Estimation of the Length and the Polynomial Order of Polynomial-based Filters

Djordje Babic[(1)], and Heinz G. Göckler[(2)]

(1) Faculty of Computer Science, University Union, Belgrade, Knez Mihailova 6/VI, 11000 Belgrade, Serbia.
(2) DISPO, Faculty of Electrical Engineering and Information Sciences, Ruhr-Universität, Bochum, Germany.
djbabic@raf.edu.rs, goeckler@nt.rub.de

Abstract:

In many signal processing applications it is beneficial to use polynomial-based interpolation filters for sampling rate conversion. Actual implementations of these filters can be performed effectively by using the Farrow structure or its modifications. In the literature, several design methods have been proposed. However, estimation formulae for the number of polynomial-segments defining the finite length of the underlying continuous-time filter impulse response and the order of polynomials have not been known. This contribution presents estimation formulae for the length and the polynomial order of polynomial-based filters for various types of requirements. The formulae presented here can save time in designing, since they provide good starting values of length and order for a given set of requirements.

1. Introduction

In many signal processing applications it is required to determine signal samples at arbitrary positions between existing samples of a discrete-time signal. In these cases, it is beneficial to use polynomial-based interpolation filters. For these filters, an efficient overall implementation can be achieved by using a continuous-time impulse response $h_a(t)$ having the following properties [1], [2]; First, $h_a(t)$ is nonzero only in a finite interval $0 \le t < NT$ with N being an integer. Second, in each subinterval $nT \le t < (n+1)T$, for $n=0, \ldots, N-1$, $h_a(t)$ is expressible as a polynomial of t of a given (low) order M. Third, $h_a(t)$ is symmetric with respect to $t = NT/2$ to guarantee phase linearity of the resulting overall system. The length of polynomial segments, T, can be selected to be equal to the input T_{in} or output T_{out} sampling interval, a fraction of the input or output sampling interval, or an integer multiple of the input or output sampling interval. The advantage of the above system lies in the fact that the actual implementation can be efficiently performed by using the Farrow structure [3] or its modifications [4], [5].

In the literature, several design methods have been proposed [1], [2], [4]. However, estimation formulae for the number N of polynomial-segments and the order M of polynomial have not been known. This contribution presents the missing estimation formulae for the length N and polynomial order M for various types of requirements. The formulae presented subsequently can save time for the filter designers, because they get suitable starting values for N and M that can be used for the given set of requirements. The formulae can also be used to estimate implementation costs of Farrow filter as subsystem of general sampling rate converters, for example, in optimal factorization of multistage decimation (interpolation).

2. Polynomial-based filters

As it has been originally suggested in [1], [2] when deriving the modified Farrow structure for interpolation, it is beneficial to construct $h_a(t)$ as follows:

$$h_a(t) = \sum_{n=0}^{N-1} \sum_{m=0}^{M} c_m(n) f_m(n,T,t) \qquad (1)$$

where the number of polynomial segments N is an integer. The basis functions $f_m(n, T, t)$, as defined in [1], are given by

$$f_m(n,T,t) = \begin{cases} \left(\dfrac{2(t-nT)}{T} - 1 \right)^m & \text{for } nT \le t < (n+1)T \\ 0 & \text{otherwise,} \end{cases} \qquad (2)$$

where the common polynomial order of all segments is M. The coefficients $c_m(n)$ are the adjustable parameters being related to each other by

$$c_m(N-1-n) = \begin{cases} c_m(n) & \text{for } m \text{ even} \\ -c_m(n) & \text{for } m \text{ odd} \end{cases} \qquad (3)$$

for $n = 0, 1, \ldots, N-1$, as consequence of the symmetry properties required above. The resulting $h_a(t)$ is characterized by the following properties: (i) $h_a(t)$ is nonzero for $0 \le t < NT$ and zero elsewhere; (ii) in each subinterval $nT \le t < (n+1)T$ for $n = 0, \ldots, N-1$, $h_a(t)$ is expressed as a polynomial of degree M; (iii) $h_a(t)$ is symmetric about $t = NT/2$, that is, $h_a(NT-t) = h_a(t)$.

