

# A non-conforming discontinuous Galerkin method for solving Maxwell's equations

H. Fahs

INRIA, 2004 Route des Lucioles, BP 93, F-06902 Sophia Antipolis Cedex, France  
E-mail: hassan.fahs@inria.fr

**Abstract** — This paper reviews the main features of a high-order non-dissipative discontinuous Galerkin (DG) method recently investigated in [1]-[3] for solving Maxwell's equations on non-conforming simplex meshes. The proposed method combines a centered approximation for the numerical fluxes at inter element boundaries, with either a second-order or a fourth-order leap-frog time integration scheme. Moreover, the interpolation degree is defined at the element level and the mesh is refined locally in a non-conforming way resulting in arbitrary-level hanging nodes.

## I. INTRODUCTION

In the recent years, several works have demonstrated the benefits of using DG methods for the simulation of time-domain electromagnetic wave propagation problems involving complex geometries and heterogeneous media. Being higher order versions of traditional finite volume method [1]-[5], Discontinuous Galerkin Time-Domain (DGTD) methods are flexible discretization methods that can handle complicated geometries, media and meshes, and achieve high-order accuracy by simply choosing suitable local basis functions. Whereas several conforming DGTD methods for the numerical resolution of the system of Maxwell equations have been developed so far [4], the design of non-conforming DGTD methods is still in its infancy. In practice, the non-conformity can result from a local refinement of the mesh (*i.e.*  $h$ -refinement), of the interpolation order (*i.e.*  $p$ -enrichment) or of both of them (*i.e.*  $hp$ -refinement).

## II. DISCONTINUOUS GALERKIN METHOD

We consider the three-dimensional Maxwell equations on a bounded domain  $\Omega$  of  $\mathbb{R}^3$ . The electric permittivity tensor  $\bar{\epsilon}(x)$  and the magnetic permeability tensor  $\bar{\mu}(x)$  are varying in space and both symmetric positive definite. The electric field  $\vec{E}$  and the magnetic field  $\vec{H}$  verify:  $\bar{\epsilon}\partial_t\vec{E} = \text{curl}\vec{H}$  and  $\bar{\mu}\partial_t\vec{H} = -\text{curl}\vec{E}$ . We consider a partition  $\Omega_h$  of  $\Omega$  into a set of tetrahedra  $\tau_i$ . To each  $\tau_i \in \Omega_h$  we assign an integer  $p_i \geq 0$  (the local interpolation order) and we collect the  $p_i$  in the vector  $p = \{p_i : \tau_i \in \Omega_h\}$ . On the domain boundary  $\partial\Omega$ , we use either a metallic boundary condition or a first-order Silver-Müller absorbing boundary condition. In the following, for a given partition  $\Omega_h$  and vector  $p$ , we seek approximate solutions to Maxwell's system in the finite dimensional subspace  $V_p(\Omega_h) := \{v \in L^2(\Omega)^3 : v|_{\tau_i} \in \mathbb{P}_{p_i}(\tau_i), \forall \tau_i \in \Omega_h\}$ , where  $\mathbb{P}_{p_i}(\tau_i)$  denotes the space of nodal polynomials of total degree at most  $p_i$  inside the element  $\tau_i$ . The DGTD method at the heart of this study is based on a totally centered numerical fluxes at the interface between elements of the mesh. We suppose that all electric (resp. magnetic) unknowns are gathered in a column vector  $\mathbb{E}$  (resp.  $\mathbb{H}$ ) of size  $d$  (the total number of degrees of freedom), then the semi-discretized DGTD method can be rewritten as (see [1]-[2]-[3] for more details):

$$\begin{cases} \mathbb{M}^\epsilon \partial_t \mathbb{E} &= \mathbb{K}\mathbb{H} - \mathbb{A}\mathbb{H} - \mathbb{B}\mathbb{H}, \\ \mathbb{M}^\mu \partial_t \mathbb{H} &= -\mathbb{K}\mathbb{E} + \mathbb{A}\mathbb{E} - \mathbb{B}\mathbb{E}, \end{cases} \quad (1)$$

where we have the following definitions and properties:

