Dans le Théorème 1.3.1 (cas Neumann), nous nous plaçons au voisinage d'un point critique U d'indice p de $f\big|_{\partial\Omega}$ tel que $\frac{\partial f}{\partial n}(U) < 0$. Cela correspond à la généralisation de point critique d'indice p dans le bord avec laquelle nous travaillons au Chapitre 4.

Dans l'autre théorème, concernant le cas Dirichlet, la généralisation de point critique d'indice p dans le bord, utilisée dans [HeNi], est différente. Ce sont dans ce cas des points critiques U d'indice $p-1$ de $f\big|_{\partial\Omega}$ tels que $\frac{\partial f}{\partial n}(U) > 0$.

Ces résultats étendent aux variétés à bord ceux déjà établis dans [HeSj4], utilisés notamment dans [HeKlNi] pour approcher les vecteurs propres du Laplacien de Witten agissant sur les fonctions. Les résultats de [HeSj4] sont également utilisés dans [HeNi] et au Chapitre 4 lorsqu'il s'agit d'approcher les vecteurs propres se concentrant autour des points critiques intérieurs à la variété.

Notons aussi qu'il s'agit uniquement ici de construire des solutions WKB approchées et qu'il n'est pas question de comparer ces solutions avec des vecteurs propres effectifs.

Cette comparaison est un point essentiel qui reste à traiter si l'on veut obtenir les valeurs propres de Laplaciens de Witten agissant sur de vraies p-formes. Elle est d'ailleurs effectuée dans [HeNi] et au Chapitre 4 pour les 1-formes.

Les preuves de ces deux théorèmes que nous donnons au Chapitre 3 sont similaires et très techniques, celle du cas Dirichlet étant plus subtile.

L'utilisation d'un "bon système de coordonnées" locales (x', x^n), en particulier adapté à la métrique et aux conditions au bord - c'est-à-dire (1.3.2)(1.3.3) pour le problème de Neumann et (1.3.5)(1.3.6) pour celui de Dirichlet - est un point crucial de la démonstration. Cela permet notamment d'exprimer très simplement les composantes normales et tangentielles $\mathbf{n}$ et $\mathbf{t}$ dans le bord et de faire de nombreux calculs en coordonnées dans le même esprit que ceux faits dans [HeSj4].

On se ramène ensuite au cas sans bord traité dans [HeSj4], la variété sans bord étant ici le bord, justement.

Parlons un peu plus en détails de ce système de bonnes coordonnées (x', x^n) et prenons $n = 2$ pour simplifier les notations. L'idée est de se ramener le plus près possible du cas plat à bord, c'est-à-dire de presque se ramener, localement, à $\mathbb{R}^2_- = \mathbb{R} \times (-\infty, 0]$ muni de sa métrique euclidienne g_e usuelle,

$$g_e = d(x^1)^2 + d(x^2)^2 \sim G_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Dans $\mathbb{R}^2_-$ muni de cette métrique, prenons

$$\omega(x)\,,\ \omega'(x) = \omega'_1(x)dx^1 + \omega'_2(x)dx^2 \quad \text{et} \quad \omega''(x) = \omega''_{12}(x)dx^1 \wedge dx^2$$

une 0-forme, une 1-forme et une 2-forme. Les composantes normales de ces formes s'écrivent simplement, sur le bord $\{x^2 = 0\}$,

$$\mathbf{n}\omega = 0\,,\ \mathbf{n}\omega' = \omega'_2(x^1, 0)dx^2 \quad \text{et} \quad \mathbf{n}\omega'' = \omega''_{12}(x^1, 0)dx^1 \wedge dx^2\,.$$

Et les composantes tangentielles sont quant à elles :

$$\mathbf{t}\omega = \omega(x^1, 0)\,,\ \mathbf{t}\omega' = \omega'_1(x^1, 0)dx^1 \quad \text{et} \quad \mathbf{t}\omega'' = 0$$

(bien sûr, le cas des 0-formes est trivial car, par définition, la partie tangentielle d'une fonction est sa valeur au bord et sa partie normale est nulle).

Pour manier aussi aisément les coordonnées tangentielles et normales, nous utilisons un système de coordonnées (x^1, x^2) telle que le bord $\partial\Omega$ corresponde localement à $\{x^2 = 0\}$ et $\frac{\partial}{\partial x^2}$ soit orthogonal à $\frac{\partial}{\partial x^1}$ en tous points. Mais nous ne pouvons pas rendre plate la métrique pour autant, à cause de la courbure naturelle de la variété. Nous ne nous ramenons donc qu'à la métrique suivante :

$$g_0 = g_{11}(x^1, x^2)d(x^1)^2 + d(x^2)^2 \sim G_e = \begin{pmatrix} g_{11}(x^1, x^2) & 0 \\ 0 & 1 \end{pmatrix} ,$$

où le terme $g_{11}(x^1, x^2)$ dépend en général de x^2 (et de x_1), comme le montre par exemple la figure ci-desous.

1.4 Problème associé aux conditions de Neumann (Chapitre 4, [Lep3])

Dans le Chapitre 4, nous nous intéressons à l'étude des petites valeurs propres de la réalisation autoadjointe du Laplacien semi-classique de Witten agissant sur les 0-formes, avec conditions au bord de Neumann. Nous y montrons que ses petites valeurs propres (plus petites que h^c pour un certain

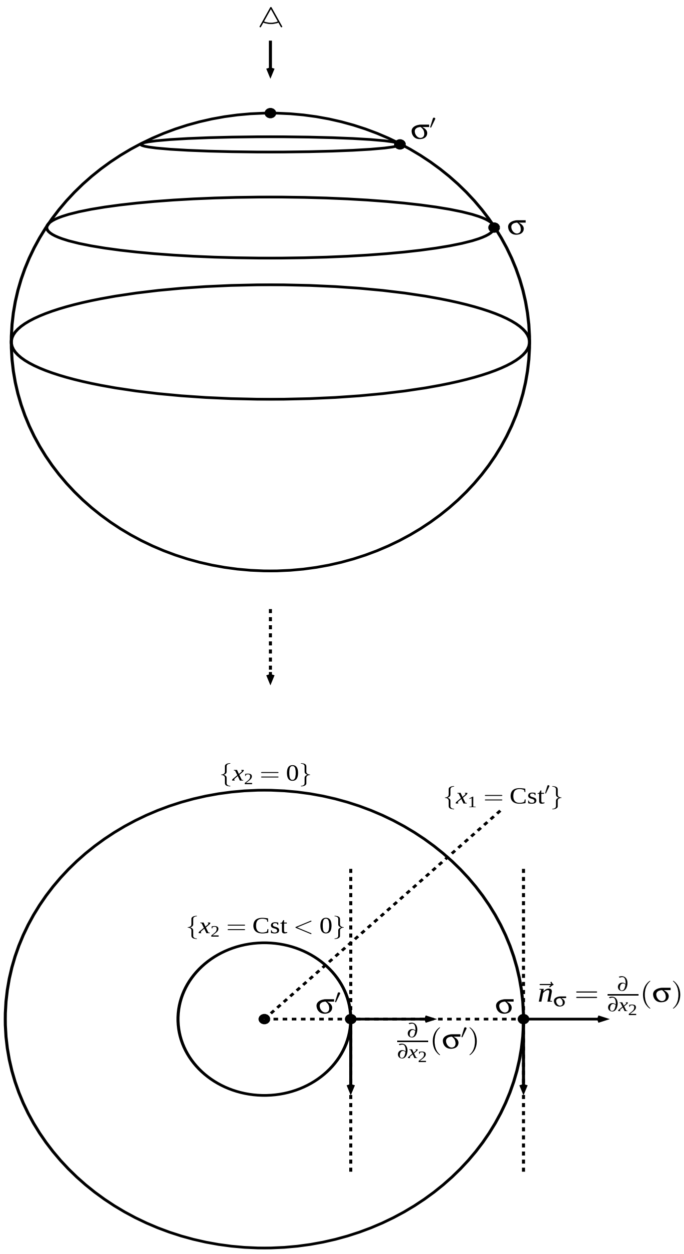


FIGURE 1.6 – La variété étudiée est ici la partie supérieure d'une sphère, délimitée par le parallèle passant par le point σ. La construction des coordonnées (x_1, x_2) faite au Chapitre 3 nous donne ici x_1 et x_2 respectivement constantes le long d'un grand axe et d'un parallèle. Si $g_{11}(x^1, x^2)$ ne dépendait que de x_1, les parallèles passant respectivement par σ et σ' auraient la même longueur.

$c > 1$) sont exponentionnellement petites et qu'elles admettent un développement asymptotique en fonction du petit paramètre $h > 0$ de l'analyse semi-classique.

Ce résultat est dans la continuation directe de ceux déjà établis dans le cas sans bord par M. Klein, B. Helffer et F. Nier dans [HeKlNi] et dans le cas à bord avec conditions de Dirichlet par B. Helffer et F. Nier dans [HeNi]. À la fin de cette partie, nous comparons d'ailleurs les résultats principaux de [HeNi] à ceux du Chapitre 4.

Le domaine de $\Delta^N_{f,h}$, la réalisation autoadjointe du Laplacien de Witten avec conditions de Neumann, est, comme nous l'avons déjà dit plus tôt,

$$D(\Delta^N_{f,h}) = \left\{\omega \in \Lambda H^2(\Omega)\,;\ \mathbf{n}\omega = 0 \text{ et } \mathbf{n}d_{f,h}\omega = 0\right\}\ .$$

Une fois ce domaine obtenu, nous montrons d'abord au Chapitre 4 le résultat suivant, relatif à la dimension des sous-espaces spectraux associés aux petites valeurs propres de $\Delta^N_{f,h}$.

Hypothèse 1.4.1. *La fonction de Morse f n'a pas de point critique sur le bord et $f\big|_{\partial\Omega}$ est également une fonction de Morse.*

Théorème 1.4.2.
Notons m^Ω_p le nombre de points critiques d'indice p de f dans Ω et $m^{\partial\Omega}_{p,-}$ celui de $f\big|_{\partial\Omega}$ dans $\partial\Omega$ en lesquels la dérivée normale de f est négative. Notons enfin $m^{\overline{\Omega},N}_p$ la somme

$$m^{\overline{\Omega},N}_p = m^\Omega_p + m^{\partial\Omega}_{p,-}\,.$$

Alors, sous l'Hypothèse 1.4.1, pour $h_0 > 0$ suffisamment petit, $h \in (0, h_0]$ et $p \in \{0, \dots, n\}$, le sous-espace spectral $F^{(p)} = \mathrm{Ran}1_{[0,h^{3/2})}(\Delta^{N,(p)}_{f,h})$ est de dimension finie $m^{\overline{\Omega},N}_p$.

Remarque 1.4.3. *Dans le cas des conditions au bord de Dirichlet étudié dans [HeNi], la dimension de $\mathrm{Ran}1_{[0,h^{3/2})}(\Delta^{D,(p)}_{f,h})$ est, pour h assez petit,*

$$m^{\overline{\Omega},D}_p = m^\Omega_p + m^{\partial\Omega}_{p-1,+}\,,$$

où $m^{\partial\Omega}_{p,+}$ est le nombre de points critiques de $f\big|_{\partial\Omega}$ (dans $\partial\Omega$) en lesquels la dérivée normale de f est positive (avec la convention $m^{\partial\Omega}_{0-1,+} = 0$).

Ce résultat est la version précise, concernant le cas Neumann, de ce que nous avons déjà évoqué plus tôt. C'est l'un des résultats de [ChLi], mais il convient de le redémontrer au Chapitre 4, car la métrique y est imposée.

Sa démonstration permet également de voir que les vecteurs propres sont effectivement concentrés près des points critiques considérés dans l'énoncé.

Ces points critiques d'indice p sont d'ailleurs ce que nous appelons au Chapitre 4 des points critiques *généralisés* d'indice p. Il s'agit, en plus des véritables points critiques d'indice p de f, des points critiques d'indice p de $f\big|_{\partial\Omega}$ en lesquels la dérivée normale de f est négative. Notons aussi que la dérivée normale de f est, en un point critique de $f\big|_{\partial\Omega}$, soit strictement positive, soit strictement négative. Cela découle simplement du fait que f est supposée ne pas avoir de point critique sur le bord.

De plus, les points critiques *généralisés* d'indice p dans le bord sont justement ceux autour desquels nous nous placions dans la section précédente, dans l'énoncé du Théorème 1.3.1, concernant les approximations WKB au bord, dans le cas Neumann. Et naturellement, les points critiques *généralisés* dans le bord, dans le cas des conditions de Dirichlet, sont ceux autour desquels nous nous placions dans le Théorème 1.3.2.

Remarquons enfin, f n'ayant pas de point critique dans le bord, que les points critiques *généralisés* d'indice 0 sont exactement les minima locaux de f (et en vertu de la Remarque 1.4.3, dans le cas de conditions au bord de Dirichlet, les points critiques *généralisés* d'indice 0 sont exactement les minima locaux à l'intérieur de la variété).

Comme cela est également fait dans les travaux antérieurs [HeKlNi] et [HeNi], il convient ensuite d'ordonner correctement les minima locaux de la fonction de Morse f, en accord avec les travaux probabilistes [BoGaKl], [BoEcGaKl] et [FrWe]. Ce rangement des minima locaux nous donne un couplage naturel entre ces minima et certains points selles *généralisés* d'indice 1.

Rappelons qu'en Physique statistique, les petites valeurs propres recherchées correspondent aux inverses de temps de sortie moyens d'états métastables, les minima de f, pour une particule soumise au champ de potentiel $-\nabla f$. Ceux-ci doivent être ordonnés de façon à ce que les temps de sortie moyens associés soient décroissants. De plus, pour aller d'un état métastable à un autre, une particule doit franchir un col, comme cela a déjà été évoqué dans la Partie 1.1.3.

Cet arrangement des minima locaux nécessite une étude des ensembles de sous-niveaux de la fonction de Morse f. Pour éviter certains problèmes techniques lors de la construction des quasimodes associés aux minima locaux et aux points critiques d'indice 1, nous nous plaçons au Chapitre 4 dans la situation suivante :

Hypothèse 1.4.4. *Toutes les valeurs critiques de f sont distinctes, ainsi que les quantités $f(U_j^{(1)}) - f(U_k^{(0)})$, où $U_j^{(1)}$ et $U_k^{(0)}$ désignent respectivement* *des points critiques généralisés d'indice* 1 *et d'indice* 0.

Nous nous retrouvons finalement avec deux suites ; l'une, $\left(U_k^{(0)}\right)_{k\in\left\{1,\ldots,m_0^{\overline{\Omega},N}\right\}}$, contenant tous les minima locaux et l'autre, $\left(U_{j(k)}^{(1)}\right)_{k\in\left\{1,\ldots,m_0^{\overline{\Omega},N}\right\}}$, contenant les points cols *généralisés* d'indice 1 associés. L'application injective j, de $\left\{1,\ldots,m_0^{\overline{\Omega},N}\right\}$ dans $\left\{1,\ldots,m_1^{\overline{\Omega},N}\right\}$, correspond à celle de la Section 1.2.

Grâce à l'Hypothèse 1.4.4, les $m_0^{\overline{\Omega},N}$ petites valeurs propres de $\Delta_{f,h}^{N,(0)}$ sont distinctes et la k-ième plus petite est de taille $A(h)e^{-2\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})}{h}}$, la plus petite étant 0, naturellement associée au vecteur propre $e^{-\frac{f}{h}}$.

La Figure 1.7 montre comment ordonner les minima locaux et obtenir le couplage minima locaux-points cols d'indice 1 dans le cas unidimensionnel.

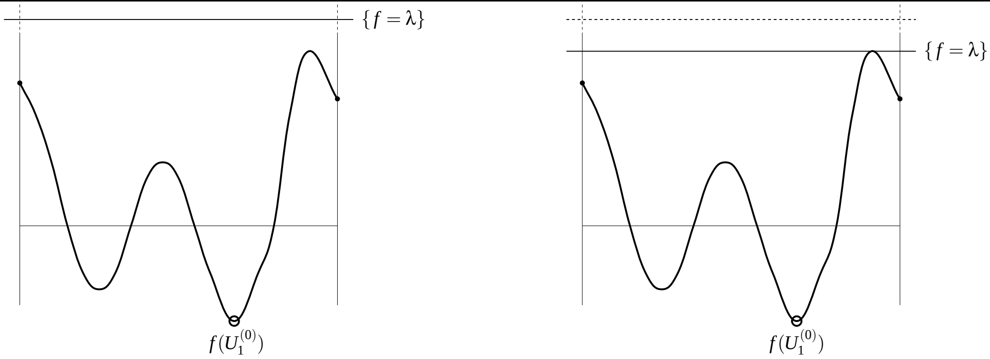
Remarque 1.4.5. *Dans la Figure 1.7, le point $U_{j(1)}^{(1)}$ n'apparaît pas. Cela est dû au fait que* 0 *est valeur propre du Laplacien de Witten associé aux conditions de Neumann. Il n'y a donc aucun point selle associé au minimum global de f, $U_1^{(0)}$. Une autre façon de le dire est que le terme $j(1)$ n'est pas défini. Pour garder des notations génériques, on pose dans ce cas $f(U_{j(1)}^{(1)})=+\infty$. La plus petite valeur propre recherchée,* 0, *est effectivement de taille $A(h)e^{-2\frac{f(U_{j(1)}^{(1)})-f(U_1^{(0)})}{h}}$.*

Une fois ces points critiques ordonnés, nous construisons deux familles de quasimodes, $\left(\psi_k^{(0)}\right)_{k\in\left\{1,\ldots,m_0^{\overline{\Omega},N}\right\}}$ pour $\Delta_{f,h}^{N,(0)}$ et $\left(\psi_j^{(1)}\right)_{j\in\left\{1,\ldots,m_1^{\overline{\Omega},N}\right\}}$ pour $\Delta_{f,h}^{N,(1)}$, $\psi_l^{(i)}$ étant concentré autour du point $U_l^{(i)}$.

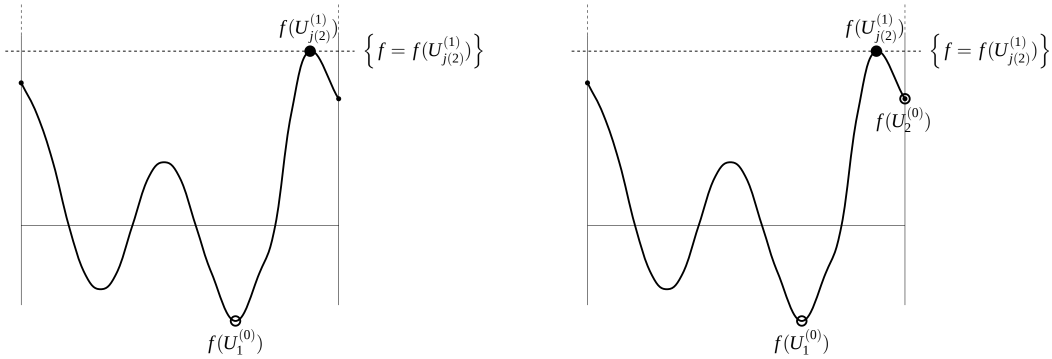
Bien sûr, dans la famille $\left(\psi_j^{(1)}\right)_{j\in\left\{1,\ldots,m_1^{\overline{\Omega},N}\right\}}$, les seules 1-formes qui vont réellement nous intéresser sont les $\psi_{j(k)}^{(1)}$, où k décrit $\left\{1,\ldots,m_0^{\overline{\Omega},N}\right\}$, ou plutôt $\left\{2,\ldots,m_0^{\overline{\Omega},N}\right\}$ d'après la remarque précédente.

Cette construction est assez délicate et il faut jongler avec plusieurs systèmes de coordonnées. Nous devons en effet tenir compte de trois géométries “imposées” : la géométrie de la variété $\overline{\Omega}$, en particulier celle du bord $\partial\Omega$, la géométrie liée à la fonction de Morse et enfin celle imposée par la métrique qui est fixée.

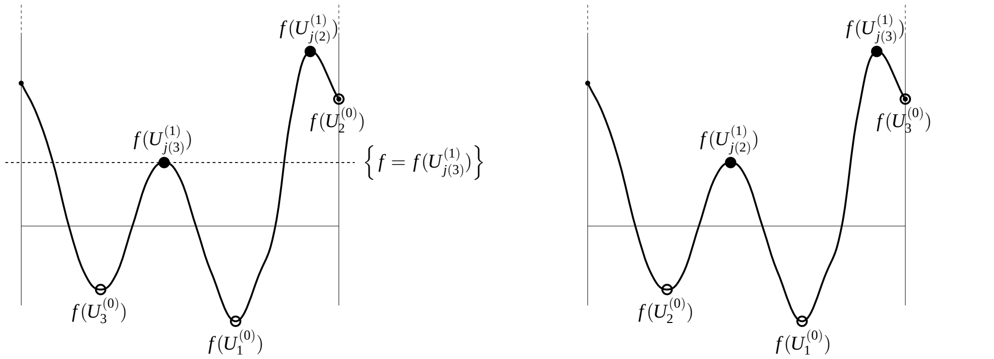
Nous devons ainsi manier des systèmes de coordonnées adaptés respectivement à la métrique et au bord (celui déjà évoqué dans la Partie 1.3 concernant le Chapitre 3), à la fonction de Morse f et au bord, à la distance



Notons $U_1^{(0)}$ le minimum global de f, $L(\lambda) = \{x \in M, f(x) < \lambda\}$ son ensemble de sous-niveau de hauteur λ et faisons décroître λ (en partant de $\lambda \sim +\infty$) jusqu'à ce que le nombre de composantes connexes de $L(\lambda)$ augmente de $+1$.



Notons $U_{j(2)}^{(1)}$ le point col de séparation des composantes connexes et $U_2^{(0)}$ le minimum global de la nouvelle composante connexe.



Réitérons le procédé (en partant de $\lambda = f(U_{j(2)}^{(1)})$) jusqu'à obtention de tous les minima locaux $U_k^{(0)}$ en notant $U_{j(k)}^{(1)}$ les cols de séparation. Réordonnons enfin les k de façon à ce que les quantités $f(U_{j(k)}^{(1)}) - f(U_k^{(0)})$ soient strictement décroissantes.

FIGURE 1.7 – Rangement des points critiques (Cas Neumann).

d'Agmon Φ et au bord, à $f+\Phi$ et au bord, et enfin à la métrique et aux fonctions f, Φ, $f+\Phi$.

Bien sûr, certains de ces systèmes peuvent coïncider, comme celui adapté à θ et au bord et celui adapté à la métrique et à θ, pour $\theta \in \{f, \Phi, f+\Phi\}$; mais ils ne peuvent cependant pas tous coïncider simultanément.

Avant de donner le résultat principal du Chapitre 4, définissons les quantités suivantes :

Définition 1.4.6. *Définissons, pour k dans $\left\{2, \ldots, m_0^{\overline{\Omega},N}\right\}$:*

$$\gamma_k(h) = \begin{cases} \dfrac{\left|\det \mathrm{Hess}\, f(U_k^{(0)})\right|^{\frac14}}{(\pi h)^{\frac{n}{4}}} & si \quad U_k^{(0)} \in \Omega \\ \left(\dfrac{-2\partial_n f(U_k^{(0)})}{h}\right)^{\frac12} \dfrac{\left|\det \mathrm{Hess}\, f|_{\partial\Omega}(U_k^{(0)})\right|^{\frac14}}{(\pi h)^{\frac{n-1}{4}}} & si \quad U_k^{(0)} \in \partial\Omega\,, \end{cases}$$

$$\delta_{j(k)}(h) = \begin{cases} \dfrac{\left|\det \mathrm{Hess}\, f(U_{j(k)}^{(1)})\right|^{\frac14}}{(\pi h)^{\frac{n}{4}}} & si \quad U_{j(k)}^{(1)} \in \Omega \\ \left(\dfrac{-2\partial_n f(U_{j(k)}^{(1)})}{h}\right)^{\frac12} \dfrac{\left|\det \mathrm{Hess}\, f|_{\partial\Omega}(U_{j(k)}^{(1)})\right|^{\frac14}}{(\pi h)^{\frac{n-1}{4}}} & si \quad U_{j(k)}^{(1)} \in \partial\Omega\,, \end{cases}$$

et

$$\theta_{j(k)}(h) = \begin{cases} \dfrac{h^{\frac12}}{\pi^{\frac12}} \dfrac{(\pi h)^{\frac{n}{2}}|\widehat{\lambda}_1^{\Omega}|^{\frac12}}{\left|\det \mathrm{Hess}\, f(U_{j(k)}^{(1)})\right|^{\frac12}} & si \quad U_{j(k)}^{(1)} \in \Omega \\ \dfrac{h^2}{-2\partial_n f(U_{j(k)}^{(1)})} \dfrac{(\pi h)^{\frac{n-2}{2}}|\widehat{\lambda}_1^{\partial\Omega}|^{\frac12}}{\left|\det \mathrm{Hess}\, f|_{\partial\Omega}(U_{j(k)}^{(1)})\right|^{\frac12}} & si \quad U_{j(k)}^{(1)} \in \partial\Omega\,, \end{cases}$$

où $\widehat{\lambda}_1^W$ est la valeur propre négative de $\mathrm{Hess}\, f|_W(U_{j(k)}^{(1)})$ pour $W = \Omega$ ou $W = \partial\Omega$.

Le théorème principal du Chapitre 4 est le suivant :

Théorème 1.4.7.
Il existe h_0 tel que, pour tout $h \in (0, h_0]$, le spectre de $\Delta_{f,h}^{N,(0)}$ inclus dans $[0, h^{\frac32})$ est constitué de $m_0^{\overline{\Omega},N}$ valeurs propres, $0 = \lambda_1(h) < \ldots < \lambda_{m_0^{\overline{\Omega},N}}(h)$, toutes de multiplicité 1.
De plus, les $m_0^{\overline{\Omega},N} - 1$ valeurs propres non nulles sont exponentiellement petites et admettent le développement asymptotique suivant :

$$\lambda_k(h) = \gamma_k^2(h)\, \delta_{j(k)}^2(h)\, \theta_{j(k)}^2(h)\, e^{-2\frac{f(U_{j(k)}^{(1)}) - f(U_k^{(0)})}{h}} \left(1 + h c_k^1(h)\right)$$

où $\gamma_k(h)$, $\delta_{j(k)}(h)$ et $\theta_{j(k)}(h)$ sont définis dans la proposition précédente et $c_k^1(h)$ admet le développement suivant : $c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m}$.

Interprétation et comparaison avec le cas Dirichlet

Commençons d'abord par donner le résulat obtenu dans [HeNi] dans le cas à bord avec conditions de Dirichlet.

Théorème 1.4.8.
Il existe h_0 tel que, pour tout $h \in (0, h_0]$, le spectre de $\Delta_{f,h}^{D,(0)}$ inclus dans $[0, h^{\frac{3}{2}})$ est constitué de $m_0^{\overline{\Omega},D}$ valeurs propres, $0 < \lambda_1(h) < \ldots < \lambda_{m_0^{\overline{\Omega},D}}(h)$, toutes de multiplicité 1, exponentiellement petites et admettant le développement asymptotique suivant :

$$\lambda_k(h) = \frac{h}{\pi}|\widehat{\lambda}_1^{\Omega}| \frac{\left|\det \operatorname{Hess} f(U_k^{(0)})\right|^{\frac{1}{2}}}{\left|\det \operatorname{Hess} f(U_{j(k)}^{(1)})\right|^{\frac{1}{2}}} \left(1 + hc_k^1(h)\right) \times e^{-2\frac{f(U_{j(k)}^{(1)}) - f(U_k^{(0)})}{h}} \quad , \quad si\ U_{j(k)}^{(1)} \in \Omega \ ,$$

et

$$\lambda_k(h) = \frac{2h^{\frac{1}{2}}\partial_n f(U_{j(k)}^{(1)})}{\pi^{\frac{1}{2}}} \frac{\left|\det \operatorname{Hess} f(U_k^{(0)})\right|^{\frac{1}{2}}}{\left|\det \operatorname{Hess} f\big|_{\partial\Omega}(U_{j(k)}^{(1)})\right|^{\frac{1}{2}}} \left(1 + hc_k^1(h)\right) \times e^{-2\frac{f(U_{j(k)}^{(1)}) - f(U_k^{(0)})}{h}} \quad , \quad si\ U_{j(k)}^{(1)} \in \partial\Omega \ ,$$

où $c_k^1(h)$ admet le développement suivant : $c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m}$.

Une première remarque que l'on peut faire au sujet de la différence entre ces deux résultats est le fait que 0 est valeur propre dans le cas Neumann mais ne l'est pas dans le cas Dirichlet, car $e^{-\frac{f}{h}}$ n'est dans le domaine que pour des conditions de Neumann (c.f. Remarque 1.4.3). Cela a déjà été évoqué précédemment et correspond au fait que dans le cas Neumann, une particule est piégée dans la variété (comme dans le cas sans bord) tandis que dans le cas Dirichlet, elle en sortira au bout d'un certain temps (c.f. Figures 1.4 et 1.5).

Observons aussi que l'énoncé du résultat dans le cas Dirichlet est plus concis ; nous n'introduisons pas dans ce cas de définition équivalente à la

Définition 1.4.6. Cela est dû au fait que pour calculer la k-ième valeur propre de $\Delta^D_{f,h}$, nous regardons seulement si le point col *généralisé* $U^{(1)}_{j(k)}$ est ou non dans le bord $\partial\Omega$. Dans ce cas, les points critiques *généralisés* de f d'indice 0 sont en effet exactement les minima locaux à l'intérieur de la variété, c'est-à-dire les points critiques de f d'indice 0 habituels. Rappelons également que les points critiques *généralisés* d'indice 1 de f dans le bord sont les minima locaux de $f\big|_{\partial\Omega}$ en lesquels la dérivée normale de f est positive.

D'ailleurs, les résultats obtenus dans les cas Neumann et Dirichlet pour la k-ième valeur propre, ou du moins pour son terme principal, sont les mêmes si les points $U^{(0)}_k$ et $U^{(1)}_{j(k)}$ sont simultanément intérieurs pour chacun de ces cas.

Dans cette situation, ce sont effectivement des points critiques d'indices respectifs 0 et 1 et, le calcul de ces valeurs propres étant effectué en utilisant des quasimodes supportés à l'intérieur de la variété, nous nous ramenons à l'étude du cas sans bord faite dans [HeKlNi].

Commentons un peu plus précisément les résulats énoncés en commençant par le cas où les points critiques $U^{(0)}_k$ et $U^{(1)}_{j(k)}$ sont intérieurs (ce qui revient à regarder le cas sans bord).

Tout d'abord, la valeur propre $\lambda_k(h)$ est d'autant plus petite que la différence $f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)$ est grande (elle est en effet proportionnelle à $e^{-2\frac{f(U^{(1)}_{j(k)})-f(U^{(0)}_k)}{h}}$). Physiquement, cela signifie que plus le puits de potentiel correspondant au minimum local $U^{(0)}_k$ est profond, plus une particule y étant piégée a du mal à en sortir. La valeur popre $\lambda_k(h)$ est également proportionnelle aux quantités

$$\left|\det \mathrm{Hess}\, f(U^{(0)}_k)\right|^{\frac{1}{2}} \quad \text{et} \quad \frac{|\widehat{\lambda}^{\Omega}_1|}{\left|\det \mathrm{Hess}\, f(U^{(1)}_{j(k)})\right|^{\frac{1}{2}}}.$$

Ainsi, plus les valeurs propres de Hess $f(U^{(0)}_k)$ sont petites et plus $\lambda_k(h)$ l'est. En d'autres termes, plus le puits de potentiel associé à $U^{(0)}_k$ est plat et plus une particule qui y est piégée y reste longtemps.

Concernant enfin le terme $\frac{|\widehat{\lambda}^{\Omega}_1|}{\left|\det \mathrm{Hess}\, f(U^{(1)}_{j(k)})\right|^{\frac{1}{2}}}$, $|\widehat{\lambda}^{\Omega}_1|$ (la valeur absolue de la valeur propre négative de Hess $f(U^{(1)}_{j(k)})$) est d'autant plus petite, et les autres valeurs propres de Hess $f(U^{(1)}_{j(k)})$, $\widehat{\lambda}^{\Omega}_2, \dots, \widehat{\lambda}^{\Omega}_n$, d'autant plus grandes, que le col $U^{(1)}_{j(k)}$ est difficile à passer. Ceci est illustré par les Figures 1.8 et 1.9.

Notons enfin que le préfacteur du développement asymptotique de la valeur propre $\lambda_k(h)$, i.e. le premier terme du développement asymptotique du

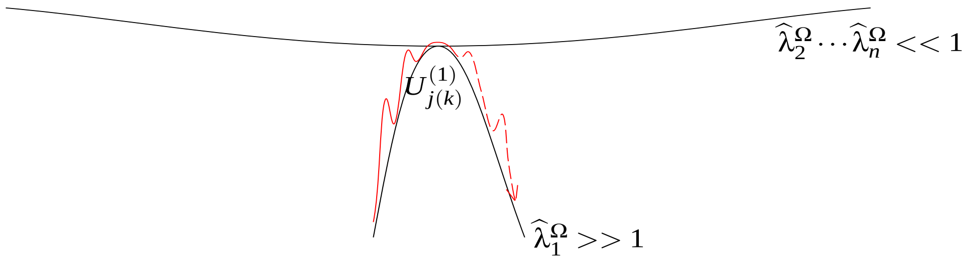


Figure 1.8 – Col “facile” à franchir, à l’intérieur.

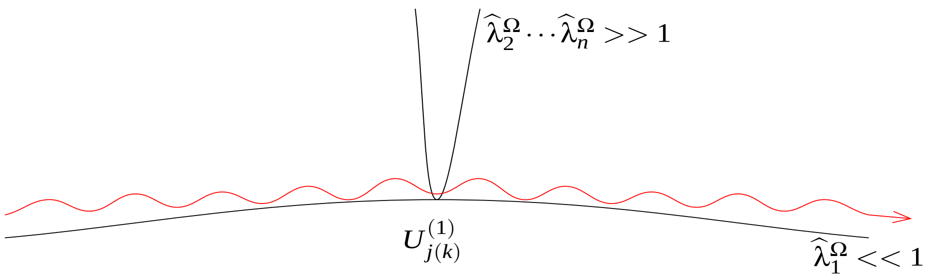


Figure 1.9 – Col “difficile” à franchir, à l’intérieur.

facteur $A(h)$ de $A(h)e^{-2\frac{f(U^{(1)}_{j(k)})-f(U^{(0)}_k)}{h}}$, est d'ordre h (ce qui avait déjà été établi dans [BoGaKl], puis [HeKlNi]).

Regardons maintenant les résultats spécifiques aux cas à bord, en particulier lorsque le point col *généralisé* $U^{(1)}_{j(k)}$ appartient à $\partial\Omega$.

Dans les deux cas, on retrouve le même terme exponentiellement petit $e^{-2\frac{f(U^{(1)}_{j(k)})-f(U^{(0)}_k)}{h}}$, dont l'interprétation physique est identique à celle donnée dans le cas intérieur.

Dans le cas Dirichlet maintenant, $\lambda_k(h)$ est aussi proportionnelle aux quantités

$$\left|\det \operatorname{Hess} f(U^{(0)}_k)\right|^{\frac{1}{2}} \quad \text{et} \quad \frac{\partial_n f(U^{(1)}_{j(k)})}{\left|\det \operatorname{Hess} f\big|_{\partial\Omega}(U^{(1)}_{j(k)})\right|^{\frac{1}{2}}}.$$

L'interprétation physique au niveau d'un minimum local de f ne change pas, un tel point étant toujours à l'intérieur de la variété. Le col est quant à lui d'autant plus long à franchir que $\partial_n f(U^{(1)}_{j(k)})$ est petit (c'est-à-dire que f arrive avec une pente faible au bord) et que les valeurs propres de Hess $f\big|_{\partial\Omega}(U^{(1)}_{j(k)})$ sont grandes. Comme $U^{(1)}_{j(k)}$ est, dans le cas Dirichlet, un minimum local de $f\big|_{\partial\Omega}$, cela signifie que ce minimum est "plat". Les Figures 1.10 et 1.11 illustrent respectivement des franchissements longs et rapides d'un tel col.

Notons aussi que le préfacteur du développement asymptotique de $\lambda_k(h)$ est ici d'ordre $h^{\frac{1}{2}} = h^{-\frac{1}{2}} \times h$.

Intéressons-nous désormais au cas Neumann. D'abord, si seul le point $U^{(1)}_{j(k)}$ est dans le bord, ce qui se passe au niveau du minimum local intérieur peut être interprété comme dans les exemples précédents. Concernant le point critique généralisé dans le bord, $\lambda_k(h)$ est proportionnelle à la quantité

$$\frac{|\widehat{\lambda}^{\partial\Omega}_1|}{-\partial_n f(U^{(1)}_{j(k)})\left|\det \operatorname{Hess} f\big|_{\partial\Omega}(U^{(1)}_{j(k)})\right|^{\frac{1}{2}}}.$$

Par rapport au cas Dirichlet, le terme $\partial_n f(U^{(1)}_{j(k)})$ se retrouve ici au dénominateur. Cependant, le même type d'interprétations physiques que celles données précédemment reste valable : le col est d'autant plus dur à passer, d'une part, si la valeur propre négative de Hess $f\big|_{\partial\Omega}(U^{(1)}_{j(k)})$ est grande et les autres petites, comme dans le cas sans bord, et d'autre part, si la dérivée

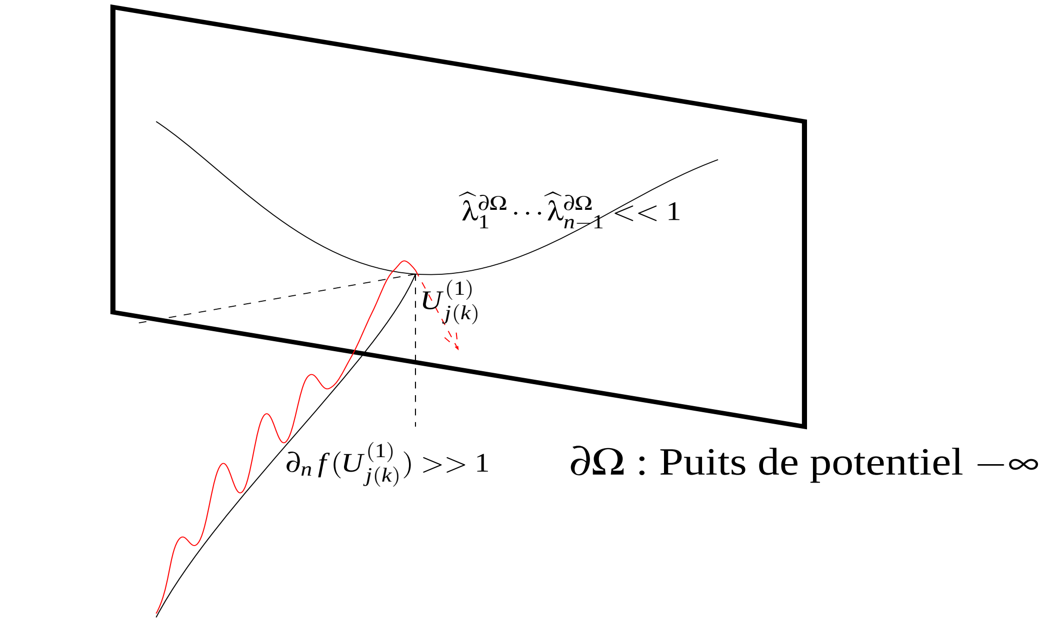


Figure 1.10 – Col “facile” à franchir dans le bord, cas Dirichlet.

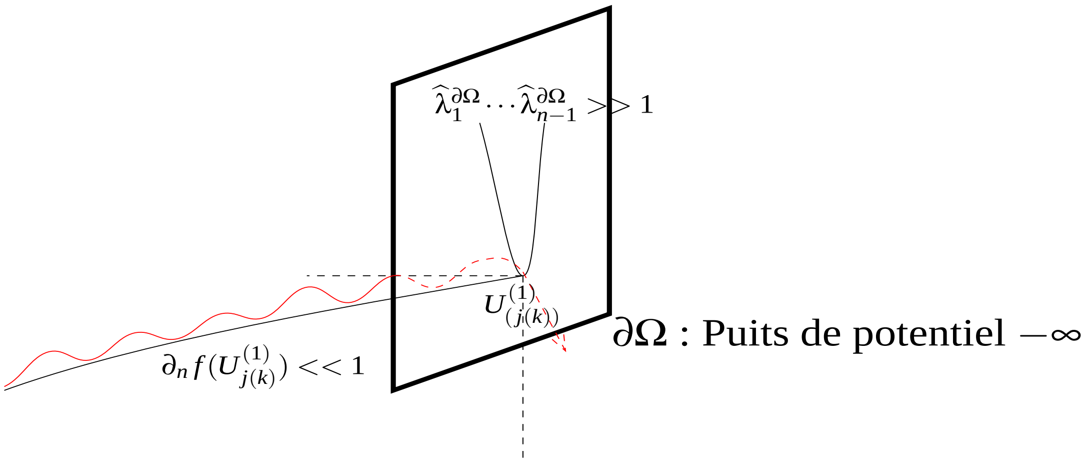


Figure 1.11 – Col “difficile” à franchir dans le bord, cas Dirichlet.

normale de f est importante au point col. Autrement dit, encore une fois, un col est d'autant plus difficile à passer que sa partie "descendante" est plate et que sa partie "ascendante" est étroite.

Quant au préfacteur du développement asymptotique de $\lambda_k(h)$, il est d'ordre $h^{\frac{3}{2}} = h^{\frac{1}{2}} \times h$.

Si $U_k^{(0)}$ est dans le bord mais pas $U_{j(k)}^{(1)}$, $\lambda_k(h)$ est proportionnelle à la quantité suivante, relative au minimum local $U_k^{(0)}$,

$$\partial_n f(U_k^{(0)}) \det \mathrm{Hess}\, f\big|_{\partial\Omega}(U_k^{(0)})\,.$$

Cela veut encore bien dire que plus ce minimum local est plat et plus la valeur propre associée est petite. Le préfacteur du développement asymptotique de $\lambda_k(h)$ est ici d'ordre $h^{\frac{1}{2}} h^{-\frac{1}{2}} \times h$.

Pour terminer avec le cas Neumann, regardons le résulat obtenu si $U_k^{(0)}$ et $U_{j(k)}^{(1)}$ sont dans le bord. L'interprétation physique au niveau du minimum local ou du point col *généralisé* correspond à ce qui vient d'être expliqué ci-dessus et le préfacteur du développement asymptotique de $\lambda_k(h)$ est d'ordre h. Plus précisément, ce préfacteur est

$$\frac{h}{\pi} |\lambda_1^{\partial\Omega}| \frac{\left|\det \mathrm{Hess}\, f\big|_{\partial\Omega}(U_k^{(0)})\right|^{\frac{1}{2}}}{\left|\det \mathrm{Hess}\, f\big|_{\partial\Omega}(U_{j(k)}^{(1)})\right|^{\frac{1}{2}}} \frac{\partial_n f(U_k^{(0)})}{\partial_n f(U_{j(k)}^{(1)})}\,,$$

c'est-à-dire le préfacteur obtenu dans le cas sans bord, pour la fonction $f\big|_{\partial\Omega}$, pondéré par la quantité

$$\frac{\partial_n f(U_k^{(0)})}{\partial_n f(U_{j(k)}^{(1)})}\,,$$

qui condense le rôle de l'intérieur Ω.

Quelques prolongements possibles

Plusieurs prolongements de ces résultats paraissent intéressants. D'abord, le cas général du Laplacien de Witten agissant sur les p-formes reste à traiter, et ce même dans le cas sans bord. Cela demande de traiter complètement la construction de p-quasimodes et constitue a priori un travail ardu.

Un autre prolongement envisageable est le calcul précis du deuxième terme du développement asymptotique des valeurs propres, que ce soit dans le cas sans bord ou bien dans les cas à bord avec conditions de Neumann ou de Dirichlet.

Nous pouvons également envisager des problèmes à bord avec conditions au bord mixtes Dirichlet-Neumann. Cela pourrait par exemple permettre de calculer les petites valeurs propres pour le problème du col de cygne de L. Pasteur déjà cité quelques fois dans cette introduction.

Quant au calcul de petites valeurs propres dans le cas non autoadjoint des opérateurs de type Kramers-Fokker-Planck, l'analyse multipuits n'en est qu'à ses débuts. Un résultat analogue à celui du Chapitre 3, par exemple, reste encore à établir dans ce cas.

Bibliographie

[BiLe] J.M. Bismut and G. Lebeau. *The hypoelliptic Laplacian and Ray-Singer metrics*, to appear in the Princeton University Press (2008).

[Bis] J.M. Bismut. The Witten complex and the degenerate Morse inequalities. J. Differ. Geom. 23, pp. 207-240 (1986).

[BoEcGaKl] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes I : Sharp asymptotics for capacities and exit times. JEMS 6(4), pp. 399-424 (2004).

[BoGaKl] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II : Precise asymptotics for small eigenvalues. JEMS 7(1), pp. 69-99 (2004).

[Bur] D. Burghelea. Lectures on Witten-Helffer-Sjöstrand theory. Gen. Math. 5, pp. 85-99 (1997).

[ChLi] K.C. Chang and J. Liu. A cohomology complex for manifolds with boundary. Topological Methods in Non Linear Analysis. Volume 5, pp. 325-340 (1995).

[CyFrKiSi] H.L Cycon, R.G Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Text and Monographs in Physics. Springer Verlag (1987).

[CoPaYc] Y. Colin de Verdière, Y. Pan, and B. Ycart Singular limits of Schrödinger operators and Markov processes. J. Operator Theory 41, no. 1, pp. 151-173 (1999).

[DiSj] M. Dimassi and J. Sjöstrand. *Spectral Asymptotics in the semi-classical limit.* London Mathematical Society. Lecture Note Series 268. Cambridge University Press (1999).

[Duf] G.F.D. Duff. Differential forms in manifolds with boundary. Ann. of Math. 56, pp. 115-127 (1952).

[DuSp] G.F.D. Duff and D.C. Spencer. Harmonic tensors on Riemannian manifolds with boundary. Ann. of Math. 56, pp. 128-156 (1952).

[Ev] L.C. Evans *An introduction to Stochastic Differential Equations.* http ://math.berkeley.edu/ evans/

[FrWe] M.I. Freidlin and A.D. Wentzell. *Random perturbations of dynamical systems. Transl. from the Russian by Joseph Szuecs. 2nd ed.* Grundlehren der Mathematischen Wissenschaften, 260, New York (1998).

[GaHuLa] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian Geometry.* Universitext, 2nd Edition, Springer Verlag (1993).

[Gil] P.B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.*
Mathematics Lecture Series, 11. Publish or Perish, Wilmington (1984).

[GoKr] I. Gohgerg and M. Krejn. *Introduction à la théorie des opérateurs linéaires non auto-adjoints dans un espace hilbertien,* Monographies Universitaires de Mathématiques, Vol. 39, Dunod (1971).

[Gol] S.I. Goldberg. *Curvature and Homology.*
Dover books in Mathematics, 3rd edition (1998).

[Gue] P. Guérini. Prescription du spectre du Laplacien de Hodge-de Rham. Annales de l'ENS, Vol. 37 (2), p. 270-303 (2004).

[Hel1] B. Helffer. Etude du Laplacien de Witten associé à une fonction de Morse dégénérée. Publications de l'université de Nantes, Séminaire EDP 1987-88.

[Hel2] B. Helffer. *Introduction to the semi-classical Analysis for the Schrödinger operator and applications.* Springer Verlag. Lecture Notes in Mathematics 1336 (1988).

[Hel3] B. Helffer. *Semi-classical analysis, Witten Laplacians and statistical mechanics.* World Scientific (2002).

[Hen] G. Henniart. Les inégalités de Morse (d'après E. Witten). Seminar Bourbaki, Vol. 1983/84. Astérisque No. 121-122, pp. 43-61 (1985) .

[HeKlNi] B. Helffer, M. Klein, and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Matematica Contemporanea, 26, pp. 41-85 (2004).

[HeNi] B. Helffer and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach : the case with boundary, Mémoire 105, Société Mathématique de France (2006).

[HeSj1] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical limit I, Comm. Partial Differential Equations 9 (4), pp. 337-408 (1984).

[HeSj2] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique II -Interaction moléculaire-Symétries-Perturbations. Ann. Inst. H. Poincaré Phys. Théor. 42 (2), pp. 127-212 (1985).

[HeSj4] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique IV -Etude du complexe de Witten -. Comm. Partial Differential Equations 10 (3), pp. 245-340 (1985).

[HeSj5] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique V - Etude des minipuits-. Current topics in partial differential equations, pp. 133-186, Kinokuniya, Tokyo (1986).

[HerNi] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to the equilibrium for the Fokker-Planck equation with high degree potential. Archive for Rational Mechanics and Analysis 171 (2), pp. 151-218 (2004).

[HerSjSt] F. Hérau, J. Sjöstrand, and C. Stolk. Semiclassical analysis for the Kramers-Fokker-Planck equation, Comm. Partial Differential Equations Vol. 30 no. 4-6, pp. 689-760 (2005).

[HerHiSj] F. Hérau, M. Hitrik, and J. Sjöstrand. Tunnel effect for Kramers-Fokker-Planck type operators, to appear in the Ann. Inst. Henri Poincaré, p.78 (2007).

[HoKuSt] R. Holley, S. Kusuoka and D. Stroock. Asymptotics of the spectral gap with applications to the theory of simulated annealing, J. Funct. Anal. 83 (2), pp. 333-347 (1989).

[KoPrSh] Nilufer Koldan, Igor Prokhorenkov, and Mikhail Shubin. Semi-classical Asymptotics on Manifolds with Boundary. Preprint (2008). http ://arxiv.org/abs/0803.2502v1

[Kol] V.N. Kolokoltsov. *Semi-classical analysis for diffusions and stochastic processes*, Lecture Notes in Mathematics 172, Springer Verlag (2000).

[KoMa] V.N. Kolokoltsov and K. Makarov. Asymptotic spectral analysis of a small diffusion operator and the life times of the corresponding diffusion process. Russian J. Math. Phys. 4 (3), pp. 341-36 (1996).

[Lau] F. Laudenbach. *Topologie différentielle.* Cours de Majeure de l'Ecole Polytechnique (1993).

[Leb] G. Lebeau. *Geometric Fokker-Planck equations*, Portugaliae Mathematica. Nova Série 62 (2005).

[Lep1] D. Le Peutrec. Small singular values of an extracted matrix of a Witten complex. To appear in Cubo, A Mathematical Journal.

[Lep2] D. Le Peutrec. Local WKB construction for boundary Witten Laplacians. IRMAR. Preprint 08-52 (2008).

[Lep3] D. Le Peutrec. Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian. IRMAR. Preprint (2008). http ://hal.archives-ouvertes.fr/hal-00297207/fr/

[Mas] W.S. Massey. *A Basic Course in Algebraic Topology.* Graduate Texts in Mathematics, Springer Verlag (1998).

[Mic] L. Miclo. Comportement de spectres d'opérateurs à basse température. Bull. Sci. Math. 119, pp. 529-533 (1995).

[Mil1] J.W. Milnor. *Morse Theory.* Princeton University press (1963).

[Mil2] J.W. Milnor. *Lectures on the h-cobordism Theorem.* Princeton University press (1965).

[Nie1] F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach, Journées "Equations aux Dérivées Partielles" Forges les eaux, Exp No VIII, Ecole Polytechnique (2004).

[Nie2] F. Nier. Hypoellipticity for Fokker-Planck operators and Witten Laplacians, IRMAR. Preprint 06-34 (2006).

[Per] A. Persson. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. Math. Scandinavica 8, pp. 143-153 (1960).

[Pet] P. Petersen. *Riemannian geometry.* Graduate Texts in Mathematics, Springer Verlag (1998).

[Ris] H. Risken. *The Fokker-Planck equation. Methods of solution and applications*, Springer-Verlag, Berlin, second edition (1989).

[Sch] G. Schwarz. *Hodge decomposition. A method for Solving Boundary Value Problems.* Lecture Notes in Mathematics 1607, Springer Verlag (1995).

[Sima] C.G. Simader. Essential self-adjointness of Schrödinger operators bounded from below. Math. Z. 159, pp. 47-50 (1978).

[Sim1] B. Simon. *Trace Ideals and their Applications*, Cambridge University Press, Lecture Notes Series vol. 35 (1979).

[Sim2] B. Simon. Semi-classical analysis of low lying eigenvalues, I. Non-degenerate minima : Asymptotic expansions. Ann. Inst. H. Poincaré Phys. Théor. 38, pp. 296-307 (1983).

[StVa] D.W. Stroock and S.R. Varadhan. *Multidimensionnal diffusion processes*, Springer-Verlag, Berlin (1979).

[Wit] E. Witten. Supersymmetry and Morse inequalities. J. Diff. Geom. 17, pp. 661-692 (1982).

[Zha] Weiping Zhang. *Lectures on Chern-Weil theory and Witten deformations.* Nankai Tracts in Mathematics. Vol. 4. World Scientific (2002).

Chapitre 2

Petites valeurs singulières d'une matrice extraite d'un complexe de Witten

Article [Lep1], rédigé en anglais, à paraître dans Cubo, A Mathematical Journal.

Small singular values of an extracted matrix of a Witten complex

Abstract: It is shown how rather tricky induction processes, used for the accurate computation of exponentially small eigenvalues of Witten Laplacians, essentially amount to some Gaussian elimination after the proper rewriting.