Based on Property (iii), it is guaranteed that the resulting overall system has a linear phase, a very attractive property for many applications. Furthermore, the generation of the above $h_a(t)$ guarantees that, in the frequency domain, the zero-phase frequency response, when omitting the linear-phase term, is expressible as (see [1] for details)

$$H_a(j2\pi f) = \sum_{n=0}^{N/2-1} \sum_{m=0}^{M} c_m(n) G_m(n,T,f), \qquad (4)$$

where $G_m(n, T, f)$ is the Fourier transform of

$$g_m(n,T,t) = (-1)^m f_m(n,T,t-NT/2) + f_m(N-1-n,T,t-NT/2). \quad (5)$$

Since the above approximating function is linear with respect to the unknown coefficients $c_m(n)$, it enables one to optimize the overall filter to meet the given criteria in a manner similar to that used for synthesizing various types of linear-phase FIR filters [6]. In the above, T, the length of the polynomial segments, can be used to define different implementation structures as discussed in [4], [5]. As seen in [4], [5], T can be chosen as $T=\beta T_{in}$ or $T=\beta T_{out}$, where β is unity, an integer, or one divided by an integer. The selection depends on whether decimation or interpolation is under consideration, and on the structural needs for efficient implementation. The actual implementation can be efficiently performed by using the Farrow structure [3] or its modifications [4], [5].

For all these structure the number of fixed coefficients depends on the number N of polynomial segments and the order M of the polynomial in each segment. The total number of multipliers, exploiting the symmetry properties of (3), is given by

$$S = \begin{cases} N \cdot (M+1)/2 & \text{for } N \text{ even} \\ (N-1)(M+1)/2 + \lceil (M+1)/2 \rceil & \text{for } N \text{ odd.} \end{cases} \quad (6)$$

For the purpose of illustration, the modified Farrow structure [1] is used with $T=T_{in}$. It should be pointed out that, in a practical realization, the coefficients' symmetry of the FIR branches will be exploited, and a single delay line can be shared with all branches.

3. Review of minimax design method

This section reviews minimax design method of polynomial-based filters of arbitrary length and order, as presented in [1], [2], for which we estimate N and M.

To this end, we assume a lowpass signal $x(n) \leftrightarrow X(e^{j\Omega in})$. Its sampling rate $F_{in}=1/T_{in}$ shall be converted by an arbitrary ration according to $F_{out}=RF_{in}$ yielding $y(l) \leftrightarrow Y(e^{j\Omega out})$. In case of $R>1$ ($R<1$) the system realizes interpolation (decimation). The ultimate aim is to determine a continuous-time, finite-length impulse response $h_a(t)$ of the sampling rate conversion system such that the Fourier transform of $h_a(t)$ meets following requirements [4] , [7]:

$$\begin{aligned} &(1-\delta_p) \le H_a(f) \le (1+\delta_p) && \text{for } |f| \le f_p = \alpha F/2 \\ &|H_a(f)| \le \delta_s && \text{for } |f| \in \Phi_s, \end{aligned} \quad (7)$$

where

$$\Phi_s = \begin{cases} [F/2, \infty] & \text{for Case A} \\ \bigcup_{k=1}^{\infty} [kF - f_p, kF + f_p] & \text{for Case B} \\ [F - f_p, \infty] & \text{for Case C.} \end{cases} \quad (8)$$

In all three cases, the signal is preserved according to the given tolerance in the passband region [0, f_p]. Furthermore, the aliasing components are attenuated in the defined manner. In Case A, all components aliasing into the baseband [0, $F/2$] are attenuated. In Case B, all components aliasing into the passband [0, f_p] are attenuated, but aliasing is allowed in the transition band [f_p, $F/2$]. In Case C, aliasing into the transition band [f_p, $F/2$] is allowed only from the band [$F/2$, $F+f_p$]. In the above discussion and in (7) and (8) F stands for F_{out} in a decimation case, and F_{in} in an interpolation case.

The minimax optimization method introduced in [1], [2] is probably the most convenient and the most flexible solution for designing polynomial-based interpolation filters:

Minimax Optimization Problem: Given N, M, and a compact subset $\Phi \subset [0,\infty)$ as well as a desired function $D(f)$ being continuous for $f \in \Phi$ and a weight function $W(f)$ being positive for $f \in \Phi$, find the $(M+1)N/2$ unknown coefficients $c_m(n)$ to minimize

