MULTICHANNEL SAMPLING OF TRANSLATED, ROTATED AND SCALED BILEVEL POLYGONS USING EXPONENTIAL SPLINES

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ABSTRACT

Recently there has been an interest in single and multichannel sampling of certain parametric signals based on rate of innovation using exponential reproducing kernels. In [5] it was shown that, using exponential reproducing kernels, we can achieve a fully symmetric multichannel sampling system where different channels receive translated versions of the input signal. For the case of bilevel polygons as the input signal considered in [5], having only translations is not practical and one may want to look at the cases of more complicated geometric transformations, such as rotation and scaling. In this paper we present a sampling theorem for multichannel sampling of translated, rotated and scaled bilevel polygons using Radon projections and generalized exponential splines.

1. INTRODUCTION

Recently, it was shown [1, 2] that it is possible to sample and perfectly reconstruct some classes of non-bandlimited signals using suitable sampling kernels. Signals that can be reconstructed using this framework are called signals with Finite Rate of Innovation (FRI) as they can be completely defined by a finite number of parameters. Stream of weighted Dirac impulses and bilevel polygons are some examples of FRI signals.

There has been a recent interest in sampling FRI signals using exponential spline [3] (E-spline) kernels. Dragotti et al. [2] showed that E-splines can be used as the sampling kernel to sample and perfectly reconstruct 1-D FRI signals. Extensions to the multidimensional case were considered in [5, 14] where we proposed sampling theorems for a stream of 2-D Dirac impulses (based on the ACMP algorithm [11]) and bilevel polygons (based on Radon projections [10]). Apart from the sampling kernels used in [5, 14], the reconstruction algorithms are also different from the ones used in the conventional multidimensional sampling theories [12, 13].

An advantage of E-spline sampling kernels over polynomial reproducing kernels such as B-splines is that, they can be employed in a fully symmetric multichannel sampling environment. By symmetric sampling, we mean that the sampling process can be evenly distributed between different acquisition devices. The inspiration and development of multichannel sampling of FRI signals is very recent and it has been looked at in [5, 6, 7, 8].

In [6] Seelamantula and Unser, by using simple RC filters, propose a simple acquisition and reconstruction method within the framework of multichannel sampling, where 1-D FRI signals such as an infinite stream of nonuniformly-spaced Dirac impulses and piecewise-constant signals can be sampled and perfectly reconstructed. In [7] Kusuma and Goyal proposed new ways of sampling 1-D Dirac impulses using a bank of integrators or B-splines. Their proposed scheme is closely related to previously known cases [1, 2] but provides a successive approximation property, which could be useful for detecting undermodelling when the number of Dirac impulses are unknown. In [8] Baboulaz and Dragotti use a multichannel sampling setup for sampling FRI signals and utilize that for image registration based on continuous moments and image super-resolution.

In [5] we illustrate that symmetric multichannel sampling of bilevel polygons can be achieved with the geometric transformations being a 2-D translation between the different signals. In practice, this is usually not the case, and in this paper we want to look at the cases of more complicated geometric transformations, such as rotation and scaling. The paper is organised as follows: In Section II we will briefly discuss the sampling setup needed for sampling 2-D FRI signals (single channel) and based on that we describe our multichannel sampling setup. In Section III we present our algorithm for sampling and perfectly reconstructing translated, rotated and scaled bilevel polygons with the use of generalized E-splines and Radon projections. In Section IV we provide simulation results to support our proposed theory.

2. MULTICHANNEL SAMPLING SETUP

Before describing the multichannel sampling framework, let us first, for the sake of clarity, show how a general 2-D sampling setup (single channel) for FRI signals is represented. Figure 1 shows a general 2-D sampling setup for FRI signals where $g(x, y)$ represents the input signal, $\varphi(x, y)$ the sampling kernel, $s_{j,k}$ the samples and Tx, Ty are the sampling intervals. From the setup shown in Figure 1, the samples $s_{j,k}$

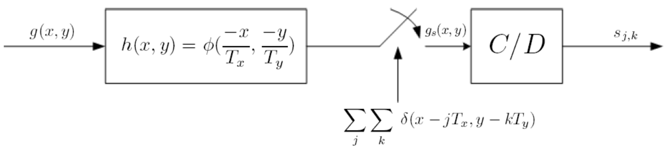

Fig. 1. 2-D sampling setup

are given by:

$$s_{j,k} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\ \varphi(\frac{x}{T_x} - j, \frac{y}{T_y} - k)\ dx\ dy \quad (1)$$

where the kernel $\varphi(x, y)$ is the time reversed version of the filter response $h(x, y)$. $\varphi(x, y)$ can easily be produced by the tensor product between $\varphi(x)$ and $\varphi(y)$, that is $\varphi(x, y) = \varphi(x) \otimes \varphi(y)$. As mentioned before, $\varphi(x, y)$ is chosen to be an exponential reproducing kernel. The theory of exponential reproducing kernels is quite recent and is based on the notion of exponential splines (E-splines) [3]. A function $\beta_{\vec{\alpha}}(x)$ with Fourier transform

$$\hat{\beta}_{\vec{\alpha}}(\omega) = \prod_{n=0}^{N} \frac{1 - e^{\alpha_n - j\omega}}{j\omega - \alpha_n}$$

is called E-spline of order N where $\vec{\alpha} = (\alpha_0, \alpha_1, \ldots, \alpha_N)$ can be real or complex. The produced spline has a compact support and can reproduce any exponential in the subspace spanned by $(e^{\alpha_0 t}, e^{\alpha_1 t}, \ldots, e^{\alpha_N t})$ which is obtained by successive convolutions of lower order E-splines ((N+1)-fold convolution). Exponential spline kernels can therefore reproduce, with their shifted versions, real or complex exponentials. That is, in 2-D form, any kernel satisfying:

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} c_{j,k}^{m,n} \varphi(x - j, y - k) = e^{\alpha_m x} e^{\beta_n y} \quad (2)$$

is an E-spline for a proper choice of the coefficients $c_{j,k}^{m,n}$. Here, $m = 0, 1, \ldots, M$, $n = 0, 1, \ldots, N$, $\alpha_m = \alpha_0 + m\lambda_1$ and $\beta_n = \beta_0 + n\lambda_2$. The values of (α_0, β_0) and (λ_1, λ_2) can be chosen arbitrarily, but too small or too large values could lead to unstable results for the reproduction of exponentials. E-splines are biorthogonal functions and the coefficients $c_{j,k}^{m,n}$ can be found using the dual of $\beta_{\vec{\alpha}}(x)$. An important property of E-splines is that they are a generalized version of B-splines. This is because, if the $\vec{\alpha}$ parameters are set to zero, then the produced spline would result in a B-spline, a polynomial reproducing spline. This property will be used to estimate the transformation parameters in Section III. The reader can refer to [5, 14] for sampling theories on single-channel sampling and perfect reconstruction of 2-D Dirac impulses and bilevel polygons using exponential splines.

We can now describe our multichannel sampling setup. A multichannel sampling system can be thought of multiple acquisition devices observing an input signal. In order to perfectly reconstruct the input signal using only one acquisition device, we normally require expensive acquisition devices with high sampling rates. By using a bank of acquisition devices (filters) and synchronizing the different channels exactly, we are able to reduce the number of samples needed from each device, resulting in a cheaper and more efficient sampling system. To model our multichannel system, consider a bank of E-spline filters to acquire FRI signals where each filter has access to a geometrically transformed version of the input signal. Figure 2 shows the described multichannel sampling scenario where the bank of filters $\varphi_1(x, y), \varphi_2(x, y), \ldots, \varphi_{N-1}(x, y)$ receive different versions of the input signal $g_0(x, y)$. Here, the geometric transformations (e.g. translation, rotation and scaling) are denoted by $\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_{N-1}$.

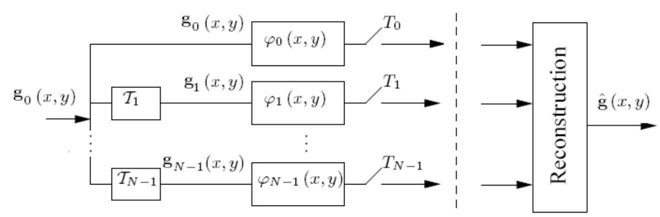

Fig. 2. Multichannel sampling setup

In [4] Baboulaz considered the use of E-splines for sampling a stream of 1-D Dirac impulses in a multichannel sampling setup described in Figure 2. He showed that if two 1-D signals are just shifted version of the other, then by setting one parameter to be common between the exponents of the E-spline sampling kernels for the two signals, one can not only estimate the shifts between the two signals, but also can linearly relate the exponential moments of the two signals (the reader can refer to [4, 5, 14] for more detailed discussion). Because of the direct relationship between the exponential moments of the two signals, we can achieve perfect reconstruction of the reference signal with fewer exponential moments required. Since less moments are required from each signal, a lower order E-spline sampling kernel would be needed, which in turn less samples from each signal are required to achieve perfect reconstruction. This is because, from [2] we know that a stream of Dirac impulses is uniquely determined from the samples if there are at most K Dirac impulses in an interval size of $2KLT$ where L is the support of the sampling kernel. Since the support of the sampling kernels is reduced in the multichannel case, we can achieve the same performance with a smaller sampling rate T.

3. ALGORITHM

Unfortunately we can not estimate the more complicated geometric transformations like the way it was done for the simple translation case in [5] with exponential reproducing kernels. Also, even if we assume that the transformation parameters are known and given, we still can not use the sampling algorithm shown in [5] for the multichannel framework. This is because introducing more complicated transforms such as rotation and/or scaling for example, would result in a non-linear relationship between the exponential moments of the different signals.