Key words and phrases: *Induction process, Witten Laplacian, exponentially small eigenvalues, Gaussian elimination.*

Contents

2.1 Introduction and motivations

The accurate computation of exponentially small eigenvalues of Witten Laplacians on 0-forms, or generators associated with reversible diffusion processes, relies on some rather tricky induction process. In [BoEcGaKl][BoGaKl], the induction scheme is modelled after the probabilistic picture of exit times. After [HeKlNi][HeNi] it appeared that this induction scheme could be extracted from its spectral analysis or probabilistic framework as a pure problem of finite dimensional linear algebra. The aim of this short text is to show that all these previous and rather involved inductions essentially amount, after a proper rewriting, to some Gaussian elimination.
We first recall that the Witten Laplacian writes

$$\Delta^{(0)}_{f,h} = d^{(0),*}_{f,h} d^{(0)}_{f,h} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x) \tag{2.1.1}$$

on functions and more generally on differential forms with arbitrary degree,

$$\Delta_{f,h} = \left(d^*_{f,h} + d_{f,h}\right)^2 \quad \text{with} \quad d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}}\,. \tag{2.1.2}$$

Since it has a square structure, the eigenvalues of $\Delta^{(0)}_{f,h}$ (resp. $\Delta_{f,h}$) are the squares of the singular values of $d^{(0)}_{f,h}$ (resp. $(d^*_{f,h} + d_{f,h})$). Remember that in the study of Witten Laplacians d denotes the exterior differential on a Riemannian manifold, d^* the codifferential, $h > 0$ is a small parameter considered in the limit $h \to 0$ and f is a Morse function. In the case when the manifold is $\mathbb{R}^n$ with the euclidean metric, recall that the Witten Laplacian on functions (2.1.1) in $L^2(\mathbb{R}^n, dx)$ is unitary equivalent to the following operator

$$-h(-2\nabla f(x).\nabla + h\Delta)$$

in $L^2(\mathbb{R}^n, e^{-2f/h}dx)$. This last operator fits better with the probabilistic presentation ([FrWe][Ris][StVa]) and the simulated annealing framework ([HoKuSt]).

The main purpose is the accurate computation of the smallest non zero eigenvalue of these operators among a finite collection of exponentially small eigenvalues, i.e. of order $e^{-\frac{C_k}{h}}$ as $h \to 0$. The inverse of this eigenvalue can be interpreted as the longest lifetime of metastable states. The issue is the suitable control of errors (which are in absolute values larger than the final result) at every step of the induction process. A usual Gramm-Schmidt type orthonormalisation process as it is used in the semiclassical multiple wells problem ([Hel2][HeSj2]) does not allow such a control.

As this was pointed out in [HeKlNi][Nie1], working with singular values rather than with eigenvalues of the square operator allows to use the Fan inequalities ([GoKr][Sim1]) in their simplest form. These multiplicative inequalities propagate the control of the small relative errors on the singular values through the induction process.

At the moment, this approach has been applied systematically only in the case of Witten Laplacian acting on functions. Some cases with higher order Witten Laplacians can be considered. The only condition is the construction of global quasimodes, which is not completely elucidated for the moment except in the case of 0-forms. Besides the simplification of previous proofs, this text aims at providing an abstract and general result to be referred to in the next future.

2.2 Result

Let $F^{(0)}$ and $F^{(1)}$ be two complex Hilbert spaces respectively of dimension $m_0 < +\infty$ and $m_1 < +\infty$. Let $\langle\,|\,\rangle$ denote the scalar product on $F^{(0)}$ or $F^{(1)}$ (without distinction), and let $\|\psi\|$ and $\|A\| = \sup_{\psi\neq 0}\frac{\|A\psi\|}{\|\psi\|}$ denote the norms of the vector ψ and of the linear application A associated with this scalar product. Let moreover h_0 and ε_0 be two positive numbers.
Consider a linear application $B(h)$ depending on $h \in (0,\, h_0]$:

$$B(h) : F^{(0)} \longrightarrow F^{(1)}\,.$$

and set

$$A_0(h) = B^*(h)B(h) \geq 0\,.$$

Let

$$A_1(h) = B(h)B^*(h) \geq 0\,,$$

and note the intertwining relation:

$$B(h)A_0(h) = A_1(h)B(h)\,.$$

Definition 2.2.1. *For a number (resp. a linear operator) $g(h)$, the notation $g(h) = \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})$ means that, for all $\varepsilon \in (0, \varepsilon_0]$, there exists a constant $C_\varepsilon > 0$ such that:*

$$\forall h \in (0,\, h_0]\;, \quad |g(h)| \leq C_\varepsilon e^{-\frac{\alpha}{h}} \;(resp.\; \|g(h)\| \leq C_\varepsilon e^{-\frac{\alpha}{h}})\,.$$

Assumption 2.2.2. *Assume that there exist two bases (of $F^{(0)}$ and $F^{(1)}$ respectively) depending on $(\varepsilon, h) \in (0, \varepsilon_0] \times (0,\, h_0]$ and a positive number α independent of $(\varepsilon\,, h) \in (0, \varepsilon_0] \times (0,\, h_0]$ such that:*

$$\begin{array}{l}\psi_k^{(0)} = \psi_k^{(0)}(\varepsilon, h)\;(k \in \{1, \ldots, m_0\})\,,\; \left\langle \psi_k^{(0)} \mid \psi_{k'}^{(0)} \right\rangle = \delta_{kk'} + \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}}) \quad ,\\ \psi_j^{(1)} = \psi_j^{(1)}(\varepsilon, h)\;(j \in \{1, \ldots, m_1\})\,,\; \left\langle \psi_j^{(1)} \mid \psi_{j'}^{(1)} \right\rangle = \delta_{jj'} + \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}}) \quad .\end{array}$$

Assumption 2.2.3. *Assume furthermore that there exist an injective map* $j : \{1, \dots, m_0\} \to \{1, \dots, m_1\}$ *, a decreasing sequence* $(\alpha_k)_{k\in\{1,\dots,m_0\}}$ *of real numbers, and a positive number* d *(independent of* $(\varepsilon\,, h) \in (0, \varepsilon_0] \times (0,\, h_0]$*) such that:*

$$\begin{aligned}&\forall \varepsilon \in (0, \varepsilon_0],\ \exists\, C_\varepsilon > 1, \forall k \in \{1, \dots, m_0\}\,,\\ &\qquad \forall h \in (0, h_0], \quad C_\varepsilon^{-1} e^{-\frac{\alpha_k + d\varepsilon}{h}} \leq \left|\left\langle \psi^{(1)}_{j(k)} \mid B(h)\psi^{(0)}_k \right\rangle\right| \leq C_\varepsilon e^{-\frac{\alpha_k - d\varepsilon}{h}}\\ &\qquad \forall h \in (0, h_0], \forall j' \neq j(k), \quad \left|\left\langle \psi^{(1)}_{j'} \mid B(h)\psi^{(0)}_k \right\rangle\right| \leq C_\varepsilon e^{-\frac{\alpha_k + \alpha}{h}}\,.\end{aligned}$$

Theorem 2.2.4. *There exist positive numbers* $h_0' \leq h_0$ *and* $\varepsilon_0' \leq \varepsilon_0$ *such that, under Assumptions 2.2.2 and 2.2.3, the eigenvalues* $0 \leq \lambda_1(h) \leq \cdots \leq \lambda_{m_0}(h)$ *of* $A_0(h)$ *satisfy:*

$$\begin{gathered} 0 < \lambda_1(h) < \cdots < \lambda_{m_0}(h)\,,\\ \forall k \in \{1, \dots, m_0\}\,, \quad \lambda_k(h) = \left|\left\langle \psi^{(1)}_{j(k)} \mid B(h)\psi^{(0)}_k \right\rangle\right|^2 (1 + \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}))\,,\end{gathered}$$

where $\eta > 0$ *is a real number independent of* $(\varepsilon,\, h) \in (0,\, \varepsilon_0'] \times (0,\, h_0']$*.*

Remark 2.2.5. *More generally, vanishing eigenvalues can be included. It suffices to allow the value* $+\infty$ *for the first values*

$$\alpha_1 = \cdots = \alpha_\ell = +\infty \quad \text{and} \quad \alpha_{m_0} < \cdots < \alpha_{\ell+1} \in \mathbb{R}\,,$$

for some given $\ell \in \{1, \dots, m_0\}$*. In this last case, the eigenvalues of* $A_0(h)$ *satisfy:*

$$\lambda_1 = \cdots = \lambda_\ell = 0 \quad \text{and} \quad 0 < \lambda_{\ell+1} < \cdots < \lambda_{m_0}\,,$$

while the above estimates hold for the non-zero eigenvalues.

This theorem, or a modified form of this theorem according to Remark 2.2.5, can be applied to simplify the final proof done in [HeKlNi] for the case of the Witten Laplacian acting on 0-forms on a Riemannian manifold without boundary or the one in [HeNi] for some Dirichlet realization in the case with a boundary. This final part of the analysis in [HeKlNi][HeNi] has been reconsidered in [Nie1], without giving all the possible simplifications. The reader can also find in [Nie1] various illustrations in practical cases of this approach .

Once the quasimodes satisfying Assumptions 2.2.2 and 2.2.3 are constructed, Theorem 2.2.4 can be applied as soon as we work with a self-adjoint operator with a square structure. The application to Witten Laplacians on 0-forms

with alternative boundary conditions is in progress. Some examples of Witten Laplacians acting on p-forms for which quasimodes are constructed can be treated with this result and Theorem 2.2.4 may be useful for a future generalization.
While working with Witten Laplacians on 0-forms, the quasimodes $\psi_k^{(0)}$'s are constructed globally after truncating $e^{-\frac{f}{h}}$, while the $\psi_j^{(1)}$'s are introduced locally via a WKB approximation around saddle points of f, $U_{j(k)}^{(1)}$. Note that the discussion in [HeKlNi][Nie1] about sending $U_{j(1)}^{(1)}$ to infinity when $\lambda_1 = 0$ is replaced by consedering $\alpha_1 = +\infty$ (according to Remark 2.2.5) with an arbitrary additionnal $\psi_{j(1)}^{(1)}$.
The application to some non self-adjoint Fokker-Planck operators with a distorted square strusture (see [BiLe][HeNi][HerNi][Leb]) seems more delicate (see Remark 2.3.7).

2.3 Proof

Let us begin by fixing the positive numbers ε_0' and η. We first choose ε_0' small enough such that:

$$\alpha' = \alpha - d\,\varepsilon_0' > 0 \quad \text{and} \quad \alpha'' = \min_{k>k'} \{\alpha_{k'} - \alpha_k - 2d\,\varepsilon_0'\} > 0\,.$$

Then, we set:

$$\eta = min\,\{\alpha,\, \alpha'\,, \alpha''\} = min\,\{\alpha'\,, \alpha''\}\ .$$

To prove Theorem 2.2.4, it will be more convenient to work with matrices. Let us give a definition and an easy application which will be very useful.

Definition 2.3.1. *A square matrix $V(h)$ is said quasi-unitary if there exists an unitary matrix U such that:*

$$V(h) = U + \mathcal{O}_\varepsilon(e^{-\eta/h})$$

Lemma 2.3.2. *The product of quasi-unitary matrices is a quasi-unitary matrix.*

Furthermore, to prove Theorem 2.2.4, we need a particular case of Fan inequalities that we recall here (we refer the reader to [Sim1] for a proof).

Lemma 2.3.3. *Let B and C be respectively a compact and a bounded linear operator on a Hilbert space $\mathcal{H}$. The inequalities*

$$\begin{aligned} \mu_n(BC) &\leq \|C\|\,\mu_n(B) \\ \mu_n(CB) &\leq \|C\|\,\mu_n(B) \quad , \end{aligned}$$

where $\mu_n(B)$ is the n-th singular value of B, hold for all $n \leq \dim \mathcal{H}$.

We apply this lemma with $\mathcal{H} = \mathcal{H}_0 \overset{\perp}{\oplus} \mathcal{H}_1$, while identifying $B : \mathcal{H}_0 \to \mathcal{H}_1$ with $J_1 B \Pi_0 \in \mathcal{L}(\mathcal{H})$, where Π_0 is the orthogonal projection $\mathcal{H} \to \mathcal{H}_0$ and J_1 the embedding $\mathcal{H}_1 \to \mathcal{H}$.

Corollary 2.3.4. *Let $\mathcal{H}_0$, $\mathcal{H}_1$ be two Hilbert spaces. Let B be a compact linear operator from $\mathcal{H}_0$ to $\mathcal{H}_1$. Assume that $C \in \mathcal{L}(\mathcal{H}_1)$ and $D \in \mathcal{L}(\mathcal{H}_0)$ are two invertible operators with:*

$$\max\left\{\|C\|\,,\,\left\|C^{-1}\right\|\,,\,\|D\|\,,\,\left\|D^{-1}\right\|\right\} \leq 1+\rho\,,$$

for some $\rho > -1$. Then the inequality

$$(1+\rho)^{-2}\mu_n(B) \leq \mu_n(CBD) \leq (1+\rho)^2\mu_n(B)$$

holds for all $n \leq \min(\dim \mathcal{H}_0,\ \dim \mathcal{H}_1)$.

Remark 2.3.5. *We will apply this corollary in the particular case when C and D depend on $h \in (0,\, h'_0]$ and are quasi-unitary:*

$$C(h) = U + \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}) \quad \textit{and} \quad D(h) = V + \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})\,,$$

where U and V are unitary matrices and $\rho = \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})$. We obtain the equivalent relations:

$$\mu_n(CBD) = \mu_n(B)(1+\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}))\ ,\ \mu_n(B) = \mu_n(CBD)(1+\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}))\,. \tag{2.3.1}$$

From $A_0(h) = B^*(h)B(h)$, we deduce that the eigenvalues of $A_0(h)$ are the squares of the singular values of $B(h)$:

$$\forall k \in \{1,\ldots,m_0\}\ ,\ \lambda_k(h) = \mu^2_{m_0+1-k}(B(h)) \quad (\,\mu_1(B(h)) = \|B(h)\|)\,.$$

In order to apply Corollary 2.3.4, it will be easier to work with the singular values of $B(h)$ than with the eigenvalues of $A_0(h)$.

Choose now two arbitrary orthonormal bases $\mathcal{B}^{(0)}$ and $\mathcal{B}^{(1)}$ (of $F^{(0)}$ and $F^{(1)}$ respectively). We make the identifications:

$$B(h) = \underset{\mathcal{B}^{(0)},\,\mathcal{B}^{(1)}}{Mat}(B(h))\,, \quad B^*(h) = (\underset{\mathcal{B}^{(0)},\,\mathcal{B}^{(1)}}{Mat}(B(h)))^*\,.$$

Let be $B'(h) = \left(\left\langle \psi_j^{(1)} \mid B(h)\psi_k^{(0)} \right\rangle\right)_{j,k} = \left(b'_{j\,k}\right)_{j,\,k}$. For $i \in \{1,\,\dots,\,m_l\}$ and $l \in \{0,\,1\}$, we set

$$C_l = \mathrm{Mat}_{\mathcal{B}^{(l)}}\left(\psi_1^{(l)} \dots \psi_{m_l}^{(l)}\right) ,$$

where $\psi_i^{(l)}$ is written as a column vector in $\mathcal{B}^{(l)}$. These change-of-coordinates matrices give $B'(h) = C_1^* B(h) C_0$.

Remark 2.3.6. *By Assumption 2.2.2, the matrices C_0 and C_1^* are quasi-unitary and Assumption 2.2.3 implies, for h'_0 small enough:*

$$\forall \quad 1 \le k' < k \le m_0 \ , \qquad b'_{j(k')\,k'} = b'_{j(k)\,k}.\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})\,, \tag{2.3.2}$$

$$\forall \quad 1 \le k \le m_0\,,\,\forall j \ne j(k) \ , \qquad b'_{j\,k} = b'_{j(k)\,k}.\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})\,. \tag{2.3.3}$$

We now simplify $B'(h)$ by Gaussian elimination *in the following order*:

Step 0: By permutating the rows, that is by left-multiplying with permutation matrices which are unitary, put the coefficients $b'_{j(k)\,k}$ (for k in $\{1,\dots,m_0\}$) on the k-th row and k-th column. The new matrix has the form:

$$B''(h) = \begin{pmatrix} b''_{1\,1} = b'_{j(1)\,1} & b'_{j(2)\,2}.\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}) & \dots & b'_{j(m_0)\,m_0}.\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}) \\ \vdots & b''_{2\,2} = b'_{j(2)\,2} & & \vdots \\ b'_{j(1)\,1}.\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}) & \vdots & \ddots & \vdots \\ \vdots & b'_{j(2)\,2}.\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}) & & b''_{m_0\,m_0} = b'_{j(m_0)\,m_0} \\ \vdots & \vdots & & \vdots \end{pmatrix} .$$

Furthermore, the matrix $B''(h)$ satisfies the structure equations (2.3.2) and (2.3.3) with the injective map $j : \{1,\dots,m_0\} \to \{1,\dots,m_1\}$ replaced by the canonical injection $i : \{1,\dots,m_0\} \to \{1,\dots,m_1\}$, $i(k) = k$.

Step 1: For $j \in \{1,\,\dots,\,m_1\} \setminus \{m_0\}$, replace the j-th row L_j by $L_j - \frac{b''_{j\,m_0}}{b''_{m_0\,m_0}} L_{m_0} = L_j - \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}).L_{m_0}$.

Step 2: Then, for $k \in \{1,\,\dots,\,m_0-1\}$, replace the k-th column C_k by $C_k - \frac{b''_{m_0\,k}}{b''_{m_0\,m_0}} C_{m_0} = C_k - \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}}).C_{m_0}$. Due to the previous operations, only the m_0-th row of the new matrix is changed by these operations.

Each operation of the two last steps preserves the structure of Assumption 2.2.3, or more precisely the structure of Remark 2.3.6 where we have replaced

the injective map j by the canonical injection i. Moreover, these operations correspond to left multiplications or right multiplications by quasi-unitary matrices.
The new matrix only contains zeros on the m_0-th row and m_0-th column except for the (m_0, m_0)-coefficient which is $b'_{j(m_0)\, m_0} = \left\langle \psi^{(1)}_{j(m_0)} \mid B(h)\psi^{(0)}_{m_0} \right\rangle$.
When $m_0 \geq 2$, iterate the Gaussian elimination, Step 1 with the reference row $m_0 - \nu$ and Step 2 with the reference column $m_0 - \nu$, by taking successively $\nu = 1, \ldots, m_0{-}2$. At the end, we obtain a diagonal matrix $D(h) \in M_{m_0,m_1}(\mathbb{C})$ such that:

$$\forall k \in \{1, \ldots, m_0\}\,,\ (D(h))_{k,k} = \left\langle \psi^{(1)}_{j(k)} \mid B(h)\psi^{(0)}_{k} \right\rangle (1 + \mathcal{O}_\varepsilon(e^{-\eta/h}))\,,$$

Moreover, by Lemma 2.3.2, there exist two quasi-unitary matrices $U(h) \in M_{m_0}(\mathbb{C})$ and $V(h) \in M_{m_1}(\mathbb{C})$ satisfying

$$D(h) = V(h)B'(h)U(h) = V(h)C_1^* B(h) C_0 U(h)\,.$$

Using again Lemma 2.3.2, $V'(h) = V(h)C_1^*$ and $U'(h) = C_0 U(h)$ are quasi-unitary. From $D(h) = V'(h)B(h)U'(h)$, we conclude using Corollary 2.3.4 and (2.3.1).

Remark 2.3.7. ***a)*** *The square self-adjoint structure $A_0(h) = B^*(h)B(h)$ is essential here to be able to conclude.*
Even a small distortion, $A_0(h) = B^*(h)CB(h)$ with $C = Id{+}r$ with $r = \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})$, in dimension 2, destroys the above arguments, due to ill-conditioning problem.
In the decomposition

$$A_0(h) = B^*(h)B(h) + B^*(h)rB(h) = B^*(h)B(h) + B^*(h)\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})B(h)\,,$$

the remainder term $B^*(h)\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})B(h)$ cannot be put in general in the form $B^*(h)B(h)\mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})$:

$$\begin{aligned} & B^*(h)rB(h) = B^*(h)B(h)\left(B(h)^{-1}rB(h)\right) \\ \textit{with} \quad & \left\|B(h)^{-1}rB(h)\right\| \leq \left\|B(h)^{-1}\right\| \|B(h)\| \|r\|\,. \end{aligned}$$

For example, take $\eta = 1$ and

$$B(h) = \begin{pmatrix} e^{-\frac{4}{h}} & 0 \\ 0 & e^{-\frac{2}{h}} \end{pmatrix} \quad \textit{and} \quad C(h) = \begin{pmatrix} 1 & e^{-\frac{1}{h}} \\ 0 & 1 \end{pmatrix}.$$

In this example the remainder factor equals

$$B(h)^{-1}rB(h) = \begin{pmatrix} 0 & e^{+\frac{1}{h}} \\ 0 & 0 \end{pmatrix}$$

with a norm of order $e^{\frac{1}{h}} = e^{\frac{2}{h}} \times e^{-\frac{\eta}{h}}$ *.*
b) *A first attempt at the extension of this analysis to the non self-adjoint case related with Kramers-Fokker-Planck type operators, studied in [HerSjSt] [HerHiSj], led to the simple distortion* $A_0(h) = B^*(h)CB(h)$ *with* $C = Id + \mathcal{O}_\varepsilon(e^{-\frac{\eta}{h}})$*. The previous remark shows that it cannot work without including some additional information about the intimate link between these non self-adjoint operators coming from kinetic theory and Witten Laplacians ([BiLe][HeNi][HerNi][Leb]).*

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Bibliography

[BiLe] J.M. Bismut and G. Lebeau. *The hypoelliptic Laplacian and Ray-Singer metrics*, to appear in the Princeton University Press (2008).

[BoEcGaKl] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times, JEMS 6(4), pp. 399-424 (2004).

[BoGaKl] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues, JEMS 7(1), pp. 66-99 (2004).

[FrWe] M.I. Freidlin and A.D. Wentzell. *Random perturbations of dynamical systems*, Springer-Verlag, New York (1984).

[GoKr] I. Gohgerg and M. Krejn. *Introduction à la théorie des opérateurs linéaires non auto-adjoints dans un espace hilbertien*, Monographies Universitaires de Mathématiques, Vol. 39, Dunod (1971).

[Hel2] B. Helffer. *Introduction to the semi-classical Analysis for the Schrödinger operator and applications.* Springer Verlag. Lecture Notes in Mathematics 1336 (1988).

[HeKlNi] B. Helffer, M. Klein, and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach, Matematica Contemporanea Vol. 26, pp. 41-85 (2004).

[HeNi] B. Helffer and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary, Mémoire 105, Société Mathématique de France (2006).

[HeSj2] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique II -Interaction moléculaire-Symétries-Perturbations. Ann. Inst. H. Poincaré Phys. Théor. 42 (2), pp. 127-212 (1985).

[HeSj4] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical analysis IV. Etude du complexe de Witten, Comm. Partial Differential Equations Vol. 10 no. 3, pp. 245-340 (1985).

[HerNi] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to the equilibrium for the Fokker-Planck equation with high degree potential, Archive for Rational Mechanics and Analysis 171 (2), pp. 151-218 (2004).

[HerSjSt] F. Hérau, J. Sjöstrand, and C. Stolk. Semiclassical analysis for the Kramers-Fokker-Planck equation, Comm. Partial Differential Equations Vol. 30, no. 4-6, pp. 689-760 (2005).

[HerHiSj] F. Hérau, M. Hitrik, and J. Sjöstrand. Tunnel effect for Kramers-Fokker-Planck type operators, to appear in the Ann. Inst. Henri Poincaré, p.78 (2007).

[HoKuSt] R. Holley, S. Kusuoka, and D. Stroock. Asymptotics of the spectral gap with applications to the theory of simulated annealing, J. Funct. Anal. 83 (2), pp. 333-347 (1989).

[Leb] G. Lebeau. *Geometric Fokker-Planck equations*, Portugaliae Mathematica. Nova Série 62, 2005.

[Nie1] F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach, Journées "Equations aux Dérivées Partielles" Forges les eaux, Exp No VIII, Ecole Polytechnique (2004).

[Ris] H. Risken. *The Fokker-Planck equation. Methods of solution and applications*, Springer-Verlag, Berlin, second edition (1989).

[Sim1] B. Simon. *Trace Ideals and their Applications*, Cambridge University Press, Lecture Notes Series vol. 35 (1979).

[StVa] D.W. Stroock and S.R. Varadhan. *Multidimensionnal diffusion processes*, Springer-Verlag, Berlin (1979).

Chapitre 3

Construction WKB locale pour Laplaciens de Witten à bord

Article [Lep2], rédigé en anglais, soumis pour publication.

Local WKB construction for boundary Witten Laplacians

Abstract: WKB p-forms are constructed as approximate solutions to boundary value problems associated with semi-classical Witten Laplacians. Naturally distorted Neumann or Dirichlet boundary conditions are considered.

Key words and phrases: *WKB expansion, Boundary value problem, Witten Laplacian.*

Contents

3.1 Introduction

In order to compute accurately the small eigenvalues, i.e. of order $\mathcal{O}(e^{-\frac{C}{h}})$ with $C > 0$, of a self-adjoint Witten Laplacian acting on 0-forms,

$$\Delta_{f,h}^{(0)} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x) ,$$

as the small parameter $h > 0$ tends to 0, we need WKB approximations of the 1-eigenforms associated with the small eigenvalues of $\Delta_{f,h}^{(1)}$, the self-adjoint Witten Laplacian acting on 1-forms. The function f is assumed to be a Morse function on some bounded domain Ω with or without boundary.

In the article of B. Helffer, M. Klein and F. Nier [HeKlNi], which improves the previous works [BoEcGaKl] and [BoGaKl] done in a probabilistic point of view, the authors compute accurately the small eigenvalues of $\Delta_{f,h}^{(0)}$ in the case of a manifold without boundary. In this case, the WKB approximations of 1-eigenforms are the one provided in the work by B. Helffer and J. Sjöstrand [HeSj4], where the analysis is done for general p-forms.
In the case without boundary, it is moreover well known, since the article by E. Witten [Wit], that the dimension of the spectral subspace associated with the small eigenvalues (i.e. smaller than h) of $\Delta_{f,h}^{(p)}$, the self-adjoint Witten Laplacian acting on p-forms, is $m_p(f)$, the number of critical points of f with index p. Furthermore, the corresponding eigenvectors are concentrated around these critical points (see also [HeSj4][HeKlNi][Hel2]).

According to [ChLi], [HeNi], [KoPrSh], and [Lep3], in the case of a compact manifold with boundary, these last statements require the introduction of *generalized* critical points of f with index p (see Definition 3.2.6). For a self-adjoint Witten Laplacian $\Delta_{f,h}^{(p)}$ with Neumann or Dirichlet type boundary conditions, $\Delta_{f,h}^{(p)}$ admits $m_p(f)$ eigenvalues, where $m_p(f)$ is the number of *generalized* critical points of f with index p. Moreover, the corresponding p-eigenforms are concentrated around these *generalized* critical points, which can belong to the boundary. The proper definition of *generalized* critical point of f relies on the additionnal assumption that f has no critical point in the boundary $\partial\Omega$ and that $f\big|_{\partial\Omega}$ is also a Morse function (see Assumption 3.2.5). This definition is different for Neumann or Dirichlet type boundary conditions, but, in both cases, the interior *generalized* critical points of f with index p are the usual critical points with index p (see again Definition 3.2.6).

Hence, in the case of a manifold with boundary, some WKB approximations of 1-eigenforms have to be constructed near some *generalized* critical

points which lie on the boundary. This was done in [HeNi] for Dirichlet type boundary conditions. Nevertheless, the construction done in [HeNi] relies on some specific trick which cannot be extended to the construction of local WKB 1-forms in the Neumann case. In order to treat this last case (see [Lep3]), a finer treatment of the three geometries involved in the boundary problem (boundary, metric, Morse function) is carried out.
It happens that the Neumann case for 1-forms contains all the technical obstructions for a general WKB ansatz for p-eigenforms. Moreover, this construction can be extended to the Dirichlet case, for general p-forms, using "dual" computations.

Therefore we show in this paper how to construct local WKB p-forms localized near the boundary in both Neumann and Dirichlet cases. However, only the construction of local WKB p-forms is considered here and the comparison with the corresponding p-eigenforms has only be treated in the case $p = 1$, in [HeNi] and [Lep3].

Our results are:

Theorem 3.1.1 (Neumann case)**.**
Let U be a generalized critical point of f with index p in the boundary, for Neumann type boundary conditions. There exists around U a local "adapted coordinate system" $x = (x', x^n)$ centered at U such that, there exists locally, in a neighborhood of $x = 0$, a $\mathcal{C}^\infty$ solution u_p^{wkb} to

$$\Delta_{f,h}^{(p)} u_p^{wkb} = e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) \tag{3.1.1}$$
$$\mathbf{n} u_p^{wkb} = 0 \text{ on } \partial\Omega \tag{3.1.2}$$
$$\mathbf{n} d_{f,h} u_p^{wkb} = 0 \text{ on } \partial\Omega\,, \tag{3.1.3}$$

where u_p^{wkb} has the form:

$$\begin{aligned} u_p^{wkb} &= a(x,h) e^{-\frac{\Phi}{h}}\,, \\ \text{with} \quad a(x,h) &\sim \sum_k a^k(x) h^k \text{ and } a^0(0) = dx^1 \wedge \cdots \wedge dx^p\,. \end{aligned}$$

The coordinates $(x^1, \ldots, x^{n-1})$ are Morse coordinates for $f\big|_{\partial\Omega}$ in the boundary, normalized at $x = 0$, and $\frac{\partial}{\partial x^n}\big|_{x=0} = \vec{n}$, the outgoing normal vector at $x = 0$.

Theorem 3.1.2 (Dirichlet case)**.**
Let U be a generalized critical point of f with index p in the boundary, for

Dirichlet type boundary conditions. There exists around U a local "adapted coordinate system" $x = (x', x^n)$ centered at U such that, there exists locally, in a neighborhood of $x = 0$, a $\mathcal{C}^\infty$ solution u_p^{wkb} to

$$\begin{aligned} \Delta_{f,h}^{(p)} u_p^{wkb} &= e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) && (3.1.4) \\ \mathbf{t} u_p^{wkb} &= 0 \text{ on } \partial\Omega && (3.1.5) \\ \mathbf{t} d_{f,h}^* u_p^{wkb} &= 0 \text{ on } \partial\Omega\,, && (3.1.6) \end{aligned}$$

where u_p^{wkb} has the form:

$$\begin{aligned} u_p^{wkb} &= a(x,h) e^{-\frac{\Phi}{h}}\,, \\ \textit{with} \quad a(x,h) &\sim \sum_k a^k(x) h^k \text{ and } a^0(0) = dx^1 \wedge \cdots \wedge dx^{p-1} \wedge dx^n\,. \end{aligned}$$

The coordinates $(x^1, \ldots, x^{n-1})$ are Morse coordinates for $f\big|_{\partial\Omega}$ in the boundary, normalized at $x = 0$, and $\frac{\partial}{\partial x^n}\big|_{x=0} = \vec{n}$.

The operators $\mathbf{n}$ and $\mathbf{t}$ are the normal and tangential components. They are defined in the next section. The function Φ is the degenerate Agmon distance to the point U associated with the function f.
Recall moreover that for a p-form $b(x,h)$, the notation $b(x,h) = \mathcal{O}(h^\infty)$ means that, for each N in $\mathbb{N}$, $b(x,h) = \mathcal{O}(h^N)$ in the following sense :

$$\forall N \in \mathbb{N}\,,\ \exists C_N > 0 \ \text{ s.t. } \ ||b(x,h)|| \leq C_N h^N\,,$$

where $||\cdot||$ is the L^2-norm over the p-forms inherited from the Riemannian structure.

The notion of "adapted coordinates" will be defined in Subsection 3.3.1. Moreover, the particular "adapted coordinates" used in these theorems will be specifed in Subsections 3.4.1 and 3.4.4 (see indeed equations (3.4.7) and (3.4.29)). The statements of Theorems 3.1.1 and 3.1.2 simply specify the polarization of $a^0(0)$ which is imposed, while solving degenerate transport equations (see Subsections 3.4.3 and 3.4.6). Again, this is more explicit later, choosing the suitable coordinate system. These theorems are respectively proved in Subsections 3.4.3 and 3.4.6.

When the metric is Euclidean, $g = \sum_{i=1}^n (dx^i)^2$, the manifold Ω is locally $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$, the boundary $\partial\Omega$ is locally $\partial\Omega = \{x^n = 0\}$, and the function f writes

$$f(x) = -x^n - \frac{|\lambda_1|}{2}(x^1)^2 - \cdots - \frac{|\lambda_p|}{2}(x^p)^2 + \frac{|\lambda_{p+1}|}{2}(x^{p+1})^2 + \frac{|\lambda_{n-1}|}{2}(x^{n-1})^2$$

(resp. $f(x) = +x^n - \frac{|\lambda_1|}{2}(x^1)^2 - \cdots - \frac{|\lambda_p|}{2}(x^p)^2 + \frac{|\lambda_{p+1}|}{2}(x^{p+1})^2 + \frac{|\lambda_{n-1}|}{2}(x^{n-1})^2$)

in the Neumann case (resp. in the Dirichlt case), the "adapted coordinates" are simply $(x^1, \ldots, x^n)$. The general case is more involved because the three geometries of the boundary, the metric (curvature), and of the level sets of the function f do not match.

Our goal consists in reducing the analysis to a boundary value problem, hence to a problem in a manifold without boundary. Once this will be done, we will be able to apply the results of [HeSj4], obtained in the case of a manifold without boundary, to this reduced problem.

3.2 Generalities about Witten Laplacians

Let $\overline{\Omega}$ be a $\mathcal{C}^\infty$ connected compact oriented Riemannian manifold with boundary $\partial\Omega$ and dimension $n \in \mathbb{N}^*$. We will denote by g_0 the given Riemannian metric on $\overline{\Omega}$; Ω and $\partial\Omega$ will denote respectively its interior and its boundary.
The cotangent (resp. tangent) bundle on Ω is denoted by $T^*\Omega$ (resp. $T\Omega$) and the exterior fiber bundle by $\Lambda T^*\Omega = \oplus_{p=0}^{n} \Lambda^p T^*\Omega$ (resp. $\Lambda T\Omega = \oplus_{p=0}^{n} \Lambda^p T\Omega$). The fiber bundles $\Lambda T\partial\Omega = \oplus_{p=0}^{n-1} \Lambda^p T\partial\Omega$ and $\Lambda T^*\partial\Omega = \oplus_{p=0}^{n-1} \Lambda^p T^*\partial\Omega$ are defined similarly. The space of $\mathcal{C}^\infty$, $\mathcal{C}_0^\infty$, L^2, H^s , etc. sections in any of these fiber bundles, E, on $O = \Omega$ or $O = \partial\Omega$, will be denoted respectively by $\mathcal{C}^\infty(O; E)$, $\mathcal{C}_0^\infty(O; E)$, $L^2(O; E)$, $H^s(O; E)$, etc..
When no confusion is possible we will simply use the short notation $\Lambda^p\mathcal{C}^\infty$, $\Lambda^p\mathcal{C}_0^\infty$, $\Lambda^p L^2$ and $\Lambda^p H^s$ for $E = \Lambda^p T^*\Omega$ or $E = \Lambda^p T^*\partial\Omega$.
Note that the L^2 spaces are those associated with the unit volume form for the Riemannian structure on Ω or $\partial\Omega$ (Ω and $\partial\Omega$ are oriented).
The notation $\mathcal{C}^\infty(\overline{\Omega}; E)$ is used for the set of $\mathcal{C}^\infty$ sections up to the boundary.

Let d be the exterior differential on $\mathcal{C}_0^\infty(\Omega; \Lambda T^*\Omega)$,

$$d^{(p)} : \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \,,$$

and d^* its formal adjoint with respect to the L^2-scalar product inherited from the Riemannian structure,

$$d^{(p),*} : \mathcal{C}_0^\infty(\Omega; \Lambda^{p+1} T^*\Omega) \to \mathcal{C}_0^\infty(\Omega; \Lambda^p T^*\Omega) \,.$$

Remark 3.2.1. *Note that d and d^* are both well defined on $\mathcal{C}^\infty(\overline{\Omega};\Lambda T^*\Omega)$.*

Set, for a function $f\in\mathcal{C}^\infty(\overline{\Omega};\mathbb{R})$ and $h>0$, the distorted operators defined on $\mathcal{C}^\infty(\overline{\Omega};\Lambda T^*\Omega)$:

$$d_{f,h}=e^{-f(x)/h}\,(hd)\,e^{f(x)/h}\quad\text{and}\quad d^*_{f,h}=e^{f(x)/h}\,(hd^*)\,e^{-f(x)/h}\,.$$

The Witten Laplacian is the differential operator defined on $\mathcal{C}^\infty(\overline{\Omega};\Lambda T^*\Omega)$ by:

$$\Delta_{f,h}=d^*_{f,h}d_{f,h}+d_{f,h}d^*_{f,h}=(d_{f,h}+d^*_{f,h})^2\,. \tag{3.2.1}$$

The last equality follows from the property $dd=d^*d^*=0$ which implies:

$$d_{f,h}d_{f,h}=d^*_{f,h}d^*_{f,h}=0. \tag{3.2.2}$$

It means, by restriction to the p-forms in $\mathcal{C}^\infty(\overline{\Omega};\Lambda^pT^*\Omega)$:

$$\Delta^{(p)}_{f,h}=d^{(p),*}_{f,h}d^{(p)}_{f,h}+d^{(p-1)}_{f,h}d^{(p-1),*}_{f,h}\ .$$

Let us give also a few relations with exterior and interior products (respectively denoted by $\wedge$ and $\mathbf{i}$), gradients (denoted by ∇) and Lie derivatives (denoted by $\mathcal{L}$) which will be very useful:

$$\begin{aligned}
(df\wedge)^* &= \mathbf{i}_{\nabla f}\quad(\text{in }L^2(\overline{\Omega};\Lambda^pT^*\Omega\,))\ , && (3.2.3)\\
d_{f,h} &= hd+df\wedge\ , && (3.2.4)\\
d^*_{f,h} &= hd^*+\mathbf{i}_{\nabla f}\ , && (3.2.5)\\
d\circ\mathbf{i}_X+\mathbf{i}_X\circ d &= \mathcal{L}_X\ , && (3.2.6)\\
\Delta_{f,h} &= h^2(d+d^*)^2+|\nabla f|^2+h\left(\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f}\right)\ , && (3.2.7)
\end{aligned}$$

where X denotes a vector field on Ω or $\overline{\Omega}$.

Remark 3.2.2. *The operators introduced depend on the Riemannian metric g_0 but we omit this dependence for conciseness.*

Definition 3.2.3. *We denote by $\vec{n}_\sigma$ the outgoing normal at $\sigma\in\partial\Omega$ and by $\vec{n}^*_\sigma$ the 1-form dual to $\vec{n}_\sigma$ for the Riemannian scalar product.*

For any $\omega\in\mathcal{C}^\infty(\overline{\Omega};\Lambda^pT^*\Omega)$, the form $\mathbf{t}\omega$ is the element of $\mathcal{C}^\infty(\partial\Omega;\Lambda^pT^*\Omega)$ defined by:

$$(\mathbf{t}\omega)_\sigma(X_1,\dots,X_p)=\omega_\sigma(X_1^T,\dots,X_p^T)\ ,\ \forall\sigma\in\partial\Omega\ ,$$

with the decomposition into the tangential and normal components to $\partial\Omega$ at σ: $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$.
Moreover,

$$(\mathbf{t}\omega)_\sigma = \mathbf{i}_{\vec{n}_\sigma}(\vec{n}_\sigma^* \wedge \omega_\sigma)\ .$$

The projected form $\mathbf{t}\omega$, which depends on the choice of $\vec{n}_\sigma$ (i.e. on g_0), can be compared with the canonical pull-back $j^*\omega$ associated with the embedding $j : \partial\Omega \to \Omega$. Actually the exact relationship is $j^*\omega = j^*(\mathbf{t}\omega)$.
The normal part of ω on $\partial\Omega$ is defined by:

$$\mathbf{n}\omega = \omega|_{\partial\Omega} - \mathbf{t}\omega \quad \in \mathcal{C}^\infty(\partial\Omega; \Lambda^p T^*\Omega).$$

Definition 3.2.4. *We denote by $\frac{\partial f}{\partial n}(\sigma)$ or $\partial_n f(\sigma)$ the normal derivative of f at σ:*

$$\frac{\partial f}{\partial n}(\sigma) = \partial_n f(\sigma) := \langle \nabla f(\sigma) \,|\, \vec{n}_\sigma \rangle\ .$$

Assumption 3.2.5. *The functions $f \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ and $f\big|_{\partial\Omega} \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R})$ are Morse functions. Moreover, the function f has no critical point on $\partial\Omega$.*

According to [ChLi], [HeNi], and [Lep3], the Neumann realization (resp. the Dirichlet realization) of the Witten Laplacian, denoted by $\Delta_{f,h}^N$ (resp. $\Delta_{f,h}^D$), is the self-adjoint realization of $\Delta_{f,h}$ whose domain is

$$D(\Delta_{f,h}^N) = \{\omega \in \Lambda H^2(\Omega)\,,\ \mathbf{n}\omega = 0\,,\ \mathbf{n}d_{f,h}\omega = 0\}$$
$$\text{(resp.} \quad D(\Delta_{f,h}^D) = \left\{\omega \in \Lambda H^2(\Omega)\,,\ \mathbf{t}\omega = 0\,,\ \mathbf{t}d_{f,h}^*\omega = 0\right\} \quad)\,.$$

Definition 3.2.6. *A point $U \in \overline{\Omega}$ is called a generalized critical point of f with index p in the Neumann case (resp. in the Dirichlet case) if:*
- *either $U \in \Omega$ and U is a critical point of f with index p,*
- *or $U \in \partial\Omega$ and U is a critical point with index p of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(U) < 0$ (resp. $U \in \partial\Omega$ and U is a critical point with index $p-1$ of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(U) > 0$).*

Remark 3.2.7. *This convention implies, in the Neumann case (resp. in the Dirichlet case), that for a generalized critical point U in the boundary with index p,*

$$p \in \{0, \dots, n-1\} \quad (\textit{resp. } p \in \{1, \dots, n\})\,.$$

We end up this section by giving the statement extending to the case of a manifold with boundary the analysis done by E. Witten in [Wit] (see [ChLi][HeNi][Lep3]).

Theorem 3.2.8. *Under Assumption 3.2.5, there exists* $h_0 > 0$*, such that the Neumann realization (resp. the Dirichlet realization) of the Witten Laplacian* $\Delta^N_{f,h}$ *(resp.* $\Delta^D_{f,h}$*) has, for* $h \in (0, h_0]$*, the following property:*
For any $p \in \{0, \ldots, n\}$*, the spectral subspace* $\mathrm{Ran}1_{[0,h^{3/2})}(\Delta^{N,(p)}_{f,h})$ *(resp.* $\mathrm{Ran}1_{[0,h^{3/2})}(\Delta^{D,(p)}_{f,h})$*) has rank* $m_p(f)$*, the number of generalized critical points of* f *with index* p *in the Neumann case (resp. in the Dirichlet case).*

Moreover, the proofs done in [HeNi] and [Lep3] show that the corresponding eigenvectors are concentrated around these critical points.

3.3 Preliminaries, coordinate systems

Since more than two geometries overlap around a generalized critical point of f with index p in the boundary and since systems of PDE are considered, the choice of the proper coordinate systems is a crucial point for making the analysis possible.

3.3.1 Existence of a local adapted coordinate system

Definition 3.3.1. *Let* σ *be a point on the boundary* $\partial\Omega$*. A local adapted coordinate system around* σ *is a local coordinate system* $(x^1, \ldots, x^n) = (x', x^n)$ *centered at* σ *satisfying the following properties:*

i) $dx^1, \ldots, dx^n$ *is an orthonormal positively oriented basis of* $T^*_\sigma(\overline{\Omega})$*, the cotangent space at* σ*.*

ii) The boundary $\partial\Omega$ *corresponds locally to* $x^n = 0$ *and the interior* Ω *to* $x^n < 0$*.*

iii) $\frac{\partial}{\partial x^n}|_{\partial\Omega} = \vec{n}$*, the outgoing normal at the boundary. Moreover,* $\frac{\partial}{\partial x^n}$ *is unitary and normal to* $\{x^n = \mathit{Constant}\}$*.*

Such a coordinate system is more specific than the one provided by the collar theorem in [Sch], [Duf], and [DuSp]. Moreover, owing to the analysis done in [Pet] pp. 117-122, it can be proven that such a system always exists. This is the aim of the following proposition.

Proposition 3.3.2. *A local coordinate system satisfying Definition 3.3.1 always exists.*

Proof. Consider indeed (see [Pet] pp. 119-120)

$$T\partial\Omega^\perp = \left\{ v \in T_\sigma\overline{\Omega} \,:\, \sigma \in \partial\Omega,\, v \in (T_\sigma\partial\Omega)^\perp \subset T_\sigma\overline{\Omega} \right\} ,$$

where $(T_\sigma\partial\Omega)^\perp$ is the orthogonal complement of $T\partial\Omega$ in $T_\sigma\overline{\Omega}$ (so for each $\sigma \in \partial\Omega$, $T_\sigma\overline{\Omega} = T_\sigma\partial\Omega \oplus^\perp (T_\sigma\partial\Omega)^\perp$). Then, the map $\exp^\perp$ introduced in [Pet] is a diffeomorphism from an open neighborhood of the zero section in $T\partial\Omega^\perp$ onto its image in $\overline{\Omega}$. It means, choosing a point σ near the boundary $\partial\Omega$, that there exists an unique geodesic ν joining σ to a point σ_b on the boundary which satisfies $\dot\nu(\sigma_b) \in T\partial\Omega^\perp$. It is equivalent to say that there exists an unique geodesic ν joining σ to σ_b with $\dot\nu(\sigma_b) = \vec{n}_{\sigma_b}$.
Set now $-x^n$ the geodesic distance to $\partial\Omega$ and take x' such that $x'\big|_{\partial\Omega}$ is a coordinate system on the boundary and x' is constant along the geodesics parametrized by x^n. The second point of the definition is then satisfied and $\frac{\partial}{\partial x^n}$ is unitary. Moreover, the choice of $x'\big|_{\partial\Omega}$ is arbitrary and we can choose it centered at U such that $dx^1, \ldots, dx^n$ is an orthonormal basis of $T^*_U(\overline{\Omega})$ positively oriented. Then the first point of the definition is also satisfied.
Verify now that the third point of the definition is fulfilled. Write

$$\begin{aligned}\frac{\partial}{\partial x^n}\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_\sigma &= \langle\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_\sigma + \langle\frac{\partial}{\partial x^n}\,|\,\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^i}\rangle_\sigma \\ &= 0 + \langle\frac{\partial}{\partial x^n}\,|\,\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^i}\rangle_\sigma \\ &= \langle\frac{\partial}{\partial x^n}\,|\,\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^n}\rangle_\sigma \\ &= \frac{1}{2}\frac{\partial}{\partial x^i}\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^n}\rangle_\sigma = 0\,,\end{aligned}$$

where we used the fact that ∇ is the Levi-Civita connexion and $\nabla_{\frac{\partial}{\partial x^n}}\frac{\partial}{\partial x^n} = 0$ since x^n is a geodesic curve. Hence,

$$\langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_\sigma = \langle\frac{\partial}{\partial x^n}\,|\,\frac{\partial}{\partial x^i}\rangle_{\sigma_b} = \langle\vec{n}_{\sigma_b}\,|\,\frac{\partial}{\partial x^i}\rangle_{\sigma_b} = 0\,,$$

which gives the third point of the definition.

∎

Remark 3.3.3. *In a local adapted coordinate system (x', x^n) around σ, remark that the metric g_0 writes*

$$g_0(x) = (dx^n)^2 + \sum_{1\le i,j<n} g_{ij}(x)dx^i dx^j\,.$$

Moreover, it can be convenient to work with matrices and we note $G_0(x) = (g_{ij}(x))_{ij}$, $G_0^{-1}(x) = (g^{ij}(x))_{ij}$. *Remember that* $g_{ij} = \langle\frac{\partial}{\partial x^i}\,|\,\frac{\partial}{\partial x^j}\rangle$,

$g^{ij} = \langle dx^i \,|\, dx^j \rangle$, and $dx^i(\frac{\partial}{\partial x^j}) = \delta_{ij}$.
Hence, $G_0^{\pm 1}(x)$ has the form, in the coordinate system (x', x^n):

$$G_0^{\pm 1}(x) = \begin{pmatrix} & & & 0 \\ & G_0^{\pm 1'}(x) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

with $G_0^{\pm 1}(0) = I_n$.

3.3.2 Separating the x^n-variable

Lemma 3.3.4. *1) Let f_1 belong in $\mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ and $U \in \partial\Omega$ be a critical point of $f_1|_{\partial\Omega}$ such that $\frac{\partial f_1}{\partial n}(U) \neq 0$. Assume furthermore $\alpha \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R})$ be a local solution to $|\nabla_T \alpha|^2 = |\nabla_T f_1|^2$ around U.*
Then there exists a neighborhood $\mathcal{V}$ of U in $\overline{\Omega}$ such that the eikonal equation

$$|\nabla \Phi_\pm|^2 = |\nabla f_1|^2 \tag{3.3.1}$$

(on the boundary, it means $|\nabla \Phi_\pm|^2 = |\partial_n \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2$; see the details in the proof)
with the boundary conditions

$$\Phi_\pm|_{\partial\Omega\cap\mathcal{V}} = \alpha \quad , \quad \partial_n \Phi_\pm|_{\partial\Omega\cap\mathcal{V}} = \pm \frac{\partial f_1}{\partial n}|_{\partial\Omega\cap\mathcal{V}}$$

admits a unique local smooth real-valued solution.

2) There exist local coordinates $(\overline{x}^1, \dots, \overline{x}^n) = (\overline{x}', \overline{x}^n)$ in a neighborhood of U in $\overline{\Omega}$ with $(\overline{x}', \overline{x}^n)(U) = 0$ where the function $\Phi_\pm$ and the metric g_0 have the form:

$$\Phi_\pm = \mp \overline{x}^n + \alpha(\overline{x}') \quad \text{and} \quad g_0 = g_{nn}(\overline{x}) \ (d\overline{x}^n)^2 + \sum_{i,j=1}^{n-1} g_{ij}(\overline{x}) d\overline{x}^i d\overline{x}^j \,.$$

Moreover, the boundary $\partial\Omega$ is locally defined by $\{\overline{x}^n = 0\}$ and Ω corresponds to $\left\{sgn\left(\frac{\partial f_1}{\partial n}(U)\right) \overline{x}^n > 0\right\}$.

Proof. 1) Take a local adapted coordinate system (x', x^n) around U in order to write (3.3.1):

$$|\partial_{x^n} \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2 = |\partial_{x^n} f_1|^2 + |\nabla_T f_1|^2$$

(see Appendix 3.A.1 for the exact meaning of ∇_T in the interior). We obtain in particular on the boundary,

$$|\partial_n \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2 = |\partial_n f_1|^2 + |\nabla_T \alpha|^2 \ .$$

The first point is then a direct consequence of the Hamilton-Jacobi theorem, due to the condition $\frac{\partial f_1}{\partial n}(U) \neq 0$.
2) Like in [HeSj4], set:

$$f_+ = \Phi_+ - \Phi_- \qquad \text{and} \qquad f_- = \Phi_+ + \Phi_- \ ,$$

and note the relations:

$$\Phi_- = -\frac{1}{2}f_+ + \frac{1}{2}f_- \ , \quad \Phi_+ = \frac{1}{2}f_+ + \frac{1}{2}f_- \ , \tag{3.3.2}$$

$$\nabla f_+ \cdot \nabla f_- = 0 \ , \tag{3.3.3}$$

$$f_+|_{\partial\Omega\cap\mathcal{V}} = 0 \ , \quad \frac{\partial f_+}{\partial n}|_{\partial\Omega\cap\mathcal{V}} = 2\frac{\partial f_1}{\partial n}|_{\partial\Omega\cap\mathcal{V}} \neq 0 \ , \tag{3.3.4}$$

$$\text{and} \qquad f_-|_{\partial\Omega\cap\mathcal{V}} = 2\alpha \ , \quad \frac{\partial f_-}{\partial n}|_{\partial\Omega\cap\mathcal{V}} = 0 \ . \tag{3.3.5}$$

Let $(\overline{x}^1, \ldots, \overline{x}^{n-1}) = \overline{x}'$ denote a set of coordinates on $\partial\Omega$ in a neighborhood of U (then contained in $\mathcal{V}$) and such that $\overline{x}^j(U) = 0$. We extend them in a neighborhood of U in $\overline{\Omega}$ as constant along the integral curve of the vector field ∇f_+. Then we take $\overline{x}^n = -\frac{1}{2}f_+(x)$ for the last coordinate.
In these coordinates, the functions $\Phi_\pm$ and the metric g_0 have the forms announced in the lemma.
We remark furthermore, by (3.3.4) and $\frac{\partial f_1}{\partial n}(U) \neq 0$, that the boundary $\partial\Omega$ is locally defined by $\{\overline{x}^n = 0\}$ and Ω corresponds to $\left\{sgn\left(\frac{\partial f_1}{\partial n}(U)\right)\overline{x}^n > 0\right\}$. ∎

In the sequel, we will apply the first result of this lemma in the Neumann case (resp. in the Dirichlet case) in order to specify the Agmon distance, associated with the function f, to a generalized critical point U with index p in the boundary.
Then, using the second result of this lemma and Proposition 4.3.16 of [Lep3] (resp. Proposition 3.3.9 of [HeNi]), $\Delta_{f,h}^{(p),N}$ (resp. $\Delta_{f,h}^{(p),D}$) can be viewed locally in $\mathcal{V}$ around $U \in \partial\Omega$ as $\mathcal{A}_N^{(p)}\big|_{\mathcal{V}}$ (resp. as $\mathcal{A}_D^{(p)}\big|_{\mathcal{V}}$) where $\mathcal{A}_N^{(p)}$ (resp. $\mathcal{A}_D^{(p)}$) is a self-adjoint Witten Laplacian on $\mathbb{R}^n_- = \mathbb{R}^{n-1} \times (-\infty, 0)$ (ever if it means choosing $-\overline{x}^n$ instead of $\overline{x}^n$) whose domain is (see also [KoPrSh])

$$D(\mathcal{A}_N) = \left\{\omega \in \Lambda H^2(\mathbb{R}^n_-)\,,\ \mathbf{n}\omega = \mathbf{n}d_{f,h}\omega = 0\right\}$$
$$(\text{resp.} \quad D(\mathcal{A}_D) = \left\{\omega \in \Lambda H^2(\mathbb{R}^n_-)\,,\ \mathbf{t}\omega = \mathbf{t}d^*_{f,h}\omega = 0\right\} \quad)\,,$$

and which satisfies

$$\dim \ \mathrm{Ker}\ \mathcal{A}_N^{(p)} = 1 \ \text{ and } \ \sigma(\mathcal{A}_N^{(p)}) \setminus \{0\} \subset \left[Ch^{6/5}, +\infty\right) \quad (3.3.6)$$

$$(\text{resp. } \dim \ \mathrm{Ker}\ \mathcal{A}_D^{(p)} = 1 \ \text{ and } \ \sigma(\mathcal{A}_D^{(p)}) \setminus \{0\} \subset \left[Ch^{6/5}, +\infty\right)). \quad (3.3.7)$$

3.4 WKB construction near the boundary for $\Delta_{f,h}^{(p)}$, with p in $\{0, \ldots, n\}$

3.4.1 Local WKB construction in the Neumann case

Let U be a generalized critical point of f with index p in the Neumann case, i.e. a critical point with index $p \in \{0, \ldots, n-1\}$ of $f\big|_{\partial\Omega}$ satisfying $\frac{\partial f}{\partial n}(U) < 0$, and take a local adapted coordinate system (x', x^n) around U.

