RATES OF CONVERGENCE TO EQUILIBRIUM STATES IN THE STOCHASTIC THEORY OF NEUTRON TRANSPORT

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Abstract. We study a class of nonlinear equations arising in the stochastic theory of neutron transport. After proving existence and uniqueness of the solution, we consider the large-time behaviour of the solution and give explicit rates of convergence of the solution towards the asymptotic state.

Contents

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1. Introduction

Classical neutron transport theory (see, e.g. [4]) deals with *expected values* of neutron populations. In order to describe the *fluctuations* from the mean value of neutron distributions, stochastic formulations of neutron chain fissions have been introduced very early (see, [1, 8, 9]) in terms of probability generating functions. More recent developments in this direction are given in [13, 14].

The first mathematical analysis (existence, uniqueness and asymptotic behaviour) of the space inhomogeneous case is given in [10–12] for constant cross sections. General situations are dealt with in [5, 7] and [6] (Chapter 10).

The existence of non trivial stationary solutions relies on monotonicity arguments (sub- and supersolutions) and spectral theory while uniqueness of such solutions relies on concavity arguments. It is known (see [6], Theorem 10.11, p. 241) that the time dependent solution converges in L^∞-norm (as $t \to +\infty$) to the non-trivial solution of the stationary equation; the latter solution being the probability of divergent chain reactions. On the other hand, no rate of convergence to this stationary solution is known today. It is the aim of this paper to fill in this gap in the case of *space-homogeneous* and *non-correlated* models. Because of its relative simplicity (in comparison to full non space-homogeneous models), we give first a direct and simpler approach to the existence-uniqueness theory and the (qualitative) trend to equilibrium states. We deal subsequently with (explicit) rates of convergence to such equilibrium states. We show then that the convergence is exponential for subcritical and supercritical equations. On the other hand, in the critical case, we show that the convergence is no longer exponential but only polynomial. We consider separately the completely homogeneous case (i.e. all cross sections are constants) for which the rates of convergence are obtained by a direct qualitative analysis. On the other hand, the case of non constant (non-correlated) cross sections is dealt with by a an entropy dissipation method.

This is a powerful strategy which has been successfully employed to obtain explicit decay rates towards equilibrium of weak solutions to Cauchy problems for dissipative or hypocoercive equations and systems (see, for example, [2, 3] for applications to transport equations).

Basically, the key point of the method is the choice of a Lyapunov functional for the problem, sometimes called *entropy*. Once proved that this (convex) functional is monotone decreasing in time (this property justifies the name given to the functional, on the analogy of the physical entropy), if some norm of the difference between the solution and the stationary state is controlled by the entropy, the method permits to deduce that the solutions decay in time towards equilibrium with an explicit convergence rate.

The structure of the paper is the following: after this Introduction, we first give, in Section 2, a complete study of the Cauchy problem that describes completely homogeneous neutron chain fissions. Subsequently, in Section 3, we consider space-homogeneous only non-correlated neutron chain fissions.

2. Completely homogeneous neutron chain fissions

This section is devoted to the complete study of the simplest possible situation, namely the fully homogeneous case.

The system is modelled by the following ordinary differential equation:

$$\begin{cases} x'(t) = -\sigma x(t) + \sigma \left[1 - c_0 - \sum_{k=1}^m c_k (1-x(t))^k\right] \\ \\ x(0) = x_0 \in [0,1] \end{cases} \tag{1}$$

where $m \geq 2$, $c_k \geq 0$, for all $k = 0, \dots, m-1$, $c_m > 0$, and

$$\sum_{k=0}^m c_k = 1. \tag{2}$$

2.1. **The stationary equation.** An interesting feature of the system concerns the equilibrium points. We hence study the stationary problem

$$1 - c_0 - \sum_{k=1}^m c_k (1-x)^k = x; \quad x \in [0,1]\,.$$

Thanks to the condition (2), this equation is equivalent to

$$\sum_{k=1}^{m} c_k \left[1-(1-x)^k\right] = x; \quad x \in [0,1]\,. \tag{3}$$

It is apparent that $x = 0$ is a trivial solution. On the other hand, if we consider the function φ defined by

$$\varphi : x \in [0,1] \to \sum_{k=1}^{m} c_k \left[1-(1-x)^k\right] \in [0,1]\,,$$

we deduce that

$$\varphi'(x) = \sum_{k=1}^{m} kc_k(1-x)^{k-1}$$

and

$$\varphi''(x) = -\sum_{k=2}^{m} k(k-1)c_k(1-x)^{k-2}.$$

The facts that $c_m > 0$ and $m \geq 2$ imply that φ is a strictly concave function. Note that

$$c := \sum_{k=1}^{m} kc_k = \varphi'(0)$$

is the mean number of neutrons produced by a fission. The equation is said to be critical (resp. subcritical, supercritical) if $\sum_{k=1}^{m} kc_k = 1$, (resp. $\sum_{k=1}^{m} kc_k < 1$, $\sum_{k=1}^{m} kc_k > 1$). The following result shows that there exists a nontrivial equilibrium point if and only if the equation is supercritical.

Theorem 2.1. *Let us consider Equation* (3)*, with* $m \geq 2$*,* $c_k \geq 0$ *for all* $k = 0, \ldots, m-1$*,* $c_m > 0$*, and*

$$\sum_{k=0}^{m} c_k = 1.$$

i) If $\sum_{k=1}^{m} kc_k \leq 1$*, then Equation* (3) *has no nontrivial solution.*

ii) If $\sum_{k=1}^{m} kc_k > 1$*, then Equation* (3) *has a unique nontrivial solution.*

Proof. i) The strict concavity of φ implies that (apart from the point $(0,0)$) its graph is strictly below the graph of its tangent at $(0,0)$ given by the equation

$$y = \left(\sum_{k=1}^{m} kc_k\right) x$$

so that, except the point $x = 0$, φ has no other fixed point. This ends the proof of *i)*.

ii) We first suppose that $c_0 > 0$. Let

$$\psi(x) := \varphi(x) - x.$$

We have $\psi(0) = 0$ and

$$\psi'(x) = \sum_{k=1}^{m} kc_k(1-x)^{k-1} - 1 \tag{4}$$

so that

$$\psi'(0) = \varphi(0) - 1 = \sum_{k=1}^{m} kc_k - 1 > 0.$$

On the other hand

$$\psi''(x) = -\sum_{k=2}^{m} k(k-1)c_k(1-x)^{k-2} < 0; \;\; x \in [0,1) \tag{5}$$

shows that ψ' decreases *strictly*. Now the fact that $\psi'(1) = c_1 - 1 < 0$ shows the existence of a unique $\overline{x} \in (0,1)$ such that $\psi'(\overline{x}) = 0$. Thus ψ increases strictly on $[0, \overline{x}]$ and decreases strictly on $[\overline{x}, 1]$. It follows that $\psi(x) > 0$ on $(0, \overline{x}]$. On the other hand,

$$\psi(1) = \sum_{k=1}^{m} c_k - 1 = -c_0 < 0$$

so that there exists a unique $\tilde{x} \in (\overline{x}, 1)$ such that $\psi(\tilde{x}) = 0$, i.e. $\varphi(\tilde{x}) = \tilde{x}$.