- $\mathbb{M}^\epsilon, \mathbb{M}^\mu$  and  $\mathbb{K}$  are  $d \times d$  block diagonal matrices with diagonal blocks equal to the local mass and stiffness matrices respectively. Therefore  $\mathbb{M}^\epsilon$  and  $\mathbb{M}^\mu$  are symmetric positive definite matrices, and  $\mathbb{K}$  is a symmetric matrix.
- $\mathbb{A}$  is also a  $d \times d$  block sparse matrix, whose non-zero blocks are equal to the internal interface matrix (corresponds to fluxes at internal interfaces of the mesh). Then  $\mathbb{A}$  is a symmetric matrix.
- $\mathbb{B}$  is a  $d \times d$  block diagonal matrix, whose non-zero blocks are equal to the metallic interface matrix (corresponds to fluxes at metallic boundary interfaces of the mesh). Then  $\mathbb{B}$  is a skew-symmetric matrix.

The DGTD- $\mathbb{P}_{p_i}$  method using centered fluxes combined with  $N$ th order leap-frog ( $\text{LF}_N$ ) time scheme [6] and arbitrary local accuracy and basis functions can be written, in function of the matrix  $\mathbb{S} = \mathbb{K} - \mathbb{A} - \mathbb{B}$ , in the general form:

$$\begin{cases} \mathbb{M}^\epsilon \frac{\mathbb{E}^{n+1} - \mathbb{E}^n}{\Delta t} &= \mathbb{S}_N \mathbb{H}^{n+\frac{1}{2}}, \\ \mathbb{M}^\mu \frac{\mathbb{H}^{n+\frac{3}{2}} - \mathbb{H}^{n+\frac{1}{2}}}{\Delta t} &= -{}^t\mathbb{S}_N \mathbb{E}^{n+1}, \end{cases} \quad (2)$$

where the matrix  $\mathbb{S}_N$  verifies:

$$\mathbb{S}_N = \begin{cases} \mathbb{S} & \text{if } N = 2, \\ \mathbb{S}(\mathbb{I} - \frac{\Delta t^2}{24} \mathbb{M}^{-\mu} {}^t\mathbb{S} \mathbb{M}^{-\epsilon} \mathbb{S}) & \text{if } N = 4. \end{cases} \quad (3)$$

## III. STABILITY AND CONVERGENCE ANALYSIS

We define the following discrete version of the electromagnetic energy.

$$\mathcal{E}^n = \frac{1}{2} ({}^t\mathbb{E}^n \mathbb{M}^\epsilon \mathbb{E}^n + {}^t\mathbb{H}^{n-\frac{1}{2}} \mathbb{M}^\mu \mathbb{H}^{n+\frac{1}{2}}). \quad (4)$$

Then we have the following Lemmas [2]:

**Lemma 1** Using the DGTD- $\mathbb{P}_{p_i}$  method (2)-(3), the total discrete electromagnetic energy  $\mathcal{E}^n$  (4) is a positive definite quadratic form of all unknowns if:

$$\Delta t \leq \frac{2}{d_N}, \quad \text{with } d_N = \|\mathbb{M}^{-\frac{\mu}{2}} {}^t\mathbb{S}_N \mathbb{M}^{-\frac{\epsilon}{2}}\|,$$

where  $\|\cdot\|$  denote the canonical norm of a matrix ( $\forall X, \|AX\| \leq \|A\|\|X\|$ ), and the matrix  $\mathbb{M}^{-\frac{\sigma}{2}}$  ( $\sigma$  stands for  $\epsilon$  or  $\mu$ ) is the inverse square root of  $\mathbb{M}^\sigma$ . Also, for a given mesh, the stability limit of the  $\text{LF}_4$  scheme is roughly 2.85 times larger than that of the  $\text{LF}_2$  scheme.

**Lemma 2** The convergence order in space and time of the DGTD- $\mathbb{P}_{p_i}$  method (2)-(3) is:

$$\mathcal{O}(Th^{\min(s,p)}) + \mathcal{O}(\Delta t^N),$$

where  $\Delta t \in [0, T]$ ,  $h$  is the mesh size and the solution belongs to  $H^s(\Omega)$  with  $s > 1/2$  a regularity parameter.