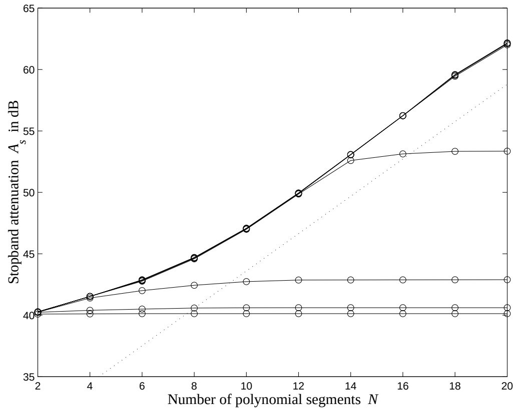
$$\delta_\infty = \max_{f \in \Phi} \left| W(f)\left[H_a(f) - D(f)\right]\right| \quad (9)$$

subject to the given time-domain conditions of $h_a(t)$. Here, $H_a(f)$ is the real-valued frequency response and $D(f)$ is the desired function according to specifications. (For details refer to [2]). The design procedure has been generalized, and modified for optimization of prolonged and transposed prolonged polynomial-based filters [4].

The minimax design method has several design parameters. First of all, the design parameters include passband and stopband regions Φ_p and Φ_s. The desired filter may have several passbands and stopbands as stated in [2]. Next, the minimum stopband attenuation δ_s, and maximum allowable passband ripple δ_p are also included. Other design parameters are the number of polynomial segments N and the order M of the polynomial, which determine the number of multipliers in the overall structure, see (6). Finally, some weighting function can be used to give different weights to passband and stopband [2]. Hence we give estimation formulae for the number N of polynomial segments and the order M of polynomial for a minimax design.

4. Estimation of *N* and *M*

In the previous section, we have seen that the number of polynomial segments N and the order M of the polynomial, are the design parameters that highly influence the performance of the filter in the frequency domain. Furthermore, the cost of realization, i.e. the number of multipliers, of a filter can be estimated by introducing the required values for N and M into (6). It would be very beneficial to estimate N and M by only using the given specifications of the filter in the frequency domain. Similar order estimation formulae exist for FIR filters, for example Kaiser order estimation [6], [8]. In the actual implementation, polynomial-based filters can be modeled as FIR filters [4]. Thus, we can start from the Kaiser formula and adapt it to polynomial-based filters. To this end, a lot of filters were designed, by using different system specifications, in order to adapt the Kaiser formula to polynomial-based case. The obtained estimation formula for the number of polynomial segments N, is rather similar to Kaiser formula for the order estimation of FIR filters. The



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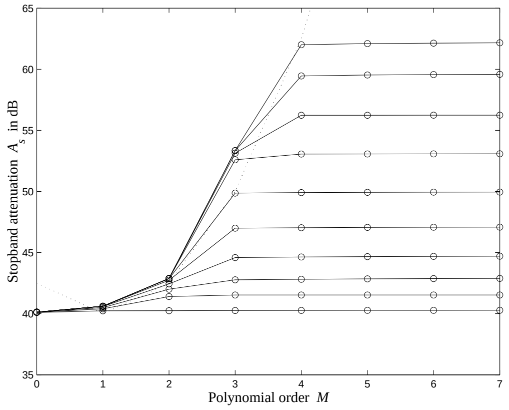


(b)

Fig. 1. *Case A specifications*: The passband and stopband edges are at f_p=0.4F_{in} and at f_s=0.5F_{in}, and stopband weighting W=100. (a) The curves are shown for M equals 0 to 7. Dashed line is plot obtained from the estimation formula for N. (b) The curves are shown for N equals 2 to 20. Dashed line is plot obtained from the estimation formula for M.

estimation formula for N, which can be found in [9], is not accurate enough. Hence, we propose the more accurate formula:

$$N_e = 2\left\lceil \frac{-20\log_{10}(\sqrt{\delta_p\delta_s}) - 8.4}{30.4(f_s - f_p)/F} \right\rceil \quad (10)$$

where δ_p and δ_s are the maximum deviations of the amplitude response from unity for $f\in[0,f_p]$ and the maximum deviation from zero for $f\in\Phi_s$, respectively. Here, $\lceil x \rceil$ stands for the smallest integer which is larger or equal to x. It has been observed that in most cases the above estimation formula is rather accurate with only a 2% error. The formula above is valid for all three types of requirements, i.e., A, B, and C, as given by (7) and (8). However, if the transition band is narrow, i.e., in the case when $(f_s-f_p)/F\leq 0.1$, the required value of N should be increased by 2. Further, in the case of very narrow transition band $((f_s-f_p)/F \leq 0.05)$ the formula can not be used.