If $c_0 = 0$, then $\psi(0) = \psi(1) = 0$. Since ψ is strictly convex in the interval $[0,1]$ by formula (5), and moreover $\psi'(0) > 0$ and $\psi'(1) = -1$ by formula (4), then $\tilde{x} = 1$ is the only non-trivial solution of Equation (3). □

Remark 2.1. *We note that the nontrivial equilibrium point $\tilde{x}$ satisfies $\overline{x} < \tilde{x} \leq 1$. Moreover, $\tilde{x} < 1$ when $c_0 > 0$.*

2.2. **The evolution equation.** We consider now the integral version of the Cauchy problem (1)

$$x(t) = e^{-\sigma t} x_0 + \sigma \int_0^t e^{-\sigma(t-s)} \left[\sum_{k=1}^{m} c_k \left[1 - (1 - x(s))^k\right] \right] ds. \tag{6}$$

We have:

Theorem 2.2. *For any $x_0 \in [0,1]$, Equation (6) has a unique global solution $x(\cdot)$ such that $x(t) \in [0,1]$.*

Proof. We fix an arbitrary $T > 0$ and define the operator

$$L : C([0,T]) \to C([0,T])$$

by

$$Lx(t) = e^{-\sigma t} x_0 + \sigma \int_0^t e^{-\sigma(t-s)} \left[\sum_{k=1}^{m} c_k \left[1 - (1 - x(s))^k\right] \right] ds.$$

We observe that if $0 \leq x(t) \leq 1$ then

$$\begin{aligned} 0 \;\leq\; & Lx(t) \leq e^{-\sigma t} x_0 + \sigma \left(\sum_{k=1}^{m} c_k \right) \int_0^t e^{-\sigma(t-s)} ds \\ \leq\; & e^{-\sigma t} + \left(\sum_{k=1}^{m} c_k \right) (1 - e^{-\sigma t}) \leq e^{-\sigma t} + (1 - e^{-\sigma t}) = 1 \end{aligned}$$

so that L maps the convex set

$$\mathcal{C} := \{x \in C([0,T]); \; 0 \leq x(t) \leq 1\}$$

into itself.

Let us consider now x_1, $x_2 \in \mathcal{C}$. We deduce that

$$\begin{aligned} |Lx_1(t) - Lx_2(t)| &\leq \sigma \sum_{k=1}^{m} c_k \int_0^t e^{-\sigma(t-s)} \left|(1-x_1(s))^k - (1-x_2(s))^k\right| ds \\ &\leq \sigma \sum_{k=1}^{m} kc_k \int_0^t e^{-\sigma(t-s)} |x_1(s) - x_2(s)|\, ds \\ &\leq \sigma \sum_{k=1}^{m} kc_k \int_0^t e^{-\sigma(t-s)} ds \sup_{s\in[0,t]} |x_1(s) - x_2(s)| \\ &\leq (1-e^{-\sigma T}) \sum_{k=1}^{m} kc_k \sup_{s\in[0,T]} |x_1(s) - x_2(s)| \end{aligned}$$

so that

$$\|Lx_1 - Lx_2\|_{C([0,T])} \leq c(1-e^{-\sigma T}) \, \|x_1 - x_2\|_{C([0,T])}$$

where $c = \sum_{k=1}^{m} kc_k$. We note that in the subcritical case $c < 1$, we can work directly in $C_b([0,+\infty))$ (endowed with the sup norm) and $L : \mathcal{C} \to \mathcal{C}$ is a strict contraction.

We hence deduce that there exists a unique global solution. If $c \geq 1$ we choose $T > 0$ such that $c(1-e^{-\sigma T}) < 1$, i.e.

$$T < \sigma^{-1} \ln(1-c^{-1})^{-1}.$$

Then $L : \mathcal{C} \to \mathcal{C}$ is a strict contraction and we obtain a unique solution in $C([0,T])$. Since the life-time of the solution is independent of the initial data, by a bootstrap argument, we can continue the solution beyond T indefinitely. □

2.3. **Convergence to equilibrium.** Since the constant trajectory

$$x(t) = 0 \quad \forall t \geq 0$$

is always a solution of Equation (3), the Cauchy problem with initial data $x_0 = 0$ is, actually, a stationary problem.

Nevertheless, in the supercritical case, by Theorem 2.1 there exists another constant trajectory

$$x(t) = \tilde{x} \quad \forall t \geq 0.$$

The following result holds:

Theorem 2.3. *Let $x(\cdot)$ be the solution of the Cauchy problem* (1) *with initial data $x_0 \in (0,1]$. Then $x(t) \to \tilde{x}$ as $t \to +\infty$ in the supercritical case.*

Proof. We exclude the elementary case where $x_0 = \tilde{x}$ (which implies that $x(t) = \tilde{x}$ $\forall t \geq 0$). We assume for instance that

$$0 < x_0 < \tilde{x}.$$

We have

$$x'(t) = \sigma \left[\sum_{k=1}^{m} c_k \left[1 - (1-x(t))^k\right] - x(t) \right] = \sigma\psi(x(t)) \tag{7}$$

where $\psi(x) := \varphi(x) - x$. According to the properties of ψ given in the proof of Theorem 2.1, we have $\psi(x_0) > 0$ so that $x'(0) = \sigma\psi(x_0) > 0$. Let

$$\bar{t} := \sup\{t \geq 0;\ x'(s) > 0\ \forall s \in [0,t]\} \leq +\infty.$$

We note that $x(\cdot)$ is strictly increasing on $\left[0,\bar{t}\right)$. Let us show that $\bar{t} = +\infty$.

If $\bar{t} < +\infty$ then, by assumption, $x'(\bar{t}) = 0$ and the choice $t = \bar{t}$ in (7) shows that $x(\bar{t}) = \tilde{x}$ and consequently $x(t) = \tilde{x}\ \forall t \geq 0$ which yields a contradiction. Thus $\bar{t} = +\infty$.

Since $x(\cdot)$ is strictly increasing on $\left[0,\bar{t}\right)$ then $x(t)$ has a limit $\widetilde{x}$ as $t \to +\infty$ and $x(t) < \widetilde{x}\ \forall t \geq 0$. Hence $\lim_{t\to+\infty} \sigma\psi(x(t)) = \sigma\psi(\widetilde{x})$ and $\lim_{t\to+\infty} x'(t) = \sigma\psi(\widetilde{x})$.