Let Φ and φ be respectively the Agmon distance to U associated with the function f and its restriction to the boundary. The Agmon distance associated with f, i.e. with the metric $|\nabla f(x)|^2 \ dx^2$, is denoted by d_{Ag}: $\Phi(x) = d_{\mathrm{Ag}}(x, U)$.
Recall that, locally,

$$|\nabla f|^2 = |\nabla \Phi|^2$$

and that Φ is smooth near U (see [HeSj1]).
Moreover, φ is nothing but the Agmon distance to U on the boundary and satisfies locally, on the boundary,

$$|\nabla_T f|^2 = |\nabla \varphi|^2 .$$

We now use the first result of Lemma 3.3.4 with $f_1 = f$ and $\alpha = \varphi$. The function Φ_+ of the lemma is consequently Φ and we have locally:

$$|\partial_n \Phi|^2 + |\nabla_T \Phi|^2 = |\nabla \Phi|^2 = |\nabla f|^2 \ , \quad (3.4.1)$$

$$\Phi|_{\partial\Omega} = \varphi \ , \quad (3.4.2)$$

$$\partial_n \Phi|_{\partial\Omega} = \frac{\partial f}{\partial n}|_{\partial\Omega} \, . \quad (3.4.3)$$

Moreover, the next relation is valid:

$$\partial^2_{x^n x^n}(f - \Phi)(0) = \partial^2_{nn}(f - \Phi)(0) = 0 \, . \quad (3.4.4)$$

Write indeed in the coordinates (x', x^n), for the metric g_0:

$$|\partial_{x^n} \Phi|^2 + |\nabla_T \Phi|^2_{g_0} = |\partial_{x^n} f|^2 + |\nabla_T f|^2_{g_0}$$

where $|\nabla_T \Phi|^2_{g_0} = \mathcal{O}(|x|^2)$ and $|\nabla_T f|^2_{g_0} = \mathcal{O}(|x|^2)$ because 0 is a critical point of $f\big|_{\partial\Omega}$ in the coordinates (x', x^n) (see indeed for example Appendix 3.A.1). Apply then ∂_{x^n} to the last equation:

$$\partial_{x^n} |\partial_{x^n}\Phi|^2 + \mathcal{O}(|x|) = \partial_{x^n} |\partial_{x^n} f|^2 + \mathcal{O}(|x|)$$

i.e., using (3.4.3),

$$2\partial^2_{x^n x^n}(f - \Phi)\partial_{x^n} f = \mathcal{O}(|x|)$$

which yields the result.
According to [HeSj4] pp. 279-280, there exist local coordinates $(\overline{x}', \overline{x}^n)$ centered at U, where $\overline{x}' = (\overline{x}^1, \dots, \overline{x}^{n-1})$ are Morse coordinates for $f\big|_{\partial\Omega}$ around U, such that $d\overline{x}^1, \dots, d\overline{x}^{n-1}, dx^n$ is orthonormal at U, and

$$f(\overline{x}', 0) = \frac{\lambda_1}{2}(\overline{x}^1)^2 + \cdots + \frac{\lambda_{n-1}}{2}(\overline{x}^{n-1})^2 + f(U) \qquad (3.4.5)$$

$$\text{and} \qquad \varphi(\overline{x}') = \frac{|\lambda_1|}{2}(\overline{x}^1)^2 + \cdots + \frac{|\lambda_{n-1}|}{2}(\overline{x}^{n-1})^2 \,. \qquad (3.4.6)$$

with $\lambda_i < 0$ for $i \in \{1, \dots, p\}$ and $\lambda_i > 0$ for $i \in \{p+1, \dots, n-1\}$.
Furthermore, the coordinates (x', x^n) can be chosen such that $dx^1, \dots, dx^{n-1}$ and $d\overline{x}^1, \dots, d\overline{x}^{n-1}$ coincide at U, and even such that $x'\big|_{\partial\Omega} = \overline{x}'\big|_{\partial\Omega}$ since $x'\big|_{\partial\Omega}$ can be chosen freely.

Specification of the coordinate system for Theorem 3.1.1
The local adapted coordinate system $x = (x', x^n)$ around U is chosen such that,

$$\forall i \in \{1, \dots, n-1\}\ , \quad dx^i = d\overline{x}^i \text{ at } U\,. \qquad (3.4.7)$$

3.4.2 First boundary conditions in the Neumann case

Let us first write, in our coordinate system,

$$a(x,h) = \sum_{I \in \mathcal{I}} a_I(x,h)\ dx^I = \sum_{I' \in \mathcal{I}'} a_{I'}(x,h) dx^{I'} + \sum_{I_n \in \mathcal{I}_n} a_{I_n}(x,h) dx^{I_n}\,, \quad (3.4.8)$$

where $\mathcal{I} := \{(i_1, \dots, i_p) \in \{1, \dots, n\}^p\,,\, i_1 < \cdots < i_p\}$,
$\mathcal{I}' := \{(i_1, \dots, i_p) \in \{1, \dots, n\}^p\,,\, i_1 < \cdots < i_p < n\}$,
$\mathcal{I}_n := \{(i_1, \dots, i_p) \in \{1, \dots, n\}^p\,,\, i_1 < \cdots < i_p = n\}$,
and $dx^{(i_1,\dots,i_p)} = dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. We will use in the sequel the Einstein summation convention in order to write (3.4.8) without the summation symbols:

$$a(x,h) = a_I(x,h)\ dx^I = a_{I'}(x,h) dx^{I'} + a_{I_n}(x,h) dx^{I_n}\,.$$

The first boundary condition simply says:

$$\forall I_n \in \mathcal{I}_n\,,\ a_{I_n}((x',0),h) \sim \sum_k a^k_{I_n}(x',0)h^k \equiv 0\,, \tag{3.4.9}$$

which is equivalent to

$$\forall k \in \mathbb{N},\, \forall I_n \in \mathcal{I}_n\,,\ a^k_{I_n}(x',0) \equiv 0\,. \tag{3.4.10}$$

The rest of this subsection specifies some consequence of these conditions. These consequences will be used in the next subsection to prove Theorem 3.1.1.

Proposition 3.4.1. *Using the notation of Appendices 3.A.1 and 3.A.2, the following relations are satisfied for every tangential p-form* $b(x) = b_I(x)dx^I$*, i.e. every p-form* $b(x)$ *satisfying* $b_{I_n}(x',0) \equiv 0$ *for all* $I_n \in \mathcal{I}_n$*:*

$$\left\{\begin{array}{lcl} \mathbf{t}\left((2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)b\right) & = & (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}^T_{Neu})b_{I'}dx^{I'} + 2\frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}db \\ \mathbf{n}\left((2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)b\right) & = & 2\left(\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n} + \ell_{I_n}(x',0)\right)dx^{I_n}\,, \end{array}\right.$$

where the ℓ_{I_n}*'s are* $\mathcal{C}^\infty(\partial\Omega)$*-linear combinations of the* $b_{I'}$*'s (for* I' *in* $\mathcal{I}'$*) which do not depend on the* b_{I_n}*'s (for* I_n *in* $\mathcal{I}_n$*) and* $\mathcal{R}^T_{Neu}$ *is a* 0*-th order differential operator given on the boundary by the following matrix, in the coordinates* (x',x^n)*:*

$$\mathcal{R}^T_{Neu}(x',0) = \left(\begin{array}{cccc} & & & 0 \\ & \mathcal{R}^{T'}_{Neu}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{array}\right)^{(p)} - \gamma(x')\,Id\,,$$

where $\beta(0) = 0$*,*

$$\gamma(0) = Tr\left(\text{Hess}\,(f\big|_{\partial\Omega} - \varphi)(0)\right)\,,$$

and

$$\mathcal{R}^{T'}_{Neu}(0) = 2\left(\text{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.$$

In particular, this is true for a^k *for* k *in* $\mathbb{N}$ *when* (3.4.10) *is fulfilled.*

The following elementary result is important to notice here and also while verifying the final compatibility conditions (see pp. 21-22).

Lemma 3.4.2. *Let $b(x)$ be a <u>tangential</u> p-form. The p-form*

$$\mathbf{i}_{\vec{n}}(db)$$

is then tangential and the following equivalence is locally valid on the boundary $\partial\Omega$:

$$\mathbf{i}_{\vec{n}}(db)=0 \Leftrightarrow \mathbf{n}db=0\,.$$

In particular, this is true for a^k for k in $\mathbb{N}$ when (3.4.10) is fulfilled.

Proof. Write indeed on the boundary $\partial\Omega$ in the coordinate system (x',x^n):

$$\begin{aligned}
\mathbf{i}_{\frac{\partial}{\partial x^n}}(db) &= \mathbf{i}_{\frac{\partial}{\partial x^n}}\mathbf{n}db+\mathbf{i}_{\frac{\partial}{\partial x^n}}\mathbf{t}db\\
&= \mathbf{i}_{\frac{\partial}{\partial x^n}}\mathbf{n}db+0\\
&= \mathbf{i}_{\frac{\partial}{\partial x^n}}(db)_{I_n}dx^{I_n}\\
&= (-1)^p(db)_{I_n}dx^{I_n\setminus\{n\}}\,,
\end{aligned}$$

which leads to the result. ∎

Lemma 3.4.3. *The following relations are satisfied for every tangential p-form $b(x)$:*

$$\left\{\begin{aligned}
\mathbf{t}\left((\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})b\right) &= \mathbf{t}\left((\mathcal{L}_{\nabla_T\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})b\right)=\frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}db\,,\\
\mathbf{n}\left((\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})b\right) &= \left(\frac{\partial b_{I_n}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}+\tilde{\ell}_{I_n}(x',0)\right)dx^{I_n}\,,
\end{aligned}\right.$$

where the $\tilde{\ell}_{I_n}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the b_{I_n}'s (for I_n in $\mathcal{I}_n$).

Proof. On the boundary $\partial\Omega$, we have the following decomposition:

$$(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})\,b=\mathcal{L}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}}b+(\mathcal{L}_{\nabla_T\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})\,b\,. \tag{3.4.11}$$

Owing to the Cartan formula (3.2.6), rewrite (3.4.11):

$$\begin{aligned}
(\mathcal{L}_{\nabla\Phi}-\mathcal{L}_{\nabla\tilde{\Phi}})\,b &= \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}}db+d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}}b)\\
&+ \mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}db+d(\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b)\,.
\end{aligned}\tag{3.4.12}$$

Using Lemma 3.4.2, the first term

$$\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}}db=\frac{\partial\Phi}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}db$$

of the r.h.s. of (3.4.12) is tangential.
Moreover, since $\nabla_T\Phi = \nabla\tilde{\Phi}$ on the boundary (see Appendix 3.A.1), the term $\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}db$ of the r.h.s. equals 0 on $\partial\Omega$.
Hence, we can write on $\partial\Omega$:

$$(\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}})\, b = \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} db + d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} b) + d(\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})} b)\,. \qquad (3.4.13)$$

Let us study in a first time the term $d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} b)$. Writing

$$b = b_I dx^I = b_{I'} dx^{I'} + b_{I_n} dx^{I_n}\,,$$

we deduce (in $\overline{\Omega}$):

$$\begin{aligned} \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} b &= b_{I_n} \mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} dx^{I_n} \\ &= (-1)^{p-1} b_{I_n} \frac{\partial\Phi}{\partial x^n} dx^{I_n\setminus\{n\}}\,, \end{aligned}$$

and, applying d to this last relation, we obtain on $\partial\Omega$ (remember that $b_{I_n} = 0$ on $\partial\Omega$)

$$\begin{aligned} d(\mathbf{i}_{(\frac{\partial\Phi}{\partial x^n})\frac{\partial}{\partial x^n}} b) &= (-1)^{p-1} \sum_{i=1}^{n} \frac{\partial}{\partial x^i}(b_{I_n} \frac{\partial\Phi}{\partial x^n}) dx^i \wedge dx^{I_n\setminus\{n\}} \\ &= (-1)^{p-1} \frac{\partial b_{I_n}}{\partial x^n} \frac{\partial\Phi}{\partial x^n} dx^n \wedge dx^{I_n\setminus\{n\}} + 0 \\ &= \frac{\partial b_{I_n}}{\partial x^n} \frac{\partial\Phi}{\partial x^n} dx^{I_n}\,. \end{aligned} \qquad (3.4.14)$$

Look now at the third term of the r.h.s. of (3.4.13) and write (remember that $\mathcal{I} \ni I = (i_1, \dots, i_p)$ with $1 \le i_1 \le \dots \le i_p \le n$ and denote by $\operatorname{ind}(i_k)$ the integer k):

$$\begin{aligned} \mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})} b_I dx^I &= b_I dx^I(\nabla_T\Phi - \nabla\tilde{\Phi}) \\ &= b_I \sum_{j\in I} (-1)^{\operatorname{ind}(j)+1} \left(\nabla_T\Phi - \nabla\tilde{\Phi}\right)_j dx^{I\setminus\{j\}} \\ &= b_I \sum_{j\in I} (-1)^{\operatorname{ind}(j)+1} \alpha_j dx^{I\setminus\{j\}}\,, \end{aligned}$$

where, due to (3.A.2)(3.A.3), for all j in $\{1, \dots, n\}$,

$$\alpha_j = \left(\nabla_T\Phi - \nabla\tilde{\Phi}\right)_j = \sum_{i=1}^{n} g^{ij} \left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x', 0)\right)\,.$$

Moreover, due to the block diagonal form of G_0^{-1}, for all j in $\{1,\ldots,n\}$, α_j satisfies (again by (3.A.2)(3.A.3)):

$$\alpha_n(x)\equiv 0 \text{ and } \forall j\in\{1,\ldots,n-1\}\ ,\alpha_j(x',0)\equiv 0\,.$$

Hence, we obtain on $\partial\Omega$,

$$\begin{aligned}
d(\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I dx^I)(x',0) &= \sum_{l=1}^{n}\sum_{j\in I}(-1)^{\,\mathrm{ind}(j)+1}\frac{\partial}{\partial x^l}(b_I\alpha_j)(x',0)dx^l\wedge dx^{I\backslash\{j\}}\\
&= 0+\sum_{j\in I}(-1)^{\,\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_I\alpha_j)(x',0)dx^n\wedge dx^{I\backslash\{j\}}\\
&= \sum_{j\in I'}(-1)^{\,\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I'}\alpha_j)(x',0)dx^n\wedge dx^{I'\backslash\{j\}}\\
&\quad+\sum_{j\in I_n\backslash\{n\}}(-1)^{\,\mathrm{ind}(j)+1}\,\frac{\partial}{\partial x^n}(b_{I_n}\alpha_j)(x',0)dx^n\wedge dx^{I_n\backslash\{j\}}\\
&= \sum_{j\in I'}(-1)^{\,\mathrm{ind}(j)+1}\frac{\partial}{\partial x^n}(b_{I'}\alpha_j)(x',0)dx^n\wedge dx^{I'\backslash\{j\}}\,,
\end{aligned}$$

where we used $\alpha_j(x',0)\equiv 0$ at the second line and $\alpha_n(x)\equiv 0$ at the second to last line.
Using again $\alpha_j(x',0)\equiv 0$ allows us to write on $\partial\Omega$:

$$\begin{aligned}
d(\mathbf{i}_{(\nabla_T\Phi-\nabla\tilde{\Phi})}b_I dx^I)(x',0) &= b_{I'}\sum_{j\in I'}(-1)^{\,\mathrm{ind}(j)+1}\frac{\partial\alpha_j}{\partial x^n}(x',0)dx^n\wedge dx^{I'\backslash\{j\}}\\
&= b_{I'}\sum_{j\in I'}(-1)^{\,\mathrm{ind}(j)+p}\frac{\partial\alpha_j}{\partial x^n}(x',0)dx^{I'\backslash\{j\}}\wedge dx^n\\
&=:\tilde{\ell}_{I_n}(x',0)dx^{I_n}\,, \qquad (3.4.15)
\end{aligned}$$

where the $\tilde{\ell}_{I_n}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the b_{I_n}'s (for I_n in $\mathcal{I}_n$).
Combining (3.4.13), (3.4.14), and (3.4.15) leads to the result announced in Lemma 3.4.3.

∎

Proof of Proposition 3.4.1.
Remember first the following relation (see indeed Subsection 3.A.2):

$$\mathcal{L}_{\nabla\Phi}-\mathcal{L}^*_{\nabla\Phi}+\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f}=2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1\,,$$

where $\mathcal{R}_1$ is a 0-th order differential operator. Writing $\mathcal{R}_1 = \mathcal{R}_1^T + \mathcal{R}_1^N$, we deduce from Remark 3.A.6, since $b_I dx^I = b_{I'} dx^{I'}$ on the boundary,

$$\left\{ \begin{array}{rcl} \mathbf{t}\left(\mathcal{R}_1(b_I dx^I)\right) & = & b_{I'}(x',0)\mathcal{R}_1^T(dx^{I'}) \\ \mathbf{n}\left(\mathcal{R}_1(b_I dx^I)\right) & = & b_{I'}(x',0)\mathcal{R}_1^N(dx^{I'}) = \tilde{\ell}'_{I_n}(x',0)dx^{I_n}\,, \end{array} \right.$$

where the $\tilde{\ell}'_{I_n}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the b_{I_n}'s (for I_n in $\mathcal{I}_n$).

Moreover, from (3.4.1)-(3.4.4), $f - \Phi$ satisfies the assumptions of Corollary 3.A.5, then $\mathcal{R}_1^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_1^T(x',0) = \left(\begin{array}{cccc} & & & 0 \\ & \mathcal{R}_1^{T'}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{array} \right)^{(p)} - \gamma(x')\, Id\,,$$

where β, γ are $\mathcal{C}^\infty$ functions which satisfy $\beta(0) = 0$, $\gamma(0) = Tr\left(\text{Hess}\,(f\big|_{\partial\Omega} - \varphi)(0)\right)$, and

$$\mathcal{R}_1^{T'}(0) = 2\left(\text{Hess}\,(f\big|_{\partial\Omega} - \varphi)(0)\right)\,.$$

Having in mind Lemma 3.4.3, look now at the term $2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_1$.
From Proposition 3.A.3, write:

$$2\mathcal{L}_{\nabla\tilde{\Phi}} = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}_3$$

where $\mathcal{R}_3 = \mathcal{R}_3^T + \mathcal{R}_3^N$ is a 0-th order differential operator such that, since $\tilde{\Phi}$ satisfies the assumptions of Corollary 3.A.5,

$$\left\{ \begin{array}{rcl} \mathbf{t}\left(\mathcal{R}_3(b_I dx^I)\right) & = & b_{I'}(x',0)\mathcal{R}_3^T(dx^{I'}) \\ \mathbf{n}\left(\mathcal{R}_3(b_I dx^I)\right) & = & b_{I'}(x',0)\mathcal{R}_3^N(dx^{I'}) = \tilde{\ell}''_{I_n}(x',0)dx^{I_n}\,, \end{array} \right.$$

where the $\tilde{\ell}''_{I_n}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the b_{I_n}'s, and $\mathcal{R}_3^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_3^T(x',0) = \left(\begin{array}{cccc} & & & 0 \\ & \mathcal{R}_3^{T'}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 0 \end{array} \right)^{(p)}\,,$$

with

$$\mathcal{R}_3^{T'}(0) = 2\left(\text{Hess}\,(\tilde{\Phi}\big|_{\partial\Omega})(0)\right) = 2\left(\text{Hess}\,(\varphi)(0)\right)\,.$$

Note, according to Remark 3.A.4, that the term of index (n,n) of the matrix is indeed 0 since $\frac{\partial^2\tilde{\Phi}}{(\partial x^n)^2} \equiv 0$.

Set $\mathcal{R}_{\text{Neu}} = \mathcal{R}_1 + \mathcal{R}_3$ and $\tilde{\ell}^{(3)}_{I_n} = \tilde{\ell}'_{I_n} + \tilde{\ell}''_{I_n}$ for I_n in $\mathcal{I}_n$. $\mathcal{R}_{\text{Neu}}$ is a 0-th order differential operator which satisfies

$$2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_1 = 2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}_{\text{Neu}}\,, \tag{3.4.16}$$

and

$$\begin{cases} \mathbf{t}\left(\mathcal{R}_{\text{Neu}}(b_I dx^I)\right) &= b_{I'}(x',0)\mathcal{R}^T_{\text{Neu}}(dx^{I'}) \\ \mathbf{n}\left(\mathcal{R}_{\text{Neu}}(b_I dx^I)\right) &= b_{I'}(x',0)\mathcal{R}^N_{\text{Neu}}(dx^{I'}) = \tilde{\ell}^{(3)}_{I_n}(x',0)dx^{I_n}\,, \end{cases} \tag{3.4.17}$$

where the $\tilde{\ell}^{(3)}_{I_n}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $b_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the b_{I_n}'s (for I_n in $\mathcal{I}_n$).
Moreover, $\mathcal{R}^T_{\text{Neu}}$ is given on the boundary, in the coordinates (x',x^n), by:

$$\mathcal{R}^T_{\text{Neu}}(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_1^{T'}(x',0) + \mathcal{R}_3^{T'}(x',0) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \beta(x') \end{pmatrix}^{(p)} - \gamma(x')\,Id\,,$$

where $\beta(0)=0$,

$$\gamma(0) = Tr\left(\text{Hess}\,(f\big|_{\partial\Omega} - \varphi)(0)\right)\,,$$

and

$$\mathcal{R}_1^{T'}(0) + \mathcal{R}_3^{T'}(0) = 2\left(\text{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.$$

Look now at the term $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id$. By the Cartan formula (3.2.6),

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id)b \;=\; db_{I'}(\nabla\tilde{\Phi})dx^{I'} + db_{I_n}(\nabla\tilde{\Phi})dx^{I_n}\,,$$

and, using the boundary condition satisfied by the b_{I_n}'s (for I_n in $\mathcal{I}_n$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain:

$$\begin{aligned} (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id)b &= \sum_{i=1}^{n-1} \frac{\partial b_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i dx^{I'} \\ &= (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id)b_{I'}dx^{I'}\,. \end{aligned} \tag{3.4.18}$$

Set $\ell_{I_n} = \tilde{\ell}_{I_n} + \frac{1}{2}\tilde{\ell}^{(3)}_{I_n}$ for I_n in $\mathcal{I}_n$. Writing

$$(2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1)\,b = 2\left(\mathcal{L}_{\nabla\Phi} - \mathcal{L}_{\nabla\tilde{\Phi}}\right)b + \left(2\mathcal{L}_{\nabla\tilde{\Phi}} + \mathcal{R}_1\right)b\,,$$

and using (3.4.16), (3.4.17), and (3.4.18), we obtain Proposition 3.4.1 after the application of Lemma 3.4.3.

∎

3.4.3 Proof of Theorem 3.1.1

We shall first consider a WKB-approximation for

$$(\Delta^{(p)}_{f,h} - E(h))u^{wkb}_p = e^{-\frac{\Phi}{h}}\mathcal{O}(h^\infty) \tag{3.4.19}$$

with $E(h) = O(h^2)$ and the boundary conditions (3.1.2)(3.1.3) and then check $E(h) = O(h^\infty)$.
Writing

$$\forall k \in \mathbb{N}\,, \quad d_{f,h}(e^{-\frac{\Phi}{h}}a^k) = e^{-\frac{\Phi}{h}}\left[hda^k + d(f-\Phi)\wedge a^k\right]\,,$$

where, due to (3.1.2) and (3.4.3), a^k and $d(f-\Phi)$ are tangential forms, the second boundary condition corresponds to

$$\forall k \in \mathbb{N}\,, \quad \mathbf{n}(da^k) = 0\,. \tag{3.4.20}$$

Let us now recall the following relation which will be very useful (see [HeSj4] for a complete proof):

$$\begin{aligned} e^{\frac{\Phi}{h}}\Delta_{f,h}e^{-\frac{\Phi}{h}} &= h^2(d+d^*)^2 + |\nabla f|^2 - |\nabla\Phi|^2 + h(\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}) \\ &= h^2(d+d^*)^2 + h(\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f})\,, \end{aligned} \tag{3.4.21}$$

and write, with the notation of Appendix 3.A.2,

$$\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} = 2\mathcal{L}_{\nabla\Phi} + \mathcal{R}_1 = 2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}\,,$$

where $\mathcal{R}$ and $\mathcal{R}_1$ are 0-th differential operators defined in Appendix 3.A.2.
By looking for $E(h) \sim \sum_{k=1}^\infty h^{k+1}E_k$, the interior equation (3.4.19) reads

$$e^{\frac{\Phi}{h}}(\Delta_{f,h} - E(h))e^{-\frac{\Phi}{h}} = h^2[(d+d^*)^2 - h^{-2}E(h)] + h\,[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}]$$

We now verify that it is possible to construct a solution u^{wkb}_p to (3.4.19) in Ω which can be extended to $\overline{\Omega}$ and satisfying the boundary conditions (3.1.2) and (3.1.3).
The construction of an interior WKB solution in Ω is standard as an inductive

Cauchy problem, once the a^k's are known on $\partial\Omega$ (see [DiSj],[Hel2]). Actually the non characteristic Cauchy problems

$$[2\mathcal{L}_{\nabla\Phi}\otimes Id+\mathcal{R}]a^k = -(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} \quad \text{in } \overline{\Omega}\,. \tag{3.4.22}$$

are solved by induction with the convention $a_{-1}=0$.
Hence the problem is reduced to the solving of the system made of the boundary conditions (3.4.10), (3.4.20) and of the compatibility equation on the boundary (see Appendix 3.A.2 for the meaning of the notations):

$$[2\mathcal{L}_{\nabla\Phi}+\mathcal{R}_1]\,a^k = -(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} \quad \text{on } \partial\Omega\,. \tag{3.4.23}$$

Owing to Proposition 3.4.1 (with the notation of Section 3.4.2) and to (3.4.3), the system (3.4.23), (3.4.10), (3.4.20) is equivalent to the differential system on $\partial\Omega$:

$$\begin{cases}-\mathbf{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id+\mathcal{R}^T_{\mathrm{Neu}})a^k_{I'}dx^{I'}+2\dfrac{\partial f}{\partial x^n}\mathbf{i}_{\frac{\partial}{\partial x^n}}da^k\\ -\mathbf{n}(d+d^*)^2a^{k-1}-2\ell_{I_n}(x',0)dx^{I_n}=2\dfrac{\partial f}{\partial n}\dfrac{\partial a^k_{I_n}}{\partial x^n}dx^{I_n}\\ \forall k\in\mathbb{N}\,,\quad \big(a^k_{I_n}|_{\partial\Omega}\equiv 0\big)+\big(\mathbf{n}(da^k)=0\big)\,,\end{cases}$$

where the ℓ_{I_n}'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I'}$'s (for I' in $\mathcal{I}'$) which do not depend on the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$).
Note also, owing to Lemma 3.4.2, that the first line of this system simply reads:

$$-\mathbf{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell}=(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id+\mathcal{R}^T_{\mathrm{Neu}})a^k_{I'}dx^{I'}\,. \tag{3.4.24}$$

Moreover, since $dx^i=d\overline{x}^i$ (for $i\in\{1,\ldots,n-1\}$) at the point U, thanks to Corollary 3.A.5, (3.4.5)-(3.4.6), and according to [HeSj4] pp. 271-275, $\mathcal{R}^T_{\mathrm{Neu}}(0)$ restricted to tangential forms is symmetric with the one dimensional kernel $\mathbb{R}dx^1\wedge\cdots\wedge dx^p$.

Since $a^k_{I'}dx^{I'}$ is tangential and $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id$ only differentiates tangentially the $a^k_{I'}$'s, since

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id)a^k_{I'}dx^{I'}=\sum_{i=1}^{n-1}\frac{\partial a^k_{I'}}{\partial x^i}(\nabla\tilde{\Phi})_i dx^{I'}\,,$$

(3.4.24) can be rewritten as a tangential system which can be solved according to the analysis of the boundaryless case done in [HeSj4].
Here are the details:
Owing to Lemma 3.4.2, the complete system (3.4.22), (3.4.23), (3.4.10), (3.4.20) becomes equivalent to

$$\begin{cases}(2\mathcal{L}_{\nabla\tilde\Phi}\otimes Id+\mathcal{R}^T_{\text{Neu}})a^k_{I'}dx^{I'}=-\mathbf{t}(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}+E_k a^0 & \text{on } \partial\Omega\\ (2\mathcal{L}_{\nabla\Phi}\otimes Id+\mathcal{R})a^k=-(d+d^*)^2a^{k-1}+\sum_{\ell=1}^{k}E_\ell a^{k-\ell} & \text{on } \overline{\Omega}\\ \qquad\forall I_n\in\mathcal{I}_n\,,\ a_{I_n}|_{\partial\Omega}\equiv 0\,. \end{cases}$$

Note that the first line is a degenerate matricial transport equation which can be solved according to [HeSj4] p. 275 and [Hel2] pp. 13-14:
For $k=0$, the boundary equation

$$(2\mathcal{L}_{\nabla\tilde\Phi}\otimes Id+\mathcal{R}^T_{\text{Neu}})a^0_{I'}dx^{I'}=0$$

admits some solution iff

$$a^0_{I'}(0)dx^{I'}\in\ \text{Ker}\,(\mathcal{R}^T_{\text{Neu}}(0))\,.$$

Take then $a^0(0)=dx^1\wedge\cdots\wedge dx^p\in\ \text{Ker}\,(\mathcal{R}^T_{\text{Neu}}(0))$. Now, for $k=1$, we have to solve the boundary equation

$$(2\mathcal{L}_{\nabla\tilde\Phi}\otimes Id+\mathcal{R}^T_{\text{Neu}})a^1_{I'}dx^{I'}=-\mathbf{t}(d+d^*)^2a^0+E_1a^0\,.$$

Choose then $a^1_{I'}(0)dx^{I'}=0$, i.e. $a^1(0)=0$ owing to $a_{I_n}|_{\partial\Omega}\equiv 0$, and E_1 such that

$$-\mathbf{t}(d+d^*)^2a^0(0)+E_1a^0(0)\in\big(\ \text{Ker}\,(\mathcal{R}^T_{\text{Neu}}(0))\big)^\perp\,,$$

or, equivalently, such that

$$E_1=\frac{\langle\mathbf{t}(d+d^*)^2a^0(0)\,|\,a^0(0)\rangle_{g_0(0)}}{||a^0(0)||^2_{g_0(0)}}\,.$$

Note that it is indeed possible since $a^0(0)\neq 0$.
Then, for each $k>2$, choose $a^{k-1}_{I'}(0)dx^{I'}=0$, i.e. $a^{k-1}(0)=0$, and E_k such that the compatibility condition

$$-\mathbf{t}(d+d^*)^2a^{k-1}(0)+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}(0)+E_ka^0(0)\in\big(\ \text{Ker}\,(\mathcal{R}^T_{\text{Neu}}(0))\big)^\perp$$

is satisfied. This means more precisely that

$$-\mathbf{t}(d+d^*)^2a^{k-1}(0)+E_ka^0(0)\in\big(\ \text{Ker}\,(\mathcal{R}^T_{\text{Neu}}(0))\big)^\perp\,,$$

or equivalently that

$$E_k = \frac{\langle \mathbf{t}(d+d^*)^2 a^{k-1}(0) \mid a^0(0)\rangle_{g_0(0)}}{||a^0(0)||^2_{g_0(0)}}\,.$$

Thus, at every step $k \in \mathbb{N}$, the first and the third line of the previous system fully determine the Cauchy data $a^k(x',0)$ and the number E_k. The first line indeed fully determines the $a_{I'}|_{\partial\Omega}$. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Lemma 3.4.2, the second trace condition (3.4.20).

Let us now check $E(h) = \mathcal{O}(h^\infty)$. We prove this by comparing with the half-space problem, for which we know by (3.3.6) that the first eigenvalue is 0 with multiplicity one and that the second one is larger than $Ch^{6/5}$. Take a cut-off function $\chi \in \mathcal{C}^\infty_0(\overline{\Omega})$, $\chi = 1$ in a neighborhood of U such that $\frac{\partial\chi}{\partial n}\big|_{\partial\Omega} = 0$ and set

$$u^K_p = \chi e^{-\frac{\Phi}{h}} \sum_{k=0}^{K} a^k h^k = \chi e^{-\frac{\Phi}{h}} A^K_h\,.$$

From $\frac{\partial\chi}{\partial n}\big|_{\partial\Omega} \equiv 0$ and

$$d_{f,h}(\chi A^K_h) = (hd + df\wedge)\chi A^K_h = hd\chi \wedge A^K_h + \chi d_{f,h} A^K_h\,,$$

the form $u^K_p \in \Lambda^1 H^2(\mathbb{R}^n_-)$ belongs to the domain of $\mathcal{A}^{(p)}_N$ and the approximations u^K_p and $E^K(h) = \sum_{k=1}^K E_k h^{k+1}$ satisfy

$$\begin{cases} [\mathcal{A}^{(p)}_N - E^K(h)]u^K_p = h^{K+2}\rho^K e^{-\frac{\Phi}{h}} - h^2\,[\Delta,\chi]\,u^K_p = \mathcal{O}(h^{K+2}) & \text{in } \overline{\mathbb{R}^n_-} \\ \mathbf{n}u^K_p = 0 & \text{on } \mathbb{R}^{n-1}\times\{0\} \\ \mathbf{n}d_{f,h}u^K_p = 0 & \text{on } \mathbb{R}^{n-1}\times\{0\}\,, \end{cases}$$

for some $\mathcal{C}^\infty$ 1-form ρ^K defined in a neighborhood of U and independent of h.

From a direct Laplace method,

$$\left\|u^K_p\right\| \sim ch^{\frac{n+1}{4}}\,,$$

and the spectral theorem then implies that there exists an eigenvalue $\lambda(h)$ of $\mathcal{A}^{(p)}_N$ such that:

$$|E^K(h) - \lambda(h)| = \mathcal{O}(h^{K+2-\frac{n+1}{4}})\,.$$

Choosing the integer number K large enough, the inclusion

$$\sigma(\mathcal{A}^{(p)}_N)\setminus\{0\} \subset [Ch^{6/5}, +\infty)$$

combined with the estimate $E^K(h) = \mathcal{O}(h^2)$ implies $\lambda(h) = 0$. The number K being arbitrary, the construction of the previous quasimode is then possible only if

$$\forall k \in \mathbb{N}^* \,, \quad E_k = 0 \,.$$

∎

3.4.4 Local WKB construction in the Dirichlet case

Let U be a generalized critical point of f with index p in the Dirichlet case, i.e. a critical point with index $p-1 \in \{0, \dots, n-1\}$ (i.e. $p \in \{1, \dots, n\}$) of $f|_{\partial\Omega}$ satisfying $\frac{\partial f}{\partial n}(U) > 0$, and take again a local adapted coordinate system (x', x^n) around U, like in Section 3.4.1.
Let φ be the Agmon distance to U on the boundary and use the first result of Lemma 3.3.4 with $f_1 = f$ and $\alpha = \varphi$. Denoting by Φ the function Φ_- of the lemma, Φ is then the Agmon distance to U and we have locally:

$$|\partial_n \Phi|^2 + |\nabla_T \Phi|^2 = |\nabla \Phi|^2 = |\nabla f|^2 \,, \tag{3.4.25}$$

$$\Phi|_{\partial\Omega} = \varphi \,, \tag{3.4.26}$$

$$\partial_n \Phi|_{\partial\Omega} = -\frac{\partial f}{\partial n}|_{\partial\Omega} \,. \tag{3.4.27}$$

Moreover, the following relation is satisfied (see indeed the proof of (3.4.4) and replace $\partial_n \Phi|_{\partial\Omega} = \partial_n f|_{\partial\Omega}$ by $\partial_n \Phi|_{\partial\Omega} = -\partial_n f|_{\partial\Omega}$):

$$\partial^2_{x^n x^n}(f + \Phi)(0) = \partial^2_{nn}(f + \Phi)(0) = 0 \,. \tag{3.4.28}$$

Like in Section 3.4.1, there exist other local coordinates $(\overline{x}', \overline{x}^n)$ centered at U, with $\overline{x}' = (\overline{x}^1, \dots, \overline{x}^{n-1})$ and $d\overline{x}^1, \dots, d\overline{x}^{n-1}, dx^n$ is orthonormal at U, such that (3.4.5) and (3.4.6) are satisfied with $\lambda_i < 0$ for $i \in \{1, \dots, p-1\}$ and $\lambda_i > 0$ for $i \in \{p, \dots, n-1\}$.
Furthermore, the coordinates (x', x^n) can be chosen such that $dx^1, \dots, dx^{n-1}$ and $d\overline{x}^1, \dots, d\overline{x}^{n-1}$ coincide at U and even such that $x'\big|_{\partial\Omega} = \overline{x}'\big|_{\partial\Omega}$.

Specification of the coordinate system for Theorem 3.1.2
The local adapted coordinate system $x = (x', x^n)$ around U is again chosen such that,

$$\forall i \in \{1, \dots, n-1\} \,, \quad dx^i = d\overline{x}^i \text{ at } U \,. \tag{3.4.29}$$

The proof is quite close to that done in the Neumann case but it appears here more natural to make "dual computations". In particular, we will work with d^* where we worked with d in the Neumann case. This leads to computations a little bit more complicated.

3.4.5 First boundary conditions in the Dirichlet case

Writing

$$a(x,h) = a_I(x,h)\ dx^I = a_{I'}(x,h)dx^{I'} + a_{I_n}(x,h)dx^{I_n}\,,$$

the first boundary condition is equivalent to:

$$\forall k \in \mathbb{N},\ \forall I' \in \mathcal{I}'\,,\ a^k_{I'}(x',0) \equiv 0\,. \tag{3.4.30}$$

The end of this subsection specifies some consequence of these conditions, in the same spirit as those specified in the Section 3.4.2 concerning the Neumann case.

About $\mathcal{L} + \mathcal{L}^*$

The following relation is obviously satisfied

$$\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} = -2\mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla(f+\Phi)} + \mathcal{L}^*_{\nabla(f+\Phi)}\,,$$

and using again Proposition 3.A.3, write:

$$\mathcal{L}^*_{\nabla(f+\Phi)} + \mathcal{L}_{\nabla(f+\Phi)} = \mathcal{R}_4\,,$$

where $\mathcal{R}_4$ is a 0-th order differential operator.

Writing $\mathcal{R}_4 = \mathcal{R}_4^T + \mathcal{R}_4^N$, we deduce from Remark 3.A.7, since $a^k_I dx^I = a^k_{I_n} dx^{I_n}$ on the boundary,

$$\begin{cases} \mathbf{t}\left(\mathcal{R}_4(a^k_I dx^I)\right) &= a^k_{I_n}(x',0)\mathcal{R}_4^N(dx^{I_n}) = \tilde{\ell}'_{I'}(x',0)dx^{I'} \\ \mathbf{n}\left(\mathcal{R}_4(a^k_I dx^I)\right) &= a^k_{I_n}(x',0)\mathcal{R}_4^T(dx^{I_n})\,, \end{cases}$$

where the $\tilde{\ell}'_{I'}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a^k_{I'}$'s (for I' in $\mathcal{I}'$).

Moreover, from (3.4.25)-(3.4.28), $f+\Phi$ satisfies here the assumptions of Corollary 3.A.5, then $\mathcal{R}_4^T$ is given on the boundary, in the coordinates (x', x^n), by:

$$\mathcal{R}_4^T(x',0) = \begin{pmatrix} & & & 0 \\ & \mathcal{R}_4^{T'}(x') & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & \delta(x') \end{pmatrix}^{(p)} - \kappa(x')\, Id\,,$$

where δ, κ are $\mathcal{C}^\infty$ functions which satisfy $\delta(0)=0$,
$\kappa(0) = Tr\left(\text{Hess}\,(f\big|_{\partial\Omega} + \varphi)(0)\right)$, and

$$\mathcal{R}_4^{T'}(0) = 2\left(\text{Hess}\,(f\big|_{\partial\Omega} + \varphi)(0)\right)\,.$$

Expression of the codifferential d^*

As it was already mentioned, in order to make a study similar to the one done in Section 3.4.2 for the Neumann case, we need to work with d^* and then, to have a handy expression of this operator.

Set, for a differential form ω, in the coordinate system (x', x^n)

$$\nabla_i = \nabla_{x^i}\,,\ \ \mathbf{a}_i^*\omega = dx^i \wedge \omega\,,\ \text{ and } \mathbf{a}_i\omega = \mathbf{i}_{\nabla x^i}\omega\,.$$

Then d and d^* write (see [CyFrKiSi] pp. 238-247):

$$d = \sum_{i=1}^{n} \mathbf{a}_i^*\nabla_i = -\sum_{i=1}^{n}(\nabla_i)^*\mathbf{a}_i^*\,, \tag{3.4.31}$$

$$d^* = -\sum_{i=1}^{n} \mathbf{a}_i\nabla_i\,. \tag{3.4.32}$$

Recall moreover the characteristic relations:

$$\forall i,j \in \{1,\dots,n\}\,,\quad \mathbf{a}_i^*\mathbf{a}_j^* + \mathbf{a}_j^*\mathbf{a}_i^* = 0\,, \tag{3.4.33}$$

$$\mathbf{a}_i\mathbf{a}_j + \mathbf{a}_j\mathbf{a}_i = 0\,, \tag{3.4.34}$$

$$\mathbf{a}_i^*\mathbf{a}_j + \mathbf{a}_j\mathbf{a}_i^* = g^{ij}\,. \tag{3.4.35}$$

Denoting by ∂_i the operator defined by components with differentiation in a fixed coordinate system,

$$\partial_i(\omega_I dx^I) = \frac{\partial \omega_I}{\partial x^i} dx^I\,,$$

∇_i writes (see [CyFrKiSi] pp. 238-247)

$$\nabla_i = \partial_i - \sum_{j,l,m} \Gamma^j_{il} g_{jm}\mathbf{a}_l^*\mathbf{a}_m\,, \tag{3.4.36}$$

where the Γ^j_{il} are the Christoffel symbols.
Then d^* writes:

$$\begin{aligned} d^* &= -\sum_i \mathbf{a}_i\partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm}\mathbf{a}_i\mathbf{a}_l^*\mathbf{a}_m \\ &= -\sum_i \mathbf{a}_i\partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm}\left(\mathbf{a}_i\mathbf{a}_l^* + \mathbf{a}_l^*\mathbf{a}_i\right)\mathbf{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm}\mathbf{a}_l^*\mathbf{a}_i\mathbf{a}_m \\ &= -\sum_i \mathbf{a}_i\partial_i + \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il}\mathbf{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm}\mathbf{a}_l^*\mathbf{a}_i\mathbf{a}_m\,. \end{aligned} \tag{3.4.37}$$

Results

Proposition 3.4.4. *Using the notation of Appendix 3.A.1 and Section 3.4.5, the following relations are satisfied for every normal p-form $b(x)=b_I(x)dx^I$, i.e. every p-form $b(x)$ satisfying $b_{I'}(x',0)\equiv 0$ for all $I'\in\mathcal{I}'$:*

$$\begin{cases} \mathbf{t}\left((-2\mathcal{L}^*_{\nabla\Phi}+\mathcal{R}_4)b\right) &=& 2\left(\frac{\partial b_{I'}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}+\ell_{I'}(x',0)\right)dx^{I'}\\ \mathbf{n}\left((-2\mathcal{L}^*_{\nabla\Phi}+\mathcal{R}_4)b\right) &=& (2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id+\mathcal{R}^T_{Dir})b_{I_n}dx^{I_n}-2\frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*b\,, \end{cases}$$

where the $\ell_{I'}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n}'s (for I_n in $\mathcal{I}_n$) which do not depend on the $b_{I'}$'s (for I' in $\mathcal{I}'$) and $\mathcal{R}^T_{Dir}$ is a 0-th order differential operator given on the boundary by the following matrix, in the coordinates (x',x^n):

$$\mathcal{R}^T_{Dir}(x',0)=\begin{pmatrix} & & & 0\\ & \mathcal{R}^{T'}_{Dir}(x') & & \vdots\\ & & & 0\\ 0 & \cdots & 0 & \delta(x')\end{pmatrix}^{(p)}-\kappa_2(x')\,Id\,,$$

where $\delta(0)=0$,

$$\kappa_2(0)=Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)\,,$$

and

$$\mathcal{R}^{T'}_{Dir}(0)=2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.$$

In particular, this is true for a^k for k in $\mathbb{N}$, when (3.4.30) *is fulfilled.*

Lemma 3.4.5. *Let $b(x)$ be a <u>normal</u> p-form. The p-form*

$$\vec{n}^*\wedge d^*b$$

is then normal and the following equivalence is locally valid on the boundary $\partial\Omega$:

$$\vec{n}^*\wedge d^*b=0\Leftrightarrow \mathbf{t}d^*b=0\,.$$

In particular, this is true for a^k for k in $\mathbb{N}$, when (3.4.30) *is fulfilled.*

Proof. Write indeed on the boundary $\partial\Omega$, in the coordinate system (x',x^n), since $dx^n=\vec{n}^*$:

$$\begin{aligned} dx^n\wedge d^*b &= dx^n\wedge\mathbf{n}d^*b+dx^n\wedge\mathbf{t}d^*b\\ &= 0+dx^n\wedge\mathbf{t}d^*b\\ &= dx^n\wedge(d^*b)_{I'}dx^{I'}\\ &= (-1)^{p-1}(d^*b)_{I'}dx^{I'}\wedge dx^n\,, \end{aligned}$$

which leads to the result.

∎

Lemma 3.4.6. *The following relations are satisfied for every tangential p-form $b(x)$:*

$$\left\{\begin{array}{rcl}\mathbf{n}\left((\mathcal{L}^*_{\nabla\Phi}-\mathcal{L}^*_{\nabla\tilde\Phi})b\right) & = & \frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*b\,,\\ \mathbf{t}\left((\mathcal{L}^*_{\nabla\Phi}-\mathcal{L}^*_{\nabla\tilde\Phi})b\right) & = & \left(-\frac{\partial b_{I'}}{\partial x^n}\frac{\partial\Phi}{\partial x^n}+\tilde\ell_{I'}(x',0)\right)dx^{I'}\,,\end{array}\right.$$

where the $\tilde\ell_{I'}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n}'s (for I_n in $\mathcal{I}_n$) which do not depend on the $b_{I'}$'s (for I' in $\mathcal{I}_n$).

Proof. Owing to (3.2.3) and to the Cartan formula (3.2.6), write in the coordinates (x',x^n) (the function $\tilde\Phi$ is defined in Appendix 3.A.1):

$$\begin{aligned}\left(\mathcal{L}^*_{\nabla\Phi}-\mathcal{L}^*_{\nabla\tilde\Phi}\right)b &= d^*(d\Phi\wedge b)+d\Phi\wedge d^*b+d^*(d\tilde\Phi\wedge b)+d\tilde\Phi\wedge d^*b\\ &= d^*(\frac{\partial\Phi}{\partial x^n}dx^n\wedge b)+\frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*b\\ &+ d^*((d_T\Phi-d\tilde\Phi)\wedge b)+(d_T\Phi-d\tilde\Phi)\wedge d^*b\,. \qquad (3.4.38)\end{aligned}$$

The second term $\frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*b$ of the r.h.s. of (3.4.38) is normal according to Lemma 3.4.5.
Moreover, since $d_T\Phi=d\tilde\Phi$ on the boundary, the term $(d_T\Phi-d\tilde\Phi)\wedge d^*b$ of the r.h.s. also equals 0 on $\partial\Omega$.
Hence, we can write on $\partial\Omega$:

$$\begin{aligned}\left(\mathcal{L}^*_{\nabla\Phi}-\mathcal{L}^*_{\nabla\tilde\Phi}\right)b &= \frac{\partial\Phi}{\partial x^n}dx^n\wedge d^*b\\ &+ d^*(\frac{\partial\Phi}{\partial x^n}dx^n\wedge b)+d^*((d_T\Phi-d\tilde\Phi)\wedge b)\,. \qquad (3.4.39)\end{aligned}$$

Let us study in a first time the term $d^*(\frac{\partial\Phi}{\partial x^n}dx^n\wedge b)$. Writing

$$b=b_Idx^I=b_{I'}dx^{I'}+b_{I_n}dx^{I_n}\,,$$

we deduce (in $\overline{\Omega}$):

$$\frac{\partial\Phi}{\partial x^n}dx^n\wedge b = \frac{\partial\Phi}{\partial x^n}b_{I'}dx^n\wedge dx^{I'}\,,$$

and, applying d^* to this last relation (see (3.4.37)), we obtain on $\partial\Omega$ (remember that $b_{I'}=0$ on $\partial\Omega$)

$$\begin{aligned}
d^*(\frac{\partial\Phi}{\partial x^n}dx^n\wedge b) &= -\sum_i \mathbf{a}_i\partial_i(\frac{\partial\Phi}{\partial x^n}b_{I'}dx^n\wedge dx^{I'}) \\
+\sum_{i,j,l,m}\Gamma^j_{il}g_{jm}g^{il}\mathbf{a}_m(\frac{\partial\Phi}{\partial x^n}b_{I'}\,dx^n\wedge dx^{I'}) &- \sum_{i,j,l,m}\Gamma^j_{il}g_{jm}\mathbf{a}_l^*\mathbf{a}_i\mathbf{a}_m(\frac{\partial\Phi}{\partial x^n}b_{I'}dx^n\wedge dx^{I'}) \\
&= -\sum_i \mathbf{a}_i\partial_i(\frac{\partial\Phi}{\partial x^n}b_{I'}dx^n\wedge dx^{I'})+0 \\
&= -\mathbf{i}_{\nabla x^n}\frac{\partial\Phi}{\partial x^n}\frac{\partial b_{I'}}{\partial x^n}dx^n\wedge dx^{I'} \\
&= -\frac{\partial\Phi}{\partial x^n}\frac{\partial b_{I'}}{\partial x^n}dx^{I'}\,. \qquad (3.4.40)
\end{aligned}$$

We used at the last line the fact that G_0^{-1} is block diagonal with $g^{nn}\equiv 1$. Look now at the third term of the r.h.s. of (3.4.39) and write:

$$\begin{aligned}
(d_T\Phi - d\tilde{\Phi})\wedge b_I dx^I &= \sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial x^i}(x)-\frac{\partial\Phi}{\partial x^i}(x',0)\right)b_I dx^i\wedge dx^I \\
&=: \sum_{i=1}^{n-1}\alpha_i b_I dx^i\wedge dx^I
\end{aligned}$$

where, for i in $\{1,\dots,n-1\}$,

$$\alpha_i = \frac{\partial\Phi}{\partial x^i}(x)-\frac{\partial\Phi}{\partial x^i}(x',0)\,.$$

Hence, for j in $\{1,\dots,n-1\}$, α_j satisfies (see (3.A.5)):

$$\forall j\in\{1,\dots,n-1\}\ ,\alpha_j(x',0)\equiv 0\,.$$

We obtain consequently on $\partial\Omega$ (see again (3.4.37)),

$$\begin{aligned}
d^*((d_T\Phi - d\tilde{\Phi}) \wedge b_I dx^I)(x',0) &= -\sum_i \mathbf{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j b_I dx^j \wedge dx^I \\
+\left(\sum_{i,j,l,m} \Gamma^j_{il} g_{jm} g^{il} \mathbf{a}_m - \sum_{i,j,l,m} \Gamma^j_{il} g_{jm} \mathbf{a}^*_l \mathbf{a}_i \mathbf{a}_m\right) &\sum_{j=1}^{n-1} \alpha_j b_I dx^j \wedge dx^I \\
&= -\sum_i \mathbf{a}_i \partial_i \sum_{j=1}^{n-1} \alpha_j b_I dx^j \wedge dx^I \\
&= -\mathbf{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_I) dx^j \wedge dx^I
\end{aligned}$$

where we used $\alpha_j(x',0) \equiv 0$ at the two last lines.
Now, since $g^{ni} = g^{in} = 0$ for i in $\{1,\dots,n-1\}$, write for all $I' \in \mathcal{I}'$:

$$\mathbf{a}_n dx^{I'} = \mathbf{i}_{\nabla x^n} dx^{I'} = 0\,.$$

It implies:

$$\begin{aligned}
d^*((d_T\Phi - d\tilde{\Phi}) \wedge b_I dx^I)(x',0) &= -\mathbf{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_I) dx^j \wedge dx^I \\
&= -\mathbf{a}_n \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_{I_n}) dx^j \wedge dx^{I_n} \\
&= (-1)^{p+1} \sum_{j=1}^{n-1} \frac{\partial}{\partial x^n}(\alpha_j b_{I_n}) dx^j \wedge dx^{I_n\setminus\{n\}} \\
&= (-1)^{p+1} \sum_{j=1}^{n-1} b_{I_n} \frac{\partial \alpha_j}{\partial x^n}(x',0) dx^j \wedge dx^{I_n\setminus\{n\}} \\
&=: \tilde{\ell}_{I'}(x',0) dx^{I'}\,, \qquad (3.4.41)
\end{aligned}$$

where the $\tilde{\ell}_{I'}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n}'s (for I_n in $\mathcal{I}_n$) which do not depend on the $b_{I'}$'s (for I' in $\mathcal{I}'$).
Combining (3.4.39), (3.4.40), and (3.4.41) leads to the result announced in Lemma 3.4.6.