The first question we need to answer is that, assuming an oracle gives us the values of the transformation parameters, can we sample and perfectly reconstruct translated, rotated and scaled bilevel polygons in a symmetric multichannel framework? It is known that for an N-sided bilevel polygon, with N+1 projections, perfect reconstruction of the polygon can be achieved. That is points that have N+1 line intersections from the N+1 back-projections correspond to the N vertices of the polygon [9]. We also know that a Radon projection at an angle ϕ of a rotated image with respect to its reference image with an angle θ, is the same projection, but scaled and translated, on the reference image at the angle $\phi + \theta$. Therefore, if all the transformation parameters are known, and assuming that the rotation angle is not zero that is, $\theta \neq 0$, then the $N + 1$ projections needed could be separated between the different channels, in order to sample and perfectly reconstruct the reference image in a symmetric manner.

The next question would be, how can we estimate the transformation parameters? We know that with the use of polynomial reproducing kernels, we can obtain the geometric moments of a signal, and geometric moments up to order 2 from two signals are enough to estimate translation, rotation and scaling parameters between the two signals. We also know that, as E-splines are a generalized version of B-splines [3], we can reproduce a combination of polynomials and exponentials from E-splines. From the polynomials moments up to order 2, we can estimate all the transformation parameters.

4. RESULTS

As an example, in [5] we showed that to achieve perfect reconstruction for a 4-sided bilevel polygon, a 2-D E-spline order of 12 is required to produce 5 projections at the angles $0, 45, 90, tan^{-1}(2)$ and $tan^{-1}(\frac{1}{2})$. With 2-D E-spline order of 7 however we can produce 3 projections at the angles $0, 45, 90$ on the reference signal, and a 2-D E-spline order of 7 on the second signal would give 3 projections for the reference signal at the angles $\theta, 45 + \theta, 90 + \theta$ where θ is the rotation parameter. Assuming θ is not zero, we would have enough projections to perfectly reconstruct the reference signal. Therefore an spline order of 7+2 = 9 (2 is needed for estimating the transformation parameters) on each signal would give us enough projections to perfectly reconstruct the reference signal. An example for a 4-sided bilevel polygon with two acquisition devices is shown in Figure 3 where the reference signal, its translated, rotated and scaled version, their samples, the E-spline sampling kernel, and the reconstructed reference signal are all shown.

5. CONCLUSION

In this paper we showed that with the use of Radon projections and generalized E-splines, symmetric multichannel sampling of translated, rotated and scaled bilevel polygons can be achieved. For estimating the geometrical transformations, we showed that as E-splines are a generalized version of B-splines, we can reproduce combination of polynomials and exponentials from E-splines. Therefore from the polynomial moments up to order 2, we can estimate all the unknown transformation parameters. For symmetric multichannel sampling of geometrically transformed bilevel polygons, we illustrated that the N+1 Radon projections needed for perfect reconstruction of an N-sided bilevel polygon, can be separated between the different channels, assuming that the rotation parameter is not zero. Our sampling and reconstruction algorithm is based on noise-free communication between the transmitter and receiver which is rather not very practical. The future research of this work is to test the stability and performance of our method in the presence of noise.

6. REFERENCES

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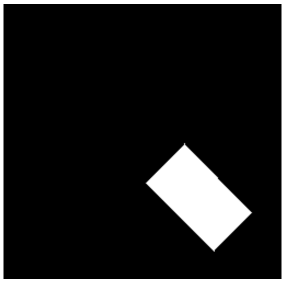
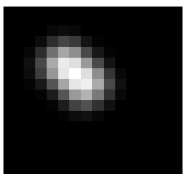
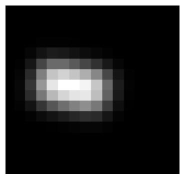
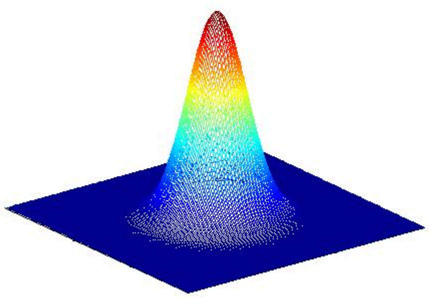


(a) (b)





(c) (d)



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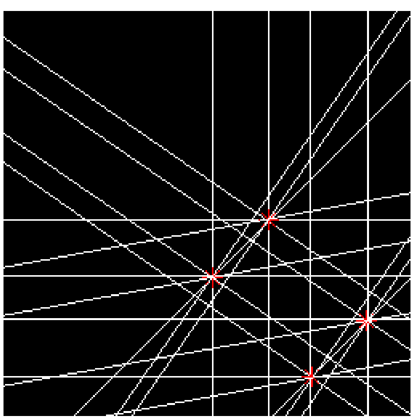

(f)

Fig. 3. **Symmetric multichannel sampling of translated, rotated and scaled bilevel polygons using E-spline sampling kernels. (a) The reference signal in a frame data size of 256×256. (b) The translated $(\triangle x = -100, \triangle y = 150)$, rotated $(\theta = 35)$ and scaled $(a = 1.1)$ version of the reference signal. (c) & (d) The 16×16 samples of both signals. (e) 2-D generalized E-spline of order 9 (f) The reconstructed vertices of the reference signal with 6 back-projections, the crosses are the actual vertices of the polygon. [Not to scale]**

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