We note that $\lim_{t\to+\infty} x'(t) > 0$ would imply that $x(t) \to +\infty$. Thus $\lim_{t\to+\infty} x'(t) = 0$ and $\psi(\widetilde{x}) = 0$, i.e. $\widetilde{x} = \tilde{x}$.

When $c_0 > 0$, we can prove in a similar way that, if $\tilde{x} < x_0$, then $x(\cdot)$ is strictly decreasing and tends to $\tilde{x}$ as $t \to +\infty$. □

Remark 2.2. *If $\sum_{k=1}^{m} kc_k \leq 1$ then $x = 0$ is the unique equilibrium point and, arguing as in the proof of Theorem 2.3, we can prove that $x(t) \to 0$ as $t \to +\infty$.*

2.4. **Convergence rates towards the asymptotic state.**

2.4.1. *The subcritical and supercritical cases.* Here we consider only the supercritical case; the other case can be dealt with similarly.

When $c_0 > 0$, we have seen in the proof of Theorem 2.1 that $\varphi'(\overline{x}) = 1$ and then $\varphi'(y) < 1\ \forall y \in (\overline{x}, 1]$. In particular $\varphi'(\tilde{x}) < 1$. Then for any α such that $\varphi'(\tilde{x}) < \alpha < 1$ there exists $\varepsilon > 0$ such that

$$\varphi'(y) \leq \alpha;\ \forall y \in [\tilde{x} - \varepsilon, \tilde{x} + \varepsilon]\,. \tag{8}$$

Theorem 2.4. *Let α and ε as in (8) and $c_0 > 0$. Then, there exists $\bar{t}_\varepsilon$ such that*

$$|x(t) - \tilde{x}| \leq \left|x(\bar{t}_\varepsilon) - \tilde{x}\right| e^{-\sigma(1-\alpha)(t-\bar{t}_\varepsilon)};\ t \geq \bar{t}_\varepsilon. \tag{9}$$

Proof. We have

$$x'(t) = \sigma\psi(x(t));\ x(0) =\ x_0$$

where (for instance) $x_0 < \tilde{x}$. Let $z(t) := \tilde{x} - x(t)$. Since $\psi(\tilde{x}) = 0$ then

$$z'(t) = \sigma\psi(\tilde{x}) - \sigma\psi(x(t)) = \sigma\psi'(\zeta(t))z(t)$$

where $\zeta(t) \in\ ]x(t), \tilde{x}[$. Thus there exists $\bar{t}_\varepsilon$ such that $\zeta(t) \in [\tilde{x} - \varepsilon, \tilde{x} + \varepsilon]$ for $t \geq \bar{t}_\varepsilon$ because $x(t) \to \tilde{x}$ as $t \to +\infty$. Hence $\psi'(\zeta(t)) = \varphi'(\zeta(t)) - 1 \leq \alpha - 1$ for $t \geq \bar{t}_\varepsilon$ so that

$$z'(t) \leq -\sigma(1 - \alpha)z(t);\ t \geq \bar{t}_\varepsilon$$

and

$$z(t) \leq z(\bar{t}_\varepsilon)e^{-\sigma(1-\alpha)(t-\bar{t}_\varepsilon)};\ t \geq \bar{t}_\varepsilon$$

which ends the proof of the estimate (9). □

Remark 2.3. *Note that α can be chosen as close to $\varphi'(\tilde{x})$ as we want so that the rate of convergence to equilibrium is "almost" as* $\exp(-\sigma(1 - \varphi'(\tilde{x}))t)$.

When $c_0 = 0$, we obtain a similar result in a direct way:

Theorem 2.5. *Let $x(t)$ be the unique solution of Equation (1) with $c_0 = 0$. Then*

$$|x(t) - 1| \leq \frac{(1 - x_0)e^{-c_m\sigma t}}{[1 - (1 - x_0)^{m-1}]^{1/(m-1)}}.$$

Proof. Since $c_0 = 0$, then $\sum_{k=1}^m c_k = 1$. In this case, $\tilde{x} = 1$ is the only non-vanishing equilibrium point (see Remark 2.1). Hence, from Equation (1) we obtain the differential equation for the unknown $y(t) = 1 - x(t) \geq 0$:

$$\frac{1}{\sigma} y'(t) = -y(t) + \sum_{k=1}^{m} c_k y^k(t),$$

which lead to the differential inequality

$$\frac{1}{\sigma} y'(t) \leq -c_m y(t) + c_m y^m(t).$$

From the previous differential inequality we can deduce exponential convergence towards the equilibrium point $\tilde{x} = 1$:

$$y(t) \leq \frac{(1-x_0)e^{-c_m \sigma t}}{[1-(1-x_0)^{m-1}]^{1/(m-1)}}.$$

□

Remark 2.4. *In the subcritical case $\sum_{k=1}^m kc_k < 1$ we can prove an estimate like inequality (9) with $\tilde{x} = 0$.*

2.4.2. *The critical case.* The situation of the critical case $\sum_{k=1}^m kc_k = 1$ is quite different. We have:

Theorem 2.6. *Let $\sum_{k=1}^m kc_k = 1$. If $c_2 > 0$, then $x(t)^2$ is integrable at infinity and we have the lower bound*

$$\frac{\sigma c_2}{\left(\int_0^{+\infty} x(s)^2 ds\right)^{-1} + \left(\frac{\sigma}{2}\sum_{k=2}^m k(k-1)c_k\right)^2 t} \leq x(t). \tag{10}$$

Proof. We already know that $x(t)$ decreases to zero as $t \to +\infty$ (see Remark 2.2).

We note that

$$\psi''(x) = -\sum_{k=2}^{m} k(k-1)c_k(1-x)^{k-2}$$

so that

$$-2c_2 \geq \psi''(x) \geq -\sum_{k=2}^{m} k(k-1)c_k.$$

Note that the criticality assumption amounts to $\varphi'(0) = 1$ (i.e. $\psi'(0) = 0$) so that

$$\psi(z) = \psi(0) + \psi'(0)z + \frac{\psi''(\zeta)}{2} z^2 = \frac{\psi''(\zeta)}{2} z^2$$

where $\zeta \in]0, z[$. Thus $\psi(x(s)) = x(s)^2 \psi''(\zeta_s)/2$ satisfies the estimate

$$c_2 x(s)^2 \leq -\psi(x(s)) \leq \left[\frac{1}{2}\sum_{k=2}^{m} k(k-1)c_k\right] x(s)^2.$$