∎

Proof of Proposition 3.4.4.
Having in mind Lemma 3.4.6, let us look at the term $-2\mathcal{L}^*_{\nabla\tilde\Phi}+\mathcal{R}_4$.
Again by Proposition 3.A.3, we can write:

$$\begin{aligned} -2\mathcal{L}^*_{\nabla\tilde\Phi} &= 2\mathcal{L}_{\nabla\tilde\Phi}+\mathcal{R}_5\\ &= 2\mathcal{L}_{\nabla\tilde\Phi}\otimes Id+\mathcal{R}_5+\mathcal{R}_6\end{aligned}$$

where $\mathcal{R}_5=\mathcal{R}_5^T+\mathcal{R}_5^N$ and $\mathcal{R}_6=\mathcal{R}_6^T+\mathcal{R}_6^N$ are 0-th order differential operators which satisfy, for $i\in\{5,6\}$ (since $b_I dx^I=b_{I_n}dx^{I_n}$ on the boundary):

$$\left\{\begin{aligned} \mathbf{t}\left(\mathcal{R}_i(b_I dx^I)\right) &= b_{I_n}(x',0)\mathcal{R}_i^N(dx^{I_n})=\tilde\ell^{i'}_{I'}(x',0)dx^{I'}\\ \mathbf{n}\left(\mathcal{R}_i(b_I dx^I)\right) &= b_{I_n}(x',0)\mathcal{R}_i^T(dx^{I_n})\,,\end{aligned}\right.$$

where the $\tilde\ell^{i'}_{I'}(x',0)$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n}'s (for I_n in $\mathcal{I}_n$) which do not depend on the $b_{I'}$'s (for I' in $\mathcal{I}'$).
Moreover, since $\tilde\Phi$ satisfies the assumptions of Corollary 3.A.5, $\mathcal{R}_5^T$ and $\mathcal{R}_6^T$ are given on the boundary, in the coordinates (x',x^n), by:

$$\mathcal{R}_5^T=\begin{pmatrix} & & & 0\\ & \mathcal{R}_5^{T'} & & \vdots\\ & & & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}^{(p)}-\zeta(x')\,Id\ \text{ and }\ \mathcal{R}_6^T=\begin{pmatrix} & & & 0\\ & \mathcal{R}_6^{T'} & & \vdots\\ & & & 0\\ 0 & \cdots & 0 & 0\end{pmatrix}^{(p)},$$

where

$$\zeta(0)=-2\,Tr\left(\text{Hess}\,(\tilde\Phi\big|_{\partial\Omega})(0)\right)=-2\,Tr\left(\text{Hess}\,(\varphi)(0)\right)\,,$$

and $\mathcal{R}_5^{T'}$, $\mathcal{R}_6^{T'}$ write at 0:

$$\mathcal{R}_5^{T'}(0)=-4\left(\text{Hess}\,(\varphi)(0)\right)\ \text{ and }\ \mathcal{R}_6^{T'}(0)=2\left(\text{Hess}\,(\varphi)(0)\right)$$

Set $\mathcal{R}_{\text{Dir}}=\mathcal{R}_4+\mathcal{R}_5+\mathcal{R}_6$ and $\tilde\ell^{(3)}_{I'}=\tilde\ell'_{I'}+\tilde\ell^{5'}_{I'}+\tilde\ell^{6'}_{I'}$ for I' in $\mathcal{I}'$. $\mathcal{R}_{\text{Dir}}$ is a 0-th order differential operator which satisfies

$$-2\mathcal{L}_{\nabla\tilde\Phi}+\mathcal{R}_4=2\mathcal{L}_{\nabla\tilde\Phi}\otimes Id+\mathcal{R}_{\text{Dir}} \tag{3.4.42}$$

and

$$\left\{\begin{aligned} \mathbf{t}\left(\mathcal{R}_{\text{Dir}}(b_I dx^I)\right) &= b_{I_n}(x',0)\mathcal{R}_{\text{Dir}}^N(dx^{I_n})=\tilde\ell^{(3)}_{I'}(x',0)dx^{I'}\\ \mathbf{n}\left(\mathcal{R}_{\text{Dir}}(b_I dx^I)\right) &= b_{I_n}(x',0)\mathcal{R}_{\text{Dir}}^T(dx^{I_n})\,,\end{aligned}\right. \tag{3.4.43}$$

where the $\tilde\ell^{(3)}_{I'}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the b_{I_n}'s (for I_n in $\mathcal{I}_n$) which do not depend on the $b_{I'}$'s (for I' in $\mathcal{I}'$).
Moreover, $\mathcal{R}_{\text{Dir}}^T$ is given on the boundary, in the coordinates (x',x^n), by:

$$\mathcal{R}^{T}_{\mathrm{Dir}}(x',0)=\begin{pmatrix} & & & 0\\ & \mathcal{R}^{T'}_{\mathrm{Dir}}(x',0) & & \vdots\\ & & & 0\\ 0 & \cdots & 0 & \delta(x')\end{pmatrix}^{(p)} -\kappa_2(x')Id\,,$$

where $\delta(0)=0$,

$$\begin{aligned}\kappa_2(0)=\kappa(0)+\zeta(0) &= Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}+\varphi)(0)\right)-2\,Tr\,(\mathrm{Hess}\,(\varphi)(0))\\ &= Tr\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}-\varphi)(0)\right)\,,\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}^{T'}_{\mathrm{Dir}}(0) &= \mathcal{R}^{T'}_{4}(0)+\mathcal{R}^{T'}_{5}(0)+\mathcal{R}^{T'}_{6}(0)\\ &= 2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega}+\varphi)(0)\right)-2\,(\mathrm{Hess}\,(\varphi)(0))\\ &= 2\left(\mathrm{Hess}\,(f\big|_{\partial\Omega})(0)\right)\,.\end{aligned}$$

Let us now look at the term $2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id$. By the Cartan formula (3.2.6),

$$(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id)b \;=\; db_{I'}(\nabla\tilde{\Phi})dx^{I'}+db_{I_n}(\nabla\tilde{\Phi})dx^{I_n}\,,$$

and, using the boundary conditions satisfied by the b_I's (for I in $\mathcal{I}$) and the fact that $\nabla\tilde{\Phi}$ is a tangential vector field, we obtain:

$$\begin{aligned}(2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id)b &= \sum_{i=1}^{n-1}\frac{\partial b_{I_n}}{\partial x^i}(\nabla\tilde{\Phi})_i dx^{I_n}\\ &= 2\mathcal{L}_{\nabla\tilde{\Phi}}\otimes Id\, b_{I_n}dx^{I_n}\,.\end{aligned}\tag{3.4.44}$$

Set $\ell_{I'}=-\tilde{\ell}_{I'}+\frac{1}{2}\tilde{\ell}^{(3)}_{I'}$ and write

$$\left(-2\mathcal{L}^{*}_{\nabla\Phi}+\mathcal{R}_3\right)b=-2\left(\mathcal{L}^{*}_{\nabla\Phi}-\mathcal{L}^{*}_{\nabla\tilde{\Phi}}\right)b+\left(-2\mathcal{L}^{*}_{\nabla\tilde{\Phi}}+\mathcal{R}_3\right)b\,.$$

Using (3.4.42), (3.4.43), (3.4.44), Proposition 3.4.4 is then a direct consequence of Lemma 3.4.6.

∎

3.4.6 Proof of Theorem 3.1.2

Although the calculations are different, the scheme of the proof is the same as for Theorem 3.1.1. Consider first a WKB-approximation for

$$(\Delta^{(p)}_{f,h} - E(h))u_p^{wkb} = e^{-\frac{\Phi}{h}}\mathcal{O}(h^\infty) \tag{3.4.45}$$

with $E(h) = O(h^2)$ and the boundary conditions (3.1.5)(3.1.6).

From

$$\forall k \in \mathbb{N}\,, \quad d^*_{f,h}(e^{-\frac{\Phi}{h}}a^k) = e^{-\frac{\Phi}{h}}\left[hd^*a^k + \mathbf{i}_{\nabla(f+\Phi)}a^k\right]\,,$$

where, due to (3.1.5) and (3.4.27), a^k is a normal form and $\nabla(f+\Phi)$ is a tangential vectorfield, the second boundary condition corresponds to

$$\forall k \in \mathbb{N}\,, \quad \mathbf{t}(d^*a^k) = 0\,. \tag{3.4.46}$$

Let us now recall, using the notation of Appendix 3.A.2 and Section 3.4.5, the following relation,

$$\begin{aligned} e^{\frac{\Phi}{h}}\Delta_{f,h}e^{-\frac{\Phi}{h}} &= h^2(d+d^*)^2 + h(2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}) \\ &= h^2(d+d^*)^2 + h(-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4)\,. \end{aligned}$$

By looking for $E(h) \sim \sum_{k=1}^{\infty} h^{k+1}E_k$, the interior equation (3.4.45) reads, like in Section 3.4.3,

$$e^{\frac{\Phi}{h}}(\Delta_{f,h} - E(h))e^{-\frac{\Phi}{h}} = h^2[(d+d^*)^2 - h^{-2}E(h)] + h\left[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}\right]\,.$$

Hence, like in Section 3.4.3, the construction of an interior WKB solution in Ω is standard as an inductive Cauchy problem, once the a^k's are known on $\partial\Omega$, since the non characteristic Cauchy problems

$$[2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}]a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{in } \overline{\Omega} \tag{3.4.47}$$

are solved by induction with the convention $a_{-1} = 0$.
The problem is then reduced to the solving of the system made of the boundary conditions (3.4.30), (3.4.46) and of the compatibility equation (see Section 3.4.5 for the meaning of the notations):

$$\left[-2\mathcal{L}^*_{\nabla\Phi} + \mathcal{R}_4\right]a^k = -(d+d^*)^2a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} \quad \text{on } \partial\Omega\,. \tag{3.4.48}$$

Owing to Proposition 3.4.4 (with the notation of Section 3.4.5) and to (3.4.27), the system (3.4.48), (3.4.30), (3.4.46) is equivalent to the differential system on $\partial\Omega$:

$$\begin{cases} -\mathbf{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}^T_{\text{Dir}}) a^k_{I_n} dx^{I_n} + 2\frac{\partial f}{\partial x^n} dx^n \wedge d^* a^k \\ -\mathbf{t}(d+d^*)^2 a^{k-1} - 2\ell_{I'}(x',0) dx^{I'} = -2\frac{\partial f}{\partial n}\frac{\partial a^k_{I'}}{\partial x^n} dx^{I'} \\ \forall k \in \mathbb{N}\,, \quad \big(a^k_{I'}|_{\partial\Omega} \equiv 0\big) + \big(\mathbf{t}(d^* a^k) = 0\big)\,, \end{cases}$$

where the $\ell_{I'}$'s are $\mathcal{C}^\infty(\partial\Omega)$-linear combinations of the $a^k_{I_n}$'s (for I_n in $\mathcal{I}_n$) which do not depend on the $a^k_{I'}$'s (for I' in $\mathcal{I}'$).
Note also, according to Lemma 3.4.5, that the first line of the last system reads:

$$-\mathbf{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} = (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}^T_{\text{Dir}}) a^k_{I_n} dx^{I_n}\,. \qquad (3.4.49)$$

Moreover, since $dx^i = d\overline{x}^i$ (for $i \in \{1,\dots,n-1\}$) at the point U, thanks to Corollary 3.A.5, (3.4.5)-(3.4.6), and according to [HeSj4] pp. 271-275, $\mathcal{R}^T_{\text{Dir}}(0)$ restricted to normal forms is symmetric with the one dimensional kernel $\mathbb{R}dx^1 \wedge \dots \wedge dx^{p-1} \wedge dx^n$.

Since $a^k_{I_n} dx^{I_n}$ is normal and $2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id$ only differentiates tangentially the $a^k_{I_n}$'s, since

$$(2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id) a^k_{I_n} dx^{I_n} = \sum_{i=1}^{n-1} \frac{\partial a^k_{I_n}}{\partial x^i} (\nabla\tilde{\Phi})_i dx^{I_n}\,,$$

(3.4.49) can be rewritten as a tangential system which can be solved according to the analysis of the boundaryless case done in [HeSj4].
Here are the details:
Owing to Lemma 3.4.5, the complete system becomes equivalent to

$$\begin{cases} (2\mathcal{L}_{\nabla\tilde{\Phi}} \otimes Id + \mathcal{R}^T_{\text{Dir}}) a^k_{I_n} dx^{I_n} = -\mathbf{n}(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k-1} E_\ell a^{k-\ell} + E_k a^0 & \text{on } \partial\Omega \\ (2\mathcal{L}_{\nabla\Phi} \otimes Id + \mathcal{R}) a^k = -(d+d^*)^2 a^{k-1} + \sum_{\ell=1}^{k} E_\ell a^{k-\ell} & \text{on } \overline{\Omega} \\ \qquad \forall I' \in \mathcal{I}'\,,\ a_{I'}|_{\partial\Omega} \equiv 0\,. \end{cases}$$

The first line is again a degenerate matricial transport equation which can be solved according to [HeSj4][Hel2]:

For $k=0$, take $a^0(0)=dx^1\wedge\cdots\wedge dx^{p-1}\wedge dx^n\in$ Ker $(\mathcal{R}^T_{\mathrm{Dir}}(0))$ and for $k>0$ choose E_k such that the compatibility condition

$$-\mathbf{n}(d+d^*)^2a^{k-1}(0)+\sum_{\ell=1}^{k-1}E_\ell a^{k-\ell}(0)+E_ka^0(0)\in\left(\text{ Ker }(\mathcal{R}^T_{\mathrm{Dir}}(0))\right)^\perp$$

is satisfied. Thus, at every step $k\in\mathbb{N}$, the first and the third line of the previous system fully determine the Cauchy data $a^k(x',0)$ and the number E_k. The second line solves the interior problem with these Cauchy data and contains, with the two other lines, thanks to Lemma 3.4.5, the second trace condition (3.4.46).
Checking $E(h)=O(h^\infty)$ is then identical to the end of the proof of Theorem 3.1.1 done in Section 3.4.3 after choosing a cut-off function χ which satisfies $\nabla\chi=\nabla_T\chi$ on the boundary $\partial\Omega$. ∎

3.A Computations in local adapted coordinate systems

We work here in a local adapted coordinate system (x',x^n) around $U\in\partial\Omega$ in order to apply indifferently the results of this section to the Neumann and Dirichlet cases.

3.A.1 A modified Agmon distance

Define $\tilde\Phi$ around U in the coordinates (x',x^n) by

$$\forall x=(x',x^n)\,,\ \tilde\Phi(x',x^n)=\Phi(x',0)\,, \tag{3.A.1}$$

and note the following relation satisfied for all x around U, in the coordinates (x',x^n), due to the form of $G_0^{\pm1}$ (see Remark 3.3.3):

$$\left\{\begin{array}{lcl} d\tilde\Phi(x) &=& d_T\tilde\Phi(x)+\frac{\partial\tilde\Phi}{\partial x^n}(x)dx^n=d_T\tilde\Phi(x)\\ \nabla\tilde\Phi(x) &=& \nabla_T\tilde\Phi(x)+\frac{\partial\tilde\Phi}{\partial x^n}(x)\frac{\partial}{\partial x^n}=\nabla_T\tilde\Phi(x)\end{array}\right. .$$

For a vector (or a vector field) $X=\sum_{i=1}^n X_i\frac{\partial}{\partial x^i}$, making the identification

$$X=\begin{pmatrix}X_1\\ \vdots\\ X_n\end{pmatrix},$$

the tangential part X_T (resp. the normal part X_N) of X is defined as:

$$X_T = \begin{pmatrix} X_1 \\ \vdots \\ X_{n-1} \\ 0 \end{pmatrix} \text{ (resp. } X_N = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X_n \end{pmatrix}).$$

Similarly, for a (n,n)-matrix $A(x) = (a_{ij}(x))_{i,j}$, define $A_T(x)$ and $A_N(x)$ by:

$$A_T = \begin{pmatrix} & A' & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} \text{ and } A_N = \begin{pmatrix} & & & a_{1n} \\ & [0] & & \vdots \\ & & & a_{n-1\,n} \\ a_{n1} & \cdots & a_{n\,n-1} & 0 \end{pmatrix}.$$

Recall moreover that, for a vector (or a vector field) X and a $\mathcal{C}^\infty$ function ψ, the identification $\langle \nabla\psi \,|\, X\rangle_{g_0} = d\psi(X)$ leads to:

$$\nabla\psi = G_0^{-1} \begin{pmatrix} \frac{\partial\psi}{\partial x^1} \\ \vdots \\ \frac{\partial\psi}{\partial x^n} \end{pmatrix}.$$

Hence, due to the form of G_0^{-1} (see Remark 3.3.3), the following relations are indeed satisfied:

$$\begin{aligned} (\nabla\psi)_T &= \nabla_T\psi = G_0^{-1} \begin{pmatrix} \frac{\partial\psi}{\partial x^1} \\ \vdots \\ \frac{\partial\psi}{\partial x^{n-1}} \\ 0 \end{pmatrix} \\ &\text{and} \\ (\nabla\psi)_N &= \frac{\partial\psi}{\partial x^n}\frac{\partial}{\partial x^n} = G_0^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{\partial\psi}{\partial x^n} \end{pmatrix}. \end{aligned}$$

In the Neumann case, we are going to compare $\mathcal{L}_{\nabla\Phi}$ and $\mathcal{L}_{\nabla\tilde{\Phi}}$ and the following relations could be convenient:

$$\nabla\Phi - \nabla\tilde{\Phi} = G_0^{-1}\begin{pmatrix} \frac{\partial\Phi}{\partial x^1}(x) - \frac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \frac{\partial\Phi}{\partial x^{n-1}}(x) - \frac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ \frac{\partial\Phi}{\partial x^n}(x) \end{pmatrix} \quad (3.A.2)$$

and

$$\nabla_T\Phi - \nabla\tilde{\Phi} = G_0^{-1}\begin{pmatrix} \frac{\partial\Phi}{\partial x^1}(x) - \frac{\partial\Phi}{\partial x^1}(x',0) \\ \vdots \\ \frac{\partial\Phi}{\partial x^{n-1}}(x) - \frac{\partial\Phi}{\partial x^{n-1}}(x',0) \\ 0 \end{pmatrix}. \quad (3.A.3)$$

At least, we are going to compare $\mathcal{L}^*_{\nabla\Phi}$ and $\mathcal{L}^*_{\nabla\tilde{\Phi}}$ in the Dirichlet case and the following relations could also be convenient:

$$d\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\right)dx^i + \frac{\partial\Phi}{\partial x^n}(x)dx^n \quad (3.A.4)$$

and

$$d_T\Phi - d\tilde{\Phi} = \sum_{i=1}^{n-1}\left(\frac{\partial\Phi}{\partial x^i}(x) - \frac{\partial\Phi}{\partial x^i}(x',0)\right)dx^i. \quad (3.A.5)$$

3.A.2 About $\mathcal{L} + \mathcal{L}^*$

For a general $\mathcal{C}^\infty$ function h

In this subsection, we give similar results to those done in [HeSj4] Appendix A.

Take h a $\mathcal{C}^\infty$ function from $\overline{\Omega}$ on $\mathbb{R}$ and write:

$$\nabla h = \sum_{i=1}^{n}(\nabla h)_i \frac{\partial}{\partial x^i}.$$

According to [HeSj4], let us give the following algebraic definition:

Definition 3.A.1. *For a Euclidean space $(E, \langle\cdot\,|\,\cdot\rangle)$ and $A \in \mathcal{L}(E)$, $A^{(p)}$ and $\Gamma^{(p)}(A)$ denote respectively the linear application $A^{(p)} \in \mathcal{L}(\Lambda^p E)$ and the application $\Gamma^{(p)}(A) = A \otimes \cdots \otimes A$:*

$$A^{(p)}(\omega_1 \wedge \cdots \wedge \omega_p) = (A\omega_1 \wedge \cdots \wedge \omega_p) + \cdots + (\omega_1 \wedge \cdots \wedge A\omega_p)$$

and

$$\Gamma^{(p)}(A)(\omega_1 \wedge \cdots \wedge \omega_p) = (A\omega_1) \wedge \cdots \wedge (A\omega_p),$$

with the obvious convention $A^{(0)} = 0$ and $\Gamma^{(0)}(A) = 1$.

Remark 3.A.2. *Under the canonical identification* $\Lambda^1 E = E$*, note that* $A^{(1)} = A$*. Moreover, if* A^* *denotes the adjoint of* A *according to the scalar product on* E*, the adjoint of* $A^{(p)}$ *is simply* $(A^{(p)})^* = (A^*)^{(p)} =: A^{(p),*}$*. Let us recall here that* $\Lambda^p E$ *is a Euclidean space with the scalar product* $\langle \cdot \,|\, \cdot \rangle_p$*:*

$$\langle \omega_1 \wedge \cdots \wedge \omega_p \,|\, \mu_1 \wedge \cdots \wedge \mu_p \rangle_p = \det\left(\langle \omega_i \,|\, \mu_j \rangle \right)_{i,j} .$$

Remark that for a p-form $a_I^k dx^I = a_{I'}^k dx^{I'} + a_{I_n}^k dx^{I_n}$, with the notation of Appendix 3.A.1, $A^{(p)} = A_T^{(p)} + A_N^{(p)}$ and:

$$\begin{cases} \mathbf{t}\left(A^{(p)}(a_I^k dx^I)\right) &= a_{I'}^k(x',0) A_T^{(p)}(dx^{I'}) + a_{I_n}^k(x',0) A_N^{(p)}(dx^{I_n}) \\ \mathbf{n}\left(A^{(p)}(a_I^k dx^I)\right) &= a_{I_n}^k(x',0) A_T^{(p)}(dx^{I_n}) + a_{I'}^k(x',0) A_N^{(p)}(dx^{I'}) . \end{cases}$$

Moreover, for any 0-th order differential operator $\mathcal{A}$ on the form $\mathcal{A} = A^{(p)} + \psi\, Id$, where ψ is a $\mathcal{C}^\infty$ function, we will denote by $\mathcal{A}^T$ and $\mathcal{A}^N$ the following 0-th order differential operators:

$$\mathcal{A}^T = A_T^{(p)} + \psi\, Id \quad \text{and} \quad \mathcal{A}^N = A_N^{(p)}$$

(notice that $\mathcal{A}^T$ (resp. $\mathcal{A}^N$) coincides with $A_T^{(p)}$ (resp. $A_N^{(p)}$) if $\psi \equiv 0$).
Furthermore, our aim is to work with tangential forms in the Neumann case (i.e. $a_I^k dx^I = a_{I'}^k dx^{I'}$ on $\partial\Omega$) and with normal forms in the Dirichlet case (i.e. $a_I^k dx^I = a_{I_n}^k dx^{I_n}$ on $\partial\Omega$). Hence, for any tangential form in the Neumann case (resp. for any normal form in the Dirichlet case), write:

$$\begin{cases} \mathbf{t}\left(\mathcal{A}(a_I^k dx^I)\right) &= a_{I'}^k(x',0) A_T^{(p)}(dx^{I'}) + \psi(x',0) a_{I'}^k(x',0) dx^{I'} \\ &= \mathbf{t}\left(\mathcal{A}^T(a_I^k dx^I)\right) \\ \\ \mathbf{n}\left(\mathcal{A}(a_I^k dx^I)\right) &= a_{I'}^k(x',0) A_N^{(p)}(dx^{I'}) = \mathbf{n}\left(\mathcal{A}^N(a_I^k dx^I)\right) \end{cases} \tag{3.A.6}$$

(resp.

$$\begin{cases} \mathbf{t}\left(\mathcal{A}(a_I^k dx^I)\right) &= a_{I_n}^k(x',0) A_N^{(p)}(dx^{I'}) = \mathbf{t}\left(\mathcal{A}^N(a_I^k dx^I)\right) \\ \\ \mathbf{n}\left(\mathcal{A}(a_I^k dx^I)\right) &= a_{I_n}^k(x',0) A_T^{(p)}(dx^{I_n}) + \psi(x',0) a_{I_n}^k(x',0) dx^{I_n} \\ &= \mathbf{n}\left(\mathcal{A}^T(a_I^k dx^I)\right) \end{cases}).(3.A.7)$$

The end of this section is devoted to the proof of the following proposition:

Proposition 3.A.3. *In the coordinates* (x', x^n)*, the following equalities are satisfied:*

$$\begin{cases} \mathcal{L}_{\nabla h} &= \mathcal{L}_{\nabla h} \otimes Id + \mathcal{R}_h \\ \mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^* &= \mathcal{R}_h + \mathcal{R}_h^* - \left(\sum_{i=1}^n \left(\frac{\partial (\nabla h)_i}{\partial x^i} + \frac{1}{2} (\nabla h)_i \frac{\partial [\det G_0]}{\partial x^i} \right) \right) Id \\ &\qquad - \sum_{i=1}^n (\nabla h)_i \big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i}\big)^{(p)} , \end{cases}$$

where $(\mathcal{L}_{\nabla h} \otimes Id) a_I^k dx^I = (\mathcal{L}_{\nabla h}(a_I^k)) dx^I$*,* $\mathcal{R}_h$ *is the* 0*-th differential operator given by the matrix:*

$$\mathcal{R}_h(x) = \left(\frac{\partial (\nabla h)_j}{\partial x^i} \right)_{i,j}^{(p)} =: A_h^{(p)} \, ,$$

and $\left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)_{i,j}$*,* $G_0 \frac{\partial [G_0^{-1}]}{\partial x^i}$ *are viewed as endomorphisms of* $T_x^* \overline{\Omega}$*. Moreover,* $\mathcal{R}_h^*$ *is given by the matrix:*

$$\mathcal{R}_h^* := A_h^{(p),*} = (G_0 {}^t\!A_h G_0^{-1})^{(p)} \, .$$

Remark 3.A.4. *Owing to the the computations done in Appendix 3.A.1,* $(\nabla h)_n = \frac{\partial h}{\partial x^n}$*. Moreover, due to the form of* $G_0^{\pm 1}$*, note that*

$$\mathcal{R}_h + \mathcal{R}_h^* - \sum_{i=1}^n (\nabla h)_i (G_0 \frac{\partial [G_0^{-1}]}{\partial x^i})^{(p)}$$

is given by the matrix:

$$\begin{pmatrix} A'_h + G'_0 {}^t\!A'_h G_0^{-1'} - \sum_{i=1}^n (\nabla h)_i G'_0 \frac{\partial [G_0^{-1'}]}{\partial x^i} & \left(\frac{\partial^2 h}{\partial x^n \partial x^i}\right)_{i,1} + G'_0 \left(\frac{\partial (\nabla h)_i}{\partial x^n}\right)_{i,1} \\ \left(\frac{\partial (\nabla h)_j}{\partial x^n}\right)_{1,j} + \left(\frac{\partial^2 h}{\partial x^n \partial x^j}\right)_{1,j} G_0^{-1'} & \frac{\partial^2 h}{(\partial x^n)^2} \end{pmatrix}^{(p)} .$$

Corollary 3.A.5. *In the coordinates* (x', x^n)*, assume that the function* h *admits a critical point at* 0*, that* $\frac{\partial h}{\partial x^n} \equiv 0$ *on the boundary* $\partial\Omega$*, and that* $\frac{\partial^2 h}{(\partial x^n)^2}(0) = 0$*. Then the following relations are true:*

$$\mathcal{R}_h(0) = \mathcal{R}_h^*(0) = \begin{pmatrix} & & 0 \\ & \mathrm{Hess}\,(h\big|_{\partial\Omega})(0) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^{(p)}$$

and

$$\left(\mathcal{L}_{\nabla h} + \mathcal{L}_{\nabla h}^*\right)(0) = 2\mathcal{R}_h(0) - Tr\left(\mathrm{Hess}\,(h\big|_{\partial\Omega})(0)\right) Id \, .$$

Proof. Since (x', x^n) are local adapted coordinates around $U \cong 0$ and 0 is a critical point of h, note first that for all i in $\{1, \ldots, n\}$,

$$(\nabla h)_i = \sum_{j=1}^n g^{ij} \frac{\partial h}{\partial x^j} = \frac{\partial h}{\partial x^i} + \mathcal{O}(|x|^2) \, .$$

This implies

$$\mathcal{R}_h(x) = \left(\frac{\partial(\nabla h)_j}{\partial x^i}\right)^{(p)}_{i,j} = (\text{Hess}\,(h))^{(p)} + \mathcal{O}(|x|)$$

and in particular at 0, since $\frac{\partial h}{\partial x^n} \equiv 0$ on the boundary and $\frac{\partial^2 h}{(\partial x^n)^2}(0) = 0$:

$$\mathcal{R}_h(0) = \begin{pmatrix} & & 0 \\ & \text{Hess}\,(h\big|_{\partial\Omega})(0) & \vdots \\ 0 & \cdots & 0 \end{pmatrix}^{(p)} .$$

Moreover, we deduce from $G_0^{\pm 1}(0) = I_n$ and the symmetry of Hess $(h\big|_{\partial\Omega})(0)$,

$$\mathcal{R}_h^*(0) = \mathcal{R}_h(0)\,.$$

At least, we obtain from $\frac{\partial^2 h}{(\partial x^n)^2}(0) = 0$,

$$-\left(\sum_{i=1}^{n} \frac{\partial(\nabla h)_i}{\partial x^i}\right) Id = -Tr\left(\text{Hess}\,(h\big|_{\partial\Omega})(0)\right) \text{ at } 0\,,$$

which leads to the end of the proof, using that for all i in $\{1,\dots,n\}$, $(\nabla h)_i(0) = \frac{\partial h}{\partial x^i}(0) = 0$. ∎

Proof of Proposition 3.A.3.
The first equality is proved in [HeSj4] pp. 334-336. There is also a proof of the second equality in [HeSj4] but we need to be more precise here.
From the first equality, let us deduce:

$$\mathcal{L}_{\nabla h}^* = (\mathcal{L}_{\nabla h} \otimes Id)^* + \mathcal{R}_h^*\,.$$

Remarking that the scalar product of two p-forms ω and η is given by

$$\langle \omega \,|\, \eta \rangle_{g_0} = \langle \omega \,|\, \Gamma^{(p)}(G_0^{-1})\eta \rangle_{g_e}\,,$$

where g_e is the Euclidean metric $\sum_{i=1}^n (dx^i)^2$, we obtain

$$\mathcal{R}_h^* = \Gamma^{(p)}(G_0)({}^t\!A_h)^{(p)}\Gamma^{(p)}(G_0^{-1}) = (G_0\,{}^t\!A_h G_0^{-1})^{(p)}\,.$$

Look now at the term $(\mathcal{L}_{\nabla h} \otimes Id)^*$.
Take first two p-forms $\alpha\omega$ and $\beta\eta$ where α, β are $\mathcal{C}_0^\infty(\Omega, \mathbb{R})$ functions, and ω,

η are two p-forms dx^I and dx^J. Denoting by $V_{g_0}(dx)$ the normalized volume form, $V_{g_0}(dx)$ satisfies:

$$V_{g_0}(dx) = (\det G_0(x))^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n =: \nu(x) dx^1 \wedge \cdots \wedge dx^n \,.$$

Hence we deduce:

$$\begin{aligned} \langle \alpha\omega \,|\, (\mathcal{L}_{\nabla h} \otimes Id)^* \beta\eta \rangle_{g_0} &= \langle \mathcal{L}_{\nabla h}(\alpha)\omega \,|\, \eta \rangle_{g_0} \\ &= \int (\mathcal{L}_{\nabla h}(\alpha))\beta \langle \omega \,|\, \eta \rangle_{g_0(x)} (\det G_0(x))^{\frac{1}{2}} dx \,. \end{aligned}$$

Using now the Cartan formula (3.2.6), $\mathcal{L}_{\nabla h}(\alpha) = d\alpha(\nabla h) = \sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i$ and we obtain:

$$\begin{aligned} \int (\mathcal{L}_{\nabla h}(\alpha))\beta \langle \omega \,|\, \eta \rangle_{g_0(x)} (\det G_0(x))^{\frac{1}{2}} dx &= \int \left(\sum_{i=1}^n \frac{\partial \alpha}{\partial x^i} (\nabla h)_i \beta \right) \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu \; dx \\ &= -\int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu \big) dx \end{aligned}$$

Moreover, write:

$$\begin{aligned} \int \alpha \sum_{i=1}^n \frac{\partial}{\partial x^i} \big((\nabla h)_i \beta \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu \big) dx &= -\int \alpha \sum_{i=1}^n \left(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu \right) dx \\ -\int \alpha \sum_{i=1}^n \left((\nabla h)_i \frac{\partial \beta}{\partial x^i} \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu \right) dx &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \,|\, \eta \rangle_{g_0(x)}) \nu \right) dx \\ &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \langle \omega \,|\, \eta \rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \right) dx \\ &= -\int \alpha \sum_{i=1}^n \left(\frac{\partial (\nabla h)_i}{\partial x^i} \beta \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu \right) dx \\ -\int \alpha (\mathcal{L}_{\nabla h}(\beta)) \langle \omega \,|\, \eta \rangle_{g_0(x)} \nu dx &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \frac{\partial}{\partial x^i} (\langle \omega \,|\, \eta \rangle_{g_0(x)}) \nu \right) dx \\ &- \int \alpha \sum_{i=1}^n \left((\nabla h)_i \beta \langle \omega \,|\, \eta \rangle_{g_0(x)} \frac{\partial \nu}{\partial x^i} \right) dx \,. \end{aligned}$$

Noting that for all i in $\{1, \ldots, n\}$,

$$\begin{aligned} \frac{\partial}{\partial x^i} \Gamma^{(p)}(G_0^{-1}) &= \Big(\frac{\partial G_0^{-1}}{\partial x^i} \otimes G_0^{-1} \otimes \cdots \otimes G_0^{-1} \Big) + \cdots + \Big(G_0^{-1} \otimes \cdots \otimes G_0^{-1} \otimes \frac{\partial G_0^{-1}}{\partial x^i} \Big) \\ &= \Gamma^{(p)}(G_0^{-1}) \Big(G_0 \frac{\partial [G_0^{-1}]}{\partial x^i} \Big)^{(p)}, \end{aligned}$$

we deduce for all i in $\{1,\dots,n\}$,

$$\frac{\partial}{\partial x^i}\langle \omega \,|\, \eta\rangle_{g_0(x)} = \langle \omega \,|\, (G_0 \frac{\partial [G_0^{-1}]}{\partial x^i})^{(p)}\eta\rangle_{g_0(x)}\,.$$

Consequently,

$$\begin{aligned}(\mathcal{L}_{\nabla h}\otimes Id)^* &= -\mathcal{L}_{\nabla h}\otimes Id - \left(\sum_{i=1}^n \left(\frac{\partial(\nabla h)_i}{\partial x^i} + \frac{(\nabla h)_i}{\nu}\frac{\partial \nu}{\partial x^i}\right)\right) Id \\ &\quad - \sum_{i=1}^n (\nabla h)_i (G_0 \frac{\partial [G_0^{-1}]}{\partial x^i})^{(p)}\,,\end{aligned}$$

which leads to the second result of Proposition 3.A.3.

∎

Application to $\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$

Let us first write

$$\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} = 2\mathcal{L}_{\nabla\Phi} + \mathcal{L}_{\nabla(f-\Phi)} + \mathcal{L}^*_{\nabla(f-\Phi)}\,.$$

By Proposition 3.A.3, we deduce the following relation:

$$\mathcal{L}^*_{\nabla(f-\Phi)} + \mathcal{L}_{\nabla(f-\Phi)} = \mathcal{R}_1\,,$$

where $\mathcal{R}_1$ is a 0-th order differential operator.

Furthermore, using now the first equality of Proposition 3.A.3,

$$2\mathcal{L}_{\nabla\Phi} = 2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}_2\,,$$

where $\mathcal{R}_2$ is a 0-th differential operator too.

Consequently, setting $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$, we obtain the following relation:

$$\mathcal{L}_{\nabla\Phi} - \mathcal{L}^*_{\nabla\Phi} + \mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} = 2\mathcal{L}_{\nabla\Phi}\otimes Id + \mathcal{R}\,,$$

where $\mathcal{R}$ is a 0-th order differential operator.

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Bibliography

[BoEcGaKl] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein: Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. JEMS 6(4), pp. 399-424 (2004).

[BoGaKl] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. JEMS 7(1), pp. 69-99 (2004).

[ChLi] K. C. Chang and J. Liu. A cohomology complex for manifolds with boundary. Topological Methods in Non Linear Analysis. Volume 5, pp. 325-340 (1995).

[CyFrKiSi] H.L Cycon, R.G Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Text and Monographs in Physics, Springer Verlag (1987).

[DiSj] M. Dimassi and J. Sjöstrand. *Spectral Asymptotics in the semi-classical limit*. London Mathematical Society, Lecture Note Series 268, Cambridge University Press (1999).

[Duf] G.F.D. Duff. Differential forms in manifolds with boundary. Ann. of Math. 56, pp. 115-127 (1952).

[DuSp] G.F.D. Duff and D.C. Spencer. Harmonic tensors on Riemannian manifolds with boundary. Ann. of Math. 56, pp. 128-156 (1952).

[Hel2] B. Helffer. *Introduction to the semi-classical Analysis for the Schrödinger operator and applications.* Lecture Notes in Mathematics 1336, Springer Verlag (1988).

[HeKlNi] B. Helffer, M. Klein, and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Matematica Contemporanea 26, pp. 41-85 (2004).

[HeNi] B. Helffer and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary. Mémoire 105, Société Mathématique de France (2006).

[HeSj1] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical limit I, Comm. Partial Differential Equations 9 (4), pp. 337-408 (1984).

[HeSj4] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique IV -Etude du complexe de Witten -. Comm. Partial Differential Equations 10 (3), pp. 245-340 (1985).

[KoPrSh] Nilufer Koldan, Igor Prokhorenkov, and Mikhail Shubin. Semi-classical Asymptotics on Manifolds with Boundary. Preprint (2008). http://arxiv.org/abs/0803.2502v1

[Lep3] D. Le Peutrec. Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian. IRMAR. Preprint (2008). http://hal.archives-ouvertes.fr/hal-00297207/fr/

[Pet] P. Petersen. *Riemannian geometry.* Graduate Texts in Mathematics, Springer Verlag (1998).

[Sch] G. Schwarz. *Hodge decomposition. A method for Solving Boundary Value Problems.* Lecture Notes in Mathematics 1607, Springer Verlag (1995).

[Wit] E. Witten. Supersymmetry and Morse inequalities. J. Diff. Geom. 17, pp. 661-692 (1982).

Chapitre 4

Petites valeurs propres de la réalisation de Neumann du Laplacien de Witten semi-classique

Article [Lep3], rédigé en anglais, soumis pour publication.

Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian

Abstract: This article follows the previous works by Helffer-Klein-Nier and by Helffer-Nier about the metastability in reversible diffusion processes via a Witten complex approach. Again, exponentially small eigenvalues of some self-adjoint realization of $\Delta_{f,h}^{(0)} = -h^2\Delta + |\nabla f(x)|^2 - h\Delta f(x)$, are considered as the small parameter $h > 0$ goes to 0. The function f is assumed to be a Morse function on some bounded domain Ω with boundary $\partial\Omega$. Neumann type boundary conditions are considered. With these boundary conditions, some simplifications possible in the Dirichlet problem (studied by Helffer-Nier) are no more possible. A finer treatment of the three geometries involved in the boundary problem (boundary, metric, Morse function) is carried out.

Key words and phrases: *Witten complex, Neumann boundary conditions, accurate asymptotics, exponentially small eigenvalues.*

Contents

4.1 Introduction and result

In this text, we are interested in the exponentially small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian $\Delta^{(0)}_{f,h}$ (acting on 0-forms) on a connected compact Riemannian manifold with regular boundary.
Our purpose is to derive with the same accuracy as in [HeKlNi] and in [HeNi] asymptotic formulas for the smallest non zero eigenvalues of the Neumann realization of $\Delta^{(0)}_{f,h}$.

A similar problem was considered by many authors via a probabilistic approach in [FrWe], [HoKuSt], [Mic], and [Kol]. More recently, in the case of $\mathbb{R}^n$, accurate asymptotic forms of the exponentially small eigenvalues were obtained in [BoEcGaKl] and [BoGaKl].
These results were improved and extended to the cases of boundaryless compact manifolds in [HeKlNi] and of compact manifolds with boundaries for the Dirichlet realization of the Witten Laplacian in [HeNi].
We want here to extend these last results to the case of compact manifolds with boundaries for the Neumann realization of the Witten Laplacian, that is with coherently deformed Neumann boundary conditions.

The function f is assumed to be a Morse function on $\overline{\Omega} = \Omega \cup \partial\Omega$ with no critical points at the boundary. Furthermore, its restriction to the boundary $f|_{\partial\Omega}$ is also assumed to be a Morse function.
From [ChLi], which completed results yet obtained in the boundaryless case (see [Sim2][Wit][CyFrKiSi][Hen][HeSj4][Hel3]), the number m_p of eigenvalues of the Neumann realization of the Witten Laplacian $\Delta^{(p)}_{f,h}$ (acting on p-forms) in some interval $[0, Ch^{\frac{3}{2}}]$ (for $h > 0$ small enough) rely closely on the number of critical points of f with index p.
In the boundaryless case, these numbers are exactly the numbers of critical points of f with index p in Ω. Like in [HeNi], they have to be increased in the case with boundary, taking into account the structure of the function f at the boundary, $f|_{\partial\Omega}$. Note furthermore that m_0 is here the number of local minima of f in $\overline{\Omega}$.
Moreover, the first eigenvalue in our case is 0 and the other small eigenvalues are actually exponentially small as $h \to 0$, i.e. of order $e^{-\frac{C}{h}}$, where C is a positive number independent of the small parameter $h > 0$.

The point of view of [HeKlNi] and [HeNi] intensively uses, together with the techniques of [HeSj4], the two facts that the Witten Laplacian is associated with a cohomology complex and that the function $x \mapsto \exp -\frac{f(x)}{h}$ is a distributional solution in the kernel of the Witten Laplacian on 0-forms allowing to construct very efficiently quasimodes.

Recall that the Witten Laplacian is defined as

$$\Delta_{f,h} = d_{f,h}d^*_{f,h} + d^*_{f,h}d_{f,h} \,, \tag{4.1.1}$$

where $d_{f,h}$ is the distorted exterior differential

$$d_{f,h} := e^{-f(x)/h}\,(hd)\,e^{f(x)/h} \,, \tag{4.1.2}$$

and where $d^*_{f,h}$ is its adjoint for the L^2-scalar product canonically associated with the Riemannian structure (see for example [GaHuLa][Gol][Sch]) The restriction of $d_{f,h}$ to p-forms is denoted by $d^{(p)}_{f,h}$. With these notations, the Witten Laplacian on functions is

$$\Delta^{(0)}_{f,h} = d^{(0)*}_{f,h}d^{(0)}_{f,h} \,. \tag{4.1.3}$$

In the Witten complex spirit and due to the relation

$$d^{(0)}_{f,h}\Delta^{(0)}_{f,h} = \Delta^{(1)}_{f,h}d^{(0)}_{f,h} \,, \tag{4.1.4}$$

it is more convenient to consider the singular values of the restricted differential $d^{(0)}_{f,h} \; : F^{(0)} \to F^{(1)}$. The space $F^{(\ell)}$ is the m_ℓ-dimensional spectral subspace of $\Delta^{(\ell)}_{f,h}$, $\ell \in \{0,1\}$,

$$F^{(\ell)} = \mathrm{Ran}\; 1_{I(h)}(\Delta^{(\ell)}_{f,h}) \,, \tag{4.1.5}$$

with $I(h) = [0, Ch^{\frac{3}{2}}]$ and the property [1]

$$1_{I(h)}(\Delta^{(1)}_{f,h})d^{(0)}_{f,h} = d^{(0)}_{f,h}1_{I(h)}(\Delta^{(0)}_{f,h}) \,. \tag{4.1.6}$$

The restriction $d_{f,h}\big|_{F^{(\ell)}}$ will be more shortly denoted by $\beta^{(\ell)}_{f,h}$

$$\beta^{(\ell)}_{f,h} := (d^{(\ell)}_{f,h})_{/F^{(\ell)}} \,. \tag{4.1.7}$$

We will mainly focus on the case $\ell = 0$.
In order to exploit all the information which can be extracted from well chosen quasimodes, working with singular values of $\beta^{(0)}_{f,h}$ appears to be more efficient than considering their squares, the eigenvalues of $\Delta^{(0)}_{f,h}$. Those quantities agree better with the underlying Witten complex structure.
Note that in our case, 0 is the smallest eigenvalue of the (deformed) Neumann realization of the Witten Laplacian on 0-forms due to the belonging of $x \mapsto \exp -\frac{f(x)}{h}$ to the domain of this operator (see Proposition 4.2.7 for the exact definition).

Let us now state the main result. Let $\mathcal{U}^{(0)}$ and $\mathcal{U}^{(1)}$ denote respectively the set

1. The right end $a(h) = Ch^{\frac{3}{2}}$ of the interval $I(h) = [0, a(h)]$ is suitable for technical reasons. What is important is that $a(h) = o(h)$. The value of $C > 0$ does not play any role.

of local minima and the set of *generalized* critical points with index 1, or *generalized* saddle points, of the Morse function f on $\overline{\Omega}$ (see Definition 4.5.1 for the exact meaning of "generalized"). The analysis requires an assumption which ensures that the exponentially small eigenvalues are simple with different logarithmic equivalent as $h \to 0$. Although it is possible to consider more general cases like in [HeKlNi] and in [HeNi], we will follow the point of view presented in [Nie1] and work directly in a generic case which avoids some technical and unnecessary considerations.

Assumption 4.1.1. *The critical values of f and $f\big|_{\partial\Omega}$ are all distinct and the quantities $f(U^{(1)}) - f(U^{(0)})$, with $U^{(1)} \in \mathcal{U}^{(1)}$ and $U^{(0)} \in \mathcal{U}^{(0)}$ are distinct.*

After this assumption, a one to one mapping j from $\mathcal{U}^{(0)} \setminus \left\{U_1^{(0)}\right\}$ when $U_1^{(0)}$ is the global minimum, into the set $\mathcal{U}^{(1)}$ can be defined. The local minima are denoted by $U_k^{(0)}$, $k \in \{1, \ldots, m_0\}$, and the generalized saddle points by $U_j^{(1)}$, $j \in \{1, \ldots, m_1\}$. The ordering of the local minima as well as the one to one mapping j will be specified in Subsection 4.5.3.
The final result will be expressed with the following quantities.

Definition 4.1.2. *For $k \in \{2, \ldots, m_0\}$, we define:*

$$\gamma_k(h) = \begin{cases} \dfrac{\left|\det \mathrm{Hess}\, f(U_k^{(0)})\right|^{\frac{1}{4}}}{(\pi h)^{\frac{n}{4}}} & \text{if} \quad U_k^{(0)} \in \Omega \\ \left(\dfrac{-2\partial_n f(U_k^{(0)})}{h}\right)^{\frac{1}{2}} \dfrac{\left|\det \mathrm{Hess}\, f|_{\partial\Omega}(U_k^{(0)})\right|^{\frac{1}{4}}}{(\pi h)^{\frac{n-1}{4}}} & \text{if} \quad U_k^{(0)} \in \partial\Omega\,, \end{cases}$$

$$\delta_{j(k)}(h) = \begin{cases} \dfrac{\left|\det \mathrm{Hess}\, f(U_{j(k)}^{(1)})\right|^{\frac{1}{4}}}{(\pi h)^{\frac{n}{4}}} & \text{if} \quad U_{j(k)}^{(1)} \in \Omega \\ \left(\dfrac{-2\partial_n f(U_{j(k)}^{(1)})}{h}\right)^{\frac{1}{2}} \dfrac{\left|\det \mathrm{Hess}\, f|_{\partial\Omega}(U_{j(k)}^{(1)})\right|^{\frac{1}{4}}}{(\pi h)^{\frac{n-1}{4}}} & \text{if} \quad U_{j(k)}^{(1)} \in \partial\Omega\,, \end{cases}$$

and,

$$\theta_{j(k)}(h) = \begin{cases} \dfrac{h^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} \dfrac{(\pi h)^{\frac{n}{2}} |\widehat{\lambda}_1^{\Omega}|^{\frac{1}{2}}}{\left|\det \mathrm{Hess}\, f(U_{j(k)}^{(1)})\right|^{\frac{1}{2}}} & \text{if} \quad U_{j(k)}^{(1)} \in \Omega \\ \dfrac{h^2}{-2\partial_n f(U_{j(k)}^{(1)})} \dfrac{(\pi h)^{\frac{n-2}{2}} |\widehat{\lambda}_1^{\partial\Omega}|^{\frac{1}{2}}}{\left|\det \mathrm{Hess}\, f|_{\partial\Omega}(U_{j(k)}^{(1)})\right|^{\frac{1}{2}}} & \text{if} \quad U_{j(k)}^{(1)} \in \partial\Omega\,, \end{cases}$$

where $\widehat{\lambda}_1^W$ is the negative eigenvalue of $\mathrm{Hess}\, f|_W(U_{j(k)}^{(1)})$ *for $W = \Omega$ or $W = \partial\Omega$.*

Theorem 4.1.3.
Under Assumption 4.1.1 and after the ordering specified in Subsection 4.5.3, there exists h_0 such that, for $h \in (0, h_0]$, the spectrum in $[0, h^{\frac{3}{2}})$ of the Neumann realization of $\Delta_{f,h}^{(0)}$ in $\overline{\Omega}$ consists of m_0 eigenvalues $0 = \lambda_1(h) < \ldots < \lambda_{m_0}(h)$ of

multiplicity 1.
Moreover, the above $m_0 - 1$ *non zero eigenvalues are exponentially small and admit the following asymptotic expansions:*

$$\lambda_k(h) = \gamma_k^2(h)\, \delta_{j(k)}^2(h)\, \theta_{j(k)}^2(h)\, e^{-2\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})}{h}} \left(1 + hc_k^1(h)\right)$$

where $\gamma_k(h)$*,* $\delta_{j(k)}(h)$*, and* $\theta_{j(k)}(h)$ *are defined in the above definition and* $c_k^1(h)$ *admits a complete expansion:* $c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m}$.

This theorem extends to the case with Neumann boundary conditions the previous result of [BoGaKl] and its improvements in [HeKlNi] and [HeNi] (see also non-rigorous formal computations of [KoMa], who look also at cases with symmetry and the books [FrWe] and [Kol] and references therein).

To prove this theorem, we will follow the same strategy as in [HeKlNi] and in [HeNi] and some intermediary results will be reused without demonstration (what will be indicated in the article). Moreover, some proofs will be improved (see for example the final proof reduced now to a simple Gaussian elimination explained in [Lep1]).
At least, the geometry of the Neumann case is different from the geometry of the above references. This leads to different results (compare Theorem 4.1.3 and the main theorem of [HeNi]) and some proofs have to be entirely reconsidered. In fact, the study of the Dirichlet realization of the Witten Laplacian done in [HeNi] agreed better with the local geometry near the boundary, which led to simpler computations (see the local WKB construction in Section 4.4 for example).

The article is organized as follows.
In the second section, we analyze in detail the boundary complex adapted to our analysis in order to keep the commutation relation (4.1.4) (a part of the answer already existed in the literature (see [Sch], [Duf], [DuSp], [Gue], and [ChLi]) in connection with the analysis of the relative or absolute cohomology as defined in [Gil]).
The third section is devoted to the proof of rough estimates (to get a precise localization of the spectrum of the Laplacian) replacing the harmonic oscillator approximation in the case without boundary.
These two sections bring no additional difficulties in comparison with what was done in [HeNi].
In the fourth section, we give the WKB construction for an eigenform of the Witten Laplacian on 1-forms localized near a critical point of the boundary, according to the analysis done in [Lep2]. Moreover, it was possible in the Dirichlet case to use only a single coordinate system in order to approximate an eigenform by a WKB construction while different coordinate systems arise naturally here. Lemma 4.3.18 will play a crucial role to juggle with these different coordinate systems.
In the first part of the fifth section we label the local minima and we construct the

above injective map j under Assumption 4.1.1.
In its second part, after having constructed adapted quasimodes to our analysis, we make some scalar estimates - using the Laplace method - which lead directly to the final proof of the theorem, using the result of [Lep1], in the sixth section. Again, we cannot use like in [HeNi] a single coordinate system and we must again call on Lemma 4.3.18 to be able to use the Laplace method. It is due to the local geometry near a *generalized* critical point with index 1 which is rather more complicated than in [HeNi].

4.2 Witten Laplacian with Neumann boundary condition

4.2.1 Introduction and notations

This section is analogous to the second section of [HeNi] and we will use the same notations that we recall here.

Let $\overline{\Omega}$ be a $\mathcal{C}^\infty$ connected compact oriented Riemannian n-dimensional manifold. We will denote by g_0 the given Riemannian metric on $\overline{\Omega}$; Ω and $\partial\Omega$ will denote respectively its interior and its boundary.
The cotangent (resp. tangent) bundle on Ω is denoted by $T^*\Omega$ (resp. $T\Omega$) and the exterior fiber bundle by $\Lambda T^*\Omega = \oplus_{p=0}^{n}\Lambda^p T^*\Omega$ (resp. $\Lambda T\Omega = \oplus_{p=0}^{n}\Lambda^p T\Omega$).
The fiber bundles $\Lambda T\partial\Omega = \oplus_{p=0}^{n-1}\Lambda^p T\partial\Omega$ and $\Lambda T^*\partial\Omega = \oplus_{p=0}^{n-1}\Lambda^p T^*\partial\Omega$ are defined similarly.
The space of $\mathcal{C}^\infty$, $\mathcal{C}_0^\infty$, L^2, H^s , etc. sections in any of these fiber bundles, E, on $O = \Omega$ or $O = \partial\Omega$, will be denoted respectively by $\mathcal{C}^\infty(O;E)$, $\mathcal{C}_0^\infty(O;E)$, $L^2(O;E)$, $H^s(O;E)$, etc.
When no confusion is possible we will simply use the short notations $\Lambda^p\mathcal{C}^\infty$, $\Lambda^p\mathcal{C}_0^\infty$, $\Lambda^p L^2$ and $\Lambda^p H^s$ for $E = \Lambda^p T^*\Omega$ or $E = \Lambda^p T^*\partial\Omega$.
Note that the L^2 spaces are those associated with the unit volume form for the Riemannian structure on Ω or $\partial\Omega$ (Ω and $\partial\Omega$ are oriented).
The notation $\mathcal{C}^\infty(\overline{\Omega};E)$ is used for the set of $\mathcal{C}^\infty$ sections up to the boundary.
Finally since $\partial\Omega$ is $\mathcal{C}^\infty$, $\mathcal{C}^\infty(\overline{\Omega};E)$ is dense in $H^s(\Omega;E)$ for $s \geq 0$ and the trace operator $\omega \to \omega|_{\partial\Omega}$ extends to a surjective operator from $H^s(\Omega;E)$ onto $H^{s-1/2}(\partial\Omega;E)$ as soon as $s > 1/2$.