The kernel of the estimation formula for the number N of polynomial segments can be expressed in a different form:

$$N_e = 2\left\lceil \frac{A_s - 10\log_{10}(W) - 8.4}{30.4(f_s - f_p)/F} \right\rceil \quad (11)$$

where $A_s=-20\log_{10}(\delta_s)$ is the required attenuation in stopband, and $W=\delta_p/\delta_s$ represents weighting between required tolerances in passband and stopband.

The next problem is to find the minimum value of the polynomial order M to meet the specifications. It has been observed that the required value of M depends on the type of requirements from (7) and (8). Never the less, it is possible to consider the following estimate as good starting point for all three types of requirements:

$$M_e = \left\lceil \sqrt{\frac{A_s - 20\cdot\log_{10}(W)}{2.5}} + \log_{10}(W) \right\rceil + 1. \quad (12)$$

It has been observed that if transition band is relatively large to the sampling frequency, that is when $(f_s-f_p)/F \geq 0.5$, the required value of polynomial order M is lowered by one. The estimation formula cannot be used when the transition band is very small, i.e., in the case when $(f_s-f_p)/F<0.1$. However, even in this border situation required value of M is always smaller than M_e given by (12). Thus, the estimation formula (12) for the polynomial order M can be used to estimate the upper border for M for all types of requirements.

5. Design Examples

This part gives several examples to illustrate the performance of the presented formulae. To illustrate this, the following specifications are considered:

Case A *specifications*: The passband and stopband edges are at f_p=0.4F_{in} and at f_s=0.5F_{in}.

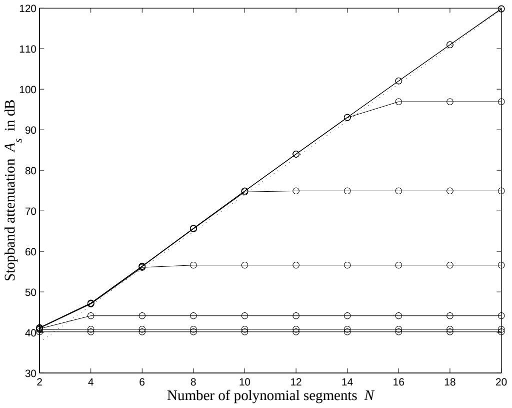
Case B *specifications*: The passband and stopband edges are at f_p=0.35F_{in} and at f_s=0.65F_{in}.

Case C *specifications*: The passband and stopband edges are at f_p=0.35F_{in} and at f_s=0.65F_{in}.

In each case, several filters have been designed in minimax sense with the passband weighting equal to unity and stopband weightings of W=100. The degree of the polynomial in each subinterval M varies from 0 to 7. The number of intervals N varies from 2 to 20. Recall that N is an even integer. Figures 1 give the results for Case A, the similar results for Case B are given in Fig. 2, and for Case C in Fig. 3. It can be observed that the estimation formulae are relatively good, as they estimate the border performance for the given set of requirements (dashed lines in Figs 1-3).

6. Conclusions

In this paper, the estimation formulae for the number N of polynomial segments and the polynomial order M are presented. It has been shown that these estimates give the border performance of the filter for the given set of specifications. Formulae for N and M can be used to estimate the starting value of these two parameters in minimax optimization. Furthermore, the formulae for N and M can be used to estimate implementation costs of



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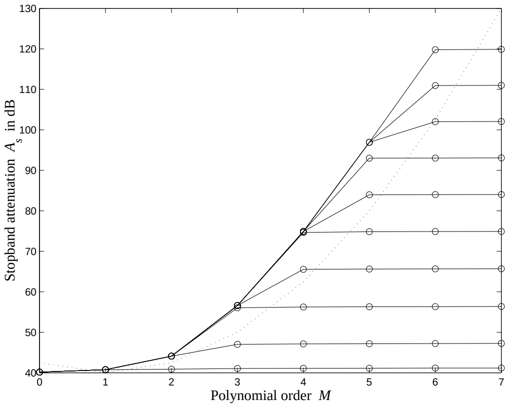
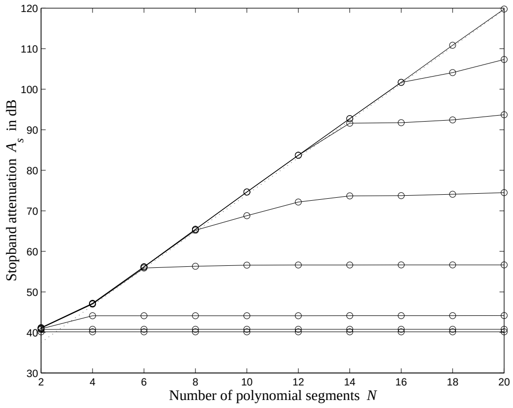