On the other hand

$$x'(t) = \sigma\psi(x(t));\ x(0) = x_0 > 0$$

and

$$x(t) - x(T) = -\sigma \int_t^T \psi(x(s))ds$$

give for $t < T$

$$\sigma c_2 \int_t^T x(s)^2 ds \leq x(t) - x(T) \leq \left[\frac{\sigma}{2} \sum_{k=2}^m k(k-1)c_k \right] \int_t^T x(s)^2 ds$$

and letting $T \to +\infty$

$$\sigma c_2 \int_t^{+\infty} x(s)^2 ds \leq x(t) \leq \sqrt{\gamma} \int_t^{+\infty} x(s)^2 ds \tag{11}$$

where

$$\sqrt{\gamma} := \frac{\sigma}{2} \sum_{k=2}^m k(k-1)c_k.$$

This shows that $x(t)^2$ is integrable at infinity. Let now

$$H(t) := \int_t^{+\infty} x(s)^2 ds;$$

then $H'(t) = -x(t)^2$ and (11) give

$$-H'(t) \leq \gamma H^2(t),$$

so that

$$H(t) \geq \frac{1}{H(0)^{-1} + \gamma t}.$$

Then (11) implies

$$\frac{\sigma c_2}{H(0)^{-1} + \gamma t} \leq x(t),$$

and this ends the proof. □

Remark 2.5. *Note that the bound* (10) *implies that*

$$\int_0^{+\infty} x(s)^2 ds \leq \frac{x_0}{\sigma c_2}.$$

3. Space-homogeneous non-correlated neutron chain fissions

The general time-dependent equation governing space-homogeneous neutron chain fissions is an evolution equation for the unknown $f = f(t, v)$, $t \in \mathbb{R}^+$ and $v \in V$, where V is the unit ball of $\mathbb{R}^n$ endowed with the normalized Lebesgue measure. It takes the form

$$\begin{aligned} \frac{\partial f}{\partial t} &= -\sigma(v) f(t,v) \\ &\quad + \sigma(v) \left[1 - c_0(v) - \sum_{k=1}^m \int_{V^k} c_k(v, v_1', \dots, v_k')(1 - f(t, v_1')) \dots (1 - f(t, v_k')) dv_1' \dots dv_k' \right] \end{aligned}$$

with initial data $f(0, v) = f_0(v)$, where $m \in \mathbb{N}$ and

$$c_0(v) + \sum_{k=1}^m \int_{V^k} c_k(v, v_1', \dots, v_k') dv_1' \dots dv_k' = 1.$$

In what follows, we suppose always that $0 \leq f_0(v) \leq 1$.

We assume here that the neutron chain fissions are *non-correlated*, i.e.

$$c_k(v, v'_1, \ldots, v'_k) = c_k(v) F(v'_1) \ldots F(v'_k)$$

where

$$\int_V F(v) dv = 1, \qquad \sum_{k=0}^{m} c_k(v) = 1.$$

The functions c_k, $k = 0, \ldots, m$, are non-negative elements of the functional space $L^1(V)$, such that

$$\sum_{k=0}^{m} c_k(v) = 1 \text{ for all } v \in V,$$

and $\hat{c}_m(v) > 0$, where

$$\hat{c}_k = \int_V c_k(v) F(v)\, dv, \qquad k = 0, \ldots, m.$$

Under these conditions, the above evolution equation becomes

$$\frac{\partial f}{\partial t} = -\sigma(v) f(t,v) + \sigma(v) \left[1 - c_0(v) - \sum_{k=1}^{m} c_k(v) \left(1 - \int_V F(v') f(t,v') dv' \right)^k \right]. \tag{12}$$

3.1. **The stationary equation.** The equilibrium states $f_\infty(v)$ are solutions of the equation

$$f_\infty(v) = \left[1 - c_0(v) - \sum_{k=1}^{m} c_k(v) \left(\int_V (1 - f_\infty(v')) F(v')\, dv' \right)^k \right], \tag{13}$$

which can be written in the equivalent form:

$$f_\infty(v) = \sum_{k=1}^{m} c_k(v) \left[1 - \left(1 - \int_V f_\infty(v') F(v')\, dv' \right)^k \right]. \tag{14}$$

The first result of this subsection concerns the existence of non-vanishing solutions of the previous equations.

Lemma 3.1. *Equation* (14) *has a non trivial solution* f_∞ *if and only if*

$$x := \int_V F(v') f_\infty(v') dv'$$

is a nontrivial solution of

$$\sum_{k=1}^{m} \hat{c}_k \left[1 - (1-x)^k \right] = x. \tag{15}$$

Proof. Let f_∞ be a non trivial solution of Equation (15). By multiplying Equation (14) by F and integrating over V we obtain that x is a nontrivial solution of (15). Note that f_∞ and x are related through the equation

$$\sum_{k=1}^{m} c_k(v) \left[1 - (1-x)^k \right] = f_\infty(v). \tag{16}$$

Conversely, let x be a non trivial solution of Equation (15). *Define* f_∞ by Equation (16). Then integrating (16) against F we get

$$\sum_{k=1}^{m} \hat{c}_k \left[1-(1-x)^k\right] = \int_V F(v')f_\infty(v')dv'.$$

Then, Equation (15) implies

$$x = \int_V F(v')f_\infty(v')dv',$$

so that f_∞ is non trivial and (16) implies (15). □

We note that Equation (15) is nothing but the completely homogeneous problem (3) with the coefficients $\hat{c}_k$ instead of c_k. Hence Theorem 2.1 implies immediately:

Theorem 3.1. *Let us consider Equation* (15)*, with* $m \geq 2$*,* $c_k \geq 0$ *for all* $k = 0, \dots, m-1$*,* $\hat{c}_m > 0$*, and*

$$\sum_{k=0}^{m} c_k = 1.$$

i) If $\sum_{k=1}^{m} kc_k \leq 1$ *then Equation* (15) *has no nontrivial solution.*

ii) If $\sum_{k=1}^{m} kc_k > 1$ *then Equation* (15) *has a unique nontrivial solution.*

It is now natural to define the subcriticality (resp. criticality, resp. supercriticality) by the condition $\sum_{k=1}^{m} k\hat{c}_k < 1$ (resp. $\sum_{k=1}^{m} k\hat{c}_k = 1$, resp. $\sum_{k=1}^{m} k\hat{c}_k > 1$).

In the supercritical case, since there is also a non-vanishing stationary solution, it is important to deduce some estimates on this non-trivial stationary profile.

These estimates, which give an upper and a lower bound when $\sum_{k=1}^{m} k\hat{c}_k > 1$, will be used, later on, for proving the exponential decay in time towards the stationary profile in the supercritical case. We have:

Lemma 3.2. *Let us suppose that the functions* c_k*,* $k = 1, \dots, m$ *in Equation* (12) *are such that*

$$\sum_{k=1}^{m} k\hat{c}_k > 1,$$

and denote with f_∞ *the unique non trivial stationary solution of Equation* (12)*. Then, if* $\hat{c}_0 > 0$*, the stationary solution satisfies the bounds*

$$1 - \left(\sum_{k=1}^{m} k\hat{c}_k\right)^{1/(1-m)} \leq \|f_\infty F\|_{L^1(V)} \leq \left(\sum_{k=1}^{m} k\hat{c}_k - 1\right) \Big/ \left(\sum_{k=1}^{m} k\hat{c}_k - 1 + \hat{c}_0\right).$$

If $\hat{c}_0 = 0$*, then* $f_\infty = 1$.