Let d be the exterior differential on $\mathcal{C}_0^\infty(\Omega;\Lambda T^*\Omega)$

$$\left(d^{(p)} : \mathcal{C}_0^\infty(\Omega;\Lambda^p T^*\Omega) \to \mathcal{C}_0^\infty(\Omega;\Lambda^{p+1} T^*\Omega) \right)$$

and d^* its formal adjoint with respect to the L^2-scalar product inherited from the Riemannian structure

$$\left(d^{(p),*} : \mathcal{C}_0^\infty(\Omega;\Lambda^{p+1}T^*\Omega) \to \mathcal{C}_0^\infty(\Omega;\Lambda^p T^*\Omega) \right) .$$

Remark 4.2.1. *Note that d and d^* are both well defined on $\mathcal{C}^\infty(\overline{\Omega};\Lambda T^*\Omega)$.*

We set, for a function $f \in \mathcal{C}^\infty(\overline{\Omega};\mathbb{R})$ and $h > 0$, the distorted operators defined on $\mathcal{C}^\infty(\overline{\Omega};\Lambda T^*\Omega)$:

$$d_{f,h} = e^{-f(x)/h}\,(hd)\,e^{f(x)/h} \quad \text{and} \quad d^*_{f,h} = e^{f(x)/h}\,(hd^*)\,e^{-f(x)/h}\,,$$

The Witten Laplacian is the differential operator defined on $\mathcal{C}^\infty(\overline{\Omega};\Lambda T^*\Omega)$ by:

$$\Delta_{f,h} = d^*_{f,h}d_{f,h} + d_{f,h}d^*_{f,h} = (d_{f,h} + d^*_{f,h})^2\,. \tag{4.2.1}$$

Remark 4.2.2. *The last equality follows from the property $dd = d^*d^* = 0$ which implies:*

$$d_{f,h}d_{f,h} = d^*_{f,h}d^*_{f,h} = 0. \tag{4.2.2}$$

It means, by restriction to the p-forms in $\mathcal{C}^\infty(\overline{\Omega};\Lambda^p T^*\Omega)$:

$$\Delta^{(p)}_{f,h} = d^{(p),*}_{f,h}d^{(p)}_{f,h} + d^{(p-1)}_{f,h}d^{(p-1),*}_{f,h}\,.$$

Note that (4.2.2) imply that, for all u in $\mathcal{C}^\infty(\overline{\Omega};\Lambda^p T^*\Omega)$,

$$\Delta^{(p+1)}_{f,h}d^{(p)}_{f,h}u = d^{(p)}_{f,h}\Delta^{(p)}_{f,h}u \tag{4.2.3}$$

and

$$\Delta^{(p-1)}_{f,h}d^{(p-1),*}_{f,h}u = d^{(p-1),*}_{f,h}\Delta^{(p)}_{f,h}u\,. \tag{4.2.4}$$

We end up this section by a few relations with exterior and interior products (respectively denoted by $\wedge$ and $\mathbf{i}$), gradients (denoted by ∇) and Lie derivatives (denoted by $\mathcal{L}$) which will be very useful:

$$(df\wedge)^* = \mathbf{i}_{\nabla f} \quad (\text{in } L^2(\overline{\Omega};\Lambda^p T^*\Omega))\,, \tag{4.2.5}$$

$$d_{f,h} = hd + df\wedge\,, \tag{4.2.6}$$

$$d^*_{f,h} = hd^* + \mathbf{i}_{\nabla f}\,, \tag{4.2.7}$$

$$d\circ\mathbf{i}_X + \mathbf{i}_X\circ d = \mathcal{L}_X\,, \tag{4.2.8}$$

$$\Delta_{f,h} = h^2(d+d^*)^2 + |\nabla f|^2 + h\left(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}\right)\,, \tag{4.2.9}$$

where X denotes a vector field on Ω or $\overline{\Omega}$.

Remark 4.2.3. *We work here on a Riemannian manifold and the operators introduced depend on the Riemannian metric g_0. Nevertheless, we have omitted here this dependence for conciseness.*

4.2.2 Stokes and Green formulas

In order to define suitably the self-adjoint Neumann realization of the Witten Laplacian $\Delta_{f,h}$ that we will study in the rest of this work, we need variants from the Stokes and the Green formulas.
For that, we use some notations and properties which are very convenient for boundary problems and which are introduced for example in [Sch] and recalled in [HeNi].

Definition 4.2.4. *We denote by $\vec{n}_\sigma$ the outgoing normal at $\sigma \in \partial\Omega$ and by $\vec{n}^*_\sigma$ the 1-form dual to $\vec{n}_\sigma$ for the Riemannian scalar product.*

For any $\omega \in \mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$, the form $\mathbf{t}\omega$ is the element of $\mathcal{C}^\infty(\partial\Omega; \Lambda^p T^*\Omega)$ defined by:

$$(\mathbf{t}\omega)_\sigma(X_1, \ldots, X_p) = \omega_\sigma(X_1^T, \ldots, X_p^T) \ , \ \forall \sigma \in \partial\Omega \ ,$$

with the decomposition into the tangential and normal components to $\partial\Omega$ at σ: $X_i = X_i^T \oplus x_i^\perp \vec{n}_\sigma$.
Moreover,

$$(\mathbf{t}\omega)_\sigma = \mathbf{i}_{\vec{n}_\sigma}(\vec{n}^*_\sigma \wedge \omega_\sigma) \ .$$

The projected form $\mathbf{t}\omega$, which depends on the choice of $\vec{n}_\sigma$ (i.e. on g_0), can be compared with the canonical pull-back $j^*\omega$ associated with the imbedding $j : \partial\Omega \to \Omega$. Actually the exact relationship is $j^*\omega = j^*(\mathbf{t}\omega)$. With an abuse of notation, the form $j^*(\mathbf{t}\omega)$ will be simply written $\mathbf{t}\omega$ for example in Stokes formula without any possible confusion.
The normal part of ω on $\partial\Omega$ is defined by:

$$\mathbf{n}\omega = \omega|_{\partial\Omega} - \mathbf{t}\omega \quad \in \mathcal{C}^\infty(\partial\Omega; \Lambda^p T^*\Omega).$$

If necessary $\mathbf{t}\omega$ and $\mathbf{n}\omega$ can be considered as elements of $\mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$ by a variant of the collar theorem (see [HeNi] or [Sch] for details).
The Hodge operator $\star$ is locally defined in a pointwise orthonormal frame $(E_1, \ldots, E_n)$ by:

$$(\star\omega_x)(E_{\sigma(p+1)}, \ldots, E_{\sigma(n)}) = \varepsilon(\sigma)\ \omega_x(E_{\sigma(1)}, \ldots, E_{\sigma(p)}) \ ,$$

for $\omega_x \in \Lambda^p T^*_x\Omega$ and with any permutation $\sigma \in \Sigma(n)$ of $\{1, \ldots, n\}$ preserving $\{1, \ldots, p\}$ ($\varepsilon(\sigma)$ denotes the signature of σ).
We recall the formulas:

$$\star(\star\omega_x) = (-1)^{p(n-p)}\omega_x \ , \qquad \forall \omega_x \in \Lambda^p T^*_x\Omega \ , \tag{4.2.10}$$

$$\langle \omega_1 \,|\, \omega_2 \rangle_{\Lambda^p L^2} = \int_\Omega \omega_1 \wedge \star\overline{\omega_2} \ , \quad \forall \omega_1, \omega_2 \in \Lambda^p L^2 \ , \tag{4.2.11}$$

and:

$$\star d^{*,(p-1)} = (-1)^p d^{(n-p)} \star \ , \quad \star d^{(p)} = (-1)^{p+1} d^{*,(n-p-1)} \star \ , \tag{4.2.12}$$

$$\star\ \mathbf{n} = \mathbf{t}\ \star \ , \qquad \star\ \mathbf{t} = \mathbf{n}\ \star \ , \tag{4.2.13}$$

$$\mathbf{t}\ d = d\ \mathbf{t} \ , \qquad \mathbf{n}\ d^* = d^*\ \mathbf{n} \ . \tag{4.2.14}$$

With the previous convention $j^*(\mathbf{t}\omega) = \mathbf{t}\omega$, the Stokes formula writes:

$$\forall \omega \in \mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega), \qquad \int_\Omega d\omega = \int_{\partial\Omega} j^*\omega = \int_{\partial\Omega} \mathbf{t}\omega \,, \tag{4.2.15}$$

and a first deformed Green formula given in [HeNi] states that

$$\begin{aligned}&\langle d_{f,h}\omega \,|\, d_{f,h}\eta\rangle_{\Lambda^{p+1}L^2} + \langle d^*_{f,h}\omega \,|\, d^*_{f,h}\eta\rangle_{\Lambda^{p-1}L^2} \\ &\quad = \langle \Delta_{f,h}\omega \,|\, \eta\rangle_{\Lambda^p L^2} + h\int_{\partial\Omega} (\mathbf{t}\overline{\eta}) \wedge (\star \mathbf{n} d_{f,h}\omega) - h\int_{\partial\Omega} (\mathbf{t} d^*_{f,h}\omega) \wedge (\star\mathbf{n}\overline{\eta})\end{aligned} \tag{4.2.16}$$

holds for all $\omega \in \Lambda^p H^2$ and $\eta \in \Lambda^p H^1$. This formulation of (4.2.16) does not depend on the choice of an orientation. If μ and $\mu_{\partial\Omega}$ denote the volume forms in Ω and $\partial\Omega$, the orientation is chosen such that $(\mu_{\partial\Omega})_\sigma(X_1, \ldots, X_{n-1}) = \mu_\sigma(\vec{n}_\sigma, X_1, \ldots, X_{n-1})$. A simple computation in normal frames (see [Sch], prop. 1.2.6) leads to:

$$\mathbf{t}\omega_1 \wedge \star\mathbf{n}\overline{\omega_2} = \langle \omega_1 \,|\, \mathbf{i}_{\vec{n}_\sigma}\omega_2\rangle_{\Lambda^p T^*_\sigma\Omega}\, d\mu_{\partial\Omega} \,, \tag{4.2.17}$$

for $\omega_1 \in \mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$ and $\omega_2 \in C^\infty(\overline{\Omega}; \Lambda^{p+1} T^*\Omega)$.

Definition 4.2.5. *We denote by $\frac{\partial f}{\partial n}(\sigma)$ or $\partial_n f(\sigma)$ the normal derivative of f at σ:*

$$\frac{\partial f}{\partial n}(\sigma) = \partial_n f(\sigma) := \langle \nabla f(\sigma) \,|\, \vec{n}_\sigma\rangle \,.$$

As a consequence of (4.2.17) we get the following useful decomposition formula.

Lemma 4.2.6. *(Normal Green Formula)*
The identity

$$\begin{aligned}\|d_{f,h}\omega\|^2_{\Lambda^{p+1}L^2} + \left\|d^*_{f,h}\omega\right\|^2_{\Lambda^{p-1}L^2} &= h^2 \|d\omega\|^2_{\Lambda^{p+1}L^2} + h^2 \|d^*\omega\|^2_{\Lambda^{p-1}L^2} \\ &+ \|\, |\nabla f|\,\omega\|^2_{\Lambda^p L^2} + h\langle (\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f})\omega \,|\, \omega\rangle_{\Lambda^p L^2} \\ &+ h\int_{\partial\Omega} \langle \omega \,|\, \omega\rangle_{\Lambda^p T^*_\sigma\Omega} \left(\frac{\partial f}{\partial n}\right)(\sigma)\ d\mu_{\partial\Omega}\end{aligned} \tag{4.2.18}$$

holds for any $\omega \in \Lambda^p H^1$ such that $\mathbf{n}\omega = 0$.

Proof.
Since $\mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$ is dense in $\Lambda^p H^1$, while both terms of the identity are continuous on $\Lambda^p H^1$, the form ω can be assumed to be in $\mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$.
We use the relation (4.2.16) with both $f = 0$ ($d_{0,h} = hd$ and $d^*_{0,h} = hd^*$) and a general $f \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R})$. We obtain:

$$\begin{aligned}&\|d_{f,h}\omega\|^2_{\Lambda^{p+1}L^2} + \left\|d^*_{f,h}\omega\right\|^2_{\Lambda^{p-1}L^2} - h^2\|d\omega\|^2_{\Lambda^{p+1}L^2} - h^2\|d^*\omega\|^2_{\Lambda^{p-1}L^2} = \\ &\langle (\Delta_{f,h} - \Delta_{0,h})\omega \,|\, \omega\rangle_{\Lambda^p L^2} + h\int_{\partial\Omega} (\mathbf{t}\overline{\omega}) \wedge \star\mathbf{n}(df \wedge \omega) - h\int_{\partial\Omega} (\mathbf{t}\mathbf{i}_{\nabla f}\omega) \wedge (\star\mathbf{n}\overline{\omega}) \\ &\qquad = \langle (\Delta_{f,h} - \Delta_{0,h})\omega \,|\, \omega\rangle_{\Lambda^p L^2} + h\int_{\partial\Omega} \langle \omega \,|\, \mathbf{i}_{\vec{n}_\sigma}(df \wedge \omega)\rangle_{\Lambda T^*_\sigma\Omega}\ d\mu_{\partial\Omega} \,.\end{aligned}$$

By (4.2.9):

$$\langle(\Delta_{f,h}-\Delta_{0,h})\omega\,|\,\omega\rangle_{\Lambda^pL^2}=\||\nabla f|\,\omega\|^2_{\Lambda^pL^2}\;+h\langle(\mathcal{L}_{\nabla f}+\mathcal{L}^*_{\nabla f})\omega\,|\,\omega\rangle_{\Lambda^pL^2}\;.$$

For the integral term, we write:

$$\begin{aligned}\mathbf{i}_{\vec{n}_\sigma}(df\wedge\omega)(X_1,\dots,X_p) &= (df\wedge\omega)\,(\vec{n}_\sigma,\,X_1,\dots,X_p)\\ &= df(\vec{n}_\sigma).\omega(X_1,\dots,X_p)\quad \text{because } \mathbf{n}\omega=0\\ &= \langle\nabla f(\sigma)\,|\,\vec{n}_\sigma\rangle.\omega(X_1,\dots,X_p)\\ &= \left(\frac{\partial f}{\partial n}(\sigma)\right)\omega(X_1,\dots,X_p)\,,\end{aligned}$$

which proves the lemma. ∎

4.2.3 Normal Neumann realization

In this subsection, we specify the self-adjoint realization of $\Delta^{(0)}_{f,h}$ in which we are interested. Like in [HeNi], we want this self-adjoint realization (denoted by $\Delta^{N}_{f,h}$) to coincide with the Neumann realization on 0-forms and to preserve the complex structure:

$$(1+\Delta^{N,(p+1)}_{f,h})^{-1}d^{(p)}_{f,h} = d^{(p)}_{f,h}(1+\Delta^{N,(p)}_{f,h})^{-1}$$

and

$$(1+\Delta^{N,(p-1)}_{f,h})^{-1}d^{(p-1),*}_{f,h} = d^{(p-1),*}_{f,h}(1+\Delta^{N,(p)}_{f,h})^{-1}$$

on the form domain of $\Delta^{N,(p)}_{f,h}$.

Having in mind the works [Sch] and [ChLi] about cohomology complexes and boundary problems, we introduce the space:

$$\Lambda^pH^1_{0,\mathbf{n}}=H^1_{0,\mathbf{n}}\,(\Omega;\Lambda^pT^*\Omega)=\left\{\omega\in H^1\,(\Omega;\Lambda^pT^*\Omega)\,;\;\mathbf{n}\omega=0\right\}. \tag{4.2.19}$$

In the case $p=0$, it coincides with the space $H^1(\Omega)$, while for $p\geq 1$ the condition says only that the form vanishes on $\partial\Omega$ when applied to non tangential p-vectors. Since the boundary $\partial\Omega$ is assumed to be regular, the space

$$\Lambda^p\mathcal{C}^\infty_{0,\mathbf{n}}=\mathcal{C}^\infty_{0,\mathbf{n}}(\Omega;\Lambda^pT^*\Omega)=\left\{\omega\in\mathcal{C}^\infty\left(\overline{\Omega},\Lambda^pT^*\Omega\right);\;\mathbf{n}\omega=0\right\}$$

is dense in $\Lambda^pH^1_{0,\mathbf{n}}$. The following construction is a variant of known results in the case $f=0$ (see [Sch]). We will use the notations:

$$\mathcal{D}_{f,h}(\omega,\eta)=\langle d_{f,h}\omega\mid d_{f,h}\eta\rangle_{\Lambda^{p+1}L^2}+\langle d^*_{f,h}\omega\mid d^*_{f,h}\eta\rangle_{\Lambda^{p-1}L^2}$$

and

$$\mathcal{D}_{f,h}(\omega)=\mathcal{D}_{f,h}(\omega,\omega)=\|d_{f,h}\omega\|^2_{\Lambda^{p+1}L^2}+\left\|d^*_{f,h}\omega\right\|^2_{\Lambda^{p-1}L^2}\;.$$

Proposition 4.2.7.
The non negative quadratic form $\omega \to \mathcal{D}_{f,h}(\omega)$ is closed on $\Lambda^p H^1_{0,\mathbf{n}}$. The associated (self-adjoint) Friedrichs extension is denoted by $\Delta^{N,(p)}_{f,h}$. Its domain is:

$$D(\Delta^{N,(p)}_{f,h}) = \left\{u \in \Lambda^p H^2; \quad \mathbf{n}\omega = 0 \ \textit{and} \ \ \mathbf{n} d_{f,h}\omega = 0\right\} \,,$$

and we have:

$$\forall \omega \in D(\Delta^{N,(p)}_{f,h}), \quad \Delta^{N,(p)}_{f,h}\omega = \Delta^{(p)}_{f,h}\omega \quad \textit{in } \Omega\,.$$

Proof.
By the same argument as in the proof of Proposition 2.4 of [HeNi], the space $\Lambda^p H^1_{0,\mathbf{n}}$ is isomorphic to the direct sum:

$$\Lambda^p H^1_0 \oplus \mathbf{t}\Lambda^p H^{1/2}(\partial\Omega; \Lambda^p T^*\Omega)$$

with continuous embedding. Hence, since $\partial\Omega$ is a regular boundaryless manifold, its dual is the direct sum of $\Lambda^p H^{-1}$ and $\mathbf{t}\Lambda^p H^{-1/2}(\partial\Omega; \Lambda^p T^*\Omega)$:

$$\left(\Lambda^p H^1_{0,\mathbf{n}}\right)' = \Lambda^p H^{-1} \oplus \mathbf{t}\Lambda^p H^{-1/2}(\partial\Omega; \Lambda^p T^*\Omega)\,.$$

We have to check that $\omega \mapsto \mathcal{D}^{(p)}_{f,h}(\omega) + C\left\|\omega\right\|^2_{\Lambda^p L^2}$ is equivalent to the square of the $\Lambda^p H^1$ norm on $\Lambda^p H^1_{0,\mathbf{n}}$. By (4.2.6)-(4.2.9) this is equivalent to the same result for $f = 0$ and $h = 1$. This last case is known as Gaffney's inequality which is a consequence of the Weitzenböck formula (see [Sch], Theorem 2.1.7).
Hence the quadratic form $\omega \to \mathcal{D}_{f,h}(\omega)$ is closed on $\Lambda^p H^1_{0,\mathbf{n}}$ and the identity

$$\forall \eta \in \Lambda^p H^1_{0,\mathbf{n}}, \quad \mathcal{D}^{(p)}_{f,h}(\omega,\eta) = \langle A^{(p)}\omega, \eta\rangle$$

defines an isomorphism $A^{(p)} : \Lambda^p H^1_{0,\mathbf{n}} \to (\Lambda^p H^1_{0,\mathbf{n}})'$.
The self-adjoint Friedrichs extension $\Delta^{N,(p)}_{f,h}$ is then defined as the operator:

$$D(\Delta^{N,(p)}_{f,h}) = \left\{\omega \in \Lambda^p H^1_{0,\mathbf{n}},\ A^{(p)}\omega \in \Lambda^p L^2\right\}, \quad \Delta^{N,(p)}_{f,h}\omega = A^{(p)}\omega\,.$$

It remains to identify this domain and the explicit action of $A^{(p)}$.
If ω belongs to $D(\Delta^{N,(p)}_{f,h})$, by the first Green formula (4.2.16) we get:

$$\forall \eta \in \Lambda^p \mathcal{C}^\infty_0, \quad \langle \omega \,|\, A^{(p)}\eta\rangle = \mathcal{D}^{(p)}_{f,h}(\omega,\eta) = \langle \omega \,|\, \Delta^{(p)}_{f,h}\eta\rangle\,.$$

The inequality:

$$|\mathcal{D}^{(p)}_{f,h}(\omega,\eta)| \leq C\left\|\omega\right\|_{\Lambda^p H^1}\left\|\eta\right\|_{\Lambda^p H^1}\,,$$

together with the density of $\Lambda^p\mathcal{C}^\infty_0$ in $\Lambda^p H^1_0$ implies that the current $\Delta^{(p)}_{f,h}\omega \in \mathcal{D}'(\Omega; \Lambda^p T^*\Omega)$ is indeed the $\Lambda^p H^{-1}$ component of $A^{(p)}\omega$.
Assume that ω belongs to $\Lambda^p H^1_{0,\mathbf{n}} \cap \Lambda^p H^2$; then the Green formula (4.2.16) gives:

$$h\int_{\partial\Omega} (\mathbf{t}\overline{\eta}) \wedge (\star\mathbf{n} d_{f,h}\omega) = \mathcal{D}^{(p)}_{f,h}(\omega,\eta) - \langle \Delta^{(p)}_{f,h}\omega \,|\, \eta\rangle_{\Lambda^p L^2}\,,\ \forall \eta \in \Lambda^p H^1_{0,\mathbf{n}}\,.$$

By density, one can define, for any ω in $\Lambda^p H^1_{0,\mathbf{n}}$ such that $\Delta^{(p)}_{f,h}\omega \in \Lambda^p L^2$, a trace of $\mathbf{n}d_{f,h}\omega$ by the previous identity, observing that the r.h.s. defines an antilinear continuous form with respect to η. With this generalized definition of $\mathbf{n}d^{(p)}_{f,h}\omega$ we claim that:

$$D(\Delta^{N,(p)}_{f,h}) = \left\{\omega \in \Lambda^p H^1_{0,\mathbf{n}},\ \Delta^{(p)}_{f,h}\omega \in \Lambda^p L^2 \text{ and } \mathbf{n}d^{(p)}_{f,h}\omega = 0\right\}.$$

The last point consists in observing that the boundary value problem

$$\Delta^{(p)}_{f,h}u = g, \quad \mathbf{n}u = g_1, \quad \mathbf{n}d^{(p)}_{f,h}u = g_2 \tag{4.2.20}$$

satisfies the Lopatinski-Shapiro conditions. At the principal symbol level ($h > 0$ fixed), these conditions are indeed the same as for

$$(dd^* + d^*d)^{(p)}u = g, \quad \mathbf{n}u = g_1, \quad \mathbf{n}d^{(p)}u = g_2.$$

This is checked in [Sch]. Hence any solution to (4.2.20) with $g \in \Lambda^p L^2$, $g_1 = g_2 = 0$ belongs to $\Lambda^p H^2$. ∎

Proposition 4.2.8.
For any $p \in \{0, \dots, n\}$, the self-adjoint unbounded operator $\Delta^{N,(p)}_{f,h}$ introduced in Proposition 4.2.7 has a compact resolvent.
Moreover, if $z \in \mathbb{C} \setminus \mathbb{R}_+$, the commutation relations

$$(z - \Delta^{N,(p+1)}_{f,h})^{-1} d^{(p)}_{f,h} v = d^{(p)}_{f,h}(z - \Delta^{N,(p)}_{f,h})^{-1} v\ ,$$

and

$$(z - \Delta^{N,(p-1)}_{f,h})^{-1} d^{(p-1),*}_{f,h} v = d^{(p-1),*}_{f,h}(z - \Delta^{N,(p)}_{f,h})^{-1} v\ ,$$

hold for any $v \in \Lambda^p H^1_{0,\mathbf{n}}$.

Proof.
The domain of the operator is contained in $\Lambda^p H^2$, which is compactly embedded in $\Lambda^p L^2$, by the Sobolev injections. This yields the first statement.
Since $\Lambda^p \mathcal{C}^\infty_{0,\mathbf{n}}$ is dense in $\Lambda^p H^1_{0,\mathbf{n}}$, it is sufficient to consider the case when $v \in \Lambda^p \mathcal{C}^\infty_{0,\mathbf{n}}$. For such a v and for $z \in \mathbb{C} \setminus \mathbb{R}_+$, we set:

$$u = (z - \Delta^{N,(p)}_{f,h})^{-1} v.$$

Due to the ellipticity of the associated boundary problem (the Lopatinski-Shapiro conditions are verified) u belongs to $\mathcal{C}^\infty(\overline{\Omega}; \Lambda^p T^*\Omega)$. The commutation relations (4.2.3) and (4.2.4) can be applied since here $f \in \mathcal{C}^\infty(\overline{\Omega}; \mathbb{R})$:

$$(z - \Delta^{(p+1)}_{f,h}) d^{(p)}_{f,h} u = d^{(p)}_{f,h}(z - \Delta^{(p)}_{f,h}) u = d^{(p)}_{f,h} v \tag{4.2.21}$$

and

$$(z - \Delta^{(p-1)}_{f,h}) d^{(p-1),*}_{f,h} u = d^{(p-1),*}_{f,h}(z - \Delta^{(p)}_{f,h}) u = d^{(p-1),*}_{f,h} v\ . \tag{4.2.22}$$

Since $u \in D(\Delta_{f,h}^{N,(p)})$, we have $\mathbf{n}u = 0$ and $\mathbf{n}d_{f,h}^{(p)}u = 0$.
Then, $\mathbf{n}d_{f,h}u = 0$ and $\mathbf{n}d_{f,h}d_{f,h}u = 0$ imply $d_{f,h}u \in D(\Delta_{f,h}^{N,(p+1)})$. So by (4.2.21) we have:

$$d_{f,h}^{(p)}(z - \Delta_{f,h}^{N,(p)})^{-1}v = d_{f,h}u = (z - \Delta_{f,h}^{N,(p+1)})^{-1}d_{f,h}^{(p)}v\ .$$

In order to show the second commutation relation, we first use the relation (4.2.14) which implies:

$$\mathbf{n}d_{f,h}^{*}u = hd^{*}\mathbf{n}u + \mathbf{n}(\mathbf{i}_{\nabla f}u) = 0.$$

For the normal trace of the differential, we write ($\Delta_{f,h}u = zu - v$):

$$\mathbf{n}d_{f,h}(d_{f,h}^{*}u) = z\mathbf{n}u - \mathbf{n}v - \mathbf{n}d_{f,h}^{*}d_{f,h}u = -d_{f,h}^{*}\mathbf{n}d_{f,h}u = 0\ .$$

Hence $d_{f,h}^{(p-1),*}u$ belongs to $D(\Delta_{f,h}^{N,(p-1)})$ and the identity (4.2.22) yields the last commutation relation to show. ∎

Definition 4.2.9. *For any Borel subset $E \subset \mathbb{R}$ and $p \in \{0, \ldots, n\}$, we will denote by $1_E(\Delta_{f,h}^{N,(p)})$ the spectral projection of $\Delta_{f,h}^{N,(p)}$ on E.*

From Proposition 4.2.8 and Stone's Formula we deduce:

Corollary 4.2.10.
For any Borel subset $E \subset \mathbb{R}$, the identities

$$1_E(\Delta_{f,h}^{N,(p+1)})d_{f,h}^{(p)}v = d_{f,h}^{(p)}1_E(\Delta_{f,h}^{N,(p)})v$$

and

$$1_E(\Delta_{f,h}^{N,(p-1)})d_{f,h}^{(p-1),*}v = d_{f,h}^{(p-1),*}1_E(\Delta_{f,h}^{N,(p)})v$$

hold for all $v \in \Lambda^p H^1_{0,\mathbf{n}}$.
In the particular case when v is an eigenvector of $\Delta_{f,h}^{N,(p)}$ corresponding to the eigenvalue λ, then $d_{f,h}^{(p)}v$ (resp. $d_{f,h}^{(p-1),*}v$) belongs to the spectral subspace Ran $1_{\{\lambda\}}(\Delta_{f,h}^{N,(p+1)})$ *(resp.* Ran $1_{\{\lambda\}}(\Delta_{f,h}^{N,(p-1)})$*).*

Proposition 4.2.8 and Corollary 4.2.10 were stated for p-forms $v \in \Lambda^p H^1_{0,\mathbf{n}}(\Omega)$, belonging to the form domain of $\Delta_{f,h}^{N,(p)}$. It is convenient to work in this framework because the multiplication by any cut-off function preserves the form domain $\Lambda H^1_{0,\mathbf{n}}(\Omega)$:

$$\big(\omega \in \Lambda H^1_{0,\mathbf{n}}(\Omega), \quad \chi \in \mathcal{C}^\infty(\overline{\Omega})\big) \Rightarrow (\chi\omega \in \Lambda H^1_{0,\mathbf{n}}(\Omega))\ ,$$

while this property is no more true for $D(\Delta_{f,h}^{N})$. In this spirit, we will often refer to the following easy consequence of the spectral theorem.

Lemma 4.2.11.
Let A be a non negative self-adjoint operator on a Hilbert space $\mathcal{H}$ with associated quadratic form $q_A(x) = (x \mid Ax)$ and with form domain $Q(A)$. Then for any $a, b \in (0, +\infty)$, the implication

$$(q_A(u) \leq a) \Rightarrow \left(\left\| 1_{[b,+\infty)}(A)u \right\|^2 \leq \frac{a}{b} \right)$$

holds for any $u \in Q(A)$.

4.3 First localization of the spectrum

4.3.1 Introduction and result

Let us first recall that we are working with the fixed Riemannian metric g_0 on $\overline{\Omega}$. Like in the third section of [HeNi] for their tangential Dirichlet realization of the Witten Laplacian, we check here that the number of eigenvalues of $\Delta_{f,h}^{N,(p)}$ smaller than $h^{3/2}$ equals a Morse index which involves in its definition the boundary conditions. To this end, we will adapt [HeNi] which uses techniques yet presented in [Sim2], [CyFrKiSi], [ChLi], [Bis], [Bur], and in [Hel1].

In order to make the connection between the normal Neumann realization of the Witten Laplacian $\Delta_{f,h}^{N}$ and the Morse theory, we assume additional properties for the function f up to the boundary $\partial\Omega$.

Assumption 4.3.1.
The real-valued function $f \in \mathcal{C}^\infty(\overline{\Omega})$ is a Morse function on Ω with no critical points in $\partial\Omega$. In addition its restriction $f|_{\partial\Omega}$ is a Morse function on $\partial\Omega$.

Remark 4.3.2.
With this assumption, the function f has a finite number of critical points with index p in Ω. Note furthermore that the assumption ensures that there is no critical point on $\partial\Omega$, which implies that the outgoing normal derivative $\frac{\partial f}{\partial n}(U)$ is not 0 when U is a critical point of $f|_{\partial\Omega}$.

Definition 4.3.3.
For $\ell \in \{0, \dots, n\}$, the integer $m_{\ell,-}^{\partial\Omega}$ is the number of critical points U of $f|_{\partial\Omega}$ with index ℓ such that $\frac{\partial f}{\partial n}(U) < 0$ (with the additional convention $m_{n,-}^{\partial\Omega} = 0$).
For $p \in \{0, \dots, n\}$, let

$$m_p^{\overline{\Omega}} = m_p^{\Omega} + m_{p,-}^{\partial\Omega} \ .$$

Remark 4.3.4.
In [HeNi], the authors worked with the tangential Dirichlet conditions ($\mathbf{t}\omega = 0$ and $\mathbf{t}d_{f,h}^{*}\omega = 0$) and the corresponding definition was similar with $m_{\ell,-}^{\partial\Omega}$ and $\frac{\partial f}{\partial n}(U) < 0$ replaced respectively by $m_{\ell-1,-}^{\partial\Omega}$ and $\frac{\partial f}{\partial n}(U) > 0$.

The aim of this section is to prove the following theorem:

Theorem 4.3.5.
Under Assumption 4.3.1, there exists $h_0 > 0$, such that the normal Neumann realization of the Witten Laplacian $\Delta^N_{f,h}$ introduced in Subsection 4.2.3 has, for $h \in (0, h_0]$, the following property:
For any $p \in \{0, \dots, n\}$, the spectral subspace $F^{(p)} = \mathrm{Ran} 1_{[0,h^{3/2})}(\Delta^{N,(p)}_{f,h})$ has rank: $\dim F^{(p)} = m^{\overline{\Omega}}_p$.

To prove this theorem, we will adapt for the normal Neumann realization of the Witten Laplacian the proof given in [HeNi] for the tangential Dirichlet realization. Many points of this demonstration do not require any modification, so we will only recall these results without any demonstration.
The theorem will be proved in the Subsection 4.3.3.

4.3.2 A few preliminary lemmas

In this subsection, we recall some results of [HeNi] that we need to prove Theorem 4.3.5.

Variationnal results for the Witten Laplacian on $\mathbb{R}^k$

Let g be a $\mathcal{C}^\infty$ metric on $\mathbb{R}^k$ which equals the Euclidean metric outside a compact set K.

Assumption 4.3.6 (g).
The function f is a Morse $\mathcal{C}^\infty$ real-valued function and there exist $C_1 > 0$ and a compact K such that, for the metric g:

$$\forall x \in \mathbb{R}^k \setminus K, \quad |\nabla f(x)| \geq C_1^{-1} \quad \textit{and} \quad |\mathrm{Hess}\, f(x)| \leq C_1 \, |\nabla f(x)|^2 \,. \tag{4.3.1}$$

Note that the above assumption ensures that f has a finite number of critical points and m_p will denote the number of critical points with index p.
Let us recall the Propositions 3.6 and 3.7 of [HeNi]. They gather consequences of Simader's Theorem in [Sima] about the essential self-adjointness of non negative Schrödinger operators, of Persson's Lemma in [Per] about the localization of the essential spectrum and of the semiclassical analysis a la Witten in [Wit] leading to Morse inequalities. We refer the reader also to [CyFrKiSi][Hen] [Hel3] or [Zha] for the Witten approach to Morse inequalities in the boundaryless case and to [Mil1] and [Lau] for a topological presentation of Morse theory.

Proposition 4.3.7.
Under Assumption 4.3.6, there exist $h_0 > 0$, $c_0 > 0$ and $c_1 > 0$ such that the following properties are satisfied for any $h \in (0, h_0]$:
i) *The Witten Laplacian $\Delta_{f,h}$ as an unbounded operator on $L^2(\mathbb{R}^k; \Lambda T^*\mathbb{R}^k)$ is*

essentially self-adjoint on $\mathcal{C}_0^\infty(\mathbb{R}^k;\Lambda T^*\mathbb{R}^k)$ *.*
ii) *For any Borel subset E in $\mathbb{R}$, the identities*

$$\begin{aligned} & 1_E(\Delta_{f,h}^{(p+1)})d_{f,h}^{(p)}u &&= d_{f,h}^{(p)}1_E(\Delta_{f,h}^{(p)})u \\ \text{and} \quad & && \\ & 1_E(\Delta_{f,h}^{(p-1)})d_{f,h}^{(p-1),*}u &&= d_{f,h}^{(p-1),*}1_E(\Delta_{f,h}^{(p)})u \end{aligned} \tag{4.3.2}$$

hold for any u belonging to the form domain of $\Delta_{f,h}^{(p)}$.
In particular, if v is an eigenvector of $\Delta_{f,h}^{(p)}$ associated with the eigenvalue λ, then $d_{f,h}^{(p)}v$ (resp. $d_{f,h}^{(p-1),*}v$) belongs to the spectral subspace Ran $1_{\{\lambda\}}(\Delta_{f,h}^{(p+1)})$ *(resp.* Ran $1_{\{\lambda\}}(\Delta_{f,h}^{(p-1)})$*).*
iii) *The essential spectrum $\sigma_{ess}(\Delta_{f,h}^{(p)})$ is contained in $[c_1,+\infty)$.*
iv) *The range of $1_{[0,c_0h)}(\Delta_{f,h}^{(p)})$ has dimension m_p , for all $h\in(0,h_0]$.*

Proposition 4.3.8.
If the Morse function f satisfies Assumption 4.3.6 and admits a unique critical point at $x=0$ with index p_0 , so $m_p=\delta_{p,p_0}$, then there exist $h_0>0$ and $c_0>0$, such that the following properties hold for $h\in(0,h_0]$:
i) *For $p\neq p_0$, $\Delta_{f,h}^{(p)}\geq c_0h\mathrm{Id}$.*
ii) *If $\psi_{p_0}^h$ is a normalized eigenvector of the one dimensional spectral subspace* Ran $1_{[0,c_0h)}(\Delta_{f,h}^{(p_0)})$ *, it satisfies*

$$d_{f,h}\psi_{p_0}^h=0\,,\quad d_{f,h}^{(p_0-1),*}\psi_{p_0}^h=0\quad\text{and}\quad \Delta_{f,h}^{(p_0)}\psi_{p_0}^h=0\,,$$

so that Ran $1_{[0,c_0h)}(\Delta_{f,h}^{(p_0)})=$ Ker $\Delta_{f,h}^{(p_0)}$ *. Moreover*

$$\sigma(\Delta_{f,h}^{(p_0)})\setminus\{0\}\subset[c_0h,\infty)\,.$$

iii) *If $\chi\in\mathcal{C}_0^\infty(\mathbb{R}^k)$ satisfies $\chi=1$ in a neighborhood of 0, then there exists $C_\chi\geq1$, such that, for all $h\in(0,h_0/C_\chi)$, the inequality,*

$$(1-\chi)\Delta_{f,h}^{(p)}(1-\chi)\geq C_\chi^{-1}\left[1-\chi\right]^2\ ,$$

holds in the sense of quadratic form on $\Lambda^pH^1(\mathbb{R}^k)$.

The model half-space problem

We work here on $\mathbb{R}^n_-=\mathbb{R}^{n-1}\times(-\infty,0)$ with a Riemannian metric $\tilde g_0$. Assume furthermore that there are coordinates $x=(x',x_n)$ such that $\tilde g_0=\sum_{i,j=1}^n\tilde g_{ij}^0(x)dx_idx_j$ satisfies

$$\tilde g_{i,n}^0=\tilde g_{n,i}^0=0\quad\text{for } i<n \tag{4.3.3}$$

and

$$\forall x\in\overline{\mathbb{R}^n_-}\setminus K_1,\quad \partial_x\tilde g_{ij}^0(x)=0\,, \tag{4.3.4}$$

for some compact set $K_1 \subset \overline{\mathbb{R}^n_-}$.
In this paragraph, the coordinates (x', x_n) are fixed while different metrics on $\overline{\mathbb{R}^n_-}$ are considered. The notation $G(\cdot)$ will be used for the matrix valued map $x \mapsto G(x) = {}^tG(x) = (g_{ij}(x))_{i,j} \in GL^n(\mathbb{R})$, which is assumed to be a $\mathcal{C}^\infty$ function. According to the standard notation, the coefficients of $G(x)^{-1}$ are written $g^{ij}(x)$.

Consider also a function f which has a specific form in the same coordinates (x', x_n).

Assumption 4.3.9.
The function $f \in \mathcal{C}^\infty(\overline{\mathbb{R}^n_-})$ satisfies:

i) *The estimates $|\nabla f(x)| \geq C^{-1}$ and $|\partial_x^\alpha f(x)| \leq C_\alpha$ hold, for all $x \in \overline{\mathbb{R}^n_-}$ and all $\alpha \in \mathbb{N}^n$, $\alpha \neq 0$.*

ii) *The function f is the sum $f(x', x_n) = -\frac{1}{2} f_+(x_n) + \frac{1}{2} f_-(x')$. Moreover, there exists $C_1 > 0$ such that*

$$\forall x_n \in (-\infty, 0)\ , \quad C_1^{-1} \leq |\partial_{x_n} f_+(x_n)| \leq C_1\ ,$$

and f_- is a Morse function on $\mathbb{R}^{n-1}$ which satisfies Assumption 4.3.6 for the metric $\sum_{i,j=1}^{n-1} \tilde{g}^0_{ij}(x', 0) dx_i dx_j$ and admits a unique critical point at $x' = 0$ with index p_0.

The boundedness of $|\partial_x^\alpha f|$, $1 \leq |\alpha| \leq 2$, avoids any subtle questions about the domains.

Proposition 4.3.10.
Under Assumption 4.3.9-i), the unbounded operator $\Delta^N_{f,h}$ on $L^2(\mathbb{R}^n_-; \Lambda T^*\mathbb{R}^n_-)$, with domain

$$D(\Delta^N_{f,h}) = \left\{\omega \in \Lambda H^2(\mathbb{R}^n_-)\,, \quad \mathbf{n}\omega = 0\,, \quad \mathbf{n} d_{f,h}\omega = 0\right\}\ ,$$

is self-adjoint.
If E is any Borel subset of $\mathbb{R}$, the relations

$$\begin{aligned} & 1_E(\Delta^{N,(p+1)}_{f,h})\, d^{(p)}_{f,h} u &&= d^{(p)}_{f,h}\, 1_E(\Delta^{N,(p)}_{f,h}) u\ , \\ \text{and} \quad & 1_E(\Delta^{N,(p-1)}_{f,h})\, d^{(p-1),*}_{f,h} u &&= d^{(p-1),*}_{f,h}\, 1_E(\Delta^{N,(p)}_{f,h}) u\ , \end{aligned} \tag{4.3.5}$$

hold for any $u \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$.

Proof.
The uniform estimate on ∇f allows the same proof as for Proposition 4.2.8 and Corollary 4.2.10 (here $\mathcal{C}^\infty_{0,\mathbf{n}}$ denotes the space of $\mathcal{C}^\infty$ compactly supported functions in $\overline{\mathbb{R}^n_-}$ with a vanishing normal component on $\{x_n = 0\}$). ∎

We are looking for a result similar to Proposition 4.3.7 and Proposition 4.3.8 for the case with normal boundary condition on $\mathbb{R}^n_-$ (this result will be stated in Subsection 4.3.2). One difficulty here comes from the metric which, although diagonal in the coordinates (x', x_n), is not constant. The general case can be reduced to a simpler situation where $g_{ij}(x) = g_{ij}(x')$ with $g_{nn} = 1$ after several steps.
We need some notations.

Definition 4.3.11.
For a metric g which satisfies (4.3.4)*, the corresponding H^s-norm on the space $\Lambda^p H^s(\overline{\mathbb{R}^n_-})$ is denoted by $\| \ \|_{\Lambda^p H^s, g}$ and the notation $\| \ \|_{\Lambda^p H^s}$ is kept for the Euclidean metric $g_e = \sum_{i=1}^n dx_i^2$.*
Similarly, the quadratic form associated with $\Delta_{f,h}^{N,(p)}$ is written

$$\mathcal{D}_{g,f,h}(\omega) = \left\| d^*_{g,f,h}\omega \right\|^2_{\Lambda^{p-1}L^2,g} + \| d_{f,h}\omega\|^2_{\Lambda^{p+1}L^2,g} \ , \quad \forall \omega \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-) \ ,$$

where the codifferential $d^*_{g,f,h}$ also depends on g.

Remark 4.3.12. *The considered metrics satisfying* (4.3.4)*, the different (L^2, g)-norms are equivalent.*

The required accuracy while comparing the quadratic forms $\mathcal{D}_{g,f,h}$ needs some care.
We will work further with partitions of unity and the following proposition, similar to the standard IMS localization formula (see [CyFrKiSi]), but in the case with boundary, will be useful.

Proposition 4.3.13. *(IMS Localization Formula)*
For $W = \Omega$ or $W = \mathbb{R}^n_-$, consider $\{\chi_k\}_{1,\dots,N}$ a partition of unity of $\overline{W}$ (i.e. satisfying $\sum_{k=1}^N \chi_k^2 = 1$ on $\overline{W}$).
Let g and f be respectively a Riemannian metric and a $\mathcal{C}^\infty$ function (satisfying Assumption 4.3.9-i) in the case $\mathbb{R}^n_-$) on $\overline{W}$.
The following IMS localization formula is then valid:

$$\forall \omega \in \Lambda H^1_{0,\mathbf{n}} \, , \quad \mathcal{D}_{g,f,h}(\omega) = \sum_{k=1}^N \mathcal{D}_{g,f,h}(\chi_k \omega) - h^2 \, \| |\nabla \chi_k| \, \omega \|^2_{\Lambda L^2, g} \ . \qquad (4.3.6)$$

Proof. For clarity, we omit the dependence on g in the proof.
Recall, from $\sum_{k=1}^N \chi_k^2 = 1$, than for any $\eta \in \Lambda H^1$:

$$\sum_{k=1}^N \chi_k d\chi_k \wedge \eta = 0 \, , \quad \text{and by duality (4.2.5),} \quad \sum_{k=1}^N \chi_k \mathbf{i}_{\nabla \chi_k} \eta = 0 \, . \qquad (4.3.7)$$

Now, for any $\omega \in \Lambda H^1_{0,\mathbf{n}}$ and $k \in \{1, \ldots, N\}$,

$$\mathcal{D}_{f,h}(\chi_k \omega) = \|d_{f,h}(\chi_k \omega)\| + \left\|d^*_{f,h}(\chi_k \omega)\right\| .$$

From (4.2.6) and (4.2.7),

$$d_{f,h}(\chi_k \omega) = h d\chi_k \wedge \omega + \chi_k d_{f,h}\omega \quad \text{and} \quad d^*_{f,h}(\chi_k \omega) = h \mathbf{i}_{\nabla \chi_k}\omega + \chi_k d^*_{f,h}\omega .$$

Hence, from $\sum_{k=1}^N \chi_k^2 = 1$, (4.2.6), and (4.2.7), for any $\omega \in \Lambda H^1_{0,\mathbf{n}}$,

$$\sum_{k=1}^N \mathcal{D}_{f,h}(\chi_k \omega) = \mathcal{D}_{f,h}(\omega) + \sum_{k=1}^N h^2 \left(\langle d\chi_k \wedge \omega \,|\, d\chi_k \wedge \omega\rangle + \langle \mathbf{i}_{\nabla \chi_k}\omega \,|\, \mathbf{i}_{\nabla \chi_k}\omega\rangle\right)$$
$$+ \sum_{k=1}^N 2\,\mathrm{Re}\, \left(\langle h d\chi_k \wedge \omega \,|\, h\chi_k d\omega + \chi_k df \wedge \omega\rangle + \langle h \mathbf{i}_{\nabla \chi_k}\omega \,|\, h\chi_k d^*\omega + \chi_k \mathbf{i}_{\nabla f}\omega\rangle\right) .$$

Using (4.3.7),

$$\sum_{k=1}^N \mathcal{D}_{f,h}(\chi_k \omega) = \mathcal{D}_{f,h}(\omega) + h^2 \sum_{k=1}^N \left(\langle d\chi_k \wedge \omega \,|\, d\chi_k \wedge \omega\rangle + \langle \mathbf{i}_{\nabla \chi_k}\omega \,|\, \mathbf{i}_{\nabla \chi_k}\omega\rangle\right) .$$

At least, the identity

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (\mathbf{i}_X \beta)$$

implies

$$\langle d\chi_k \wedge \omega \,|\, d\chi_k \wedge \omega\rangle + \langle \mathbf{i}_{\nabla \chi_k}\omega \,|\, \mathbf{i}_{\nabla \chi_k}\omega\rangle = \langle \mathbf{i}_{\nabla \chi_k}(d\chi_k \wedge \omega) + d\chi_k \wedge (\mathbf{i}_{\nabla \chi_k}\omega) \,|\, \omega\rangle$$
$$= \langle (\mathbf{i}_{\nabla \chi_k} d\chi_k)\omega \,|\, \omega\rangle = \left\langle |\nabla \chi_k|^2 \omega \,|\, \omega \right\rangle ,$$

which proves the proposition. ∎

Let us give now two lemmas whose proofs are the same than the proofs of Lemmas 3.11 and 3.12 of [HeNi].
The first lemma provides a reduction to the case $\partial_{x_n} G = 0$ and the second allows us to consider again a simpler metric with $g_{nn} = 1$.

Lemma 4.3.14.
Let g_1 and g_2 be two metrics which satisfy (4.3.4) *and coincide on $\{x_n = 0\}$. Let f be a function satisfying Assumption 4.3.9. There exist constants $C_{12} \geq 1$ and $h_0 > 0$ such that the inequality,*

$$\mathcal{D}_{g_2,f,h}(\omega) \geq (1 - C_{12} h^{2/5}) \mathcal{D}_{g_1,f,h}(\omega) - C_{12} h^{7/5} \|\omega\|^2_{\Lambda^p L^2, g_1} , \tag{4.3.8}$$

holds for $\omega \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$, with $p \in \{0, \ldots, n\}$ and $h \in (0, h_0)$, as soon as $\mathrm{supp}\,\omega \subset \left\{x_n \geq -C_0 h^{2/5}\right\}$.

Lemma 4.3.15.
Let g_1 and g_2 be two conformal metrics (which satisfy (4.3.4)*) in the sense:*

$$g_2 = e^{\varphi(x)} g_1 \ .$$

Let f be a function satisfying Assumption 4.3.9. Then there exist constants $C_{12} \geq 1$ and $h_0 > 0$, such that the inequality,

$$\forall \omega \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-), \quad \mathcal{D}_{g_2,f,h}(\omega) \geq C_{12}^{-1} \mathcal{D}_{g_1,f,h}(\omega) - C_{12} h^2 \left\|\omega\right\|^2_{\Lambda^p L^2, g_1} \ , \qquad (4.3.9)$$

holds, for all $p \in \{0, \dots, n\}$ and all $h \in (0, h_0)$.

Small eigenvalues for the model half-space problem

Before giving the proof of Theorem 4.3.5, we state the main result for the model half-space problem which is similar to Proposition 4.3.7 and Proposition 4.3.8.

Proposition 4.3.16.
Assume that the metric $\tilde{g}_0$ satisfies (4.3.3) *and* (4.3.4) *and let f be a Morse function satisfying Assumption 4.3.9 for some $p_0 \in \{0, \dots, n\}$. Then there exist constants $h_0 > 0$, $c_0 > 0$ and $c_1 > 0$, such that the self-adjoint operator $\Delta^N_{f,h}$ satisfies the following properties for $h \in (0, h_0]$:*
i) *For $p \in \{0, \dots, n\}$, the essential spectrum $\sigma_{ess}(\Delta^{N,(p)}_{f,h})$ is contained in $[c_1, +\infty)$.*
ii) *For $p \in \{0, \dots, n\}$, the range of $1_{[0,c_0 h)}(\Delta^{N,(p)}_{f,h})$ has dimension*

$$\begin{cases} \delta_{p,p_0} & \text{if} \quad \partial_{x_n} f(0) = -\frac{1}{2}\partial_{x_n} f_+(0) < 0 \ , \\ 0 & \text{if} \quad \partial_{x_n} f(0) = -\frac{1}{2}\partial_{x_n} f_+(0) > 0 \ . \end{cases}$$

iii) *In the first case,*

$$\text{Ran } 1_{[0,c_0 h)}(\Delta^{N,(p_0)}_{f,h}) = \text{ Ker } \Delta^{N,(p_0)}_{f,h} = \mathbb{C}\varphi^h \ ,$$

where

$$\|\varphi^h - \left(e^{f_+(x_n)/2h}\right) \psi^h_{p_0}\|_{\Lambda^p L^2} = \mathcal{O}(h^{1/10}) \ ,$$

and $\psi^h_{p_0}$ belongs to the kernel of a $(n-1)$-dimensional Witten Laplacian $\Delta^{(p_0)}_{g',f_-/2,h}$ in a metric g', which is conformal to $\tilde{g}'_0 = \sum_{i,j=1}^{n-1} \tilde{g}^0_{ij}(x',0) dx_i dx_j$ on $\mathbb{R}^{n-1}$.
iv) *For any $\chi \in \mathcal{C}^\infty_0(\overline{\mathbb{R}^n_-})$ such that $\chi = 1$ in a neighborhood of 0, there exists $C_\chi > 0$ such that the lower bounds*

$$(1-\chi)\Delta^{N,(p)}_{f,h}(1-\chi) \geq C_\chi^{-1} \left[1-\chi\right]^2 , \ 0 \leq p \leq n \ ,$$

hold, for any $h \in (0, h_0/C_\chi)$, in the sense of quadratic forms on $\Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$.

Remark 4.3.17.
This proposition is an adaptation of Proposition 3.13 of [HeNi] in the case with normal boundary conditions: we have mainly replaced $f_+(x_n)$ by $-f_+(x_n)$ and p_0+1 by p_0 and the proof is similar.