## IV. NUMERICAL EXPERIMENTS

In the following, for a given non-conforming mesh, we assign to coarse (*i.e.* non refined) elements a high polynomial degree  $p_1$  and to the refined region a low polynomial degree  $p_2$ . The resulting scheme is referred to as  $\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$ . If  $p_1 = p_2 = p$ , the scheme is simply called  $\text{DGTD-}\mathbb{P}_p$ .

### A. eigenmode in a PEC cavity

The first test case that we consider is the propagation of an eigenmode in a unitary PEC cavity with  $\epsilon = \mu = 1$ .

*The 2D case:* We compare the  $\text{LF}_2$  and  $\text{LF}_4$  time schemes using the  $\text{DGTD-}\mathbb{P}_p$  method. Numerical simulations make use of a non-conforming triangular mesh which consists of 782 triangles and 442 nodes (36 of them are hanging nodes) as shown on Fig. 1 right. We plot on Fig. 1 left the time evolution of the overall  $L^2$  error of the  $\text{DGTD-}\mathbb{P}_p$  method using the  $\text{LF}_2$  and  $\text{LF}_4$  schemes. Tab. I gives the  $L^2$  error, the CPU time in minutes to reach 105 periods, and the convergence order " $\mathcal{O}$ ". It can be observed from Fig. 1 and Tab. I that the  $\text{LF}_4$  is more accurate and requires less CPU time than the  $\text{LF}_2$  scheme. Moreover, the convergence order is bounded by  $N$  which confirms the result of Lemma 2.

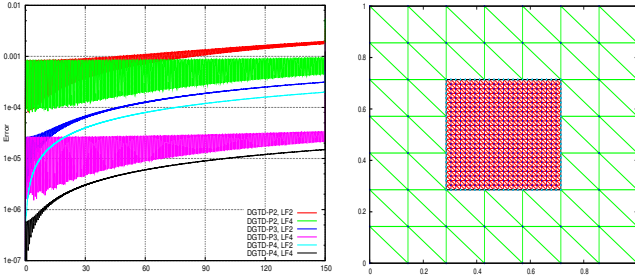


Fig. 1. Error of the  $\text{DGTD-}\mathbb{P}_p$  method (left) on the non-conforming triangular mesh (right).

TABLE I. COMPARISON BETWEEN  $\text{LF}_2$  AND  $\text{LF}_4$   $\text{DGTD-}\mathbb{P}_p$  METHOD

$p$	$\text{LF}_2$ scheme			$\text{LF}_4$ scheme		
	$L^2$ error	CPU	$\mathcal{O}$	$L^2$ error	CPU	$\mathcal{O}$
2	1.8E-03	11	2.28	5.5E-04	8	2.32
3	3.1E-04	39	2.33	2.4E-05	28	2.97
4	1.9E-04	98	2.10	1.5E-05	70	3.99

*The 3D case:* Here we compare the  $\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$  with several  $\text{DGTD-}\mathbb{P}_p$  methods using the  $\text{LF}_2$  time scheme. Numerical simulations make use of an unstructured mesh which consists of 4406 tetrahedra and 962 nodes. Tab. II gives the  $L^2$  error, the # DOF and the CPU time in minutes to reach 30 periods. Note that, the low polynomial degree  $p_2$  is used in only 5% of the tetrahedra of the mesh. We can deduce from Tab. II that to achieve a given accuracy, the  $\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$  requires less CPU time than the  $\text{DGTD-}\mathbb{P}_p$ . Moreover, for a given CPU time, the  $\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$  method is roughly 8 times more efficient than the  $\text{DGTD-}\mathbb{P}_p$  method.