Proof. Let us consider first the case $\hat{c}_0 > 0$. We multiply Equation (14) by $F(v)$ and then integrate with respect to the velocity variable v on V. Since $f_\infty > 0$ by Theorem 3.1, we deduce an equation for the L^1-norm of the product $f_\infty F$:

$$\|f_\infty F\|_{L^1(V)} = \sum_{k=1}^{m} \hat{c}_k \left[1 - \left(1 - \|f_\infty F\|_{L^1(V)}\right)^k\right].$$

It is well known that, for all $a \in (0,1)$,

$$\sum_{j=0}^{k-1} a^j = \frac{1-a^k}{1-a}.$$

Hence, the previous equation can be written in the form

$$1 = \sum_{k=1}^{m} \hat{c}_k \sum_{j=0}^{k-1} \left(1 - \|f_\infty F\|_{L^1(V)}\right)^j. \tag{17}$$

We first deduce the upper bound for $\|f_\infty F\|_{L^1(V)}$: since $\|f_\infty F\|_{L^1(V)} \leq 1$ by hypothesis, the previous equation implies that

$$1 \leq \sum_{k=1}^{m} \hat{c}_k \left[1 + \sum_{j=1}^{k-1} \left(1 - \|f_\infty F\|_{L^1(V)}\right)\right].$$

This inequality permits to deduce that

$$\|f_\infty F\|_{L^1(V)} \leq \left(\sum_{k=1}^{m} k\hat{c}_k - 1\right) / \left(\sum_{k=1}^{m} k\hat{c}_k - 1 + \hat{c}_0\right).$$

The proof of the lower bound is also obtained starting from Equation (17): we easily obtain that

$$1 \geq \sum_{k=1}^{m} \hat{c}_k \sum_{j=0}^{k-1} \left(1 - \|f_\infty F\|_{L^1(V)}\right)^{k-1} \geq \sum_{k=1}^{m} k\hat{c}_k \left(1 - \|f_\infty F\|_{L^1(V)}\right)^{m-1}.$$

Hence, we can conclude that

$$\|f_\infty F\|_{L^1(V)} \geq 1 - \left(\sum_{k=1}^{m} k\hat{c}_k\right)^{1/(1-m)}.$$

If $\hat{c}_0 = 0$, by direct inspection, we see that $f_\infty = 1$ is a solution of Equation (13), which is the unique non-vanishing stationary solution of the equation by Theorem 3.1. □

3.2. **The evolution equation.** From now on, we deal with the evolution equation under the assumption that σ is a *constant*.

We consider the integral version of (12)

$$f(t,v) = e^{-\sigma t} f_0(v) + \int_0^t e^{-\sigma(t-s)} \sigma \left[\sum_{k=1}^{m} c_k(v) \left[1 - \left(1 - \int_V F(v') f(s,v') dv'\right)^k\right]\right] ds \tag{18}$$

with $0 \leq f_0(v) \leq 1$. Arguing as in Lemma 3.1 we obtain:

Lemma 3.3. *Let us suppose that $\sigma > 0$ is constant. Then Equation* (18) *has a solution f if and only if*

$$x(t) := \int_V F(v') f(t,v') dv'$$

solves

$$x(t) = \int F(v) e^{-\sigma t} f_0(v) dv + \int_0^t e^{-\sigma(t-s)} \sigma \left[\sum_{k=1}^{m} \hat{c}_k \left[1 - (1 - x(s))^k\right]\right] ds. \tag{19}$$

Note that $f(t,v)$ and $x(t)$ are related by

$$f(t,v) = e^{-\sigma t} f_0(v) + \int_0^t e^{-\sigma(t-s)} \sigma \left[\sum_{k=1}^m c_k(v) \left[1 - (1-x(s))^k \right] \right] ds. \tag{20}$$

We can solve (19) *uniquely* by using a contraction fixed point argument exactly as in the proof of Theorem 2.2, where the term $e^{-\sigma t} x_0$ is replaced by

$$\int F(v) e^{-\sigma t} f_0(v) dv.$$

Actually, to deal with time asymptotic behaviour of the solution, we give here other existence proofs. We consider the operator

$$\Lambda : x(\cdot) \in Z \to \int F(v) e^{-\sigma t} f_0(v) dv + \int_0^t e^{-\sigma(t-s)} \sigma \left[\sum_{k=1}^m \hat{c}_k \left[1 - (1-x(s))^k \right] \right] ds,$$

where Z denotes the space of measurable functions from $[0,+\infty)$ into $[0,1]$. We note that

$$\begin{aligned} \Lambda x(t) &\leq e^{-\sigma t} + \left(\sum_{k=1}^m \hat{c}_k \right) \int_0^t e^{-\sigma(t-s)} \sigma \, ds \\ &= e^{-\sigma t} + \left(\sum_{k=1}^m \hat{c}_k \right) (1 - e^{-\sigma t}) \leq 1 \end{aligned}$$

so that Λ maps Z into itself. It is also clear that $\Lambda x \leq \Lambda y$ if $x \leq y$ so that Λ is an nondecreasing operator on Z. We note that $\Lambda 1 \leq 1$ so that the sequence $\{\overline{\varphi}_n\}_n$ defined inductively as

$$\overline{\varphi}_0 = 1, \quad \overline{\varphi}_{n+1} = \Lambda \overline{\varphi}_n$$

is *nonincreasing* since $\overline{\varphi}_1 = \Lambda \overline{\varphi}_0 = \Lambda 1 \leq 1 = \overline{\varphi}_0$ and Λ is an nondecreasing operator. We can hence pass to the limit in

$$\overline{\varphi}_{n+1}(t) = \int F(v) e^{-\sigma t} f_0(v) dv + \int_0^t e^{-\sigma(t-s)} \sigma \left[\sum_{k=1}^m \hat{c}_k \left[1 - (1-\overline{\varphi}_n(s))^k \right] \right] ds$$

and obtain *the* solution to (19).

