

Scalar conservation laws with stochastic forcing

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Abstract

We show that the Cauchy Problem for a randomly forced, periodic multi-dimensional scalar first-order conservation law with additive or multiplicative noise is well-posed: it admits a unique solution, characterized by a kinetic formulation of the problem, which is the limit of the solution of the stochastic parabolic approximation.

Keywords: Stochastic partial differential equations, conservation laws, kinetic formulation, entropy solutions.

1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t), (\beta_k(t)))$ be a stochastic basis and let $T > 0$. In this paper, we study the first-order scalar conservation law with stochastic forcing

$$du + \operatorname{div}(A(u))dt = \Phi(u)dW(t), \quad x \in \mathbb{T}^N, t \in (0, T). \quad (1)$$

The equation is periodic in the space variable x : $x \in \mathbb{T}^N$ where \mathbb{T}^N is the N -dimensional torus. The flux function A in (1) is supposed to be of class C^2 : $A \in C^2(\mathbb{R}; \mathbb{R}^N)$ and its derivatives have at most polynomial growth. We assume that W is a cylindrical Wiener process: $W = \sum_{k \geq 1} \beta_k e_k$, where the β_k are independent brownian processes and $(e_k)_{k \geq 1}$ is a complete orthonormal system in a Hilbert space H . For each $u \in \mathbb{R}$, $\Phi(u): H \rightarrow L^2(\mathbb{T}^N)$ is defined by $\Phi(u)e_k = g_k(u)$ where $g_k(\cdot, u)$ is a regular function on \mathbb{T}^N . More precisely, we assume $g_k \in C(\mathbb{T}^N \times \mathbb{R})$, with the bounds

$$\mathbf{G}^2(x, u) = \sum_{k \geq 1} |g_k(x, u)|^2 \leq D_0(1 + |u|^2), \quad (2)$$

$$\sum_{k \geq 1} |g_k(x, u) - g_k(y, v)|^2 \leq D_1(|x - y|^2 + |u - v|h(|u - v|)), \quad (3)$$



where $x, y \in \mathbb{T}^N$, $u, v \in \mathbb{R}$, and h is a continuous non-decreasing function on \mathbb{R}_+ with $h(0) = 0$. Note in particular that, for each $u \in \mathbb{R}$, $\Phi(u): H \rightarrow L^2(\mathbb{T}^N)$ is Hilbert-Schmidt since $\|g_k(\cdot, u)\|_{L^2(\mathbb{T}^N)} \leq \|g_k(\cdot, u)\|_{C(\mathbb{T}^N)}$ and thus

$$\sum_{k \geq 1} \|g_k(\cdot, u)\|_{L^2(\mathbb{T}^N)}^2 \leq D_0(1 + |u|^2).$$

The Cauchy Problem, *resp.* the Cauchy-Dirichlet Problem, for the stochastic equation (1) in the case of an additive noise (Φ independent on u) has been studied in [Kim03], *resp.* [VW10]. Existence and uniqueness of entropy solutions are proved in both papers. The Cauchy Problem for the stochastic equation (1) in case where the noise is multiplicative (and satisfies (2)-(3) above) has been studied in [FN08]. In [FN08], uniqueness of (strong) entropy solution is proved in any dimension, existence in dimension 1.

Our purpose here is to solve the Cauchy Problem for (1) in any dimension. To that purpose, we use a notion of kinetic solution, as developed by Lions, Perthame, Tadmor for deterministic first-order scalar conservation laws [LPT94]. A very basic reason to this approach is the fact that no pathwise L^∞ a priori estimates are known for (1). Thus, viewing (1) as an extension of the deterministic first-order conservation law, we have to turn to the L^1 theory developed for the latter, for which the kinetic formulation, once conveniently adapted, is slightly better suited than the renormalized-entropy formulation (developed in [CW99] for example).

There is also a definite technical advantage to the kinetic approach, for it allows to keep track of the dissipation of the noise by solutions. For entropy solutions, part of this information is lost and has to be recovered at some stage (otherwise, the classical approach à la Kruzhkov [Kru70] to Comparison Theorem fails): accordingly, Feng and Nualart need to introduce a notion of “strong” entropy solution, *i.e.* *entropy solution* satisfying the extra property that is precisely lacking [FN08]. This technical difference between the notions of kinetic and entropy solution already appears in the context of degenerate parabolic equations: in the comparison of entropy solutions, it is necessary to recover in a preliminary step the quantitative entropy dissipation due to the second-order part in non-degeneracy zones (see Lemma 1 in [Car99]). For kinetic solutions, this preliminary step is unnecessary since this dissipation is already encoded in the structure of the kinetic measure, (see Definition 2.2 in [CP03]).