(b)

Fig. 2. *Case* B *specifications*: The passband and stopband edges are at f_p=0.35F_{in} and at f_s=0.65F_{in}, and stopband weighting *W*=100. (a)The curves are shown for *M* equals 0 to 7. Dashed line is plot obtained from the estimation formula for *N*. (b) The curves are shown for *N* equals 2 to 20. Dashed line is plot obtained from the estimation formula for *M*.



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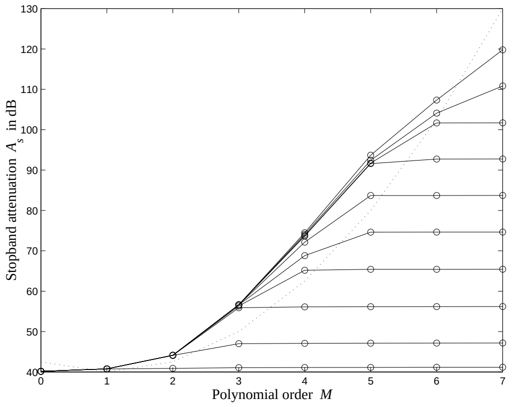

(b)

Fig. 3. *Case C specifications*: The passband and stopband edges are at f_p=0.35F_{in} and at f_s=0.65F_{in}, and stopband weighting *W*=100. (a) The curves are shown for *M* equals 0 to 7. Dashed line is plot obtained from the estimation formula for *N*. (b) The curves are shown for *N* equals 2 to 20. Dashed line is plot obtained from the estimation formula for *M*.

the Farrow based filters for the given set of requirements. Formulae can also be used to estimate implementation costs of composed sampling rate converters containing Farrow, for example, in optimal factorization for multistage decimation (interpolation).

References:

[1] J. Vesma and T. Saramäki, “Interpolation filters with arbitrary frequency response for all-digital receivers,” in *Proc. 1996 IEEE Int. Symp. Circuits and Systems*, Atlanta, Georgia, May 1996, pp. 568–571.

[2] J. Vesma and T. Saramäki, “Polynomial-based interpolation Filters - Part I: Filter synthesis," *Circuits, Systems, and Signal Processing*, vol. 26, no. 2, pp. 115-146, March/April 2007.

[3] C. W. Farrow, “A continuously variable digital delay element,”in *Proc. 1988 IEEE Int. Symp. Circuits and Systems*, Espoo, Finland, June 1988, pp. 2641–2645.

[4] D. Babic, T. Saramäki, M. Renfors, “Conversion between arbitrary sampling rates using polynomial-based interpolation filters,” in *Proc. 2nd Int. TICSP Workshop on Spectral Methods and Multirate Signal Processing SMMSP’02* , Toulouse, France, September 2002, pp. 57–64.

[5] D. Babic, *Techniques for sampling rate conversion by arbitrary factors with applications in flexible communications receivers*, Doctoral Thesis, Tampere University of Technology, 2004.

[6] T. Saramäki, “Finite impulse response filter design,” Chapter 4 in *Handbook for Digital Signal Processing*, edited by S. K. Mitra and J. F. Kaiser, John Wiley & Sons, New York, 1993.

[7] D. Babic, J. Vesma, T. Saramäki, M. Renfors, “Implementation of the transposed Farrow structure,” in *Proc. 2002 IEEE Int. Symp. Circuits and Systems*, Scotsdale, Arizona, USA, 2002, vol. 4, pp. 4–8.

[8] J.F. Kaiser, "Nonrecursive Digital Filter Design Using the - sinh Window Function," *Proc. 1974 IEEE Symp. Circuits and Systems*, (April 1974), pp.20-23.

[9]T. Saramäki, "Multirate Signal Processing," *Lecture Notes*, http://www.cs.tut.fi/~ts/