In the supercritical case, we can obtain the solution to (19) by means of a *nondecreasing* sequence $\left\{ \underline{\psi}_n \right\}_n$. Indeed: let

$$\varphi : x \in [0,1] \to \sum_{k=1}^m \hat{c}_k \left[1 - (1-x)^k \right] \in [0,1].$$

We have $\varphi(0) = 0$ and

$$\varphi'(0) = \sum_{k=1}^m k \hat{c}_k.$$

Then

$$\varphi(x) = \varphi(x) - \varphi(0) = x \varphi'(\zeta) \ (\zeta \in [0,x])$$

so

$$\varphi(x) = x + x(\varphi'(\zeta) - 1)$$

and then, since $\varphi'(\zeta) \to \varphi'(0) = \sum_{k=1}^m k\hat{c}_k > 1$ as $x \to 0$, there exists $\varepsilon > 0$ such that

$$\varphi(x) \geq x \quad \forall x \leq \varepsilon.$$

It follows that for a constant $x \leq \varepsilon$ we have

$$\begin{aligned}
& \int F(v)e^{-\sigma t} f_0(v)dv + \int_0^t e^{-\sigma(t-s)}\sigma \left[\sum_{k=1}^m \hat{c}_k \left[1-(1-x)^k\right]\right] ds \\
\geq\ & \int F(v)e^{-\sigma t} f_0(v)dv + \left(\int_0^t e^{-\sigma(t-s)}\sigma ds\right) x \\
=\ & e^{-\sigma t}\int F(v)f_0(v)dv + (1-e^{-\sigma t})x = e^{-\sigma t}\left(\int F(v)f_0(v)dv - x\right) + x \geq x
\end{aligned}$$

if

$$x \leq \int F(v)f_0(v)dv.$$

Thus for a *nontrivial* initial data f_0 and

$$\varepsilon \leq \int F(v)f_0(v)dv,$$

by the choice $\underline{\psi}_0 = x$ we have $\Lambda\underline{\psi}_0 \geq \underline{\psi}_0$. We can then define inductively a *nondecreasing* sequence $\left\{\underline{\psi}_n\right\}_n$ by

$$\underline{\psi}_{n+1} = \Lambda\underline{\psi}_n = \int F(v)e^{-\sigma t} f_0(v)dv + \int_0^t e^{-\sigma(t-s)}\sigma \left[\sum_{k=1}^m \hat{c}_k \left[1-(1-\underline{\psi}_n(s))^k\right]\right] ds$$

and then passing to the limit we obtain again the solution of (19).

Writing $\Lambda x(t)$ as

$$\int F(v)e^{-\sigma t} f_0(v)dv + \int_0^t e^{-\sigma\tau}\sigma \left[\sum_{k=1}^m \hat{c}_k \left[1-(1-x(t-\tau))^k\right]\right] d\tau$$

one sees that if $x(t) \to p$ as $t \to +\infty$ then $\Lambda x(t) \to \sum_{k=1}^m \hat{c}_k \left[1-(1-p)^k\right]$ as $t \to +\infty$. Thus, since $\varphi_0(t) \to 1$ as $t \to +\infty$ it follows that for all n, $\varphi_n(t) \to \overline{x}_n$ as $t \to +\infty$ where

$$\overline{x}_0 = 1,\ \overline{x}_{n+1} = \sum_{k=1}^m \hat{c}_k \left[1-(1-\overline{x}_n)^k\right].$$

Similarly, in the supercritical case, since $\underline{\psi}_0(t) \to x$ as $t \to +\infty$ it follows that for all n, $\underline{\psi}_n(t) \to \underline{x}_n$ as $t \to +\infty$ where

$$\underline{x}_0 = 1,\ \underline{x}_{n+1} = \sum_{k=1}^m \hat{c}_k \left[1-(1-\underline{x}_n)^k\right].$$

Lemma 3.4. *Let us suppose that $\sigma > 0$ is constant.*

i) If $\sum_{k=1}^m k\hat{c}_k \leq 1$ then $\overline{x}_n \to 0$ as $n \to +\infty$.

ii) $\sum_{k=1}^m k\hat{c}_k > 1$ then both $\overline{x}_n$ and $\underline{x}_n$ tend to the nontrivial solution of (15).

Proof. We note that

$$\Lambda_\infty : x \in [0,1] \to \sum_{k=1}^{m} \hat{c}_k \left[1-(1-x)^k\right] \in [0,1]$$

is nonincreasing so that the above sequence $\{\overline{x}_n\}_n$ is is nonincreasing since $\overline{x}_1 = \Lambda_\infty \overline{x}_0 = \Lambda_\infty 1 \leq 1 = \overline{x}_0$ and Λ_∞ is a nondecreasing map. Thus $\overline{z} := \lim \overline{x}_n$ satisfies

$$\overline{z} = \sum_{k=1}^{m} \hat{c}_k \left[1-(1-\overline{z})^k\right].$$

By Theorem 3.1, $\overline{z} = 0$ if $\sum_{k=1}^{m} k\hat{c}_k \leq 1$. Suppose now that $\sum_{k=1}^{m} k\hat{c}_k > 1$ and let x be the unique nontrivial solution to (15). We have

$$x = \sum_{k=1}^{m} \hat{c}_k \left[1-(1-x)^k\right] = \Lambda_\infty x \leq 1 = \overline{x}_0$$

so that $x = \Lambda_\infty x \leq \Lambda_\infty \overline{x}_0 = \overline{x}_1$ and by induction $x \leq \overline{x}_n$ $\forall n$ so that $x \leq \overline{z}$. Finally $x = \overline{z}$ by uniqueness of the nontrivial solution. Similarly the *nondecreasing* sequence $\underline{x}_n$ converges to x. □

We are ready to prove the following result:

Theorem 3.2. *Let f be the solution of the integral equation* (18)*.*

i) If $\sum_{k=1}^{m} k\hat{c}_k \leq 1$ then $f(t,v) \to 0$ as $t \to +\infty$ uniformly in v.

ii) If the initial data f_0 is not zero and $\sum_{k=1}^{m} k\hat{c}_k > 1$, then $f(t,v)$ tends to the nontrivial solution of Equation (14) *uniformly in v.*

Proof. According to Equation (20)

$$f(t,v) = e^{-\sigma t} f_0(v) + \int_0^t e^{-\sigma\tau}\sigma \left[\sum_{k=1}^{m} c_k(v)\left[1-(1-x(t-\tau))^k\right]\right] d\tau$$

and then it suffices to show, in the case *i)*, that $x(t) \to 0$ as $t \to +\infty$ and in the case *ii)* that $x(t)$ tends to the non trivial solution of (15) as $t \to +\infty$.

i) We know that $x(t) \leq \overline{\varphi}_n(t)$ for all $n \in \mathbb{N}$ so that

$$\limsup_{t\to+\infty} x(t) \leq \overline{x}_n$$

for all $n \in \mathbb{N}$. Hence $x(t) \to 0$ as $t \to +\infty$ since $\overline{x}_n \to 0$ as $n \to +\infty$.

ii) We know that $\underline{\psi}_n \leq x(t) \leq \overline{\varphi}_n(t)$for all $n \in \mathbb{N}$ so that

$$\underline{x}_n \leq \liminf_{t\to+\infty} x(t) \leq \limsup_{t\to+\infty} x(t) \leq \overline{x}_n$$

for all $n \in \mathbb{N}$. Consequently $x(t)$ tends to the non trivial solution of (15) as $t \to +\infty$ since both $\underline{x}_n$ and $\overline{x}_n$ tend to the non trivial solution of (15). □

3.3. **Convergence rates towards the asymptotic state.** In this section, we will study the speed of convergence towards equilibrium for the Cauchy problem (12) with initial data $f_0 \in L^\infty(V)$, $0 \le f_0 \le 1$, and give some quantitative bounds.