In the case of an additive noise, Kim [Kim03] and Vallet and Wittbold [VW10] introduce the auxiliary unknown $w := u - \Phi W$ that satisfies the first-order

scalar conservation law

$$\partial_t w + \operatorname{div}(B(x, t, w)) = 0, \quad (4)$$

where the flux $B(x, t, w) := A(w + \Phi(x)W(t))$ is non-autonomous and has limited (pathwise Hölder-) regularity with respect to the variable t . Then entropy solutions are defined on the basis of (4). In this way it is actually possible to avoid the use of Itô stochastic calculus.

In the case of an equation with a multiplicative noise, Feng and Nualart define a notion of entropy solution by use of regular entropies and Itô Formula [FN08]. They also define a notion of *strong* entropy solution, which is an entropy solution satisfying an additional technical criterion. This additional criterion is required to prove a comparison result between entropy and strong entropy solution. As already mentioned, they are able to prove existence of strong entropy solutions only in dimension one.

In all three papers [Kim03, FN08, VW10], existence is proved *via* approximation by stochastic parabolic equation. We will proceed similarly, *cf.* Theorem 19. Consequently, our notion of solution, defined in Definition 2, happen to be equivalent to the notion of entropy solution used in [Kim03, FN08, VW10], provided the convergence of the vanishing viscosity method has been proved, hence in the context of [Kim03, VW10] or in [FN08] in dimension 1¹. In fact, we prove that our notion of kinetic solution is also equivalent to the notion of (mere – not strong) entropy solution of [FN08], whatever the dimension, see section 3.3.

Our main results states that under assumptions (2) and (3), there exists a unique kinetic solution in any space dimension. Due to the equivalence with entropy solution, we fill the gap left open in [FN08]. Moreover, the use of kinetic formulation considerably simplifies the arguments. For instance, to construct a solution, only weak compactness of the viscous solutions is necessary.

There are related problems to (1). We refer to the references given in, *e.g.* [Kim03, VW10], in particular concerning the study of the deterministic inviscid Burgers equation with random initial datum. One of the important question in the analysis of (1) (and, more precisely, in the analysis of the evolution of the *law* of the solution process $u(t)$) is also the existence (uniqueness, ergodic character, etc.) of an invariant measure. This question has

¹note that we consider periodic boundary conditions here, unlike [Kim03, FN08, VW10]. However, our results extend to the whole Cauchy Problem or to the Cauchy-Dirichlet Problem.

been fully addressed in [EKMS00] for the inviscid periodic Burgers equation in dimension 1 by use of the Hopf-Lax formula.

Our analysis of (1) uses the tools developed over the past thirty years for the analysis of deterministic first-order scalar conservation laws, in particular the notion of generalized solution. Thus, in Section 2, we introduce the notion of solution to (1) by use of the kinetic formulation, and complement it with a notion of *generalized* solution. In Section 3, we prove Theorem 11, which gives uniqueness (and comparison results) for solutions and also shows that a generalized solution is actually necessarily a solution. This result is used in Section 4: we study the parabolic approximation to (1) and show that it converges to a generalized solution, hence to a solution. This gives existence and uniqueness of a solution, Theorem 19.

2 Kinetic solution

2.1 Definition

Definition 1 (Kinetic measure) *We say that a map m from Ω to the set of non-negative finite measures over $\mathbb{T}^N \times [0, T] \times \mathbb{R}$ is a kinetic measure if*

1. *m is measurable, in the sense that for each $\phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$, $\langle m, \phi \rangle: \Omega \rightarrow \mathbb{R}$ is,*
2. *m vanishes for large ξ : if $B_R^c = \{\xi \in \mathbb{R}, |\xi| \geq R\}$, then*

$$\lim_{R \rightarrow +\infty} \mathbb{E}m(\mathbb{T}^N \times [0, T] \times B_R^c) = 0, \quad (5)$$

3. *for all $\phi \in C_b(\mathbb{T}^N \times \mathbb{R})$, the process*

$$t \mapsto \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi)$$

is predictable.

Definition 2 (Solution) *Let $u_0 \in L^\infty(\mathbb{T}^N)$. A measurable function $u: \mathbb{T}^N \times [0, T] \times \Omega \rightarrow \mathbb{R}$ is said to be a solution to (1) with initial datum u_0 if $(u(t))$ is predictable, for all $p \geq 1$, there exists $C_p \geq 0$ such that for a.e. $t \in [0, T]$,*

$$\|u(t)\|_{L^p(\Omega \times \mathbb{T}^N)} \leq C_p \quad (6)$$

and if there exists a kinetic measure m such that $f := \mathbf{1}_{u>\xi}$ satisfies: for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} & \int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} g_k(x, u(x, t)) \varphi(x, t, u(x, t)) dx d\beta_k(t) \\ & \quad - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \partial_\xi \varphi(x, t, u(x, t)) \mathbf{G}^2(x, u(x, t)) dx dt + m(\partial_\xi \varphi), \quad (7) \end{aligned}$$

a.s., where $\mathbf{G}^2 := \sum_{k=1}^{\infty} |g_k|^2$ and $a(\xi) := A'(\xi)$.