Proof.
The clue of this result is an accurate lower bound for the quadratic form $\mathcal{D}_{\tilde{g}_0,f,h}(\eta)$, when evaluated for η such that $\operatorname{supp}\eta \subset \left\{x_n \geq -C_0 h^{2/5}\right\}$. By Lemmas 4.3.14 and 4.3.15, one can find a metric g, which satisfies (4.3.3) and (4.3.4), with $G(x) = G(x')$ independent of the x_n-coordinate, $g_{nn} = 1$ and a constant $C > 1$ such that

$$\mathcal{D}_{\tilde{g}_0,f,h}(\eta) \geq C^{-1}\mathcal{D}_{g,f,h}(\eta) - Ch^{7/5}\left\|\eta\right\|^2_{\Lambda L^2,g} . \tag{4.3.10}$$

Take two cut-off functions $\tilde{\chi}_i \in \mathcal{C}^\infty(\mathbb{R})$, such that $\tilde{\chi}_1 \in \mathcal{C}^\infty_0(\mathbb{R})$, $\tilde{\chi}_1 = 1$ in a neighborhood of 0 such that $\tilde{\chi}_1^2 + \tilde{\chi}_2^2 = 1$.
By the IMS localization formula (4.3.6), for any $\omega \in \Lambda H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$,

$$\mathcal{D}_{\tilde{g}_0,f,h}(\omega) \geq \mathcal{D}_{\tilde{g}_0,f,h}(\tilde{\chi}_1(h^{-2/5}x_n)\omega) + \mathcal{D}_{\tilde{g}_0,f,h}(\tilde{\chi}_2(h^{-2/5}x_n)\omega) - Ch^{6/5}\left\|\omega\right\|^2_{\Lambda L^2,\tilde{g}_0} .$$

By (4.2.18), since $\left|\nabla f(x)\right|^2 \geq C^{-1}$ on $\mathbb{R}^n_-$, the second term of the r.h.s. is bounded from below by a constant times $\left\|\tilde{\chi}_2(h^{-2/5}x_n))\omega\right\|^2_{\Lambda L^2,\tilde{g}_0}$ and we get:

$$\mathcal{D}_{\tilde{g}_0,f,h}(\omega) \geq \mathcal{D}_{\tilde{g}_0,f,h}(\tilde{\chi}_1(h^{-2/5}x_n)\omega) - Ch^{6/5}\left\|\tilde{\chi}_1(h^{-2/5}x_n)\omega\right\|^2_{\Lambda L^2,\tilde{g}_0} + \frac{C^{-1}}{2}\left\|\tilde{\chi}_2(h^{-2/5}x_n)\omega\right\|^2_{\Lambda L^2,\tilde{g}_0} .$$

Finally after changing the constant $C \geq 1$, the inequality (4.3.10) yields

$$\mathcal{D}_{\tilde{g}_0,f,h}(\omega) \geq C^{-1}\mathcal{D}_{g,f,h}(\tilde{\chi}_1(h^{-2/5}x_n)\omega) - Ch^{6/5}\left\|\tilde{\chi}_1(h^{-2/5}x_n)\omega\right\|^2 + C^{-1}\left\|\tilde{\chi}_2(h^{-2/5}x_n)\omega\right\|^2 , \tag{4.3.11}$$

where the L^2-norms in the r.h.s. can be computed with the metric g or $\tilde{g}_0$ while possibly adapting the constant C, owing to Remark 4.3.12. Here and in the sequel, we omit the subscript $(\Lambda L^2, g)$ for L^2-norms.
Now the problem is reduced to the analysis of $\mathcal{D}_{g,f,h}$ with the metric g. The product structure of the metric g allows an explicit analysis of the spectrum.

(a) The case $n = 1$.

We have $x = x_n \in \mathbb{R}_-$, $f(x) = -\frac{1}{2} f_+(x_n)$. Here the metric is $g = dx_n^2$. We keep the reference to the index n for the later application.
The spaces $\Lambda^0 H^1_{0,\mathbf{n}}(\mathbb{R}_-)$ and $\Lambda^1 H^1_{0,\mathbf{n}}(\mathbb{R}_-)$ are respectively $H^1(\mathbb{R}_-)$ and $\left\{\beta(x_n)\ dx_n\ ,\ \beta \in H^1_0(\mathbb{R}_-)\right\}$.

By identity (4.2.18), for any 1-form $\beta\ dx_n$ with $\beta \in H^1_0(\mathbb{R}_-)$:

$$\mathcal{D}_{g,-f_+/2,h}(\beta\ dx_n) = h^2 \left\|\partial_{x_n}\beta\right\|^2 + \frac{1}{4}\left\|\partial_{x_n} f_+\beta\right\|^2 - \frac{h}{2}\langle\partial^2_{x_n} f_+(x_n)\beta \mid \beta\rangle\ . \quad (4.3.12)$$

From (4.3.12), we get:

$$\mathcal{D}_{g,-f_+/2,h}(\beta\ dx_n) \geq (C^{-2} - hC)\left\|\beta\right\|^2\ ,$$

and deduce that there exist $c_1(\partial_{x_n} f_+, \partial^2_{x_n} f_+) = c_1 > 0$ and $h_0 > 0$ such that, for all $h \in (0, h_0]$,

$$\Delta^{N,(1)}_{g,-f_+/2,h} \geq c_1 \mathrm{Id}\ . \quad (4.3.13)$$

Again by identity (4.2.18), we have for any 0-form $\alpha \in H^1(\mathbb{R}_-)$:

$$\mathcal{D}_{g,-f_+/2,h}(\alpha) = h^2 \left\|\partial_{x_n}\alpha\right\|^2 + \frac{1}{4}\left\|\partial_{x_n} f_+\,\alpha\right\|^2 + \frac{h}{2}\langle\partial^2_{x_n} f_+(x_n)\alpha \mid \alpha\rangle - \frac{h}{2}\partial_{x_n} f_+(0)\left|\alpha(0)\right|^2\ , \quad (4.3.14)$$

and there are two subcases:

(a1) Subcase $\partial_{x_n} f_+(0) < 0$:

In this case, identity (4.3.14) implies:

$$\forall \alpha \in \Lambda^0 H^1_{0,\mathbf{n}},\quad \mathcal{D}_{g,-f_+/2,h}(\alpha) \geq (C^{-2} - hC)\left\|\alpha\right\|^2\ ,$$

which provides the existence of $c_1(\partial_{x_n} f_+, \partial^2_{x_n} f_+) = c_1 > 0$ and $h_0 > 0$ such that:

$$\Delta^{N,(0)}_{-f_+/2,h} \geq c_1 \mathrm{Id}\ ,\ \forall h \in (0, h_0]\ . \quad (4.3.15)$$

(a2) Subcase $\partial_{x_n} f_+(0) > 0$:

If $\Delta^{N,(0)}_{-f_+/2,h}(\alpha) = \lambda_h \alpha$, with $\lambda_h < c_1$, we have by Proposition 4.3.10(4.3.5):

$$\Delta^{N,(1)}_{-f_+/2,h}(d_{-f_+/2,h}\alpha) = \lambda_h d_{-f_+/2,h}\alpha\,,$$

which implies, by (4.2.6):

$$d_{-f_+/2,h}\alpha = h\partial_{x_n}\alpha - \frac{1}{2}\left(\partial_{x_n} f_+\right)\alpha = 0\ .$$

Hence:

$$\alpha(x_n) = C\, e^{f_+(x_n)/2h}\ .$$

The 0-form $e^{f_+(x_n)/2h}$ belongs to Ker $(\Delta^{N,(0)}_{-f_+/2,h})$, so $\lambda_h = 0$.

(b) The case $n > 1$.

First note that any $\omega \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$ is a sum

$$\omega = \sum_{\#I=p-1} \alpha_I(x) dx'^I \wedge dx_n + \sum_{\#J=p} \beta_J(x) dx'^J =: \alpha \wedge dx_n + \beta\,,$$

with $\alpha_I, \beta_J \in H^1(\mathbb{R}^n_-)$, $\alpha_I(x', 0) = 0$, while $dx'^I = dx'_{i_1} \wedge \cdots \wedge dx'_{i_{\#I}}$, $I = \{i_1 < \ldots < i_{\#I}\} \subset \{1, \ldots, n-1\}$ and $J = \{j_1 < \ldots < j_{\#J}\} \subset \{1, \ldots, n-1\}$. If in addition $\omega \in \Lambda^p H^2(\mathbb{R}^n_-)$, the condition $\mathbf{n} d\omega = 0$ reads, with the metric g, $\partial_{x_n} \beta_J(x', 0) = 0$.
Secondly, we remind the reader that with the product metric g the Riemannian connection, the Riemann tensor and therefore the Hodge Laplacian, owing to the Weitzenböck formula, split like direct sums:

$$\begin{aligned}
\nabla_X Y &= \nabla^n_{X_n} Y_n + \nabla'_{X'} Y'\,,\\
Riem(x, y, z, t) &= Riem^n(x_n, y_n, z_n, t_n) + Riem'(x', y', z', t')\,,\\
R_{(4)} &= \sum_{ijkl} Riem_{ijkl}(dx_i \wedge) \circ \mathbf{i}_{\nabla x_j} \circ (dx_k \wedge) \circ \mathbf{i}_{\nabla x_\ell} = R^n_{(4)} + R'_{(4)}\,,\\
(d + d^*)^2 &= (d_{x_n} + d^*_{x_n})^2 + (d_{x'} + d^*_{x'})^2\,.
\end{aligned}$$

We refer the reader to [GaHuLa] (p. 110 and p. 70) for details and more general statements.

Thirdly, the decomposition $f(x) = -\frac{1}{2} f_+(x_n) + \frac{1}{2} f_-(x')$ with the product metric g gives

$$\begin{aligned}
|\nabla f|^2 &= |\nabla_{x_n} f|^2 + |\nabla_{x'} f|^2\\
\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f} &= -\frac{1}{2}\left(\mathcal{L}_{\nabla f_+} + \mathcal{L}^*_{\nabla f_+}\right) + \frac{1}{2}\left(\mathcal{L}_{\nabla f_-} + \mathcal{L}^*_{\nabla f_-}\right)\,.
\end{aligned}$$

For $\omega = \alpha \wedge dx_n + \beta \in D(\Delta^N_{f,h})$ (with the product metric g), we have

$$\mathcal{D}_{g,f,h}(\omega) = \langle \omega \,|\, \Delta_{f,h} \omega \rangle = \left\langle \omega \,|\, \Delta^n_{-f_+/2,h} \omega \right\rangle + \left\langle \omega \,|\, \Delta'_{f_-/2,h} \omega \right\rangle\,.$$

Since the two operators $\Delta^n_{-f_+/2,h}$ (acting only in the variable x_n) and $\Delta'_{f_-/2,h}$ (acting only in the variable x') preserve the partial degree in dx_n, we get

$$\begin{aligned}
\mathcal{D}_{g,f,h}(\omega) = &\left\langle \alpha \wedge dx_n \,|\, \Delta^n_{-f_+/2,h}(\alpha \wedge dx_n) \right\rangle + \left\langle \beta \,|\, \Delta^n_{-f_+/2,h} \beta \right\rangle\\
&+ \left\langle \alpha \wedge dx_n \,|\, \Delta'_{f_-/2,h}(\alpha \wedge dx_n) \right\rangle + \left\langle \beta \,|\, \Delta'_{f_-/2,h} \beta \right\rangle \qquad (4.3.16)
\end{aligned}$$

Hence the variables (x', x_n) can be separated. The equivalence between the norms $\left\|\sum_J \gamma_J(x')\, dx'^J\right\|$ and $\sum_J \|\gamma_J(x')\|$ on $\Lambda^p T^* \mathbb{R}^n_-$, where

$J = \{j_1 < \ldots < j_{\#J}\} \subset \{1, \ldots, n-1\}$, leads to [2]:

$$\begin{aligned} \mathcal{D}_{g,f,h}(\omega) &\geq \\ \frac{1}{c}\int_{\mathbb{R}^{n-1}} \Big[\sum_{\#I=p-1} \mathcal{D}^n_{-f_+/2,h}(\alpha_I(x',.)\ dx_n) &+ \sum_{\#J=p} \mathcal{D}^n_{-f_+/2,h}(\beta_J(x',.)) \Big]\ d\lambda(x') \\ &+ \int_{-\infty}^0 \mathcal{D}'_{f_-/2,h}(\alpha(.,x_n)) + \mathcal{D}'_{f_-/2,h}(\beta(.,x_n))\ dx_n\ , \end{aligned} \quad (4.3.17)$$

where we used the notations $\mathcal{D}'_{f_-/2,h}$ for the quadratic form of the Witten Laplacian on $\mathbb{R}^{n-1}$ and $\mathcal{D}^n_{-f_+/2,h}$ for the quadratic form of the 1-dimensional Witten Laplacian on $\mathbb{R}_-$ with boundary conditions. The measure $d\lambda(x')$ simply equals $(\det G(x'))^{1/2}\ dx'$. The absence of $\alpha - \beta$ cross product term is due to (4.3.16).

Again there are two subcases.

(b1) Subcase $\partial_{x_n} f_+(0) < 0$:

The analysis of the one dimensional problem implies the existence of $c_1 > 0$ independent of x' such that:

$$\mathcal{D}^n_{-f_+/2,h}(\alpha_I(x',.)\ dx_n) \geq c_1 \left\|\alpha_I(x',.)\right\|^2$$

and

$$\mathcal{D}^n_{-f_+/2,h}(\beta_J(x',.)) \geq c_1 \left\|\beta_J(x',.)\right\|^2 .$$

Hence there exists $c_2 > 0$ such that:

$$\forall \omega \in \Lambda^p H^1_{0,\mathbf{n}}\,, \quad \mathcal{D}_{g,f,h}(\omega) \geq c_2 \left\|\omega\right\|^2$$

and

$$\Delta^{N,(p)}_{f,h} \geq c_2 \mathrm{Id}\ ,\ \forall p \in \{0, \ldots, n\}\ .$$

(b2) Subcase $\partial_{x_n} f_+(0) > 0$:

Then there exists $c_1 > 0$ such that

$$\mathcal{D}_{g,f,h}(\omega) \geq \frac{1}{c}\int_{\mathbb{R}^{n-1}} \sum_{\#J=p} \mathcal{D}^n_{-f_+/2,h}(\beta_J(x',.))\ d\lambda(x') + \int_{-\infty}^0 \mathcal{D}'_{f_-/2,h}(\beta(.,x_n)) + c_1 \left\|\alpha\right\|^2\ . \quad (4.3.18)$$

2. In [HeNi], at this level of the proof, one should read "$\mathcal{D}_{\tilde{g},f,h}(\omega) \geq \frac{1}{c}\int_{\mathbb{R}^{n-1}} \cdots$" instead of "$\mathcal{D}_{\tilde{g},f,h}(\omega)$ equals $\int_{\mathbb{R}^{n-1}} \cdots$" accordingly to 2.

If ω is a p-form with $p \neq p_0$ ($\deg \beta = \deg \omega$), the lower bound

$$\mathcal{D}'_{f_-/2,h}(\beta) \geq C_1^{-1} h \left\|\beta\right\|^2 \ ,$$

which was given in Proposition 4.3.8, yields:

$$\mathcal{D}_{g,f,h}(\omega) \geq C^{-1} h \left\|\omega\right\|^2 \ ,$$

while the equality $\mathcal{D}_{g,f,h}(\omega) = 0$ implies that $p = p_0$ and that $\omega = c\left(e^{f_+(x_n)/2h}\right) \psi^h_{p_0}$, where ψ_{p_0} belongs to the kernel of the $(n-1)$-dimensional Witten Laplacian associated with the metric

$$g' = \sum_{i,j=1}^{n-1} g_{i,j}(x',0) dx_i dx_i \ .$$

We have now all the ingredients to check every statement for the metric $\tilde{g}_0$. We focus on the subcase $\partial_{x_n} f_+(0) > 0$, which covers all possibilities.

Statements i) and iv)
Statement i) is a consequence of iv) together with Persson's Lemma in [Per]. It is sufficient to check that, for all $R > 0$, there exists $c_R > 0$, such that, for all $\omega \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$ supported in $\{\min(|x'|, |x_n|) > R\}$, one has

$$\mathcal{D}_{\tilde{g}_0,f,h}(\omega) \geq c_R \left\|\omega\right\|^2 \ .$$

The inequalities (4.3.11) and (4.3.18), together with the estimate

$$\mathcal{D}'_{f_-/2,h}(\beta(\cdot,x_n)) \geq c'_R \left\|\beta(\cdot,x_n)\right\|^2 \qquad \text{if } \operatorname{supp}\omega \subset \{|x'| > R\} \ ,$$

provided by Proposition 4.3.8-iii), yield the result.

Statements ii) and iii)
If $p \neq p_0$ the inequalities (4.3.11), (4.3.18) and the inequality

$$\mathcal{D}'_{f_-/2,h}(\beta(.,x_n)) \geq C^{-1} h \left\|\beta(.,x_n)\right\|^2 \ ,$$

imply

$$\mathcal{D}_{\tilde{g}_0,f,h}(\omega) \geq c_0 h \left\|\omega\right\|^2 \ ,$$

and

$$\Delta^{N,(p)}_{f,h} \geq c_0 h \,\mathrm{Id} \ . \qquad (4.3.19)$$

If $p = p_0$, by Proposition 4.3.10, the only possibility for $\lambda_h \in [0, c_0 h)$ to be an eigenvalue of $\Delta^{N,(p_0)}_{f,h}$ is $\lambda_h = 0$.

Assume indeed $\Delta_{f,h}^{N,(p_0)} u_h = \lambda_h u_h$ with $\lambda_h \in [0, c_0 h)$ and $\|u_h\| = 1$.
By Proposition 4.3.10(4.3.5) and (4.3.19), $d_{f,h}^{(p_0)} u_h = d_{f,h}^{(p_0-1),*} u_h = 0$. Thus:

$$\lambda_h = \left\langle \Delta_{f,h}^{N,(p_0)} u_h \,|\, u_h \right\rangle = \mathcal{D}_{\tilde{g}_0,f,h}(u_h) = 0\,.$$

When the metric is g, the corresponding spectral subspace is one dimensional and equals $\mathbb{C}\left(e^{f_+(x_n)/2h}\right)\psi_{p_0}^h$.
For the metric $\tilde{g}_0$, the equation $\Delta_{\tilde{g}_0,f,h}^{N,(p_0)}\omega = 0$ with $\|\omega\| = 1$ (which implies $\mathcal{D}_{\tilde{g}_0,f,h}(\omega) = 0$) and the inequality (4.3.11) lead to:

$$C^2 h^{6/5} \left\|\tilde{\chi}_1(h^{-2/5}x_n)\omega\right\|^2 \geq \mathcal{D}_{g,f,h}(\tilde{\chi}_1(h^{-2/5}x_n)\omega) + \left\|\tilde{\chi}_2(h^{-2/5}x_n)\omega\right\|^2 \,.$$

Without the last term, Lemma 4.2.11 implies:

$$\mathrm{dist}_{\,L^2}(\tilde{\chi}_1(h^{-2/5}x_n)\omega, \mathbb{C}\left(e^{f_+(x_n)/2h}\right)\psi_{p_0}^h) \leq C h^{1/10}\,.$$

The upper bound of the last term,

$$\left\|\tilde{\chi}_2(h^{-2/5}x_n)\omega\right\|^2 \leq C^2 h^{6/5}\,,$$

implies:

$$\mathrm{dist}_{\,L^2}(\omega, \mathbb{C}\left(e^{f_+(x_n)/2h}\right)\psi_{p_0}^h) = \mathcal{O}(h^{1/10})\,.$$

It remains to check that $\mathrm{Ker}\,\Delta_{f,h}^{N,(p_0)}$ is not reduced to $\{0\}$. The statements of Lemma 4.3.14 and Lemma 4.3.15 are symmetric with respect to the choice of the metric. Hence the reverse inequality of (4.3.11) (with exchange of g and $\tilde{g}_0$),

$$\mathcal{D}_{g,f,h}(\omega) \geq C^{-1}\mathcal{D}_{\tilde{g}_0,f,h}(\tilde{\chi}_1(h^{-2/5}x_n)\omega) - C h^{6/5}\left\|\tilde{\chi}_1(h^{-2/5}x_n)\omega\right\|^2 + C^{-1}\left\|\tilde{\chi}_2(h^{-2/5}x_n)\omega\right\|^2\,, \quad (4.3.20)$$

also holds for any $\omega \in \Lambda H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$. We apply it with $\omega = \left(e^{f_+(x_n)/2h}\right)\psi_{p_0}^h$ and this leads to:

$$\mathcal{D}_{\tilde{g}_0,f,h}(\tilde{\chi}_1(h^{-2/5}x_n)\omega^h) \leq C h^{6/5}\left\|\tilde{\chi}_1(h^{-2/5}x_n)\omega\right\|^2\,.$$

The Min-Max principle then says that $\Delta_{f,h}^{N,(p_0)}$ admits an eigenvalue smaller than $Ch^{6/5}$. It has to be 0 due to the above argument. ∎

4.3.3 Proof of Theorem 4.3.5

We end here the proof of Theorem 4.3.5 by introducing, after a partition of unity, convenient coordinates which allow the comparison with the model half-space problem.
That proof is almost the same as the proof of the corresponding theorem in [HeNi], but we recall it for completeness.

Proof of Theorem 4.3.5.
Let $\{U_k\,,\, 1\leq k\leq K\}$ denote the union of the critical points of f and $f|_\Omega$. Consider a partition of unity of $\overline{\Omega}$, $\sum_{k=1}^N \chi_k^2=1$, such that the $\mathcal{C}_0^\infty(\overline{\Omega})$ function χ_k identically equals 1 in a neighborhood of U_k when $1\leq k\leq K$. A refinement of this partition of unity will be specified later by the local construction of adapted coordinates.
We recall that the operator $\Delta_{f,h}^N$ is the Friedrichs extension associated with the quadratic form:

$$\mathcal{D}_{g_0,f,h}(\omega)=\|d_{f,h}\omega\|^2_{\Lambda L^2,g_0}+\left\|d^{*,g_0}_{f,h}\omega\right\|^2_{\Lambda L^2,g_0}\ ,$$

on $\Lambda H^1_{0,\mathbf{n}}(\Omega)$. The IMS localization formula (4.3.6) gives, for any $\omega\in\Lambda H^1_{0,\mathbf{n}}$,

$$\mathcal{D}_{g_0,f,h}(\omega)=\sum_{k=1}^N\mathcal{D}_{g_0,f,h}(\chi_k\omega)-h^2\,\||\nabla\chi_k|\,\omega\|^2_{\Lambda L^2,g_0}\ .$$

If $\operatorname{supp}\chi_k$ **does not meet the boundary,** the term $\mathcal{D}_{g_0,f,h}(\chi_k\omega)$ behaves like in the boundaryless case (see [HeKlNi] for details):

– If $k>K$, then we have

$$\forall\omega\in\Lambda H^1\ ,\qquad \mathcal{D}_{g_0,f,h}(\chi_k\omega)\geq C^{-1}\,\|\chi_k\omega\|^2_{\Lambda L^2,g_0}\ .$$

– If $k\leq K$ and U_k is a critical point of f with index $p_k\neq p$, then

$$\forall\omega\in\Lambda H^1\ ,\qquad \mathcal{D}_{g_0,f,h}(\chi_k\omega)\geq C^{-1}h\,\|\chi_k\omega\|^2_{\Lambda L^2,g_0}\ .$$

– If $k\leq K$ and U_k is a critical point of f with index $p_k=p$, then there exists a fixed 1-dimensional space $F_k^{(p)}$ (determined by $\operatorname{Hess} f(U_k)$) such that,

$$\forall\omega\in\Lambda H^1\ ,\qquad \mathcal{D}_{g_0,f,h}(\chi_k\omega)\leq C^{-1}h^{6/5}\,\|\chi_k\omega\|^2_{\Lambda^pL^2,g_0}$$

implies

$$\forall\omega\in\Lambda H^1\ ,\qquad \operatorname{dist}\,(\chi_k\omega,F_k^{(p)})\leq Ch^{1/10}\,\|\omega\|_{\Lambda^pL^2,g_0}\ .$$

Again like in the proof of Proposition 4.3.16-iii), this last statement refers to Lemma 4.2.11 at the level of quadratic forms.

Consider now the case when $\operatorname{supp}\chi_k \cap \partial\Omega \neq \emptyset$, with the support of χ_k centered around a point $U_0 \in \partial\Omega$. There are two cases: U_0 is a critical point of $f|_{\partial\Omega}$ with $\frac{\partial f}{\partial n}(U_0) < 0$ which is equivalent to $-\frac{\partial f}{\partial n}(U_0) = |\nabla f(U_0)|$ or U_0 is not a critical points of $f|_{\partial\Omega}$ with $\frac{\partial f}{\partial n} < 0$ which is equivalent to $(-\frac{\partial f}{\partial n})(U_0) < |\nabla f(U_0)|$. Indeed, U_0 is either a critical point of $f|_{\partial\Omega}$ with $\frac{\partial f}{\partial n}(U_0) > 0$, i.e. $\frac{\partial f}{\partial n}(U_0) = |\nabla f(U_0)|$ or U_0 is not a critical point of $f|_{\partial\Omega}$, i.e. $\left|\frac{\partial f}{\partial n}(U_0)\right| < |\nabla f(U_0)|$.

Case 1) $(-\frac{\partial f}{\partial n})(U_0) < |\nabla f(U_0)|$.
Then the cut-off χ_k is chosen so that, in a neighborhood $\mathcal{V}$ of $\operatorname{supp}\chi_k$,

$$\forall x \in \mathcal{V} \cap \partial\Omega, \quad (-\frac{\partial f}{\partial n})(x) < (1-\delta)\,|\nabla f(x)|\ ,$$

for some $\delta > 0$. Locally it is possible to construct a function $\hat{f}$ such that $-\partial_n \hat{f} = \left|\nabla \hat{f}\right|$ in $\mathcal{V}\cap\partial\Omega$ and $\left|\nabla \hat{f}\right| = |\nabla f|$ in $\mathcal{V}$. By setting $\tilde{\omega} = \chi_k \omega$ for $\omega \in \Lambda H^1_{0,\mathbf{n}}$, the Green formula (4.2.18) and the inequality $\mathcal{D}_{g_0,\hat{f},h}(\tilde{\omega}) \geq 0$ imply ($\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}$ being a tensor)

$$\begin{aligned}
&-h\int_{\partial\Omega} \langle \tilde{\omega}\,|\,\tilde{\omega}\rangle_{\Lambda^p T^*_\sigma\Omega}\left(\frac{\partial f}{\partial n}\right)(\sigma)\ d\sigma \leq -(1-\delta)h\int_{\partial\Omega} \langle \tilde{\omega}\,|\,\tilde{\omega}\rangle_{\Lambda^p T^*_\sigma\Omega}\left(\frac{\partial \hat{f}}{\partial n}\right)(\sigma)\ d\sigma\\
&\leq (1-\delta)\left[h^2\left\|d\tilde{\omega}\right\|^2_{\Lambda^{p+1}L^2} + h^2\left\|d^*\tilde{\omega}\right\|^2_{\Lambda^{p-1}L^2} + \left\||\nabla f|\,\tilde{\omega}\right\|^2_{\Lambda^p L^2} + C_1 h\left\|\tilde{\omega}\right\|^2_{\Lambda^p L^2}\right].
\end{aligned}$$

– If $k > K$,

$$\begin{aligned}
\forall \omega \in \Lambda H^1_{0,\mathbf{n}}\ , \qquad \mathcal{D}_{g_0,f,h}(\chi_k\,\omega) = \mathcal{D}_{g_0,f,h}(\tilde{\omega}) \quad &\geq \quad \frac{\delta}{2}\left\||\nabla f|\,\tilde{\omega}\right\|^2_{\Lambda^p L^2}\\
&\geq \quad C_{\mathcal{V}}^{-1}\left\|\chi_k\omega\right\|^2_{\Lambda^p L^2}\ .
\end{aligned}$$

Case 2) $-\frac{\partial f}{\partial n}(U_0) = |\nabla f(U_0)|$.
In this case we will conclude by applying Proposition 4.3.16. We recall that $U_0 \in \partial\Omega$ is a critical point of $f|_{\partial\Omega}$ with $\frac{\partial f}{\partial n}(U_0) < 0$ and with index p_0. Around U_0, we introduce adapted local coordinates, denoted by $\overline{x} = (\overline{x}', \overline{x}_n)$. This coordinate system is provided by Lemma 4.3.18 below, applied with $f_1 = f$ and $\alpha = f|_{\partial\Omega\cap\mathcal{V}_0}$. Then the function Φ_+ of Lemma 4.3.18 is nothing but f and has the form $f(\overline{x}) = -\overline{x}_n + \frac{1}{2}f_-(\overline{x}')$. Moreover, $\overline{\Omega}$ corresponds locally to $\{\overline{x}_n \leq 0\}$.
In order to apply Proposition 4.3.16, it remains to check that the function f can be extended to $\mathbb{R}^n_-$, so that it satisfies Assumption 4.3.9 where U_0 is a critical point of $f|_{\partial\Omega}$.
We recall that we have not specified the choice of $\overline{x}'$ in the boundary. The function $f|_{\partial\Omega\cap\mathcal{V}_0}$ being a Morse function, we can choose in a small neighborhood $\mathcal{V}_0' \subset \partial\Omega$ of $U_0 = (0,\ldots,0)$ Morse coordinates $\overline{x}' = (\overline{x}_1,\ldots,\overline{x}_{n-1})$ for f_- which are normal at U_0 for the metric $\sum_{i,j<n} g_{ij}(\overline{x}',0)d\overline{x}_i d\overline{x}_j$. With these coordinates, f has the

form, in a small neighborhood $\mathcal{V}_0''$ of 0:

$$f(\overline{x}) = -\overline{x}_n + \sum_{j=1}^{n-1} \lambda_j \overline{x}_j^2 + f(U_0)\,. \tag{4.3.21}$$

We choose χ_k such that $\operatorname{supp}\chi_k \subset \mathcal{V}_0''$.
Choosing a cut-off $\chi^{n-1} \in \mathcal{C}_0^\infty(\mathbb{R}^{n-1})$, $\chi^{n-1} = 1$ near $\operatorname{supp}\chi_k \cap \partial\Omega$, f is extended to $\mathbb{R}^n_-$ by:

$$\tilde{f}(\overline{x}) = -\overline{x}_n + \left[\chi^{n-1}(\overline{x}') + \frac{1-\chi^{n-1}(\overline{x}')}{|\overline{x}'|}\right]\left[\sum_{j=1}^{n-1}\lambda_j\overline{x}_j^2\right] + f(U_0)\,. \tag{4.3.22}$$

Moreover, choosing another cut-off $\chi^n \in \mathcal{C}_0^\infty(\overline{\mathbb{R}^n_-})$, $\chi^n = 1$ near $\operatorname{supp}\chi_k$, we extend g_0 to $\mathbb{R}^n_-$ by:

$$\tilde{g} = \chi^n\, g_0 + (1-\chi^n) g_e\,, \tag{4.3.23}$$

where g_e is the Euclidian metric on $\mathbb{R}^n_-$.
With these coordinates, the quantity $\mathcal{D}_{\tilde{g},\tilde{f},h}(\chi_k\omega) = \mathcal{D}_{g_0,f,h}(\chi_k\omega)$ attains the form discussed in Proposition 4.3.16.
We can now discuss the lower bound of $\mathcal{D}_{\tilde{g},\tilde{f},h}(\chi_k\omega)$, depending on the localization by the cut-off χ_k, such that $\operatorname{supp}\chi_k \cap \partial\Omega \neq \emptyset$.

- If $k \leq K$, the origin of the coordinate system is $U_0 = U_k$. If U_k is not a critical point of $f|_{\partial\Omega}$ with index $p_k = p$ and $\frac{\partial f}{\partial n}(U_k) < 0$, then

$$\forall \omega \in \Lambda^p H^1_{0,\mathbf{n}}\,, \qquad \mathcal{D}_{\tilde{g},\tilde{f},h}(\chi_k\omega) \geq C^{-1} h \, \|\chi_k\omega\|^2_{\Lambda L^2,g}\,.$$

- If $k \leq K$ and U_k is a critical point of $f|_{\partial\Omega}$ with index $p_k = p$ and $\frac{\partial f}{\partial n}(U_k) < 0$, then according to Proposition 4.3.16-iii) there exists a fixed 1-dimensional space $F_k^{(p)}$ such that the inequality,

$$\forall \omega \in \Lambda^p H^1_{0,\mathbf{n}}\,, \qquad \mathcal{D}_{\tilde{g},\tilde{f},h}(\chi_k\omega) \leq C^{-1} h^{6/5} \, \|\chi_k\omega\|^2_{\Lambda^p L^2,g}$$

  implies:

$$\operatorname{dist}\,(\chi_k\omega, F_k^{(p)}) \leq C\, h^{1/10}\, \|\chi_k\omega\|_{\Lambda^p L^2,g}\,.$$

We now introduce the set A_p of indices k, $1 \leq k \leq K$, such that

- either U_k is a critical point of f with index p,
- or U_k is a critical point of $f|_{\partial\Omega}$ with index p such that $\frac{\partial f}{\partial n}(U_k) < 0$.

For $\omega \in \Lambda^p H^1_{0,\mathbf{n}}(\Omega)$ with $\|\omega\|_{\Lambda^p L^2,g} = 1$, we get

$$\left(\mathcal{D}_{g_0,f,h}(\omega) \leq C^{-1} h^{6/5}\right) \Rightarrow \left(\operatorname{dist}\,(\omega, \sum_{k\in A_p} F_k^{(p)}) \leq C h^{1/10}\right)\,.$$

Hence the dimension of the spectral subspace,

$$F^{(p)} = \mathrm{Ran}1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(p)}) \subset \mathrm{Ran}1_{[0,ch^{6/5})}(\Delta_{f,h}^{N,(p)}) ,$$

is at most $\#A_p = m_p^{\overline{\Omega}}$.
We next verify that $\dim F^{(p)} \geq \#A_p = m_p^{\overline{\Omega}}$. According to the Min-Max principle, it suffices to find an orthonormal set of p-forms $\omega_k^h \in \Lambda^p H^1_{0,\mathbf{n}}(\Omega)$, $k \in A_p$, such that

$$\mathcal{D}_{g_0,f,h}(\omega_k^h) = o(h^{3/2}) .$$

Indeed it is enough to take a truncated element of the kernel of the local model for $\Delta_{f,h}^{N,(p)}$ around U_k, $k \in A_p$. We give the details for the case $U_k \in \partial\Omega$.
Take two cut-off $\chi_{1,k} \in \mathcal{C}_0^\infty(\overline{\mathbb{R}^n_-})$, $\chi_{1,k} = 1$ near 0 (with $\mathrm{supp}\,\chi_{1,k} \subset \mathrm{supp}\,\chi_k$) and $\chi_{2,k}$ such that $\chi_{1,k}^2 + \chi_{2,k}^2 = 1$. With the same coordinate system as above, we write on $\mathbb{R}^n_-$,using the IMS localization formula (4.3.6) and Proposition 4.3.16-iv),

$$\mathcal{D}_{\tilde{g}_k,\tilde{f}_k,h}(\omega) \geq \mathcal{D}_{\tilde{g}_k,\tilde{f}_k,h}(\chi_{1,k}\omega) + C^{-1}\left\|\chi_{2,k}\omega\right\|^2 - Ch^2 \sum_{i=1,2} \left\||\nabla \chi_{i,k}|\,\omega\right\|^2 ,$$

where $\tilde{g}_k$ and $\tilde{f}_k$ are defined on $\mathbb{R}^n_-$ according to the previous construction and coincide with g_0 and f in a neighborhood of $\mathrm{supp}\,\chi_k$. According to Proposition 4.3.16, there exists $\eta_k^h \in \Lambda^p H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$ in the domain of the associated Witten Laplacian, such that $\mathcal{D}_{\tilde{g}_k,\tilde{f}_k,h}(\eta_k^h) = 0$. By taking $\omega_k^h = \left\|\chi_{1,k}\,\eta_k^h\right\|^{-1} \chi_{1,k}\,\eta_k^h$, we obtain the existence of $h_0 > 0$, C' and C'' such that, for $h \in (0, h_0]$:

$$\left\|\chi_{2,k}\,\eta_k^h\right\|^2 \leq C'h^2 \left\|\eta_k^h\right\|^2 ,$$

and, consequently,

$$\mathcal{D}_{g_0,f,h}(\omega_k^h) \leq C'h^2 \frac{\left\|\eta_k^h\right\|^2}{\left\|\chi_{1,k}\,\eta_k^h\right\|^2} \leq C''h^2 .$$

∎

The following lemma, which provides in different situations the suitable coordinate systems, simply makes use of the standard solution to Hamilton-Jacobi equations in the non characteristic case. It is proved in [Lep2].

Lemma 4.3.18. *1) Let be $f_1 \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R})$ and $U_0 \in \partial\Omega$ a critical point of $f_1|_{\partial\Omega}$ with $\frac{\partial f_1}{\partial n}(U_0) \neq 0$.*
Assume furthermore $\alpha \in \mathcal{C}^\infty(\partial\Omega, \mathbb{R})$ be a local solution to $|\nabla_T \alpha|^2 = |\nabla_T f_1|^2$ around U_0.
Then there exists a neighborhood $\mathcal{V}_0$ of U_0 in $\overline{\Omega}$ such that the eikonal equation:

$$|\nabla \Phi_\pm|^2 = |\nabla f_1|^2$$

(on the boundary, it means $|\partial_n \Phi_\pm|^2 + |\nabla_T \Phi_\pm|^2 = |\partial_n f_1|^2 + |\nabla_T f_1|^2$)
with the boundary conditions

$$\Phi_\pm|_{\partial\Omega\cap\mathcal{V}_0} = \alpha \quad , \quad \partial_n \Phi_\pm|_{\partial\Omega\cap\mathcal{V}_0} = \pm \frac{\partial f_1}{\partial n}|_{\partial\Omega\cap\mathcal{V}_0}$$

admits a unique local smooth real-valued solution.

2) There exists local coordinates $(x_1, \ldots, x_n) = (x', x_n)$ in a neighborhood of U_0 in $\overline{\Omega}$ with $(x', x_n)(U_0) = 0$ where the function $\Phi_\pm$ and the metric g_0 have the form:

$$\Phi_\pm = \mp x_n + \alpha(x') \quad \textit{and} \quad g_0 = g_{nn}(x)\ dx_n^2 + \sum_{i,j=1}^{n-1} g_{ij}(x)\ dx_i dx_j\ .$$

Moreover, the boundary $\partial\Omega$ is locally defined by $\{x_n = 0\}$ and Ω corresponds to $\left\{sgn\left(\frac{\partial f_1}{\partial n}(U_0)\right)\right) x_n > 0\right\}$.

Remark 4.3.19. *Lemma 4.3.18 will be used with various functions f_1 and α and will provide several coordinate systems:*

- *We have already introduced the coordinate system $\overline{x} = (\overline{x}, \overline{x}_n)$ associated with $f_1 = f$ and $\alpha = f\big|_{\partial\Omega}$.*
- *The coordinate system denoted simply by $x = (x', x_n)$ will be associated with $f_1 = f$ and $\alpha = \varphi$, where φ is the Agmon distance along the boundary. This system will be used to give the simple form $\Phi = \Phi_+ = -x_n + \varphi(x')$ to the Agmon distance Φ, solving $|\nabla\Phi|^2 = |\nabla f|^2$ with the boundary condition $\partial_n \Phi = \partial_n f$. Agmon distances are specified in Section 4.4 below.*
- *Finally the coordinate system $\tilde{x} = (\tilde{x}', \tilde{x}_n)$ will be associated with $f_1 = (f+\Phi)$ and $\alpha = f\big|_{\partial\Omega} + \varphi$ and will be used in the final application of the Laplace method.*

4.4 Accurate WKB analysis near the boundary for $\Delta^{(1)}_{f,h}$

4.4.1 Introduction

We work here under Assumption 4.3.1. Like in [HeNi], we have shown that for $0 \leq p < n$, some quasimodes of $\Delta^{N,(p)}_{f,h}$ being near the spectral subspace in $1_{[0,h^{\frac{3}{2}})}(\Delta^{N,(p)}_{f,h})$ are localized near the boundary $\partial\Omega$ and more precisely near critical points of $f|_{\partial\Omega}$ with index p such that $\frac{\partial f}{\partial n} < 0$. In the boundaryless case ([HeKlNi]) and in the case with tangential Dirichlet boundary conditions ([HeNi]), the WKB analysis done in [HeSj4] and in [HeNi] says that the small eigenvalues are of order

$\mathcal{O}(e^{-C/h})$ and provides an accurate approximate basis of $\mathrm{Ran}1_{[0,h^{3/2})}(\Delta^{(p)}_{f,h})$.
In order to get a similar result, we need an accurate WKB analysis at the boundary, and like in [HeNi], we restrict our attention on the case $p = 1$ because our motivation is to analyze the Witten Laplacian on 0-forms.
For an accurate comparison between eigenvectors and WKB quasimodes near a critical point U_1 of $f|_{\partial\Omega}$ with index 1 and $\frac{\partial f}{\partial n}(U_1) < 0$, we introduce another self-adjoint realization of $\Delta^{(1)}_{f,h}$ in a neighborhood $\Omega_{U_1,\rho}$ with mixed boundary conditions: Neumann boundary conditions on $\partial\Omega_{U_1,\rho}\cap\partial\Omega$ and full Dirichlet boundary conditions on $\partial\Omega_{U_1,\rho}\setminus\partial\Omega$.

4.4.2 Local WKB construction

Take U_1 a critical point of $f\big|_{\partial\Omega}$ with index 1 such that $\frac{\partial f}{\partial n}(U_1) < 0$. According to [Lep2], there exists a local coordinate system $(\underline{x}_1, \dots, \underline{x}_n) = (\underline{x}', \underline{x}_n)$ which satisfies the following properties:

i) $d\underline{x}_1, \dots, d\underline{x}_n$ is an orthonormal basis of $T^*_{U_1}(\overline{\Omega})$ positively oriented.

ii) The boundary $\partial\Omega$ corresponds locally to $\underline{x}_n = 0$ and the interior Ω to $\underline{x}_n < 0$.

iii) $\frac{\partial}{\partial \underline{x}_n}|_{\partial\Omega} = \vec{n}$, the outgoing normal at the boundary. Moreover, $\frac{\partial}{\partial \underline{x}_n}$ is unitary and normal to $\{\underline{x}_n = \text{Constant}\}$.

Moreover, the choice of the coordinates $(\underline{x}_1, \dots, \underline{x}_{n-1})$ (centered at U_1 such that $d\underline{x}_1, \dots, d\underline{x}_n$ is an orthonormal basis of $T^*_{U_1}(\overline{\Omega})$) in the boundary is arbitrary.
Let φ be the Agmon distance to U_1 on the boundary (i.e. associated with the metric $|\nabla_{\underline{x}'} f(\underline{x}', 0)| d\underline{x}'^2$). Recall that φ satisfies

$$|\nabla_T f|^2 = |\nabla\varphi|^2$$

on the boundary and that φ is smooth near U_1 (see [HeSj1]). Apply now the first point of Lemma 4.3.18 with $f_1 = f$ and $\alpha = \varphi$ and denote by Φ the function Φ_+ of the lemma (Φ is the Agmon distance to U_1, i.e. associated with the metric $|\nabla_{\underline{x}} f(\underline{x})| d\underline{x}^2$). Hence the following equalities are locally satisfied:

$$\begin{aligned} |\partial_n\Phi|^2 + |\nabla_T\Phi|^2 &= |\nabla\Phi|^2 = |\nabla f|^2 , \\ \Phi|_{\partial\Omega} &= \varphi , \\ \partial_n\Phi|_{\partial\Omega} &= \frac{\partial f}{\partial n}|_{\partial\Omega} . \end{aligned}$$

According to [HeSj4] pp. 279–280, there exist Morse coordinates $(v_1, \dots, v_{n-1})$ for $f\big|_{\Omega}$ centered at U_1 and such that $dv_1(U_1), \dots, dv_{n-1}(U_1), \vec{n}^*_{U_1}$ is orthonormal and positively oriented. With these coordinates

$$f(v, 0) = \frac{\lambda_1}{2}v_1^2 + \cdots + \frac{\lambda_{n-1}}{2}v_{n-1}^2 + f(U_1) \tag{4.4.1}$$

and

$$\varphi(v) = \frac{|\lambda_1|}{2}v_1^2 + \cdots + \frac{|\lambda_{n-1}|}{2}v_{n-1}^2 , \tag{4.4.2}$$

with $\lambda_1 < 0$.
Moreover, $(x_1, \dots, x_{n-1})$ can be chosen equal to $(v_1, \dots, v_{n-1})$ in the boundary. Hence, the theorem of [Lep2] given in the Neumann case implies the following proposition:

Proposition 4.4.1. *Consider around U_1 the above system of coordinates $\underline{x} = (\underline{x}', \underline{x}_n)$ which satisfies (4.4.1)(4.4.2) with $\lambda_1 < 0$. There exists locally, in a neighborhood of $\underline{x} = 0$, a $\mathcal{C}^\infty$ solution u_1^{wkb} to*

$$\begin{aligned} \Delta_{f,h}^{(1)} u_1^{wkb} &= e^{-\frac{\Phi}{h}} \mathcal{O}(h^\infty) && (4.4.3)\\ \mathbf{n} u_1^{wkb} &= 0 \text{ on } \partial\Omega && (4.4.4)\\ \mathbf{n} d_{f,h} u_1^{wkb} &= 0 \text{ on } \partial\Omega\,, && (4.4.5) \end{aligned}$$

where u_1^{wkb} has the form:

$$\begin{aligned} u_1^{wkb} &= a(\underline{x}, h) e^{-\frac{\Phi}{h}}\,, \\ \text{with} \quad a(\underline{x}, h) &\sim \sum_k a^k(\underline{x}) h^k \text{ and } a^0(0) = d\underline{x}_1\,. \end{aligned}$$

4.4.3 Another local Neumann realization of $\Delta_{f,h}^{(1)}$

Let U_1 be a critical point of $f|_{\partial\Omega}$ with index 1 and $\frac{\partial f}{\partial n}(U_1) < 0$ and let us introduce a new system of local coordinates.
We apply Lemma 4.3.18 with $f_1 = f$ and $\alpha = \varphi$, the Agmon distance to U_1 on the boundary. The function Φ_+ of the lemma is then Φ, the Agmon distance to U_1 and we have the existence of local coordinates (x', x_n) around U_1 where Φ and the metric g_0 have the form:

$$\Phi = -x_n + \varphi(x') \quad \text{and} \quad g_0 = g_{nn}(x)\, dx_n^2 + \sum_{i,j=1}^{n-1} g_{ij}(x)\, dx_i dx_j\,.$$

Moreover, the boundary $\partial\Omega$ is locally defined by $\{x_n = 0\}$ and Ω corresponds to $\{x_n < 0\}$.

We work now with the local coordinate system defined above and $x \mapsto |x|$ is the Euclidean norm in these coordinates.
As in [HeNi], we consider the domain, for $\rho > 0$,

$$\Omega_{U_1,\rho} = \left\{ |x - (0,1)|^2 < \rho^2 + 1\,,\ x_n < 0 \right\}\,,$$

which has the shape of a thin lens stuck on $\partial\Omega$ with radius ρ and thickness $\mathcal{O}(\rho^2)$. Its boundary splits into

$$\Gamma_D := \partial\Omega_{U_1,\rho} \cap \Omega = \left\{ |x - (0,1)|^2 = \rho^2 + 1, x_n \leq 0 \right\}$$

and

$$\Gamma_{ND} := \partial\Omega_{U_1,\rho} \cap \partial\Omega = \left\{ |x'| < \rho, x_n = 0 \right\}\,.$$

On this domain, we introduce the functional space

$$\Lambda^1 H^1_{0;0,\mathbf{n}}(\Omega_{U_1,\rho}) = \left\{u \in \Lambda^1 H^1(\Omega_{U_1,\rho});\ \mathbf{n}u|_{\Gamma_{ND}} = 0, \quad u|_{\Gamma_D} = 0\right\} .$$

The Friedrichs extension associated with the quadratic form:

$$\Lambda^1 H^1_{0;0,\mathbf{n}}(\Omega_{U_1,\rho}) \ni \omega \mapsto \mathcal{D}^N_{g,f,h}(\omega) = \|d_{f,h}\omega\|^2 + \left\|d^*_{f,h}\omega\right\|^2 ,$$

is denoted by $\Delta^{N,D,(1)}_{f,h}$. The domain of $\Delta^{N,D,(1)}_{f,h}$ is contained in $\Lambda^1 H^2(\Omega_{U_1}, \rho')$ for any $0 < \rho' < \rho$.
An element $\omega \in D(\Delta^{N,D,(1)}_{f,h})$ satisfies indeed:

$$\langle \Delta^{N,D,(1)}_{f,h}\omega \mid \eta\rangle = \langle d_{f,h}\omega \mid d_{f,h}\eta\rangle + \langle d^*_{f,h}\omega \mid d^*_{f,h}\eta\rangle =: \mathcal{D}^N_{g,f,h}(\omega,\eta) ,$$

for all $\eta \in \Lambda^1 H^1_{0;0,\mathbf{n}}$. By testing with $\eta \in \mathcal{C}^\infty_0(\Omega_{U_1,\rho})$, this gives $\Delta_{f,h}\omega \in \Lambda^1 L^2(\Omega_{U_1,\rho})$ and therefore ω admits a second trace on Γ_{ND} . By testing with any $\eta \in \mathcal{C}^\infty_{0;0,\mathbf{n}}(\Omega_{U_{0,\rho}})$, we get:

$$\mathbf{n}d_{f,h}\omega|_{\Gamma_{ND}} = 0 .$$

Along Γ_{ND} , ω solves an elliptic boundary value problem $\Delta^{(1)}_{f,h}\omega \in \Lambda^1 L^2$, $\mathbf{n}\omega = 0$, $\mathbf{n}d_{f,h}\omega = 0$, which provides the H^2 regularity in $\Omega_{U_{0,\rho'}}$ for any $\rho' < \rho$.
We now prove the

Proposition 4.4.2.
For $\rho > 0$ small enough, there exist $h_\rho > 0$ and $C_\rho > 0$, such that the self-adjoint operator $\Delta^{N,D,(1)}_{f,h}$ satisfies the following properties:
a) *For $h \in (0, h_\rho]$, the spectral projection $1_{[0,h^{3/2})}(\Delta^{N,D,(1)}_{f,h})$ has rank 1.*
b) *Any family of L^2-normalized eigenvectors $(u^h)_{h\in(0,h_\rho]}$ of $\Delta^{N,D,(1)}_{f,h}$ such that the corresponding eigenvalue $E(h)$ is $\mathcal{O}(h)$, satisfies*

$$\begin{aligned}&\forall \rho' < \rho, \forall \alpha \in \mathbb{N}^n, \exists N_\alpha \in \mathbb{N}, \exists C_{\alpha,\rho'} > 0 \text{ such that, } \forall x \in \Omega_{U_1,\rho'} ,\\ &\left|\partial^\alpha_x u^h(x)\right| \le C_{\alpha,\rho'} h^{-N_\alpha} \exp\left(-\frac{\Phi(x)}{h}\right) .\end{aligned} \tag{4.4.6}$$

c) *There exists $\varepsilon_\rho > 0$ such that the first eigenvalue $E_1(h)$ of $\Delta^{N,D,(1)}_{f,h}$ satisfies*

$$E_1(h) = \mathcal{O}(e^{-\varepsilon_\rho/h}) .$$

d) *If u^h_1 denotes the eigenvector of $\Delta^{N,D,(1)}_{f,h}$ associated with eigenvalue $E_1(h)$ and normalized by the condition $\mathbf{t}u^h_1(0) = \mathbf{t}u^{wkb}_1(0)$, then*

$$\begin{aligned}&\forall \rho' < \rho, \forall \alpha \in \mathbb{N}^n, \forall N \in \mathbb{N}, \exists C_{N,\alpha,\rho'} > 0 \text{ such that, } \forall x \in \Omega_{U_1,\rho'} ,\\ &\left|\partial^\alpha_x (u^h_1 - u^{wkb}_1)(x)\right| \le C_{N,\alpha,\rho'} h^N \exp\left(-\frac{\Phi(x)}{h}\right) .\end{aligned} \tag{4.4.7}$$

Once this is proved, one easily gets rough exponentially small upper bounds for the $m_\ell^{\overline{\Omega}}$ first eigenvalues of $\Delta_{f,h}^{N,(\ell)}$ ($\ell \in \{0,1\}$) on Ω, by constructing quasimodes suitably localized near each of the critical points.

The next subsections are devoted to the proof of Proposition 4.4.2. A fondamental ingredient for the proof is a variant of the integration by parts formula of Lemma 4.2.6.

Lemma 4.4.3.
Let $\rho > 0$ and let ψ be a real-valued Lipschitz function on $\overline{\Omega_{U_1,\rho}}$. The relation

$$\begin{aligned}\mathrm{Re}\, \mathcal{D}_{g,f,h}^{N}(\omega, e^{2\frac{\psi}{h}}\omega) = h^2 \left\| de^{\frac{\psi}{h}}\omega\right\|^2_{\Lambda^2 L^2} + h^2 \left\| d^* e^{\frac{\psi}{h}}\omega\right\|^2_{\Lambda^0 L^2} \\ + \langle (|\nabla f|^2 - |\nabla\psi|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}^*_{\nabla f})e^{\frac{\psi}{h}}\omega \,|\, e^{\frac{\psi}{h}}\omega\rangle_{\Lambda^1 L^2} \\ + h\int_{\Gamma_{ND}} \langle \omega \,|\, \omega\rangle_{\Lambda^1 T^*_\sigma\Omega}\, e^{2\frac{\psi(\sigma)}{h}} \left(\frac{\partial f}{\partial n}\right)(\sigma)\, d\sigma \quad (4.4.8)\end{aligned}$$

holds for any $\omega \in \Lambda^1 H^1_{0;0,\mathbf{n}}(\Omega_{U_1,\rho})$.
Moreover, when $\omega \in D(\Delta_{f,h}^{N,D,(1)})$, the left-hand side equals $\mathrm{Re}\,\langle e^{2\frac{\psi}{h}}\Delta^{(1)}_{f,h}\omega \mid \omega\rangle$.