The asymptotic behaviour is governed by the quantity $\sum_{k=1}^m k\hat{c}_k$, which governs not only the equilibrium state itself, but also the speed of convergence towards the asymptotic state.

The proof of the speed of convergence towards the steady-state profile will be deduced by studying the time evolution of a suitable functional of the system. In particular, we will consider the weighted L^1-norm of the difference between the solution and the corresponding asymptotic state:

$$H(t) = \int_V |f - f_\infty| F(v)\, dv.$$

We note that, since $\|f\|_{L^\infty(V)} \le 1$ uniformly in time by Theorem 3.3, the entropy is well defined and $0 \le H(t) \le 1$ uniformly for all $t \in \mathbb{R}^+$.

The following theorem holds:

Theorem 3.3. *Consider the unique non-negative global solution of the Cauchy problem for Equation* (12)*, with $c_k(v) \ge 0$, $k = 1, \dots, m$, and $\hat{c}_m > 0$, with non-negative initial condition f_0 satisfying the bound $\|f_0\|_{L^\infty(V)} \le 1$.*

i) Let $\sum_{k=1}^m k\hat{c}_k > 1$ and $\hat{c}_0 > 0$. Then, for all $\eta > 0$ there exists a time $t_\eta > 0$ such that, for all $t \ge t_\eta$, the solution of (12) *decays exponentially fast in time towards the non-vanishing stationary state accordingly to the following estimate:*

$$\|(f - f_\infty)F\|_{L^1(V)}(t) \le \|(f(t_\eta, \cdot) - f_\infty)F\|_{L^1(V)} \exp\left[-\sigma \beta_m^\eta t\right],$$

where

$$\beta_m^\eta = \left[\hat{c}_0^{m-1}\left(\sum_{k=1}^m k\hat{c}_k - 1 + \hat{c}_0\right)^{1-m}\left(1 - \left(\sum_{k=1}^m \hat{c}_k k\right)^{1/(1-m)} - \eta\right)\sum_{k=2}^m (k-1)\hat{c}_k\right].$$

ii) Let $\sum_{k=1}^m k\hat{c}_k > 1$ and $\hat{c}_0 = 0$. Then,

$$\|(f - f_\infty)F\|_{L^1(V)}(t) \le \frac{\|(f_0 - f_\infty)F\|_{L^1(V)}}{\left[1 - \|(f_0 - f_\infty)F\|_{L^1(V)}^{m-1}\right]^{1/(m-1)}} \exp\left[-\hat{c}_m \sigma t\right].$$

iii) Let $\sum_{k=1}^m k\hat{c}_k < 1$. Then the solution of (12) *decays exponentially fast in time towards the trivial stationary solution accordingly to the following estimate:*

$$\|fF\|_{L^1(V)} \le \|f_0 F\|_{L^1(V)} \exp\left[-\sigma\left(1 - \sum_{k=1}^m k\hat{c}_k\right) t\right].$$

iv) Let $\sum_{k=1}^m k\hat{c}_k = 1$. Then, $\hat{c}_0 > 0$ and the solution of (12) *decays in time towards the trivial stationary solution with an algebraic speed of convergence and the following estimate holds:*

$$\|fF\|_{L^1(V)} \le \frac{1}{\sigma c_0 t + \|f_0 F\|_{L^1(V)}^{-1}}.$$

Proof. i) We treat first the supercritical case, namely $\sum_{k=1}^m k\hat{c}_k > 1$. We suppose that $\hat{c}_0 > 0$. We hence consider the difference between Equation (12) and Equation (13), then multiply the obtained equation by $\text{sign}(f - f_\infty)F(v)$ and integrate with respect to v in V.

We hence obtain

$$\frac{1}{\sigma}H'(t) = -H(t)+$$

$$\sum_{k=1}^m \int_V c_k(v)F(v)\,\text{sign}(f - f_\infty)\,dv\left[(1 - \|f_\infty F\|_{L^1(V)})^k - (1 - \|fF\|_{L^1(V)})^k\right].$$

The previous equation can be written in the following form:

$$\frac{1}{\sigma}H'(t) = -H(t) + \sum_{k=1}^m \Bigg[\int_V c_k(v)F(v)\,\text{sign}(f - f_\infty)\,dv \times$$

$$\sum_{j=0}^{k-1}(1 - \|f_\infty F\|_{L^1(V)})^j(1 - \|fF\|_{L^1(V)})^{k-1-j}(\|fF\|_{L^1(V)} - \|f_\infty F\|_{L^1(V)})\Bigg].$$

Since $\|fF\|_{L^1(V)} \leq 1$ and $\|f_\infty f\|_{L^1(V)} \leq 1$, we can deduce that

$$\frac{1}{\sigma}H' \leq -H + H\sum_{k=1}^m \hat{c}_k \sum_{j=0}^{k-1}(1 - \|f_\infty F\|_{L^1(V)})^j(1 - \|fF\|_{L^1(V)})^{k-j-1}.$$

The previous inequality can be written in the form

$$\frac{1}{\sigma}H' \leq -H + H\sum_{k=1}^m \hat{c}_k \sum_{j=0}^{k-1}(1 - \|f_\infty F\|_{L^1(V)})^j -$$

$$H\sum_{k=1}^m \hat{c}_k \sum_{j=0}^{k-1}(1 - \|f_\infty F\|_{L^1(V)})^j\left[1 - (1 - \|fF\|_{L^1(V)})^{k-j-1}\right].$$

We use now the elementary formula

$$\sum_{j=0}^{k-1} a^j = \frac{1 - a^k}{1 - a}, \qquad a \in (0, 1).$$

Hence,

$$\sum_{k=1}^m \hat{c}_k \sum_{j=0}^{k-1}(1 - \|f_\infty F\|_{L^1(V)})^j = \sum_{k=1}^m \hat{c}_k \frac{1 - (1 - \|f_\infty F\|_{L^1(V)})^k}{\|f_\infty F\|_{L^1(V)}}.$$