In (7), $f_0(x, \xi) = \mathbf{1}_{u_0(x)>\xi}$. We have used the brackets $\langle \cdot, \cdot \rangle$ to denote the duality between $C_c^\infty(\mathbb{T}^N \times \mathbb{R})$ and the space of distributions over $\mathbb{T}^N \times \mathbb{R}$. In what follows, we will denote similarly the integral

$$\langle F, G \rangle = \int_{\mathbb{T}^N} \int_{\mathbb{R}} F(x, \xi) G(x, \xi) dx d\xi, \quad F \in L^p(\mathbb{T}^N \times \mathbb{R}), G \in L^q(\mathbb{T}^N \times \mathbb{R}),$$

where $1 \leq p \leq +\infty$ and q is the conjugate exponent of p . In (7) also, we have indicated the dependence of g_k and \mathbf{G}^2 on u , which is actually absent in the additive case and we have used (with $\phi = \partial_\xi \varphi$) the shorthand $m(\phi)$ for

$$m(\phi) = \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \phi(x, t, \xi) dm(x, t, \xi), \quad \phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R}).$$

Equation (7) is the weak form of the equation

$$(\partial_t + a(\xi) \cdot \nabla) \mathbf{1}_{u>\xi} = \delta_{u=\xi} \dot{W} + \partial_\xi (m - \frac{1}{2} \mathbf{G}^2 \delta_{u=\xi}). \quad (8)$$

We present now a *formal* derivation of equation (8) from (1) *in the case* $m = 0$ (see also Section 4.1, where we give a rigorous derivation of the kinetic formulation at the level of the viscous approximation): it is essentially a consequence of Itô Formula. Indeed, by the identity $(\mathbf{1}_{u>\xi}, \theta') := \int_{\mathbb{R}} \mathbf{1}_{u>\xi} \theta'(\xi) d\xi = \theta(u) - \theta(-\infty)$, satisfied for $\theta \in C^\infty(\mathbb{R})$, and by Itô Formula, we have

$$\begin{aligned} d(\mathbf{1}_{u>\xi}, \theta') &= \theta'(u)(-a(u) \cdot \nabla u dt + dW) + \frac{1}{2} \theta''(u) \mathbf{G}^2 dt \\ &= -\operatorname{div} \left(\int^u a(\xi) \theta'(\xi) d\xi \right) dt + \frac{1}{2} \theta''(u) \mathbf{G}^2 dt + \theta'(u) dW \\ &= -\operatorname{div}((a \mathbf{1}_{u>\xi}, \theta')) dt - \frac{1}{2} (\partial_\xi (\mathbf{G}^2 \delta_{u=\xi}), \theta') dt + (\delta_{u=\xi}, \theta') dW. \end{aligned}$$

Taking $\theta(\xi) = \int_{-\infty}^{\xi} \varphi$, we then obtain the kinetic formulation with $m = 0$. The measure m is sometimes (quite improperly if no action, or Lagrangian, is precisely defined) interpreted as a Lagrange multiplier for the evolution of f by $\partial_t + a \cdot \nabla$ under the constraint $f = \text{graph} = \mathbf{1}_{u > \xi}$. It comes into play only when u becomes discontinuous (occurrence of shocks); in particular, it does not appear in the computation above that requires some regularity of u with respect to x to apply the chain-rule of differentiation.

2.2 Generalized solutions

With the purpose to prepare the proof of existence of solution, we introduce the following definitions.

Definition 3 (Young measure) *Let (X, λ) be a finite measure space. Let $\mathcal{P}_1(\mathbb{R})$ denote the set of probability measures on \mathbb{R} . We say that a map $\nu: X \rightarrow \mathcal{P}_1(\mathbb{R})$ is a Young measure on X , if, for all $\phi \in C_b(\mathbb{R})$, the map $z \mapsto \nu_z(\phi)$ from X to \mathbb{R} is measurable. We say that a Young measure ν vanishes at infinity if, for every $p \geq 1$,*

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) < +\infty. \quad (9)$$

Definition 4 (Kinetic function) *Let (X, λ) be a finite measure space. A measurable function $f: X \times \mathbb{R} \rightarrow [0, 1]$ is said to be a kinetic function if there exists a Young measure ν on X that vanishes at infinity such that, for λ -a.e. $z \in X$, for all $\xi \in \mathbb{R}$,*

$$f(z, \xi) = \nu_z(\xi, +\infty).$$

If $f: X \times \mathbb{R} \rightarrow [0, 1]$ is a kinetic function, we denote by \bar{f} the conjugate function $\bar{f} := 1 - f$.