Proof.
For ω in $\Lambda^1 H^1_{0;0,\mathbf{n}}(\Omega_{U_1,\rho})$, we have $\tilde{\omega} := e^{2\frac{\psi}{h}}\omega$ in $\Lambda^1 H^1_{0;0,\mathbf{n}}(\Omega_{U_1,\rho})$ and the same computations as the ones done in [HeNi] to prove Lemma 4.3 lead to:

$$\begin{aligned}\mathcal{D}_{g,f,h}^{N}(\omega, e^{2\frac{\psi}{h}}\omega) &= \mathcal{D}_{g,f,h}^{N}(\tilde{\omega},\tilde{\omega}) - \langle |\nabla\psi|^2\,\tilde{\omega} \mid \tilde{\omega}\rangle \\ &\quad - \langle d\psi\wedge\tilde{\omega} \mid d_{f,h}\tilde{\omega}\rangle + \langle d_{f,h}\tilde{\omega} \mid d\psi\wedge\tilde{\omega}\rangle \\ &\quad + \langle \mathbf{i}_{\nabla\psi}\tilde{\omega} \mid d^*_{f,h}\tilde{\omega}\rangle - \langle d^*_{f,h}\tilde{\omega} \mid \mathbf{i}_{\nabla\psi}\tilde{\omega}\rangle\,.\end{aligned}$$

By taking the real part, we obtain:

$$\mathrm{Re}\,\mathcal{D}_{g,f,h}^{N}(\omega, e^{2\frac{\psi}{h}}\omega) = \mathcal{D}_{g,f,h}^{N}(\tilde{\omega},\tilde{\omega}) - \langle |\nabla\psi|^2\,\tilde{\omega} \mid \tilde{\omega}\rangle\,.$$

We conclude by applying Lemma 4.2.6 . ∎

4.4.4 Exponential decay of eigenvectors of $\Delta_{f,h}^{N,D,(1)}$

As in [HeNi], the pointwise estimate, $\partial_x^\alpha u^h(x) = \mathcal{O}(h^{-N\alpha}e^{-\frac{\Phi(x)}{h}})$, which is stated in Proposition 4.4.2-b), will be proved in several steps. We will first consider H^1-estimates and deduce afterwards higher order estimates from elliptic regularity. Even for H^1-estimates we need two steps: we prove first the exponential decay along the boundary Γ_{ND} by applying Lemma 4.4.3 with the function ψ similar to φ introduced above ; then the exponential decay in the interior of $\Omega_{U_1,\rho}$ is obtained with ψ similar to Φ once the boundary term is well controlled.

Proof of a) and b) in Proposition 4.4.2.
Statement a)
Actually it is a simple comparison with the full half-space problem via Min-Max principle as we did for Theorem 4.3.5. Any $\omega \in \Lambda^1 H^1_{0;0,\mathbf{n}}(\Omega_{U_1,\rho})$ can indeed be viewed as an element of $\Lambda^1 H^1_{0,\mathbf{n}}(\mathbb{R}^n_-)$ by setting $\omega = 0$ on $\mathbb{R}^n_- \setminus \Omega_{U_1,\rho}$.

Statement b)
Let $u^h \in D(\Delta^{N,D,(1)}_{f,h})$ satisfy

$$\Delta^{(1)}_{f,h} u^h = E(h) u^h \ , \ E(h) = \mathcal{O}(h) \ , \ \left\| u^h \right\| = 1 \ .$$

We will use the notation

$$\tilde{u}^h = e^{\frac{\psi^h}{h}} u^h \ .$$

The integration by parts formula (4.4.8) will be applied with $\psi = \psi^h$ where ψ^h will be similar to φ or similar to Φ .
Let us recall

$$|\nabla f|^2 = |\nabla \Phi|^2 \ , \quad \frac{\partial f}{\partial n} = \frac{\partial \Phi}{\partial n} \quad \text{and} \quad \Phi(x', x_n) = -x_n + \varphi(x'), \tag{4.4.9}$$

where $x' = 0$ is a local minimum for φ with $\varphi(0) = 0$. Moreover we have $\nabla x_n \cdot \nabla \varphi(x') = 0$ so that:

$$|\nabla f|^2 = |\nabla \Phi|^2 = |\nabla x_n|^2 + |\nabla \varphi|^2 \ . \tag{4.4.10}$$

We will first show the decay along the boundary before we propagate the decay in the normal direction inside Ω (see [HeSj5] and [HeNi] for references).

Step 1: Decay along Γ_{ND} .

We take:

$$\psi^h(x', x_n) = \begin{cases} \varphi(x') - Ch \log \frac{\varphi(x')}{h} \ , & \text{if } \varphi(x') > Ch \\ \varphi(x') - Ch \log C \ , & \text{if } \varphi(x') \leq Ch \ , \end{cases}$$

where the constant $C > 1$ will be fixed later.
We associate the sets:

$$\Omega^h_- = \left\{ x = (x', x_n) \in \Omega_{U_1,\rho} \ ; \ \varphi(x') < Ch \right\} \ ,$$

and

$$\Omega^h_+ = \left\{ x = (x', x_n) \in \Omega_{U_1,\rho} \ ; \ \varphi(x') > Ch \right\} \ .$$

The condition $E(h) = \mathcal{O}(h)$ the formula (4.4.8), (4.4.9) and (4.4.10) imply the existence of $C_1 > 0$ such that:

$$C_1 h \left\|\tilde{u}^h\right\|^2_{\Lambda^1 L^2(\Omega^h_-)} \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0 L^2} + \langle|\nabla x_n|^2\tilde{u}^h \mid \tilde{u}^h\rangle_{\Lambda^1 L^2}$$
$$-h\int_{\Gamma_{ND}} \langle\tilde{u}^h \mid \tilde{u}^h\rangle_{\Lambda^1 T^*_\sigma \Omega} \left(\frac{\partial x_n}{\partial n}\right)(\sigma)\, d\sigma + \langle(|\nabla\varphi|^2 - |\nabla\psi^h|^2)\tilde{u}^h \mid \tilde{u}^h\rangle$$
$$-C_1 h\langle 1_{\Omega^h_+}(x)\tilde{u}^h \mid \tilde{u}^h\rangle\ , \tag{4.4.11}$$

with C_1 determined by f and the upper bound of $E(h)$.
Furthermore,

$$\nabla\psi^h = \nabla\varphi - 1_{\Omega^h_+}(x)\frac{Ch\nabla\varphi}{\varphi}\,,$$

so we have:

$$\left|\nabla\psi^h\right|^2 = |\nabla\varphi|^2 + 1_{\Omega^h_+}(x)\left(-2Ch\frac{|\nabla\varphi|^2}{\varphi} + C^2h^2\frac{|\nabla\varphi|^2}{\varphi^2}\right).$$

Consequently,

$$C_1 h \left\|\tilde{u}^h\right\|^2_{\Lambda^1 L^2(\Omega^h_-)} \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0 L^2} + \langle|\nabla x_n|^2\tilde{u}^h \mid \tilde{u}^h\rangle_{\Lambda^1 L^2}$$
$$- h\int_{\Gamma_{ND}} \langle\tilde{u}^h \mid \tilde{u}^h\rangle_{\Lambda^1 T^*_\sigma \Omega} \left(\frac{\partial x_n}{\partial n}\right)(\sigma)\, d\sigma$$
$$+ \langle\left[|\nabla\varphi|^2\left(\frac{2Ch}{\varphi} - \frac{C^2h^2}{\varphi^2}\right) - C_1 h\right] 1_{\Omega^h_+}(x)\tilde{u}^h \mid \tilde{u}^h\rangle\,.$$

For $x \in \Omega^h_+$,

$$\frac{2Ch}{\varphi} - \frac{C^2h^2}{\varphi^2} \geq \frac{Ch}{\varphi} \ (\text{ since } 2a - a^2 \geq a\ \forall\, a \in [0,1])$$

then, φ being a positive Morse function, there exists $C_2 > 0$ which is determined by φ such that, for all $x \in \Omega^h_+$,

$$C_2 \geq \frac{|\nabla\varphi(x')|^2}{\varphi(x')} \geq C_2^{-1}$$

and we get:

$$C_1 h \left\|\tilde{u}^h\right\|^2_{\Lambda^1 L^2(\Omega^h_-)} \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0 L^2}$$
$$+\langle|\nabla x_n|^2\tilde{u}^h \mid \tilde{u}^h\rangle_{\Lambda^1 L^2} - h\int_{\Gamma_{ND}} \langle\tilde{u}^h \mid \tilde{u}^h\rangle_{\Lambda^1 T^*_\sigma \Omega} \left(\frac{\partial x_n}{\partial n}\right)(\sigma)\, d\sigma$$
$$+ \left(CC_2^{-1} - C_1\right) h\langle 1_{\Omega^h_+}(x)\tilde{u}^h \mid \tilde{u}^h\rangle\,. \tag{4.4.12}$$

Since $\partial_n f(U_1) = \partial_n x_n(U_1) \neq 0$, we can choose ρ small enough such that:

$$C_3 \geq |\nabla x_n|^2 \geq C_3^{-1} \text{ on } \Omega_{U_1,\rho}\,,$$

where C_3 is a stricly positive constant.

Hence we get, by adding the term $(CC_2^{-1} - C_1)h\langle 1_{\Omega^h_-}(x)\tilde{u}^h\,|\tilde{u}^h\rangle$ to (4.4.12):

$$\begin{aligned} CC_2^{-1}h\left\|\tilde{u}^h\right\|^2_{\Lambda^1L^2(\Omega^h_-)} \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0L^2} \\ + (1+2\delta(C)h)\,\langle|\nabla x_n|^2\tilde{u}^h\,|\,\tilde{u}^h\rangle_{\Lambda^1L^2} \\ - h\int_{\Gamma_{ND}}\langle\tilde{u}^h\,|\,\tilde{u}^h\rangle_{\Lambda^1T^*_\sigma\Omega}\left(\frac{\partial x_n}{\partial n}\right)(\sigma)\;d\sigma\,, \end{aligned}$$

where $\delta(C) = \frac{1}{2}C_3^{-1}(CC_2^{-1} - C_1) \to +\infty$ when $C \to +\infty$.

At least, we have on Ω^h_- by the definitions:

$$\left|\tilde{u}^h\right| \leq e^C|u^h| \text{ a.e.}$$

and the condition $\left\|u^h\right\| = 1$ leads to:

$$\begin{aligned} \tilde{\delta}(C)h \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0L^2} + (1+2\delta(C)h)\,\langle|\nabla x_n|^2\tilde{u}^h\,|\,\tilde{u}^h\rangle_{\Lambda^1L^2} \\ - h\int_{\Gamma_{ND}}\langle\tilde{u}^h\,|\,\tilde{u}^h\rangle_{\Lambda^1T^*_\sigma\Omega}\left(\frac{\partial x_n}{\partial n}\right)(\sigma)\;d\sigma\,, \end{aligned} \tag{4.4.13}$$

where $\tilde{\delta}(C) = e^{2C}CC_2^{-1}$.

We now apply (4.4.8) to $\tilde{u}^h$ with $\psi = 0$, f and h replaced respectively by $-x_n$ and $\frac{h}{1+\delta(C)h}$, in order to get,

$$\begin{aligned} (1+\delta(C)h)^{-1}\left\|hd\tilde{u}^h\right\|^2_{\Lambda^2L^2} + (1+\delta(C)h)^{-1}\left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0L^2} \\ + (1+\delta(C)h)\langle|\nabla x_n|^2\tilde{u}^h\,|\,\tilde{u}^h\rangle - h\int_{\Gamma_{ND}}\langle\tilde{u}^h\,|\,\tilde{u}^h\rangle_{\Lambda^1T^*_\sigma\Omega}\left(\frac{\partial x_n}{\partial n}\right)(\sigma)\;d\sigma \\ + hC_4\,||\tilde{u}^h||^2_{\Lambda^1L^2} \geq 0\,, \end{aligned} \tag{4.4.14}$$

with $C_4 > 0$ independent of C.

The difference (4.4.13)$-$(4.4.14) yields:

$$\begin{aligned} \frac{\delta(C)h^3}{1+\delta(C)h}\left[\left\|d\tilde{u}^h\right\|^2_{\Lambda^2L^2} + \left\|d^*\tilde{u}^h\right\|^2_{\Lambda^0L^2}\right] - hC_4\,||\tilde{u}^h||^2_{\Lambda^1L^2} \\ + \delta(C)h\langle|\nabla x_n|^2\tilde{u}^h\,|\,\tilde{u}^h\rangle \leq \tilde{\delta}(C)h\,. \end{aligned}$$

We choose $C > 1$ large enough such that $\delta(C)C_3^{-1} - C_4 > 0$.
This leads, after choosing $h_0 > 0$ small enough, to the existence of a constant $C_5 > 0$ such that, for all $h \in (0, h_0]$,

$$C_5 h \geq h^3 \left\|\tilde{u}^h\right\|^2_{\Lambda^1 H^1} .$$

Since $\psi^h \geq \varphi + \tilde{C}h\log h$ (for all $\tilde{C} > C$), we have proved the existence of $N_0 > 0$ such that:

$$\left\|e^{\frac{\varphi}{h}} u^h\right\|_{\Lambda^1 H^1} \leq C_6 h^{-N_0} . \tag{4.4.15}$$

Remember that $\varphi \geq 0$ vanishes only at $x' = 0$. Using the trace theorem, this also leads to:

$$\left\|e^{\frac{\varphi}{h}} u^h|_{\Gamma_{ND}}\right\|_{\Lambda^1 H^{1/2}(\Gamma_{ND})} \leq C_7\, h^{-N_0} . \tag{4.4.16}$$

Step 2: Normal decay inside Ω.

We follow a very similar approach by working with the function Φ.
We take:

$$\psi^h(x', x_n) = \begin{cases} \Phi - Ch\log\frac{\Phi}{h}\,, & \text{if } \Phi > Ch \\ \Phi - Ch\log C\,, & \text{if } \Phi \leq Ch\,, \end{cases}$$

where the constant $C > 1$ will be fixed later.
We associate the sets:

$$\Omega^h_- = \left\{x = (x', x_n) \in \Omega_{U_1,\rho}\,;\ \Phi < Ch\right\}$$

and

$$\Omega^h_+ = \left\{x = (x', x_n) \in \Omega_{U_1,\rho}\,;\ \Phi > Ch\right\} .$$

The formula (4.4.8) is used like in Step 1, with $\tilde{u}^h = e^{\frac{\psi^h}{h}} u^h$ and $E(h) = \mathcal{O}(h)$. The difference comes from the fact that the boundary term is already estimated with (4.4.16).

We have indeed on the boundary $x_n = 0$ the inequality: $e^{\frac{\psi^h}{h}} \leq e^{\frac{\varphi}{h}}$, due to the relation $\Phi|_{x_n=0} = \varphi$.
From (4.4.8) used like in Step 1 (see (4.4.11)) we get the existence of $C_1 > 0$ such that:

$$C_1 h \left\|\tilde{u}^h\right\|^2_{\Lambda^1 L^2(\Omega^h_-)} + C_1 h \left\|e^{\frac{\varphi}{h}} u\right\|^2_{H^{1/2}(\Gamma_{ND};\Lambda^1 T^*\Omega_{U_1,\rho})} \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0 L^2}$$
$$+ \langle(|\nabla f|^2 - |\nabla\psi^h|^2)\tilde{u}^h \mid \tilde{u}^h\rangle - C_1 h\langle 1_{\Omega^h_+}(x)\tilde{u}^h \mid \tilde{u}^h\rangle .$$

Moreover, from (4.4.16) and the inequality

$$|\tilde{u}^h(x)| \leq e^C |u^h(x)| \quad \text{a.e. in } \Omega^h_- ,$$

we get, for any $C > 1$, the existence of $\tilde{\delta}(C) > 0$ such that the following estimate is satisfied:

$$\begin{aligned}\tilde{\delta}(C)h^{1-2N_0} &\geq C_1 h \left\|\tilde{u}^h\right\|^2_{\Lambda^1 L^2(\Omega^h_-)} + C_1 h \left\|e^{\frac{\varphi}{h}} u\right\|^2_{H^{1/2}(\Gamma_{ND};\Lambda^1 T^*\Omega_{U_1,\rho})} \\ &\geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0 L^2} + \langle(|\nabla f|^2 - |\nabla\psi^h|^2)\tilde{u}^h \mid \tilde{u}^h\rangle \\ &\qquad -C_1 h\langle 1_{\Omega^h_+}(x)\tilde{u}^h \mid \tilde{u}^h\rangle\,. \qquad (4.4.17)\end{aligned}$$

Since $|\nabla f|^2 = |\nabla\Phi|^2$ and Φ is a positive function without critical points, we can use the same computations as the ones done in Step 1 with φ replaced by Φ to get:

$$\begin{aligned}|\nabla f|^2 - |\nabla\psi^h|^2 &= 1_{\Omega^h_+}(x)\left(2Ch\frac{|\nabla\Phi|^2}{\Phi} - C^2h^2\frac{|\nabla\Phi|^2}{\Phi^2}\right) \\ &\geq \frac{Ch|\nabla\Phi|^2}{\Phi} \geq C_2^{-1}Ch\,,\end{aligned}$$

with $C_2 > 0$ independent of C.

We take $C \geq 2C_1C_2$. By adding the estimated term $(C_2^{-1}C - C_1)h\langle 1_{\Omega^h_-}(x)\tilde{u}^h\,|\tilde{u}^h\rangle$ to (4.4.17) we get:

$$\tilde{\delta}_2(C)h^{1-2N_0} \geq \left\|hd\tilde{u}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{u}^h\right\|^2_{\Lambda^0 L^2} + (C_2^{-1}C - C_1)h\left\|\tilde{u}^h\right\|_{\Lambda^1 L^2}\,,$$

which gives, by analogy with Step 1, the existence of $C_3 > 0$ and $N_1 > 0$ such that:

$$\left\|e^{\frac{\Phi}{h}}u^h\right\|_{\Lambda^1 H^1(\Omega_{U_1,\rho})} \leq C_3\, h^{-N_1}\,. \qquad (4.4.18)$$

Step 3: Elliptic regularity.

We now set $\tilde{u}^h = e^{\frac{\Phi}{h}}u^h$. For $\rho' < \rho$, we take a cut-off $\chi \in \mathcal{C}^\infty(\Omega_{U_1,\rho})$ with compact support in $\Omega_{U_1,\rho}\cup\Gamma_{ND}$ and such that $\chi = 1$ on a neighborhood of $\Omega_{U_1,\rho'}$. The form $v^h = \chi\tilde{u}^h$ satisfies the boundary value problem:

$$\begin{cases} v^h - \Delta v^h = r_0^h & \text{in } \mathbb{R}^n_-\,, \\ \mathbf{n}v^h = 0 \text{ and } \mathbf{n}dv^h = r_1^h & \text{on } \{x_n = 0\}\,, \end{cases}$$

$$\text{with } \left\|r_0^h\right\|_{\Lambda^1 L^2(\mathbb{R}^n_-)} = \mathcal{O}(h^{-N_1}) \text{ and } \left\|r_1^h\right\|_{\Lambda^2 H^{1/2}(\mathbb{R}^{n-1})} = \mathcal{O}(h^{-N_1})\,.$$

This implies, by [Sch], the existence of $N_2 > 0$ such that:

$$\left\|v^h\right\|_{\Lambda^1 H^2} = \mathcal{O}(h^{-N_2})\,.$$

We conclude by induction for any finite decreasing sequence $(\rho_k)_{0\leq k\leq K}$ with $\rho_K > \rho'$ and associated cut-offs χ_k , with $\chi_k = 1$ in a neighborhood of Ω_{U_1,ρ_k} and $\operatorname{supp}\chi_k \subset \{\chi_{k-1} = 1\}$, using the Sobolev injections. ∎

4.4.5 Small eigenvalues are exponentially small

We now check that the eigenvalue $E_1(h)$ of $\Delta_{f,h}^{N,D,(1)}$ lying in $[0, h^{3/2})$ is actually of order $\mathcal{O}(e^{-\varepsilon_\rho/h})$ for some $\varepsilon_\rho > 0$. We prove this by comparison with the half-space problem as it is done in [Lep2] at the end of the proof of Proposition 4.4.1.

Proof of Proposition 4.4.2-c).
Again we introduce in a neighborhood of U_1, the coordinate system $\overline{x} = (\overline{x}', \overline{x}_n)$ leading to (4.3.21). The function f and the metric g_0 are extended according to (4.3.22) and (4.3.23) so that Proposition 4.3.16 can be applied. Consequently, the half-space Witten Laplacian, $\Delta_{\tilde{f},h}^{N,(1)}$, has a one dimension kernel and its second eigenvalue is larger than $Ch^{6/5}$.
Let u^h be a normalized eigenvector of $\Delta_{f,h}^{N,D,(1)}$ associated with the first eigenvalue $E_1(h)$, which belongs to the interval $(0, h^{3/2}]$. Let $\chi \in \mathcal{C}^\infty(\overline{\Omega_{U_1,\rho}})$ be a cut-off function with compact support in $\Omega_{U_1,\rho} \cup \Gamma_{ND}$ and such that $\chi = 1$ in a neighborhood of 0 with $\frac{\partial\chi}{\partial n}\big|_{\partial\Omega} \equiv 0$.
The form $v^h = \chi u^h \in \Lambda^1 H^2(\mathbb{R}^n_-)$ belongs to the domain of $\Delta_{\tilde{f},h}^{N,(1)}$, i.e. $\mathbf{n}v^h = \mathbf{n}d_{\tilde{f},h}v^h = 0$. Moreover, v^h satisfies

$$(\Delta_{\tilde{f},h}^{(1)} - E_1(h))v^h = -h^2[\Delta, \chi]u^h \text{ in } \mathbb{R}^n_-$$

and the 1-form $r^h = -h^2[\Delta, \chi]u^h$ vanishes in a neighborhood $\mathcal{V}_1$ of $\overline{x} = 0$. Due to the exponential decay of u^h stated in Proposition 4.4.2-b), there exist C and N_0, such that r^h also satisfies

$$\left|r^h(\overline{x})\right| \leq Ch^{-N_0} \left[\sum_{1\leq|\beta|\leq 2} |\partial_{\overline{x}}^\beta \chi(\overline{x})|\right] e^{-\frac{\Phi(\overline{x})}{h}} \leq e^{-\frac{c_\chi}{h}} .$$

With $\left\|v^h\right\|_{\Lambda^1 L^2} = 1 + \mathcal{O}(e^{-c/h})$, $\left\|r^h\right\|_{\Lambda^1 L^2} = \mathcal{O}(e^{-c/h})$ and the a priori estimate $E_1(h) = \mathcal{O}(h^{3/2})$, the spectral theorem implies $|E_1(h) - 0| = \mathcal{O}(e^{-c/h})$ like in the proof of Proposition 4.4.1 given in [Lep2].

∎

4.4.6 Accurate comparison with the WKB solution

We now compare the eigenvector associated with an exponentially small eigenvalue with its WKB approximation. We adapt the method presented in [Hel2, HeSj2] and in [HeNi] by following the same strategy as in Subsection 4.4.4. The H^1-estimates are done in two steps with ψ^h similar to φ and then with ψ^h similar to Φ. Finally the elliptic regularity is used for the $\mathcal{C}^\infty$-estimates.

Proof of Proposition 4.4.2-d).
Let $u_1^h \in D(\Delta^{N,D,(1)}_{f,h})$ be an eigenvector associated with the first eigenvalue $E_1(h)$ of $\Delta^{N,D,(1)}_{f,h}$:

$$\Delta^{N,D,(1)}_{f,h} u_1^h = E_1(h) u_1^h \;,\; \left\| u_1^h \right\| = 1 \;.$$

According to Proposition 4.4.2-c), we know that $E_1(h) = \mathcal{O}(e^{-\frac{\varepsilon_\rho}{h}})$, with $\varepsilon_\rho > 0$, while the second eigenvalue of $\Delta^{N,D,(1)}_{f,h}$ is larger than $h^{3/2}$.
By taking $\rho > 0$ small enough, the WKB approximation u_1^{wkb} presented in Subsection 4.4.2 satisfies

$$\begin{cases} \Delta^{(1)}_{f,h} u_1^{wkb} = \mathcal{O}(h^\infty)\, e^{-\frac{\Phi(x)}{h}} & \text{in } \Omega_{U_1,\rho} \;, \\ \mathbf{n} u_1^{wkb}|_{\Gamma_{ND}} = 0 \;, \\ \mathbf{n} d_{f,h} u_1^{wkb}|_{\Gamma_{ND}} = 0 \;, \end{cases}$$

and there exists $c > 0$, such that for any $\rho' > 0$, we have

$$\left\| u_1^{wkb} \right\|_{\Lambda^1 L^2(\Omega_{U_1,\rho'})} \sim c h^{\frac{n+1}{4}}$$

(see indeed further the proof of Proposition 4.5.19).
The cut-off function $\chi \in \mathcal{C}^\infty(\overline{\Omega_{U_1,\rho}})$ is supported in $\Omega_{U_1,\rho/2} \cup \Gamma_{ND}$ and satisfies $\chi = 1$ on $\Omega_{U_1,\rho'}$ with $0 < \rho' < \rho/2$, $\frac{\partial \chi}{\partial n}\big|_{\partial\Omega} \equiv 0$. Later, we will take $\rho' > 0$ small enough, so that χ can be taken in the form

$$\chi(x', x_n) = \chi_1(x')\chi_n(x_n) \;.$$

Like in Lemma 4.2.11 (replace $1_{[b,+\infty)}(A)$ by $A^{1/2}1_{[b,+\infty)}(A)$ and $\frac{a}{b}$ by $a = O(h^\infty)$ here), the real constant factor $c(h)$ in the truncated WKB approximation $v_1^{wkb} = c(h)\chi u_1^{wkb}$ can be chosen so that

$$\left\| v_1^{wkb} - u_1^h \right\|_{\Lambda^1 H^1} = \mathcal{O}(h^\infty)$$

and, due to the exponential decay of u_1^h and u_1^{wkb},

$$\left\| \chi(u_1^h - c(h) u_1^{wkb}) \right\|_{\Lambda^1 H^1} = \mathcal{O}(h^\infty) \;.$$

Set

$$w^h = \chi(u_1^h - c(h) u_1^{wkb}) \;.$$

The 1-form w^h satisfies in $\Omega_{U_1,\rho}$

$$\begin{aligned}(\Delta^{(1)}_{f,h} - E_1(h))w^h &= \chi(x)(\Delta^{(1)}_{f,h} - E_1(h))(u_1^h - c(h)u_1^{wkb}) \\ &\quad +[\Delta^{(1)}_{f,h}, \chi](u_1^h - c(h)u_1^{wkb}) \\ &= \tilde{r}^h\, e^{-\frac{\Phi(x)}{h}} + r^h\,,\end{aligned} \tag{4.4.19}$$

where $\tilde{r}^h$ and r^h satisfy, according to Proposition 4.4.2-b),

$$\tilde{r}^h = \mathcal{O}(h^\infty)\,, \quad \operatorname{supp} r^h \subset \operatorname{supp} \nabla\chi \quad \text{and} \quad r^h = \mathcal{O}(h^{-N_0})e^{-\frac{\Phi(x)}{h}}\,.$$

The last estimate can be done for any $\mathcal{C}^{k_0}$-norm, with $k_0 \in \mathbb{N}$.
On the boundary $\partial\Omega_{U_1,\rho} = \Gamma_{ND} \cup \Gamma_D$, we have simply

$$\begin{aligned} &\mathbf{n}w^h|_{\Gamma_{ND}} = 0, \quad w^h|_{\Gamma_D} = 0\,, \\ \text{and} \quad &\mathbf{n}d_{f,h}w^h|_{\Gamma_{ND}} = 0\,.\end{aligned}$$

With the different of choices for ψ^h given below, we will use the notation

$$\tilde{w}^h = e^{\frac{\psi^h}{h}} w^h\,.$$

The 1-forms w and $\tilde{w}$ belong to $\Lambda^1 H^2(\Omega_{U_1,\rho})$ and their supports do not meet Γ_D. Hence the integration by parts formula (4.2.16) can be used in addition to (4.4.8).

Step 1: Comparison along Γ_{ND}.

Like in the proof of Proposition 4.4.2-b) presented in Subsection 4.4.4, we introduce the sets

$$\begin{aligned} &\Omega_-^h = \left\{x = (x', x_n) \in \Omega_{U_1,\rho}\,; \quad \varphi(x') < Ch\right\}\,, \\ \text{and} \quad &\Omega_+^h = \left\{x = (x', x_n) \in \Omega_{U_1,\rho}\,; \quad \varphi(x') > Ch\right\}\,.\end{aligned}$$

For any $N \in \mathbb{N}$, we take:

$$\varphi_N^h(x') = \min\left\{\varphi^h(x') + Nh\log h^{-1}\,, \psi(x')\right\}\,,$$

$$\text{where} \quad \varphi^h(x') = \begin{cases} \varphi(x') - Ch\log\frac{\varphi(x')}{h}\,, & \text{if } \varphi(x') > Ch \\ \varphi(x') - Ch\log C\,, & \text{if } \varphi(x') \leq Ch\,, \end{cases}$$

$$\text{and} \quad \psi(x') = \min\left\{\varphi_-^h(y') + (1-\varepsilon)|\varphi(x') - \varphi(y')|\,,\ y' \in \operatorname{supp}\nabla\chi_1\right\}\,.$$

We recall that the cut-off χ writes $\chi(x', x_n) = \chi_1(x')\chi_n(x_n)$. The constant $C \geq 1$ will be fixed at the end like in the proof of Proposition 4.4.2-b). The constants $\rho' \in (0, \rho/2)$ and $\varepsilon > 0$ are chosen so that, for $h \in (0, h_{N,\rho',\varepsilon})$,

$$\varphi_N^h(x') = \varphi^h(x') + Nh\log h^{-1} \quad \text{in } \Omega_{U_1,\rho'}\,. \tag{4.4.20}$$

Consequently, φ being the Among distance on the boundary,

$$\varphi^h_N(x') = \varphi^h(x') + Nh\log h^{-1} = \varphi(x') - Ch\log C + Nh\log h^{-1} \quad \text{on } \Omega^h_- . \quad (4.4.21)$$

Note furthermore the inequalities:

$$\begin{aligned} &\varphi^h_N(x) \leq \varphi(x) + Nh\log h^{-1} \quad \text{in} \quad \Omega_{U_1,\rho} \\ &\varphi^h_N(x) \leq \varphi(x) \leq \Phi(x)\ ,\ \text{if} \quad x' \in \operatorname{supp}\nabla\chi_1\ , \\ \text{and} \quad &\varphi^h_N(x) \leq \varphi(x) + Nh\log h^{-1} \leq \Phi(x)\ ,\ \text{if} \quad x_n \in \operatorname{supp}\chi'_n\ . \end{aligned}$$

In particular, we have for $h \in (0, h_{N,\rho',\varepsilon})$,

$$\varphi^h_N(x) \leq \Phi(x)\ ,\ \text{for } x \in \operatorname{supp}\nabla\chi\ ,$$

which implies

$$\left\| e^{\frac{\varphi^h_N}{h}} r^h \right\|_{\Lambda^1 L^2} = \mathcal{O}_N(h^{-N_0})\ .$$

We apply the integration by parts formula (4.4.8), where the left-hand side is computed with (4.2.16), and we obtain for the form $\tilde{w}^h = e^{\frac{\varphi^h_N}{h}} w^h$, by analogy with the proof of Proposition 4.4.2-b), using (4.4.19) and $E_1(h) = \mathcal{O}(h^\infty) = \mathcal{O}(h)$:

$$\begin{aligned} C_1 h \left\|\tilde{w}^h\right\|_{\Lambda^1 L^2(\Omega^h_-)} + \left\|\tilde{r}^h + e^{\frac{\varphi^h_N(x)}{h}} r^h\right\|_{\Lambda^1 L^2} \left\|\tilde{w^h}\right\|_{\Lambda^1 L^2} \geq \left\|hd\tilde{w}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{w}^h\right\|^2_{\Lambda^0 L^2} \\ + \langle |\nabla x_n|^2 \tilde{w}^h \,|\, \tilde{w}^h\rangle_{\Lambda^1 L^2} + h\int_{\Gamma_{ND}} \langle \tilde{w}^h \,|\, \tilde{w}^h\rangle_{\Lambda^1 T^*_\sigma\Omega} \left(\frac{\partial x_n}{\partial n}\right)(\sigma)\, d\sigma \\ + \langle (|\nabla\varphi|^2 - |\nabla\varphi^h_N|^2)\tilde{w}^h \,|\, \tilde{w}^h\rangle - C_1 h \langle 1_{\Omega^h_+}(x)\tilde{w}^h \,|\, \tilde{w}^h\rangle\ , \end{aligned}$$

where the constant $C_1 > 0$ is determined by f and $\tilde{r}^h = \mathcal{O}(h^\infty)$.

In Ω^h_- the weight $e^{\frac{\varphi^h_N(x)}{h}}$ is bounded by $C_2(C)h^{-N}$ and this provides

$$\left\|\tilde{w}^h\right\|_{\Lambda^1 L^2(\Omega^h_-)} \leq C_2(C)h^{-N} \left\|w^h\right\|_{\Lambda^1 L^2(\Omega^h_-)} \leq C_3(C,N)\ ,$$

due to $\left\|w^h\right\|_{\Lambda^1 H^1} = \mathcal{O}(h^\infty)$.

We obtain:

$$\begin{aligned} \tilde{\delta}(C,N)(h^{-N_0}\left\|\tilde{w}^h\right\|_{\Lambda^1 H^1} + 1) \geq \left\|hd\tilde{w}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{w}^h\right\|^2_{\Lambda^0 L^2} \\ + \langle |\nabla x_n|^2 \tilde{w}^h \,|\, \tilde{w}^h\rangle_{\Lambda^1 L^2} + h\int_{\Gamma_{ND}} \langle \tilde{w}^h \,|\, \tilde{w}^h\rangle_{\Lambda^1 T^*_\sigma\Omega} \left(\frac{\partial x_n}{\partial n}\right)(\sigma)\, d\sigma \\ + \langle (|\nabla\varphi|^2 - |\nabla\varphi^h_N|^2)\tilde{w}^h \,|\, \tilde{w}^h\rangle - C_1 h \langle 1_{\Omega^h_+}(x)\tilde{w}^h \,|\, \tilde{w}^h\rangle\ . \end{aligned}$$

In Ω^h_-, $|\nabla\varphi|^2 = \left|\nabla\psi^h_N\right|^2$, using (4.4.21).
In Ω^h_+, the point x fulfills almost surely one of the two possibilities:

- **Either** $\nabla\varphi_N^h = \nabla\psi$, and we get

$$|\nabla\varphi|^2 - \left|\nabla\psi_N^h\right|^2 \geq (2\varepsilon - \varepsilon^2)\,|\nabla\varphi|^2 \geq \delta_{\rho,\varepsilon}\,,$$

  where the last lower bound is due to the fact that $\varphi_N(x) = \psi(x)$ cannot occur in a neighborhood of $x' = 0$ for $\varepsilon > 0$ small enough and $h \in (0, h_{N,\rho',\varepsilon})$;
- **or** $\nabla\varphi_N^h = \nabla\varphi(1 - \frac{Ch}{\varphi})$.

So we get, similarly to the proof of Proposition 4.4.2-b), for C big enough and $h \in (0, h_{N,\rho',\epsilon}]$, with $h_{N,\rho',\epsilon} > 0$ small enough:

$$\begin{aligned}\tilde{\delta}_2(C,N)(h^{-N_0}\left\|\tilde{w}^h\right\|_{\Lambda^1 H^1} + 1) \geq \\ \left\|hd\tilde{w}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{w}^h\right\|^2_{\Lambda^0 L^2} + (1 + 2\delta(C)h)\langle|\nabla x_n|^2\tilde{w}^h \mid \tilde{w}^h\rangle_{\Lambda^1 L^2} \\ +h\int_{\Gamma_{ND}} \langle\tilde{w}^h \mid \tilde{w}^h\rangle_{\Lambda^1 T^*_\sigma\Omega}\left(\frac{\partial x_n}{\partial n}\right)(\sigma)\,d\sigma\,.\end{aligned}$$

After treating the r.h.s. like in the proof of Proposition 4.4.2-b)-Step 1, we obtain, for a constant $N_0 > 0$,

$$\left\|\tilde{w}^h\right\|_{\Lambda^1 H^1(\Omega_{U_1,\rho})} \leq C_4\,h^{-N_0}\,.$$

Our choice of (ε, ρ') imply

$$\forall x \in \Omega_{U_1,\rho'}, \quad \varphi_N^h \geq \varphi(x) + Nh\log h^{-1} + \tilde{C}h\log h\,.$$

We have proved the existence of N_1 and ρ'_0, such that, for any $N \in \mathbb{N}$ and $\rho' \in (0, \rho'_0]$, there exists $h_{N,\rho'} > 0$ and $C_{N,\rho'} > 0$, such that:

$$\left\|e^{\frac{\varphi}{h}}(u_1^h - c(h)u_1^{wkb})\right\|_{\Lambda^1 H^1(\Omega_{U_1,\rho'})} \leq C_{N,\rho'}\,h^{N-N_1}$$

holds for any $h \in (0, h_{N,\rho'})$.
This last estimate and $\Phi|_{\Gamma_{ND}} = \varphi$ imply

$$\left\|e^{\frac{\Phi}{h}}(u_1^h - c(h)u_1^{wkb})\right\|_{\Lambda^1 H^{1/2}(\Omega_{U_1,\rho'}\cap\Gamma_{ND})} = \mathcal{O}(h^\infty)\,.$$

Step 2: Comparison in the normal direction.

After replacing ρ' by ρ , Step 1 provides the estimate

$$\left\|e^{\frac{\varphi}{h}}(u_1^h - c(h)u_1^{wkb})\right\|_{\Lambda^1 H^1} = \mathcal{O}(h^\infty)\,. \tag{4.4.22}$$

We work in $\Omega_{U_1,\rho}$ with the above estimate and $\rho' \in (0, \rho/2)$ will be taken again small enough.

In order to get the interior estimate with the weight $e^{\frac{\Phi}{h}}$, we modify the previous analysis like in the proof of Proposition 4.4.2-b). The sets $\Omega_{\pm}^h$ are now given by

$$\Omega_-^h = \left\{x = (x', x_n) \in \Omega_{U_1,\rho}\ ;\ \Phi < Ch\right\}\ ,$$
$$\text{and}\qquad \Omega_+^h = \left\{x = (x', x_n) \in \Omega_{U_1,\rho}\ ;\ \Phi > Ch\right\}\ .$$

The function φ_N^h, $N \in \mathbb{N}$, is given by

$$\varphi_N^h(x) = \min\left\{\varphi^h(x) + Nh\log h^{-1}, \psi(x)\right\}\ ,$$
$$\text{with}\qquad \varphi^h(x) = \begin{cases} \Phi(x) - Ch\log\frac{\Phi(x)}{h}\ , & \text{if } \Phi > Ch\ ,\\ \Phi(x) - Ch\log C\ , & \text{if } \Phi \leq Ch\ ,\end{cases}$$
$$\text{and}\qquad \psi(x) = \min\left\{\varphi^h(y) + (1-\varepsilon)d_{Ag}(x,y),\ y \in \operatorname{supp}\nabla\chi\right\}\ .$$

We recall that the Agmon distance $d_{Ag}(x, y)$ is the distance between x and y for the metric $|\nabla f|^2\, dx^2$ and $\Phi(x) = d_{Ag}(x, U_1)$.
Again, the constant $C \geq 1$ will be fixed in the end like in the proof of Proposition 4.4.2-b), while the constants $\rho' \in (0, \rho/2)$ and $\varepsilon > 0$ are chosen so that:

$$\varphi_N^h(x) = \varphi^h(x) + Nh\log h^{-1} \text{ in } \Omega_{U_1,\rho'}\ .$$

Again, this implies:

$$\varphi_N^h(x) = \varphi^h(x) + Nh\log h^{-1} \quad \text{on } \Omega_-^h$$

Now we have the inequalities

$$\varphi_N^h(x) \leq \Phi(x) + Nh\log h^{-1} \quad \text{in } \Omega_{U_1,\rho}$$
$$\text{and}\qquad \varphi_N^h(x) \leq \Phi(x) \quad \text{in } \operatorname{supp}\nabla\chi\ .$$

Hence the estimate

$$\left\|e^{\frac{\varphi_N^h}{h}} r^h\right\|_{\Lambda^1 L^2} = \mathcal{O}(h^{-N_0})$$

is still valid.
Inequality (4.4.22) implies that the L^2-norm of the trace of $\tilde{w}^h$ on Γ_{ND} is $\mathcal{O}(h^\infty)$ and we have the following estimate:

$$\left\|\tilde{w}^h\right\|_{\Lambda^1 L^2(\Omega_-^h)} \leq C_2(C)h^{-N}\left\|w^h\right\|_{\Lambda^1 L^2(\Omega_-^h)} \leq C_3(C,N)\ .$$

With these estimates, the integration by parts formula (4.4.8) and (4.2.16) lead to:

$$\tilde{\delta}(C,N)(h^{-N_0}\left\|\tilde{w}^h\right\|_{\Lambda^1 L^2} + 1) \geq \left\|hd\tilde{w}^h\right\|^2_{\Lambda^2 L^2} + \left\|hd^*\tilde{w}^h\right\|^2_{\Lambda^0 L^2}$$
$$+ \langle(|\nabla\varphi|^2 - |\nabla\varphi_N^h|^2 - C_1 h)1_{\Omega_+^h}(x)\tilde{w}^h \mid \tilde{w}^h\rangle\ .$$

Finally, for almost all $x \in \Omega_+^h$ we have:
either: $\nabla\varphi_N^h(x) = \nabla\psi(x)$
and

$$|\nabla f|^2 - \left|\nabla\varphi_N^h\right|^2 = (2\varepsilon - \varepsilon^2)\,|\nabla f(x)|^2 \geq \delta_{\rho,\varepsilon} > 0\ ;$$

or: $\nabla\varphi_N^h(x) = \nabla\psi^h(x)$
and we get like in the proof of Proposition 4.4.2-b)

$$|\nabla f|^2 - \left|\nabla\varphi^h\right|^2 \geq C_4 Ch\ .$$

By taking C big enough, we get that $||e^{\frac{\varphi_N^h}{h}} w^h|| = \mathcal{O}(h^{-N_0})$ for some $N_0 > 0$.
Like in Step 1, this leads to

$$\left\|e^{\frac{\Phi}{h}}(u_1^h - c(h)u_1^{wkb})\right\|_{\Lambda^1 H^1(\Omega_{U_1,\rho'})} = \mathcal{O}(h^\infty)\ ,$$

for $\rho' \in (0, \rho/2)$ small enough.

Step 3:
The estimates in higher order Sobolev spaces are done like in the proof of Proposition 4.4.2-b) by a bootstrap argument after writing a boundary value problem for $\chi(u_1^h - c(h)u_1^{wkb})$ in $\mathbb{R}^n_-$. ∎

4.5 Labelling of local minima and construction of the quasimodes

4.5.1 Preliminaries

Here we adapt to our case with Neumann boundary condition the method of selecting the proper critical points with index 1 which was used in [HeKlNi] and in [HeNi]. We recall that the intuition for getting the good labelling of local minima, which is useful even to state properly the assumptions and results, comes from the probabilistic approach. The local minima have to be labelled according to the decreasing order of exit times. We refer to [BoGaKl], [BoEcGaKl] and [FrWe] for details.
Note that a similar strategy has independently been considered in [CoPaYc] for the spectral analysis on Markov processes on graphs.

The existence of such a labelling is an assumption which is generically satisfied. After this, it is possible to construct accurately quasimodes leading, with the help of the Witten complex structure, to accurate asymptotic expansions of the low lying eigenvalues.

4.5.2 Generalized critical points and local structure of the level sets of a Morse function

We recall that we work here on a compact connected oriented Riemannian manifold $\overline{\Omega} = \Omega \cup \partial\Omega$ with boundary and that the function f satisfies Assumption 4.3.1. According to our preliminary results on the Witten Laplacian $\Delta^N_{f,h}$ in Theorem 4.3.5, we introduce the following definition of generalized critical points with index p.

Definition 4.5.1.
A point $U \in \overline{\Omega}$ will be called a generalized critical point of f with index p if:
- *either $U \in \Omega$ and U is a critical point of f with index p,*
- *or $U \in \partial\Omega$ and U is a critical point with index p of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(U) < 0$ ($\vec{n}$ being the outgoing normal vector).*

Remark 4.5.2. *In particular, for $p = 0$, we get that the generalized minima are simply the local minima.*

The set of generalized critical points with index p is denoted by $\mathcal{U}^{(p)}$. We recall that we want to analyze the Witten Laplacian on 0-forms so we restrict our attention to the cases $p = 0$ and $p = 1$. From now on, we will use the notation:

$$m_p = \#\mathcal{U}^{(p)} \text{ for } p = 0, 1 \tag{4.5.1}$$

instead of $m_p^{\overline{\Omega}}$.
Finally it is convenient to call $\mathcal{U}$ the union of all critical points of f and $f|_{\partial\Omega}$.

Before labelling the local minima, let us recall a few remarks coming from the local analysis of a Morse function which satisfies Assumption 4.3.1 (we refer to [Mil1], [HeKlNi], and [HeNi]).

Local structure of the level sets of a Morse function.

In order to analyze the local situation near a point x_0 of $\overline{\Omega}$, let us introduce:

$$A_f^<(x_0) := \left\{x \in \overline{\Omega}\,;\, f(x) < f(x_0)\right\} \cap B_{x_0} ,$$

where B_{x_0} is a ball centered at x_0. Similarly, we can introduce

$$A_f^\leq(x_0) := \left\{x \in \overline{\Omega}\,;\, f(x) \leq f(x_0)\right\} \cap B_{x_0} .$$

Interior points:
First we observe that, near a non critical point $x_0 \in \Omega$ of f, one can find B_{x_0} and a set of local coordinates such that

$$A_f^<(x_0) = \{y_1 < 0\} \cap B_{x_0} .$$

Secondly, if x_0 is a critical point with index p, then there exists a ball B_{x_0} around x_0 and a set of local coordinates centered at x_0 such that

$$A_f^{<}(x_0) = \left\{ -\sum_{\ell=1}^{p} y_\ell^2 + \sum_{\ell=p+1}^{n} y_\ell^2 < 0 \right\} \cap B_{x_0} \,,$$

and

$$A_f^{\leq}(x_0) = \left\{ -\sum_{\ell=1}^{p} y_\ell^2 + \sum_{\ell=p+1}^{n} y_\ell^2 \leq 0 \right\} \cap B_{x_0} \,.$$

We now observe that

1. When $p = 0$ (local minimum), $A_f^{<}(x_0)$ is empty and $A_f^{\leq}(x_0)$ is reduced to $\{x_0\}$.
2. When $p = 1$, $A_f^{<}(x_0)$ has two connected components and x_0 belongs to the closure of each of the two components. This property will be crucial in the discussion.
3. When $p \geq 2$, $A_f^{<}(x_0)$ is (arcwise) connected.

Points on the boundary:
If x_0 belongs to $\partial\Omega$, Assumption 4.3.1 leads to two cases:
First case.
If x_0 is not a critical point of $f|_{\partial\Omega}$, then the hypersurfaces $\{x \mid f(x) = f(x_0)\}$ and $\partial\Omega$ intersect transversally in a neighborhood of x_0. Hence there is a ball B_{x_0} around x_0 and a set of local coordinates such that

$$A_f^{<}(x_0) = \{y_1 < 0, y_n \leq 0\} \cap B_{x_0} \,,$$

and

$$A_f^{\leq}(x_0) = \{y_1 \leq 0, y_n \leq 0\} \cap B_{x_0} \,,$$

with $\Omega \cap B_{x_0} = \{y_n < 0\} \cap B_{x_0}$.
Second case.
If x_0 is a critical point of $f|_{\partial\Omega}$ with index p and with $\pm\frac{\partial f}{\partial n}(x_0) > 0$, there are local coordinates $(y_1, \ldots, y_{n-1}, y_n)$, constructed from the second point of Lemma 4.3.18, such that $(y_1, \ldots, y_{n-1})$ are Morse coordinates for $f|_{\partial\Omega}$ and such that

$$A_f^{<}(x_0) = \left\{ \pm y_n - \sum_{i=1}^{p} y_i^2 + \sum_{i=p+1}^{n-1} y_i^2 < 0 \,,\ y_n \leq 0 \right\} \cap B_{x_0} \,,$$

and

$$A_f^{\leq}(x_0) = \left\{ \pm y_n - \sum_{i=1}^{p} y_i^2 + \sum_{i=p+1}^{n-1} y_i^2 \leq 0 \,,\ y_n \leq 0 \right\} \cap B_{x_0} \,.$$

These local models allow to see that

1. If x_0 is a local minimum of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(x_0) < 0$, then $A_f^{<}(x_0) = \emptyset$ and $A_f^{\leq}(x_0) = \{x_0\}$.
2. If x_0 is a local minimum of $f|_{\partial\Omega}$ such that $\frac{\partial f}{\partial n}(x_0) > 0$, then $A_f^{<}(x_0) \cap \partial\Omega = \emptyset$ and $A_f^{\leq}(x_0) \cap \partial\Omega = \{x_0\}$. Moreover, $A_f^{<}(x_0)$ is connected.
3. If $p = 1$ and $\frac{\partial f}{\partial n}(x_0) < 0$ (i.e. if $x_0 \in \mathcal{U}^{(1)} \cap \partial\Omega$), $A_f^{<}(x_0)$ has two connected components with a non-empty intersection with $\partial\Omega$ and x_0 belongs to the closure of each of the two components. Again, this property will be crucial in the discussion.
4. In all other cases, $A_f^{<}(x_0)$ is connected with a non-empty intersection with $\partial\Omega$.

4.5.3 Labelling of local minima and first consequence

Remember our main Assumption 4.1.1:
The function f has $\#\mathcal{U}$ distinct critical values and the quantities $f(U^{(1)}) - f(U^{(0)})$, with $U^{(1)} \in \mathcal{U}^{(1)}$ and $U^{(0)} \in \mathcal{U}^{(0)}$ are distinct.

Definition 4.5.3. *For $\lambda \in \mathbb{R}$, we define $H^0(\{f < \lambda\})$ as the number of connected components of the level set $L(\lambda) = f^{-1}((-\infty, \lambda))$.*

Due to local structure of the level sets of a Morse function and to Assumption 4.1.1, the function $H^0(\{f < \lambda\})$ of $\lambda \in \mathbb{R}$ is a step function which satisfies, with λ decreasing from $+\infty$:

- $H^0(\{f < \lambda\})$ decreases by 1 around every $\lambda = f(U^{(0)})$ with $U^{(0)} \in \mathcal{U}^{(0)}$.
- wherever $H^0(\{f < \lambda\})$ increases by 1, it is around a $\lambda = f(U^{(1)})$ with $U^{(1)} \in \mathcal{U}^{(1)}$.
- $H^0(\{f < \lambda\})$ is locally constant away from those points.

Remark 4.5.4. *$\overline{\Omega}$ is connected and compact so $H^0(\{f < \lambda\})$ equals respectively 1 or 0 for $\lambda \geq \lambda_f$ or $\lambda \leq -\lambda_f$ for some $\lambda_f > 0$.*
Consequently, the previous discussion implies that the number of critical values of f with index 1 where $H^0(\{f < \lambda\})$ increases (by 1) is equal to $m_0 - 1$ and so that $m_1 + 1 \geq m_0$.

We now label the local minima of f as follow:

1) We set $U_1^{(0)} = \min_{x \in \overline{\Omega}} f$, $z_1 = \infty$, $f(z_1) = z_1 = \infty$ and we consider $H^0(\{f < \lambda\})$ for λ decreasing from $f(z_1) = +\infty$.

2) When $U_k^{(0)}$ and z_k are defined for $k = 1, \ldots, K-1$, decrease λ from $f(z_{K-1})$ until $H^0(\{f < \lambda\})$ increases by 1. Denote by λ_K this value.

3) By Assumption 4.1.1 and by the previous discussion, there exists a unique point in $\mathcal{U}^{(1)}$, that we denote by z_K, satisfying $f(z_K) = \lambda_K$. Then we denote by $U_K^{(0)}$ the global minimum of the new connected component.

4) We iterate **2)** and **3)** until all the local minima have been considered.

5) At least we permute the k's to make the sequence $\left(f(z_k) - f(U_k^{(0)})\right)_{k\in\{1,\ldots,m_0\}}$ strictly decreasing, which is possible by Assumption 4.1.1.

Definition 4.5.5. *(The map j)*
If the generalized critical points with index 1 are numbered $U_j^{(1)}$, $j = 1, \ldots, m_1$, we set $U_1^{(1)} = z_1 = \infty$ and we define the application $k \mapsto j(k)$ from $\{1, \ldots, m_0\}$ to $\{0, 1, \ldots, m_1\}$ by:

$$\begin{cases} j(1) = 0 \text{ and } U_{j(1)}^{(1)} = z_1 \\ \quad \forall\, k \geq 2\,,\; U_{j(k)}^{(1)} = z_k\,. \end{cases}$$

Definition 4.5.6.
For $k \in \{1, \ldots, m_0\}$, we denote by E_k the connected component of $U_k^{(0)}$ in

$$f^{-1}((-\infty, f(U_{j(k)}^{(1)})]) \setminus \{U_{j(k)}^{(1)}\}\,.$$

Remark 4.5.7.
By the previous construction, $U_k^{(0)}$ is the global minimum of E_k.

Proposition 4.5.8.
Under Assumption 4.1.1, the following properties are satisfied:
a) *The sequence $\left(f(U_{j(k)}^{(1)}) - f(U_k^{(0)})\right)_{k\in\{1,\ldots,m_0\}}$ is strictly decreasing.*
b) *$E_1 = \overline{\Omega}$ is compact and for any $k > 1$ the set E_k is a relatively compact subset of $f^{-1}((-\infty, f(U_{j(k)}^{(1)})])$ satisfying $\overline{E_k} = E_k \cup \left\{U_{j(k)}^{(1)}\right\}$.*
c) *For any $(k, j) \in \{1, \ldots, m_0\} \times \{0, 1, \ldots, m_1\}$, the relation $U_j^{(1)} \in E_k$ implies:*

$$\text{either} \quad (j = j(k') \text{ for some } \quad k' > k) \;\text{ or }\quad j \notin j(\{1, \ldots, m_0\})\,.$$

d) *For any $k \neq k' \in \{1, \ldots, m_0\}$, the relation $U_{k'}^{(0)} \in E_k$ implies:*

$$\left(k' > k \quad \text{and} \quad f(U_{k'}^{(0)}) > f(U_k^{(0)})\right)\,.$$

e) *The application $j : \{1, \ldots, m_0\} \to \{0, 1, \ldots, m_1\}$ is injective.*

Proof.
By Assumption 4.1.1 and by construction, the points **a)**, **b)** and **e)** are obvious.
c) Assume now $U_{j(k')}^{(1)} \in E_k$.
Since $U_{j(k)}^{(1)} \notin E_k$, one has $k \neq k'$. Moreover, by definition of E_k and by Assumption 4.1.1, we have the inequality $f(U_{j(k')}^{(1)}) < f(U_{j(k)}^{(1)})$ which implies that $E_{k'}$ is contained in E_k, by connectedness of E_k and $E_{k'}$.