Thanks to the stationary equation (13), we deduce that

$$\sum_{k=1}^m \hat{c}_k\left(1 - \|f_\infty F\|_{L^1(V)}\right)^k = 1 - \hat{c}_0 - \|f_\infty F\|_{L^1(V)},$$

and, since

$$\sum_{k=0}^m \hat{c}_k = \sum_{k=0}^m c_k = 1$$

by the properties of the family $c_k(v)$ and F, we finally deduce that

$$\sum_{k=1}^{m} \hat{c}_k \sum_{j=0}^{k-1} (1 - \|f_\infty F\|_{L^1(V)})^j = 1.$$

This implies that

$$\frac{1}{\sigma} H' \leq -H \sum_{k=1}^{m} \hat{c}_k \sum_{j=0}^{k-1} (1 - \|f_\infty F\|_{L^1(V)})^j \left[1 - (1 - \|fF\|_{L^1(V)})^{k-j-1}\right]$$

and hence

$$\frac{1}{\sigma} H' \leq -H(1 - \|f_\infty F\|_{L^1(V)})^{m-1} \|fF\|_{L^1(V)} \sum_{k=2}^{m} (k-1)\hat{c}_k.$$

Since $\|f\|_{L^1(V)} \to \|f_\infty\|_{L^1(V)}$ thanks to the results of Theorem 3.2 then, for all $\eta > 0$ there exists $t_\eta > 0$ such that, for all $t \geq t_\eta$,

$$\left| \|f\|_{L^1(V)} - \|f_\infty\|_{L^1(V)} \right| < \eta.$$

Hence, for all $t > t_\eta$,

$$\frac{1}{\sigma} H' \leq -H(1 - \|f_\infty F\|_{L^1(V)})^{m-1} (\|f_\infty F\|_{L^1(V)} - \eta) \sum_{k=2}^{m} (k-1)\hat{c}_k.$$

Thanks to Lemma 3.2, we finally obtain

$$H' \leq -\sigma \left[\hat{c}_0^{m-1} \left(\sum_{k=1}^{m} k\hat{c}_k - 1 + \hat{c}_0 \right)^{1-m} \left(1 - \left(\sum_{k=1}^{m} \hat{c}_k k \right)^{1/(1-m)} - \eta \right) \sum_{k=2}^{m} (k-1)\hat{c}_k \right] H.$$

We hence deduce asymptotic exponential convergence in time towards equilibrium for the weighted L^1-norm $H(t)$:

$$H(t) \leq H(t_\eta) \exp\left(-\sigma \beta_m^\eta t\right),$$

where

$$\beta_m^\eta = \left[\hat{c}_0^{m-1} \left(\sum_{k=1}^{m} k\hat{c}_k - 1 + \hat{c}_0 \right)^{1-m} \left(1 - \left(\sum_{k=1}^{m} \hat{c}_k k \right)^{1/(1-m)} - \eta \right) \sum_{k=2}^{m} (k-1)\hat{c}_k \right].$$

for all $t \geq t_\eta$.

ii) We treat now the supercritical case, namely $\sum_{k=1}^{m} k\hat{c}_k > 1$, when $\hat{c}_0 = 0$. By Lemma 3.2, the stationary solution is $f_\infty = 1$.

We consider the time evolution of the weighted L^1-norm of the difference between the solution of Equation (12) and its stationary state

$$H(t) = \int_V (1 - f)F(v)\, dv = \int_V |f - 1| F(v)\, dv.$$

Thanks to Equation (12), we have that

$$\frac{dH}{dt}(t) = -\sigma H(t) + \sigma \sum_{k=1}^{m} \hat{c}_k H^k(t).$$

Since $\sum_{k=1}^m \hat{c}_k = 1$, we deduce from the previous equation that

$$\frac{1}{\sigma}\frac{dH}{dt}(t) \le -\hat{c}_m H(t) + \hat{c}_m H^m(t).$$

Hence it is easy to conclude that

$$H(t) \le \frac{H(0)}{(1-H(0)^{m-1})^{1/(m-1)}} e^{-\hat{c}_m \sigma t}.$$

iii) In the subcritical case, that is $\sum_{k=1}^m k\hat{c}_k < 1$, we have that the only stationary solution is $f_\infty = 0$ (see Theorem 3.1). Since $f \ge 0$, it is easy to see that the time evolution of the quantity $\|fF\|_{L^1(V)}$ is governed by the following ordinary differential equation:

$$\frac{1}{\sigma}\frac{d}{dt}\|fF\|_{L^1(V)} = -\|fF\|_{L^1(V)} +$$

$$\sum_{k=1}^m \int_V c_k(v)F(v)\left[1-(1-\|fF\|_{L^1(V)})^k\right] dv =$$

$$-\|fF\|_{L^1(V)} + \sum_{k=1}^m \hat{c}_k(v) \sum_{j=0}^{k-1} (1-\|fF\|_{L^1(V)})^j \|fF\|_{L^1(V)}.$$

Since $\|f\|_{L^1(V)} \le 1$, we can deduce that

$$\frac{1}{\sigma}\frac{d}{dt}\|fF\|_{L^1(V)} \le -\|fF\|_{L^1(V)} + \sum_{k=1}^m k\hat{c}_k (1-\|fF\|_{L^1(V)})\|fF\|_{L^1(V)}. \tag{21}$$

We now use the hypothesis of subcriticity: we finally obtain

$$\|fF\|_{L^1(V)} \le \|f_0 F\|_{L^1(V)} \exp\left[-\sigma\left(1-\sum_{k=1}^m k\hat{c}_k\right)t\right].$$

The previous differential inequality means exponential convergence in time towards zero.

iv) The situation $\sum_{k=1}^m k\hat{c}_k = 1$ is quite different, since the speed of convergence towards the stationary solution is no more exponential.

We first notice that, in this case, $\hat{c}_0 > 0$: indeed, by contradiction, if $\hat{c}_0 = 0$ (which is equivalent to say that $c_0(v) = 0$ for all $v \in V$), then we would have

$$\sum_{k=1}^m k\hat{c}_k = 1 = \sum_{k=1}^m \hat{c}_k,$$

a result which is false for all $m > 1$.

We consider hence, as in case *iii)*, the time evolution of the L^1-norm of (fF). The same computations as before lead to Equation (21).

Thanks to the hypothesis $\sum_{k=1}^m k\hat{c}_k = 1$, we finally deduce

$$\frac{1}{\sigma}\frac{d}{dt}\|fF\|_{L^1(V)} \le -\|fF\|^2_{L^1(V)}.$$

Hence, the thesis of the theorem follows. □

Remark 3.1. *The strategy employed in the previous proof does not give the best possible constant of decay but, nevertheless, it permits to deduce that, qualitatively, the decay towards equilibrium is exponential in both cases* $\sum_{k=1}^m k\hat{c}_k < 1$ *and* $\sum_{k=1}^m k\hat{c}_k > 1$.

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