### B. Scattering of a plane wave by a dielectric cylinder

In this problem, a plane wave impinges on a dielectric cylinder, experiencing reflection and refraction at the material interface. The material is non-magnetic, and the material exterior to the cylinder is assumed to be vacuum. The cylinder has a radius  $r_0 = 0.6$  and bounds a material of relative permittivity  $\epsilon_r = 2.25$ . The computational domain is chosen

TABLE II. # DOF,  $L^2$  ERRORS AND CPU TIME IN MINUTES

$p$	0	1	2	3
$L^2$ error	7.2E-01	2.0E-01	1.4E-02	8.0E-04
CPU	4	40	213	859
# DOF	4406	17624	44060	88120

$(p_1, p_2)$	(2,0)	(2,1)	(3,1)	(3,2)
$L^2$ error	3.6E-02	1.3E-02	1.0E-03	8.8E-04
CPU	35	106	260	499
# DOF	42908	43676	87096	86030

as a cylinder of radius 1.5 centered at  $(0, 0)$ . At the artificial boundary, we apply a first-order Silver-Müller absorbing boundary condition. Here, we compare the  $\text{DGTD-}\mathbb{P}_p$  method using a conforming mesh with the  $\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$  using a non-conforming mesh. To this end, we construct a conforming mesh consisting of 11920 triangles and 6001 nodes. Then, a non-conforming mesh is obtained by locally refining (two refinement levels) the cylindrical zone  $0.55 \leq r \leq 0.65$  of a coarse conforming mesh. The resulting non-conforming mesh consists of 5950 triangles and 3151 nodes (300 of them are hanging nodes). Tab. III shows the relative  $L^2$  error, the # DOF and the CPU time in minutes to reach  $t = 5$ . As expected, the gain in CPU time between the  $\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$  and the conforming  $\text{DGTD-}\mathbb{P}_p$  methods is considerable. For instance, to achieve an error level of 5%, the  $\text{DGTD-}\mathbb{P}_{(2,0)}$  scheme requires 3 times less DOF and 21 times less CPU time than the  $\text{DGTD-}\mathbb{P}_2$  scheme.

TABLE III. # DOF,  $L^2$  ERRORS AND CPU TIME IN MINUTES

$\text{DGTD-}\mathbb{P}_p$ : Conforming triangular mesh				
$p$	0	1	2	3
$L^2$ error	13.6%	7.15%	5.20%	5.22%
CPU	20	178	542	1817
# DOF	11920	35760	71520	119200

$\text{DGTD-}\mathbb{P}_{(p_1,p_2)}$ : Non-conforming triangular mesh				
$(p_1, p_2)$	(1,0)	(2,0)	(2,1)	(3,2)
$L^2$ error	11.6%	5.36%	5.39%	5.37%
CPU	9	25	33	179
# DOF	11450	19700	26100	46700

## REFERENCES

- [1] H. Fahs, "Development of a  $hp$ -like discontinuous Galerkin time-domain method on non-conforming simplicial meshes for electromagnetic wave propagation", *IJNAM*, accepted, 2008.
- [2] H. Fahs and S. Lanteri, "A high-order non-conforming discontinuous Galerkin method for time-domain electromagnetics", *submitted*, 2008.
- [3] H. Fahs, L. Fezoui, S. Lanteri, and F. Rapetti, "Preliminary investigation of a non-conforming discontinuous Galerkin method for solving the time-domain Maxwell equations", *IEEE Trans. on Magnet.*, vol. 44, pp. 1254–1257, 2008.
- [4] L. Fezoui, S. Lanteri, S. Lohrengel, and S. Piperno, "Convergence and stability of a discontinuous Galerkin time-domain method for the heterogeneous Maxwell equations on unstructured meshes", *ESAIM: Math. Model. and Numer. Anal.*, vol. 39, pp. 1149–1176, 2005.
- [5] J.S. Hesthaven and T. Warburton, "Nodal high-order methods on unstructured grids. I. Time-domain solution of Maxwell's equations", *J. Comput. Phys.*, vol. 181, pp. 186–221, 2002.
- [6] H. Spachmann, R. Schuhmann, and T. Weiland, "High order explicit time integration schemes for Maxwell's equations", *Int. J. Numer. Model.*, vol. 15, pp. 419–437, 2002.