Consequently, $U_{k'}^{(0)} \in E_k$ and by Assumption 4.1.1, $f(U_{k'}^{(0)}) > f(U_k^{(0)})$ (because $U_k^{(0)}$ is the global minimum of f on $\overline{E_k}$) which yields:

$$f(U_{j(k')}^{(1)}) - f(U_{k'}^{(0)}) < f(U_{j(k)}^{(1)}) - f(U_k^{(0)})$$

and the point **a)** gives $k' > k$.
d) Assume $U_{k'}^{(0)} \in E_k$ for $k \neq k'$.
Again one has $f(U_{k'}^{(0)}) > f(U_k^{(0)})$ which implies $k' \neq 1$ (then $U_{j(k')}^{(1)} \in \overline{\Omega}$) and there are two possible cases:

$$U_{j(k')}^{(1)} \in \overline{E_k} \quad \text{or} \quad U_{j(k')}^{(1)} \notin \overline{E_k}\,.$$

In the second case, let us look at $E_{k'}$. $E_{k'}$ is connected and $U_{k'}^{(0)}$ is the global minimum of f on $E_{k'}$. Moreover, $U_{k'}^{(0)} \in E_{k'} \cap E_k$ and $U_{j(k')}^{(1)} \in \overline{E_{k'}} \setminus \overline{E_k}$ imply, by connectedness, that $\partial E_k \cap E_{k'} \neq \emptyset$.
$\overline{E_k}$ is then contained in $E_{k'}$ and $U_k^{(0)} \in E_{k'}$, which cannot occur.
Consequently, $U_{j(k')}^{(1)} \in \overline{E_k}$ and the points **b)** and **c)** imply $k' > k$. ∎

4.5.4 Construction of the quasimodes

Like in [HeKlNi] and in [HeNi], we associate with every $U_k^{(0)}$ $(k \in \{1, \dots, m_0\})$ a quasimode for $\Delta_{f,h}^{N,(0)}$ which is approximately supported in E_k, while the quasimodes for $\Delta_{f,h}^{N,(1)}$ will be supported in the balls $B(U_j^{(1)}, 2\,\varepsilon_1)$ $(j \in \{1, \dots, m_1\})$. A ball $B(U, \rho)$, with $U \in \overline{\Omega}$ and $\rho > 0$, is a geodesic ball and the geodesic distance is denoted by d_Ω. The parameter $\varepsilon_1 > 0$ is fixed so that:

- $d_\Omega(U, U') \geq 10\,\varepsilon_1$ for U, $U' \in \mathcal{U}$, $U \neq U'$.
- For all $U \in \mathcal{U}$ and all $k \in \{1, \dots, m_0\}$, $U \notin \overline{E_k}$ implies

$$d_\Omega(U, E_k) \geq 10\,\varepsilon_1\,.$$

- The construction of the WKB approximation of Subsection 4.4.6 is possible in the ball $B(U_j^{(1)}, 2\,\varepsilon_1)$. If $U_j^{(1)}$ is a boundary point, this means the introduction of the coordinates (x', x_n) used in Section 4.4.3 and the existence of Φ. Recall that in these coordinates, Φ and g_0 have the form:

$$\Phi = -x_n + \varphi(x') \quad \text{and} \quad g_0 = g_{nn}(x)\, dx_n^2 + \sum_{i,j=1}^{n-1} g_{ij}(x)\, dx_i dx_j\,. \qquad (4.5.2)$$

The parameter $\varepsilon_1 > 0$ will be kept fixed, while we need another parameter $\varepsilon \in (0, \varepsilon_0)$ which will be fixed in the final step of the proof.

Like in [HeNi], the construction presented in [HeKlNi] has to be adapted when $U^{(1)}_{j(k)} \in \partial\Omega$ or $U^{(0)}_k \in \partial\Omega$ (recall that in [HeNi], the case $U^{(0)}_k \in \partial\Omega$ did not occur) and we focus on these changes.
However, note that in [HeNi] the set $\overline{E_k}$ intersected $\partial\Omega$ at most at one point ($\overline{E_k} \cap \partial\Omega \subset \left\{U^{(1)}_{j(k)}\right\}$). It is not the case here and we cannot use the same construction when $U^{(1)}_{j(k)} \in \partial\Omega$.

For every $k \in \{1, \dots, m_0\}$ and $\varepsilon > 0$, we introduce the set:

$$\widetilde{\Omega}_k(\varepsilon, \delta) = \left\{x \in \overline{\Omega}, d_\Omega\left(x, \overline{E_k} \setminus B(U^{(1)}_{j(k)}, \varepsilon)\right) < \delta\right\} \cup B(U^{(1)}_{j(k)}, \varepsilon) ,$$

with $\delta \in (0, \delta_\varepsilon)$, $\delta_\varepsilon > 0$ small enough.

The cut-off function $\tilde{\chi}_{k,\varepsilon} \in \mathcal{C}^\infty_0(\overline{\Omega})$, $0 \leq \tilde{\chi}_{k,\varepsilon} \leq 1$ is chosen so that:

$$\operatorname{supp} \tilde{\chi}_{k,\varepsilon} \subset \widetilde{\Omega}_k(\varepsilon, \delta_\varepsilon) \quad \text{and} \quad \tilde{\chi}_{k,\varepsilon}|_{\widetilde{\Omega}_k(\varepsilon,\delta_\varepsilon/2)\setminus B(U^{(1)}_{j(k)},\varepsilon)} = 1 .$$

Around $U^{(1)}_{j(k)}$, the cut-off function $\tilde{\chi}_{k,\varepsilon}$ is chosen (more accurately below when $U^{(1)}_{j(k)} \in \partial\Omega$) so that $U^{(1)}_{j(k)} \notin \operatorname{supp} \tilde{\chi}_{k,\varepsilon}$ and

$$\forall x \in B(U^{(1)}_{j(k)}, \varepsilon) , \ \left(\tilde{\chi}_{k,\varepsilon}(x) \neq 0 \text{ , and } f(x) < f(U^{(1)}_{j(k)})\right) \Rightarrow x \in E_k \ . \tag{4.5.3}$$

Remark 4.5.9. *The cut-off functions $\tilde{\chi}_{k,\varepsilon}$ are used in the construction of quasi-modes for $\Delta^{N,(0)}_{f,h}$.*
Moreover, in the case $k = 1$, we have by construction $\tilde{\chi}_{k,\varepsilon} \equiv 1$ in $\overline{\Omega}$ because $U^{(1)}_{j(1)} \notin \overline{\Omega}$. This case provides directly the eigenvector $\left\|e^{-f(x)/h}\right\|^{-1} e^{-f(x)/h}$ (of $\Delta^{N,(0)}_{f,h}$) with the eigenvalue 0.

Like in [HeKlNi] and in [HeNi], we deduce from Proposition 4.5.8 the following properties for $\tilde{\chi}_{k,\varepsilon}$.

Proposition 4.5.10.
By taking $\delta = \delta_\varepsilon$ with $\varepsilon \in (0, \varepsilon_0]$, $0 < \varepsilon_0 \leq \varepsilon_1$ small enough, the cut-off functions $\tilde{\chi}_{k,\varepsilon}$ ($k \in \{1, \dots, m_0\}$) satisfy the following properties:

a) *If x belongs to $\operatorname{supp} \tilde{\chi}_{k,\varepsilon}$ and $f(x) < f(U^{(1)}_{j(k)})$, then $x \in E_k$.*

b) *There exist $C > 0$ and, for any $\varepsilon \in (0, \varepsilon_0]$, a constant $K_\varepsilon > 0$, such that, for $x \in \operatorname{supp} \nabla\tilde{\chi}_{k,\varepsilon}$,*

$$\begin{array}{llll} \textit{either} & x \notin B(U^{(1)}_{j(k)}, \varepsilon) & \textit{and} & f(U^{(1)}_{j(k)}) + K_\varepsilon^{-1} \leq f(x) \leq f(U^{(1)}_{j(k)}) + K_\varepsilon \ , \\ \textit{or} & x \in B(U^{(1)}_{j(k)}, \varepsilon) & \textit{and} & \left|f(x) - f(U^{(1)}_{j(k)})\right| \leq C\varepsilon \ . \end{array}$$

c) *For any* $U \in \mathcal{U}$, $U \neq U^{(1)}_{j(k)}$, *the distance* $d_\Omega(U, \operatorname{supp} \nabla \tilde{\chi}_{k,\varepsilon})$ *is bounded from below by* $3\varepsilon_1 > 0$. *If in addition* $U \in \operatorname{supp} \tilde{\chi}_{k,\varepsilon}$, *then* $U \in E_k$.

d) *If, for some* $k' \in \{1, \ldots, m_0\}$, $U^{(0)}_{k'}$ *belongs to* $\operatorname{supp} \tilde{\chi}_{k,\varepsilon}$, *then* $k' \geq k$ *and*

$$f(U^{(0)}_{k'}) > f(U^{(0)}_k), \quad f(U^{(1)}_{j(k')}) \leq f(U^1_{j(k)}) , \quad \text{if } k \neq k' .$$

e) *For any* $j \in \{1, \ldots, m_1\}$, *such that* $U^{(1)}_j \in \operatorname{supp} \tilde{\chi}_{k,\varepsilon}$,

$$\begin{array}{ll} \text{either} & j \notin j(\{1, \ldots, m_0\}) , \\ \text{or} & j = j(k') , \ \text{for some } \ k' \geq k \text{ and } U^{(0)}_{k'} \in \operatorname{supp} \tilde{\chi}_{k,\varepsilon} . \end{array}$$

The quasimodes for $\Delta^{N,(1)}_{f,h}$ associated with the $U^{(1)}_j \in \Omega$ are constructed like in [HeKlNi] and in [HeNi] (and rely on the approximation by the Dirichlet problem in small balls $B(U^{(1)}_j, 2\varepsilon_1)$). We will not recall the complete construction here.
In the same spirit as in [HeNi], the quasimodes associated with the $U^{(1)}_j \in \partial\Omega$ will rely on the approximation by the Neumann realization associated with the neighborhood $\Omega_{U^{(1)}_j,\rho}$ ($\rho > 0$ small enough) which was studied in Subsection 4.4.6.
Once $\rho > 0$ is fixed uniformly for all $U^{(1)}_j \in \partial\Omega$, the parameter $\varepsilon_1 > 0$ is reduced so that $B(U^{(1)}_j, 2\varepsilon_1) \subset \Omega_{U_1,\rho}$ for all $U^{(1)}_j \in \partial\Omega$.
For all $j \in \{1, \ldots, m_1\}$, u_j denotes a normalized eigenvector associated with the first (exponentially small) eigenvalue of this Dirichlet or Neumann realization. The cut-off function $\theta_j \in \mathcal{C}^\infty_0(B(U^{(1)}_j, 2\varepsilon_1))$ is taken such that $\theta_j = 1$ on $B(U^{(1)}_j, \varepsilon_1)$ and $\frac{\partial\theta_j}{\partial n}\big|_{\partial\Omega} \equiv 0$ for boundary points $U^{(1)}_j \in \partial\Omega$.
Note that the function $\tilde{\chi}_{k,\varepsilon}$ depends on $\varepsilon \in (0, \varepsilon_0]$, while θ_j is kept fixed like $\varepsilon_1 > 0$.

Definition 4.5.11. *For cut-off* $\chi_{k,\varepsilon}$ *satisfying the properties of Proposition 4.5.10 like* $\tilde{\chi}_{k,\varepsilon}$, *introduce the following quasimodes.*
For any $k \in \{1, \ldots, m_0\}$, *the* (ε, h)*-dependent function* $\psi^{(0)}_k$ *is defined by*

$$\psi^{(0)}_k(x) = \left\| \chi_{k,\varepsilon}(x) e^{-(f(x)-f(U^{(0)}_k))/h} \right\|^{-1} \chi_{k,\varepsilon}(x) e^{-(f(x)-f(U^{(0)}_k))/h} .$$

For any $j \in \{1, \ldots, m_1\}$, *the* h*-dependent* 1*-form* $\psi^{(1)}_j$ *is defined by*

$$\psi^{(1)}_j(x) = \left(\|\theta_j u_j\|^{-1} \right) \theta_j(x) u_j(x) .$$

We set $\lambda^{app}_1(\varepsilon, h) = 0$, *and for any* $k \in \{2, \ldots, m_0\}$:

$$\lambda^{app}_k(\varepsilon, h) = \left| \left\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h} \psi^{(0)}_k \right\rangle \right|^2 .$$

Remark 4.5.12.
a) *In the case* $U^{(1)}_{j(k)} \in \Omega$, $\chi_{k,\varepsilon}$ *is* $\tilde{\chi}_{k,\varepsilon}$ *with additional properties (see [HeKlNi] for details) and we will still denote it here by* $\tilde{\chi}_{k,\varepsilon}$*. In the case* $U^{(1)}_{j(k)} \in \partial\Omega$*, the real choice of* $\chi_{k,\varepsilon}$ *will be fixed further (see Definition 4.5.20). Moreover,* $\chi_{k,\varepsilon}$ *also satisfies the properties of Proposition 4.5.10.*
b) *For the sake of conciseness, we omit the* (ε, h)*- and* h*- dependence in the notations* $\psi^{(0)}_k$ *and* $\psi^{(1)}_j$.
c) *We will show in the next section that the* $\lambda^{app}_k(\varepsilon, h)$*'s are approximated values of the small eigenvalues of* $\Delta^{N,(0)}_{f,h}$.
By Remark 4.5.9, this definition is coherent for $k = 1$ *and* $\psi^{(0)}_1$ *is the normalized eigenvector associate with the eigenvalue* 0.
d) *Due to the condition* $\frac{\partial\theta_j}{\partial n}\big|_{\partial\Omega} \equiv 0$, $\psi^{(1)}_j$ *belongs to* $D(\Delta^{N,(1)}_{f,h})$ *and this, even if* $U^{(1)}_j$ *belongs to* $\partial\Omega$.

4.5.5 Quasimodal estimates

We end this section by reviewing the quasimodal estimates which are derived from Propositions 4.5.8 and 4.5.10. The asymptotic expansion of the quantity $\left\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \right\rangle$ has also be done in [HeKlNi] when $U^{(0)}_k$ and $U^{(1)}_{j(k)} \in \Omega$ are interior points. Like in [HeNi], we will simply complete this analysis by establishing the asymptotic expansion of $\left\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \right\rangle$, when $U^{(0)}_k$ or $U^{(1)}_{j(k)}$ is in $\partial\Omega$.

Remark 4.5.13. *In this subsection, we make computations with different coordinate systems* $v = (v_1, \dots, v_n)$ *(around* $U = U^{(0)}_k$ *or* $U = U^{(1)}_{j(k)}$*) all given given by Lemma 4.3.18.*
Looking at the proof of Lemma 4.3.18 given in [Lep2], notice that the coordinates $(v_1, \dots, v_{n-1})$ *in the boundary can be chosen freely. Moreover, according to [HeSj4] pp. 279-280, they can be chosen such that* $dv_1(U), \dots, dv_{n-1}(U), \vec{n}^*_U$ *is orthonormal and positively oriented and*

$$f(v, 0) = \frac{\lambda_1}{2}v_1^2 + \dots + \frac{\lambda_{n-1}}{2}v_{n-1}^2 + f(U) \quad \text{and} \quad \varphi(v) = \frac{|\lambda_1|}{2}v_1^2 + \dots + \frac{|\lambda_{n-1}|}{2}v_{n-1}^2\,, \tag{4.5.4}$$

with $\lambda_1 < 0$ *when* $U = U^{(1)}_j$*. Hence all the coordinates systems around* $U \in \partial\Omega$ *will coincide on* $\partial\Omega$ *while they may differ in* Ω *according to the case when a normal form is used for* f, Φ *or* $f + \Phi$ *in* Ω.

Remind that the parameter $\varepsilon_1 > 0$ is fixed, while ε_0 and $\varepsilon \in (0, \varepsilon_0]$ may have to be adapted during the proof. **We shall denote by α a generic positive constant which is independent of $\varepsilon \in (0, \varepsilon_0]$.**
Introduce the following notation which will be very useful:

Definition 4.5.14. *The notation $g(h) = \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})$ means that, for all $\varepsilon \in (0, \varepsilon_0]$, there exists a constant $C_\varepsilon > 0$ such that:*

$$\forall h \in (0,\, h_0]\ , \qquad |g(h)| \leq C_\varepsilon e^{-\frac{\alpha}{h}}\ .$$

From Proposition 4.5.8-d) and the good localization of $\nabla\chi_{k,\varepsilon}$, we deduce the following estimates for $\psi_k^{(0)}$.

Proposition 4.5.15.
The system of (ε, h)-dependent functions $(\psi_k^{(0)})_{k\in\{1,\dots,m_0\}}$ of Definition 4.5.11 is almost orthogonal with

$$\left(\langle\psi_k^{(0)} \mid \psi_{k'}^{(0)}\rangle\right)_{k,k'\in\{1,\dots,m_0\}} = \mathrm{Id}_{\mathbb{C}^{m_0}} + \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})\ ,$$

and there exists $\alpha > 0$ and, for any $\varepsilon \in (0, \varepsilon_0]$, $C(\varepsilon)$ and $h_0(\varepsilon)$ such that, for any $h \in (0, h_0(\varepsilon)]$,

$$\langle\Delta_{f,h}^{N,(0)}\psi_k^{(0)} \mid \psi_k^{(0)}\rangle = \left\|d_{f,h}^{(0)}\psi_k^{(0)}\right\|^2 \leq C(\varepsilon)e^{-2\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})-\alpha\varepsilon}{h}}\ .$$

Corollary 4.5.16.
There exists $\varepsilon_0 > 0$ and $\alpha > 0$ such that, for any choice of ε in $(0, \varepsilon_0]$ and for all $k \in \{1, \dots, m_0\}$, the (ε, h)-dependent quasimodes $\psi_k^{(0)}$ satisfy the estimate

$$\langle\Delta_{f,h}^{N,(0)}\psi_k^{(0)} \mid \psi_k^{(0)}\rangle = \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})\ .$$

The exponential decay of the first eigenvector u_j, associated with an exponentially small eigenvalue, of the Dirichlet realization of $\Delta_{f,h}^{(1)}$ around $U_j^{(1)}$, provides the following estimates for $\psi_j^{(1)}$. We refer the reader to [HeKlNi] or [HeSj4] for $U_j^{(1)} \in \Omega$ and to Subsection 4.4.6 for $U_j^{(1)} \in \partial\Omega$.

Proposition 4.5.17.
The system of h-dependent 1-forms, $\left(\psi_j^{(1)}\right)_{j\in\{1,\dots,m_1\}}$ given in Definition 4.5.11 is orthonormal and there exists $\alpha > 0$ independent of ε such that

$$\langle\Delta_{f,h}^{N,(1)}\psi_j^{(1)} \mid \psi_j^{(1)}\rangle = \mathcal{O}(e^{-\frac{\alpha}{h}})\ ,$$

for all $j \in \{1, \dots, m_1\}$.

Let us now compute some asymptotic expansions.

Proposition 4.5.18. *For k in $\{2, \dots, m_0\}$ and x in $\overline{\Omega}$,*

$$\psi_k^{(0)}(x) = \gamma_k(h)(1 + a_k(h))\chi_{k,\varepsilon}(x)e^{-\frac{f(x)-f(U_k^{(0)})}{h}}\ ,$$

where $\gamma_k(h)$ is defined in Definition 4.1.2 and $a_k(h) \sim \sum_{\ell=1}^{\infty} a_{k,\ell}h^\ell$.

Proof. In the case $U_k^{(0)} \in \Omega$, we refer the reader to [HeKlNi].
If $U_k^{(0)} \in \partial\Omega$, we use again, in a neighborhood of $U_k^{(0)}$, the coordinate system $(\overline{x}', \overline{x}_n)$ introduced in the second part of the Section 4.3.3 (with $\overline{x}(U_k^{(0)}) = 0$). In this coordinate system, f and g_0 equal:

$$f(\overline{x}) = -\overline{x}_n + f|_{\partial\Omega}(\overline{x}') = -\overline{x}_n + f(U_k^{(0)}) + \varphi(\overline{x}')\,, \tag{4.5.5}$$

$$g_0 = g_{nn}(\overline{x})\ d\overline{x}_n^2 + \textstyle\sum_{i,j=1}^{n-1} g_{ij}(\overline{x})\ d\overline{x}_i d\overline{x}_j\,, \tag{4.5.6}$$

where $\varphi = f|_{\partial\Omega} - f(U_k^{(0)})$ is the Agmon distance to $U_k^{(0)}$ on the boundary. We denote by $V_{g_0}(d\overline{x})$ the normalized volume form:

$$V_{g_0}(d\overline{x}) = (\det G_0(\overline{x}))^{1/2} d\overline{x}' \wedge d\overline{x}_n =: \nu(\overline{x}', \overline{x}_n) d\overline{x}' \wedge d\overline{x}_n\,.$$

From (4.5.5),

$$d\overline{x}_n(U_k^{(0)}) = -\frac{\partial f}{\partial n}(U_k^{(0)})\ \vec{n}^*_{U_k^{(0)}} \text{ and } \nu(0,0) = \left(-\frac{\partial f}{\partial n}(U_k^{(0)})\right)^{-1}\,. \tag{4.5.7}$$

For some constants $\eta > 0$ and $\delta_\eta > 0$,

$$\begin{aligned}\left\|\chi_{k,\varepsilon} e^{-\frac{f(x)-f(U_k^{(0)})}{h}}\right\|^2 &= \int_{\overline{\Omega}} \chi_{k,\varepsilon}^2 e^{-2\frac{f(x)-f(U_k^{(0)})}{h}} V_{g_0}(dx)\\ &= \int_{B(0,\eta)} e^{2\frac{\overline{x}_n}{h}} e^{-2\frac{\varphi(\overline{x}')}{h}} \nu(\overline{x}', \overline{x}_n) d\overline{x}' \wedge d\overline{x}_n + \mathcal{O}(e^{-\frac{\delta_\eta}{h}})\,.\end{aligned}$$

According to (4.5.4),

$$\left\|\chi_{k,\varepsilon} e^{-\frac{f(x)-f(U_k^{(0)})}{h}}\right\|^2 = \int_{B(0,\eta)} e^{2\frac{\overline{x}_n}{h}} e^{-\frac{|\lambda_1|\overline{x}_1^2 + \cdots + |\lambda_{n-1}|\overline{x}_{n-1}^2}{h}} \nu(\overline{x}', \overline{x}_n) d\overline{x} \wedge d\overline{x}_n + \mathcal{O}(e^{-\frac{\delta_\eta}{h}})\,.$$

By expanding $\nu(\overline{x}', \overline{x}_n)$ to a Taylor Series of arbitrary order $k \in \mathbb{N}^*$, we can separate the variables $\overline{x}'$ and $\overline{x}_n$ in the last integral term.
Hence, using the Laplace Method for each term, we obtain an asymptotic expansion of arbitrary order of $\left\|\chi_{k,\varepsilon} e^{-\frac{f(x)-f(U_k^{(0)})}{h}}\right\|^2$.
Moreover, from (4.5.7), the first term is:

$$\left(-\frac{\partial f}{\partial n}(U_k^{(0)})\right)^{-1} \frac{h}{2} \frac{(\pi h)^{\frac{n-1}{2}}}{\left|\det \text{Hess}\, f|_{\partial\Omega}(U_k^{(0)})\right|^{\frac{1}{2}}} = (\gamma_k(h))^{-2}\,.$$

∎

Proposition 4.5.19. *In $B(U^{(1)}_{j(k)}, \varepsilon_1)$, choose the coordinate system x which satisfies (4.5.2) and (4.5.4) with $\lambda_1 < 0$. For k in $\{2, \dots, m_0\}$, the equality*

$$\psi^{(1)}_{j(k)}(x) = \delta_{j(k)}(h) b(x,h) e^{-\frac{\Phi(x)}{h}} \,,$$

holds up to a phase factor, when $\delta_{j(k)}(h)$ is defined according to Definition 4.1.2, $b(x,h) \sim \sum_{\ell=0}^{\infty} b_{k,\ell}(x) h^{\ell}$, $b_{k,\ell}(x) = \sum_{i=1}^{n} b^{i}_{k,\ell}(x) dx_i$, and $b^{i}_{k,0}(0) = \delta_{1\,i}$.

Proof. In Section 4.4, we found a WKB approximation u_1^{wkb} of an eigenvector u_1^h such that,

$$\begin{array}{l} e^{\frac{\Phi(x)}{h}} u_1^{wkb} = \sum_{i=1}^{n} a_i^0(x) dx_i + h a^1(x,h) \,, \\ a_i^0(0) = \delta_{1\,i} \,,\ a^1(x,h) \sim \sum_{\ell} h^{\ell} a_{\ell}(x) \,, \end{array}$$

and

$$\forall x \in B(U^{(1)}_{j(k)}, 2\varepsilon_1), \quad e^{\frac{\Phi(x)}{h}} \left| \partial_x^{\alpha} (u_1^h(x) - u_1^{wkb}(x)) \right| \leq C_{\alpha,N} h^N \,.$$

The WKB approximation u_1^{wkb} was initially constructed in another coordinate system $(\underline{x}_1, \dots, \underline{x}_n)$. Remark 4.5.13 recalls that the tangential coordinates $\underline{x}_1, \dots, \underline{x}_{n-1}$ and $x_1, \dots, x_{n-1}$ can coincide in $\partial\Omega$ with different deformations as entering into Ω.

The normalized eigenvector that we take here is

$$u_{j(k)} = \frac{u_1^h}{\left\| u_1^h \right\|} \,.$$

Let us first compute accurately:

$$\left\| u_1^h \right\| = \left\| \theta_{j(k)} u_1^h \right\| + \mathcal{O}(h^{\infty}) = \left\| \theta_{j(k)} u_1^{wkb} \right\| + \mathcal{O}(h^{\infty}) \,.$$

Moreover,

$$\left\| \theta_{j(k)} u_1^{wkb} \right\|^2 = \int \theta_{j(k)}(x)^2 \langle a(x,h) \,|\, a(x,h) \rangle e^{-\frac{2\Phi(x)}{h}} V_{g_0}(dx) \,,$$

where the integral is over $x_n \leq 0$. Note furthermore that,

$$dx_n(U^{(1)}_{j(k)}) = -\frac{\partial \Phi}{\partial n}(U^{(1)}_{j(k)}) \, \vec{n}^{*}_{U^{(1)}_{j(k)}} = -\frac{\partial f}{\partial n}(U^{(1)}_{j(k)}) \, \vec{n}^{*}_{U^{(1)}_{j(k)}} \,.$$

Proceeding like in the proof of Proposition 4.5.18, we obtain, using the Laplace method, a full asymptotic expansion of $\left\| \theta_{j(k)} u_1^{wkb} \right\|^2$. The first term is given by the first term of

$$\int \theta_{j(k)}(x)^2 \langle a^0(x) \,|\, a^0(x) \rangle e^{\frac{2x_n}{h}} e^{-2\frac{\varphi(x')}{h}} V_{g_0}(dx) \,,$$

and from $\langle a^0(x) \,|\, a^0(x) \rangle(0) = 1$, we conclude like in the proof of Proposition 4.5.18. ∎

Before stating the next result, let us specify the choice of $\chi_{k,\varepsilon}$ when $U^{(1)}_{j(k)} \in \partial\Omega$. We assume $\varepsilon \in (0, \varepsilon_0)$, with $0 < \varepsilon < \frac{\varepsilon_1}{10}$. We introduce locally near $U^{(1)}_{j(k)}$ a new coordinate system $(\tilde{x}_1, \ldots, \tilde{x}_n)$ by application of Lemma 4.3.18 with $f_1 = f + \Phi$ and $\alpha = (f + \Phi)|_{\partial\Omega}$.
Hence, we can write in $B(U^{(1)}_{j(k)}, 2\varepsilon_1)$, choosing ε_1 small enough:

$$(f + \Phi)(\tilde{x}) = -\tilde{x}_n + (f + \Phi)|_{\partial\Omega}(\tilde{x}') = -\tilde{x}_n + f(\tilde{x}', 0) + \varphi(\tilde{x}')$$

with an arbitrary choice of $\tilde{x}'$ in the boundary.
Remark moreover that in this case,

$$d\tilde{x}_n(U^{(1)}_{j(k)}) = -2\frac{\partial f}{\partial n}(U^{(1)}_{j(k)})\, \vec{n}^*_{U^{(1)}_{j(k)}} .$$

We choose the coordinate system $\tilde{x}'$ in the boundary like it was chosen in the boundaryless case (see [HeSj4][HeKlNi]) according to the geometry of stable and unstable manifolds in order to write $(f + \Phi)|_{\partial\Omega}$ as a function of $n - 2$ coordinates:

$$(f + \Phi)|_{\partial\Omega}(\tilde{x}') = f(\tilde{x}', 0) + \varphi(\tilde{x}') = (f + \Phi)|_{\partial\Omega}(\tilde{x}_2, \ldots, \tilde{x}_{n-1}) . \tag{4.5.8}$$

Definition 4.5.20. *For any $k \in 1, \ldots, m_0$ we define the cut-off $\chi_{k,\varepsilon}$ by:*
• *If $U^{(1)}_{j(k)} \in \Omega$, $\chi_{k,\varepsilon} = \tilde{\chi}_{k,\varepsilon}$.*
• *If $U^{(1)}_{j(k)} \in \partial\Omega$, we first construct near $\partial\Omega \cap E_k$ the cut-off $\chi^{\partial\Omega}_{k,\varepsilon}$ like it was constructed in the boundaryless case (see [HeKlNi] pp. 26-29).*
Then, choosing a cut-off

$$\chi_n(\tilde{x}_n) \in \mathcal{C}^\infty_0(\mathbb{R}_-),\ \chi_n = 1 \text{ on } (-\delta_\varepsilon, 0]$$

we take for $\chi_{k,\varepsilon}$:

$$\chi_{k,\varepsilon}(\tilde{x}) = \chi_n(\tilde{x}_n)\chi^{\partial\Omega}_{k,\varepsilon} + (1 - \chi_n(\tilde{x}_n))\tilde{\chi}_{k,\varepsilon} .$$

Note that $\chi_{k,\varepsilon}$, for δ_ε small enough, satisfies the same properties as $\tilde{\chi}_{k,\varepsilon}$ in Proposition 4.5.10 and we make that choice. Moreover, according to [HeKlNi] p.28, in a neighborhood of $\{\tilde{x}_1 = 0\} \cap \partial\Omega$, the cut-off $\chi_{k,\varepsilon}$ only depends on $\tilde{x}_1$: $\chi_{k,\varepsilon} = \chi_{k,\varepsilon}(\tilde{x}_1)$.

Proposition 4.5.21.
There exist ε_0 and sequences $(c_{k,m})_{m\in\mathbb{N}^*}$, such that the (ε, h)-dependent and h-dependent quasimodes $\psi^{(0)}_k$ and $\psi^{(1)}_j$ ($(k, j) \in \{1, \ldots, m_0\} \times \{1, \ldots, m_1\}$ and $\varepsilon \in (0, \varepsilon_0]$) satisfy:

$$|\langle \psi^{(1)}_j \mid d^{(0)}_{f,h}\psi^{(0)}_k \rangle| = 0 \quad \text{if } j \neq j(k) ,$$

$$|\langle \psi^{(1)}_{j(k)} \mid d^{(0)}_{f,h}\psi^{(0)}_k \rangle| = \gamma_k(h)\delta_{j(k)}(h)\theta_{j(k)}(h)e^{-\frac{f(U^{(1)}_{j(k)}) - f(U^{(0)}_k)}{h}} \left(1 + hc^1_k(h)\right)$$

where $\gamma_k(h)$, $\delta_{j(k)}(h)$, and $\theta_{j(k)}(h)$ are defined in Definition 4.1.2 and $c_k(h) \sim \sum_{\ell=0}^\infty c_{k,\ell}h^\ell$.

Proof. The first statement for $j \neq j(k)$ is a consequence of our choice of $\varepsilon_1 > 0$ and $\chi_{k,\varepsilon}$ which gives according to Proposition 4.5.10-c) $\operatorname{supp}\psi_j^{(1)} \cap \operatorname{supp}\nabla\chi_{k,\varepsilon} = \emptyset$. We conclude with $d_{f,h}^{(0)}\psi_k^{(0)} = C_{\varepsilon,h}\left(d^{(0)}\chi_{k,\varepsilon}\right)e^{-f/h}$.
The second case was completely treated in [HeKlNi] when $U_{j(k)}^{(1)} \in \Omega$ and $U_k^{(0)} \in \Omega$. Moreover, in the case when $U_{j(k)}^{(1)} \in \Omega$ and $U_k^{(0)} \in \partial\Omega$, the proof done in [HeKlNi] remains valid if we take the convenient $\gamma_k(h)$.
Show now the cases when $U_{j(k)}^{(1)} \in \partial\Omega$ and $U_k^{(0)} \in \Omega \cup \partial\Omega$ by adapting the proofs done in [HeKlNi] and [HeNi].
From Proposition 4.5.18, Proposition 4.5.19, and

$$d_{f,h}^{(0)}\left(\chi_{k,\varepsilon}e^{-\frac{f(x)}{h}}\right) = e^{-\frac{f(x)}{h}}hd^{(0)}\chi_{k,\varepsilon}\ ,$$

we obtain the existence, for any $\varepsilon > 0$, of $\sigma_\varepsilon > 0$ such that

$$\begin{aligned}\left\langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)}\right\rangle &= h\gamma_k(h)\delta_{j(k)}(h)\\ &\times\int_{B(U_{j(k)}^{(1)},\varepsilon)}\langle b(x,h)\,|\,d\chi_{k,\varepsilon}\rangle(x)e^{-\frac{(\Phi(x)+f(x)-f(U_k^{(0)}))}{h}}\ V_{g_0}(dx)\\ &\qquad + \mathcal{O}_\varepsilon\left(e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\sigma_\varepsilon}{h}}\right)\ ,\end{aligned}$$

with $b(x,h)$ defined in Proposition 4.5.19.
Using the coordinate system $\tilde{x}$, with the choice of $\chi_{k,\varepsilon}$,

$$\begin{aligned}\left\langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)}\right\rangle &= h\gamma_k(h)\delta_{j(k)}(h)\\ &\times\int_{\widetilde{B(U_{j(k)}^{(1)},\varepsilon)}}\langle b(\tilde{x},h)\,|\,d\chi_{k,\varepsilon}(\tilde{x})\rangle(\tilde{x})e^{-\frac{-\tilde{x}_n+\varphi(\tilde{x}')+f(\tilde{x}',0)-f(U_k^{(0)})}{h}}\ V_{g_0}(d\tilde{x})\\ &\qquad + \mathcal{O}_\varepsilon\left(e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\sigma_\varepsilon}{h}}\right)\\ &= r(h)\int_{\mathcal{C}_\varepsilon}\langle b(\tilde{x},h)\,|\,d\tilde{x}_1\rangle\chi_{k,\varepsilon}'(\tilde{x}_1)e^{\frac{\tilde{x}_n-\varphi(\tilde{x}')-(f(\tilde{x}',0)-f(U_{j(k)}^{(1)}))}{h}}\ V_{g_0}(d\tilde{x})\\ &\qquad + \mathcal{O}_\varepsilon\left(e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\sigma'_\varepsilon}{h}}\right)\ ,\end{aligned}$$

where

$$r(h) = h\gamma_k(h)\delta_{j(k)}(h)e^{\frac{f(U_k^{(0)})-f(U_{j(k)}^{(1)})}{h}}$$

and $\mathcal{C}_\varepsilon$ is a cylinder $|\tilde{x}'| < c_\varepsilon$, $-c_\varepsilon < \tilde{x}_n < 0$. Expanding $\langle b(\tilde{x},h)\,|\,d\tilde{x}_1\rangle$ to a Taylor Series (of arbitrary order), we can obtain, using the Laplace method, an asymptotic expansion (of arbitrary order) for $\left\langle \psi_{j(k)}^{(1)} \mid d_{f,h}^{(0)}\psi_k^{(0)}\right\rangle$.

Moreover, the first term in the expansion of $\langle b(\tilde{x},h)\,|\,d\ \tilde{x}_1\rangle$ equals at $\tilde{x}=0$, $\langle b_{k,0}(\tilde{x})\,|\,d\,\tilde{x}_1\rangle(0)=1$. After recalling (4.5.8) which says that the exponent $f(\tilde{x}',0)+\varphi(x')$ does not depend on $\tilde{x}_1$, the first term of the wanted expression is then given by

$$r(h)\int e^{\frac{\tilde{x}_n}{h}}\,d\tilde{x}_n\int e^{-\frac{\varphi(\tilde{x}')+(f(\tilde{x}',0)-f(U^{(1)}_{j(k)}))}{h}}d\tilde{x}_2\ldots d\tilde{x}_{n-1}\int\chi'_{k,\varepsilon}(\tilde{x}_1)d\tilde{x}_1\,.$$

Using the Laplace method and

$$\int_{\mathbb{R}}\chi'_{k,\varepsilon}(x_1)\ dx_1=-1\,,$$

we find

$$r(h)\frac{h}{2\frac{\partial f}{\partial n}(U^{(1)}_{j(k)})}\frac{|\widehat{\lambda}^{\partial\Omega}_1(U^{(1)}_{j(k)})|^{\frac{1}{2}}}{|\det\operatorname{Hess}f|_{\partial\Omega}(U^{(1)}_{j(k)})|^{\frac{1}{2}}}(\pi h)^{\frac{n-2}{2}}\,.$$

∎

4.6 Final proof

4.6.1 Main result

Recall first some notations.
The generalized critical points with index 0, $\left\{U^{(0)}_k\,,\ k\in\{1,\ldots,m_0\}\right\}$, are labelled according to Subsection 4.5.3 and the generalized critical points with index 1, $\left\{U^{(1)}_{j(k)}\,,\ k\in\{2,\ldots,m_0\}\right\}$, are those introduced in Definition 4.5.5.
Moreover, the quantity $\lambda^{app}_k(\varepsilon,h)$ introduced in Definition 4.5.11 is associated with the quasimodes $\psi^{(0)}_k$ and $\psi^{(1)}_{j(k)}$:

$$\lambda^{app}_k(\varepsilon,h)=\left|\left\langle\psi^{(1)}_{j(k)}\mid d^{(0)}_{f,h}\psi^{(0)}_k\right\rangle\right|^2\,.$$

At a generalized critical point U with index i ($i\in\{0,1\}$), the Hessians $\operatorname{Hess}f(U)$ or $\operatorname{Hess}f\big|_{\partial\Omega}$ are computed in orthonormal coordinates for the metric g_0, while considering only the tangential coordinates $x'=(x_1,\ldots,x_{n-1})$ for the second case.
At least, for a generalized critical point $U\in W$ with index 1 for $W=\Omega$ or $W=\partial\Omega$, $\widehat{\lambda}^W_1(U)$ denotes the negative eigenvalue of $\operatorname{Hess}f|_W(U)$.

With these notations, we have the following theorem, which implies Theorem 4.1.3:

Theorem 4.6.1. *Under Assumptions 4.3.1 and 4.1.1, the first eigenvalue* $\lambda_1(h)$ *of* $\Delta^{N,(0)}_{f,h}$ *is* 0 *and its* m_0-1 *first non zero eigenvalues* $\lambda_2(h),\ldots,\lambda_{m_0}(h)$ *admit* *the following asymptotic expansion. There exist* $\varepsilon_0 > 0$ *and* $\alpha > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0]$,

$$\forall k \in \{2, \dots, m_0\}, \quad \lambda_k(h) = \lambda_k^{app}(\varepsilon, h)\left(1 + \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})\right) .$$

Recall also that, from Proposition 4.5.21, for any $\varepsilon \in (0, \varepsilon_0]$,

$$\lambda_k^{app}(\varepsilon, h) = \gamma_k^2(h)\, \delta_{j(k)}^2(h)\, \theta_{j(k)}^2(h)\, e^{-2\frac{f(U_{j(k)}^{(1)}) - f(U_k^{(0)})}{h}} \left(1 + hc_k^1(h)\right)$$

where $\gamma_k(h)$, $\delta_{j(k)}(h)$, and $\theta_{j(k)}(h)$ are defined in Definition 4.1.2 and $c_k^1(h)$ admits a complete expansion: $c_k^1(h) \sim \sum_{m=0}^{\infty} h^m c_{k,m}$.

4.6.2 Finite dimensional reduction and final proof

Set first, for $\ell \in \{0, 1\}$:

$$\forall i \in \{1, \dots, m_\ell\}, \quad v_i^{(\ell)} = 1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(\ell)})\psi_i^{(\ell)}, \tag{4.6.1}$$

where the $\psi_i^{(\ell)}$ are the (ε, h)- and h- dependent quasimodes introduced in Definition 4.5.11.

Remark 4.6.2.
Note that here again we omit the (ε, h)*-dependence (resp. h-dependence) of the functions* $v_k^{(0)}$ *(resp. 1-forms* $v_j^{(1)}$*) in the notation.*

Recall furthermore the definition of the space $F^{(\ell)}$ given in introduction ($\ell \in \{0, 1\}$),

$$F^{(\ell)} = \text{Ran}\ 1_{[0,h^{\frac{3}{2}})}(\Delta_{f,h}^{(\ell)}),$$

which has dimension m_ℓ according to Theorem 4.3.5.
According to Lemma 4.2.11, Corollary 4.5.16 (for $\ell = 0$) and Proposition 4.5.17 (for $\ell = 1$), $\left\|1_{[h^{3/2},+\infty)}(\Delta_{f,h}^{N,(\ell)})\psi_i^{(\ell)}\right\|$ is estimated from above by $\mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})$, which implies the two following propositions:

Proposition 4.6.3. *For* $\ell \in \{0, 1\}$*, the* ℓ*-forms* $(v_i^{(\ell)})_{i\in\{1,\dots,m_\ell\}}$ *satisfy:*

$$\left\|v_i^{(\ell)} - \psi_i^{(\ell)}\right\| = \mathcal{O}_\varepsilon(e^{-\frac{\alpha}{h}})$$

for some $\alpha > 0$ *independent of* $\varepsilon \in (0, \varepsilon_0]$.

Proposition 4.6.4.
For $\ell \in \{0, 1\}$*, the system* $\left(v_i^{(\ell)}\right)_{i\in\{1,\dots,m_\ell\}}$ *is a basis of* $F^{(\ell)}$ *satisfying:*

$$V^{(\ell)} := \left(\langle v_i^{(\ell)}\, | v_{i'}^{(\ell)}\rangle\right)_{i,i'\in\{1,\dots,m_\ell\}} = \text{Id}_{\mathbb{C}^{m_\ell}} + \mathcal{O}_\varepsilon(e^{-\frac{\alpha'}{h}}),$$

for some $\alpha > 0$ *independent of* $\varepsilon \in (0, \varepsilon_0]$.

Finally, we can also establish:

Proposition 4.6.5.
There exist $\varepsilon_0'>0$ and $\alpha'>0$ such that, for all $\varepsilon\in(0,\varepsilon_0']$, the estimates

$$\left|\langle v_j^{(1)}\mid d_{f,h}^{(0)}v_k^{(0)}\rangle\right|\leq C_\varepsilon e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})+\alpha'}{h}}\quad,\quad \text{if } j\neq j(k)\ ,$$

and

$$\langle v_{j(k)}^{(1)}\mid d_{f,h}^{(0)}v_k^{(0)}\rangle=\langle \psi_{j(k)}^{(1)}\mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle\left(1+\mathcal{O}_\varepsilon(e^{-\frac{\alpha'}{h}})\right)\ ,$$

hold for all $(k,j)\in\{1,\dots,m_0\}\times\{1,\dots,m_1\}$.

Proof. Remark first, $1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(1)})$ being a spectral projector and using Corollary 4.2.10:

$$\begin{aligned}\langle v_j^{(1)}\mid d_{f,h}^{(0)}v_k^{(0)}\rangle&=\langle 1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(1)})v_j^{(1)}\mid d_{f,h}^{(0)}1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(0)})\psi_k^{(0)}\rangle\\&=\langle 1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(1)})v_j^{(1)}\mid 1_{[0,h^{3/2})}(\Delta_{f,h}^{N,(1)})d_{f,h}^{(0)}\psi_k^{(0)}\rangle=\langle v_j^{(1)}\mid d_{f,h}^{(0)}\psi_k^{(0)}\rangle\ .\end{aligned}$$

The end of the proof is a straightforward consequence of Proposition 4.5.15, which gives

$$\left\|d_{f,h}^{(0)}\psi_k^{(0)}\right\|\leq C_\varepsilon e^{-\frac{f(U_{j(k)}^{(1)})-f(U_k^{(0)})-\alpha''\varepsilon}{h}}\ ,$$

Propositions 4.5.21 and 4.6.3. ∎

Proof of Theorem 4.6.1.
By Propositions 4.6.4 and 4.6.5, the bases $(v_i^{(\ell)})_{i\in\{1,\dots,m_\ell\}}$ of $F^{(\ell)}$, for $\ell\in\{0,1\}$, satisfy Assumptions 2.2.2 and 2.2.3 of [Lep1]. Theorem 2.2.4 of [Lep1] then implies Theorem 4.6.1 (which immediately implies Theorem 4.1.3).

Remark 4.6.6. *The conditions of [Lep1] are not exactly satisfied here because the one to one map j should act from $\{1,\dots,m_0\}$ to $\{1,\dots,m_1\}$, with $\dim F^{(i)}=m_i$. We can easily reduce the study to this last case, by setting:*

$$\overline{m}_0=m_0\quad,\quad\overline{m}_1=m_1+1\,,$$

and,

$$\overline{F^{(0)}}=F^{(0)}\quad,\quad\overline{F^{(1)}}=F^{(1)}\oplus^\perp\mathbb{C}v_{m_1+1}^{(1)}\,.$$

Setting in addition $j(1)=m_1+1$ instead of $j(1)=0$, the conditions of [Lep1] are fulfilled.
Note furthermore that the decreasing sequence $(\alpha_k)_{k\in\{1,\dots,\overline{m}_0\}}$ of [Lep1] is then here $\left(f(U_{j(k)}^{(1)})-f(U_k^{(0)})\right)_{k\in\{1,\dots,\overline{m}_0\}}$ whose first term is by definition $+\infty$.

∎

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Bibliography

[Bis] J.M. Bismut. The Witten complex and the degenerate Morse inequalities. J. Differ. Geom. 23, pp. 207-240 (1986).

[BoEcGaKl] A. Bovier, M. Eckhoff, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes I: Sharp asymptotics for capacities and exit times. JEMS 6(4), pp. 399-424 (2004).

[BoGaKl] A. Bovier, V. Gayrard, and M. Klein. Metastability in reversible diffusion processes II: Precise asymptotics for small eigenvalues. JEMS 7(1), pp. 69-99 (2004).

[Bur] D. Burghelea. Lectures on Witten-Helffer-Sjöstrand theory. Gen. Math. 5, pp. 85-99 (1997).

[ChLi] K.C. Chang and J.Liu. A cohomology complex for manifolds with boundary. Topological Methods in Non Linear Analysis. Volume 5, pp. 325-340 (1995).

[CyFrKiSi] H.L Cycon, R.G Froese, W. Kirsch, and B. Simon. *Schrödinger operators with application to quantum mechanics and global geometry.* Text and Monographs in Physics. Springer Verlag (1987).

[CoPaYc] Y. Colin de Verdière, Y. Pan, and B. Ycart. Singular limits of Schrödinger operators and Markov processes. J. Operator Theory 41, no. 1, pp. 151–173 (1999).

[DiSj] M. Dimassi and J. Sjöstrand. *Spectral Asymptotics in the semi-classical limit.* London Mathematical Society. Lecture Note Series 268. Cambridge University Press (1999).

[Duf] G.F.D. Duff. Differential forms in manifolds with boundary. Ann. of Math. 56, pp. 115-127 (1952).

[DuSp] G.F.D. Duff and D.C. Spencer. Harmonic tensors on Riemannian manifolds with boundary. Ann. of Math. 56, pp. 128-156 (1952).

[FrWe] M.I. Freidlin and A.D. Wentzell. *Random perturbations of dynamical systems. Transl. from the Russian by Joseph Szuecs. 2nd ed.* Grundlehren der Mathematischen Wissenschaften. 260. New York (1998).

[GaHuLa] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian Geometry.* Universitext, 2nd Edition, Springer Verlag (1993).

[Gil] P.B. Gilkey. *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.*
Mathematics Lecture Series, 11. Publish or Perish, Wilmington (1984).

[Gol] S.I. Goldberg. *Curvature and Homology.*
Dover books in Mathematics, 3rd edition (1998).

[Gue] P. Guérini. Prescription du spectre du Laplacien de Hodge-de Rham. Annales de l'ENS, Vol. 37 (2), pp. 270-303 (2004).

[Hel1] B. Helffer. Etude du Laplacien de Witten associé à une fonction de Morse dégénérée. Publications de l'université de Nantes, Séminaire EDP 1987-88.

[Hel2] B. Helffer. *Introduction to the semi-classical Analysis for the Schrödinger operator and applications.* Springer Verlag. Lecture Notes in Mathematics 1336 (1988).

[Hel3] B. Helffer. *Semi-classical analysis, Witten Laplacians and statistical mechanics.* World Scientific (2002).

[Hen] G. Henniart. Les inégalités de Morse (d'après E. Witten). Seminar Bourbaki, Vol. 1983/84. Astérisque No. 121-122, pp. 43–61 (1985).

[HeKlNi] B. Helffer, M. Klein, and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Matematica Contemporanea, 26, pp 41-85 (2004).

[HeNi] B. Helffer and F. Nier. Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary, Mémoire 105, Société Mathématique de France (2006).

[HeSj1] B. Helffer and J. Sjöstrand. Multiple wells in the semi-classical limit I, Comm. Partial Differential Equations 9 (4), pp. 337-408 (1984).

[HeSj2] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique II -Interaction moléculaire-Symétries-Perturbations. Ann. Inst. H. Poincaré Phys. Théor. 42 (2), pp. 127-212 (1985).

[HeSj4] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique IV -Etude du complexe de Witten -. Comm. Partial Differential Equations 10 (3), pp. 245-340 (1985).

[HeSj5] B. Helffer and J. Sjöstrand. Puits multiples en limite semi-classique V -Etude des minipuits-. Current topics in partial differential equations, pp. 133-186, Kinokuniya, Tokyo (1986).

[HoKuSt] R. Holley, S. Kusuoka, and D. Stroock. Asymptotics of the spectral gap with applications to the theory of simulated annealing. J. Funct. Anal. 83 (2), pp. 333-347 (1989).

[Kol] V.N. Kolokoltsov. *Semi-classical analysis for diffusions and stochastic processes*, Lecture Notes in Mathematics 1724. Springer Verlag (2000).

[KoMa] V.N. Kolokoltsov and K. Makarov. Asymptotic spectral analysis of a small diffusion operator and the life times of the corresponding diffusion process. Russian J. Math. Phys. 4 (3), pp. 341-36 (1996).

[Lau] F. Laudenbach. *Topologie différentielle.* Cours de Majeure de l'Ecole Polytechnique (1993).

[Lep1] D. Le Peutrec. Small singular values of an extracted matrix of a Witten complex. To appear in Cubo, A Mathematical Journal.

[Lep2] D. Le Peutrec. Local WKB construction for boundary Witten Laplacians. IRMAR. Preprint 08-52 (2008).

[Mic] L. Miclo. Comportement de spectres d'opérateurs à basse température. Bull. Sci. Math. 119, p. 529-533 (1995).

[Mil1] J.W. Milnor. *Morse Theory.* Princeton University press (1963).

[Mil2] J.W. Milnor. *Lectures on the h-cobordism Theorem.* Princeton University press (1965).

[Nie1] F. Nier, Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach, Journées "Equations aux Dérivées Partielles", Exp No VIII, Ecole Polytechnique (2004).

[Per] A. Persson. Bounds for the discrete part of the spectrum of a semi-bounded Schrödinger operator. Math. Scandinavica 8, pp. 143-153 (1960).

[Sch] G. Schwarz. *Hodge decomposition. A method for Solving Boundary Value Problems.* Lecture Notes in Mathematics 1607, Springer Verlag (1995).

[Sima] C.G. Simader. Essential self-adjointness of Schrödinger operators bounded from below. Math. Z. 159, pp. 47-50 (1978).

[Sim2] B. Simon. Semi-classical analysis of low lying eigenvalues, I. Nondegenerate minima: Asymptotic expansions. Ann. Inst. H. Poincaré Phys. Théor. 38, pp. 296-307 (1983).

[Wit] E. Witten. Supersymmetry and Morse inequalities. J. Diff. Geom. 17, pp. 661-692 (1982).

[Zha] Weiping Zhang. *Lectures on Chern-Weil theory and Witten deformations.* Nankai Tracts in Mathematics. Vol. 4. World Scientific (2002).

Résumé

Dans cette thèse, nous nous intéressons à l'étude précise de valeurs propres exponentiellement petites du Laplacien de Witten. Plus particulièrement, nous considérons la réalisation autoadjointe du Laplacien de Witten agissant sur les fonctions, sur une variété à bord, avec conditions au bord de type Neumann. Cette étude prolonge et complète des travaux de B. Helffer, M. Klein et F. Nier dans le cas sans bord, et de B. Helffer et F. Nier dans le cas d'une variété à bord, avec conditions au bord de type Dirichlet. La prise en compte de conditions au bord de type Neumann demande de traiter l'analyse au bord avec un niveau de généralité plus large que dans les travaux antérieurs. En particulier la construction de solutions WKB doit être abordée dans le cadre général des p-formes.

Abstract

In this PhD thesis, the exponentially small eigenvalues of some self-adjoint realization of the Witten Laplacian are accurately computed. More precisely, the Neumann type realization of the Witten Laplacian on a manifold with boundary is considered. This study continues previous works by B. Helffer, M. Klein and F. Nier in the case without boundary, and by B. Helffer and F. Nier in the case of Dirichlet type boundary conditions. Moreover, a fine treatment near the boundary has been needed to complete properly this analysis. It has notably required the construction of WKB approximations of the eigenvectors localized near the boundary for general p-forms.