We also denote by χ_f the function defined by $\chi_f(z, \xi) = f(z, \xi) - \mathbf{1}_{0 > \xi}$. Contrary to f , this modification is integrable. Actually, it is decreasing faster than any power of ξ at infinity. Indeed,

$$\chi_f(z, \xi) = \begin{cases} - \int_{(-\infty, \xi]} d\nu_z, & \xi < 0, \\ \int_{(\xi, +\infty)} d\nu_z, & \xi > 0. \end{cases}$$

Therefore

$$|\xi|^p \int_X |\chi_f(z, \xi)| d\lambda(z) \leq \int_X \int_{\mathbb{R}} |\zeta|^p d\nu_{x,t}(\zeta) d\lambda(z) < \infty, \quad (10)$$

for all $\xi \in \mathbb{R}$, $1 \leq p < +\infty$.

We have the following compactness results (the proof is classical and reported to appendix).

Theorem 5 (Compactness of Young measures) *Let (X, λ) be a finite measure space. Let (ν^n) be a sequence of Young measures on X satisfying (9) uniformly for some $p \geq 1$:*

$$\sup_n \int_X \int_{\mathbb{R}} |\xi|^p d\nu_z^n(\xi) d\lambda(z) < +\infty. \quad (11)$$

Then there exists a Young measure ν on X and a subsequence still denoted (ν^n) such that, for all $h \in L^1(X)$, for all $\phi \in C_b(\mathbb{R})$,

$$\lim_{n \rightarrow +\infty} \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z^n(\xi) d\lambda(z) = \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z(\xi) d\lambda(z). \quad (12)$$

Corollary 6 (Compactness of kinetic functions) *Let (X, λ) be a finite measure space. Let (f_n) be a sequence of kinetic functions on $X \times \mathbb{R}$: $f_n(z, \xi) = \nu_z^n(\xi, +\infty)$ where ν^n are Young measures on X satisfying (11). Then there exists a kinetic function f on $X \times \mathbb{R}$ such that $f_n \rightharpoonup f$ in $L^\infty(X \times \mathbb{R})$ weak-**.

Note that if f is a kinetic function then $\partial_\xi f = -\nu$ is non-negative. Observe also that, in the context of Definition 2, setting $f = \mathbf{1}_{u > \xi}$, we have $\partial_\xi f = -\delta_{u=\xi}$ and $\nu := \delta_{u=\xi}$ is a Young measure on $\Omega \times \mathbb{T}^N \times (0, T)$. The measure ν vanishes at infinity (it even satisfies the stronger condition (13) below). Therefore any solution will also be a *generalized solution*, according to the definition below.

Definition 7 (Generalized solution) *Let $f_0: \Omega \times \mathbb{T}^N \times \mathbb{R} \rightarrow [0, 1]$ be a kinetic function. A measurable function $f: \Omega \times \mathbb{T}^N \times [0, T] \times \mathbb{R} \rightarrow [0, 1]$ is said to be a generalized solution to (1) with initial datum f_0 if $(f(t))$ is predictable and is a kinetic function such that: for all $p \geq 1$, there exists $C_p \geq 1$ such that for a.e. $t \in (0, T)$*

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi) dx \leq C_p, \quad (13)$$

where $\nu := -\partial_\xi f$, and if there exists a kinetic measure m such that for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,

$$\begin{aligned} & \int_0^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx d\beta_k(t) \\ & - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, t, \xi) \mathbf{G}^2(x, \xi) d\nu_{(x,t)}(\xi) dx dt + m(\partial_\xi \varphi), \quad a.s. \end{aligned} \quad (14)$$

Observe that, if f is a generalized solution such that $f = \mathbf{1}_{u > \xi}$, then $u(t, x) = \int_{\mathbb{R}} \chi_f(x, t, \xi) d\xi$, hence u is predictable. Moreover, $\nu = \delta_{u=\xi}$ and

$$\mathbb{E} \int_{\mathbb{T}^N} |u(t, x)|^p dx = \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}(\xi) dx.$$

Condition (6) is thus contained in the condition (13).

We conclude this chapter with a remark about the time continuity of the solution (see also [CG10] and references therein on this subject). Generalized solutions are a useful and natural tool for the analysis of *weak* solutions to (1), *i.e.* solutions that are weak with respect to space *and* time, but the process of relaxation that generalizes the notion of solution introduces additional difficulties regarding the question of time continuity of solutions. To illustrate this fact, let us consider for example the following equation (the ‘‘Collapse’’ equation in the Transport-Collapse method of Brenier [Bre81, Bre83])

$$\partial_t f(t) = \mathbf{1}_{u(t) > \xi} - f, \quad u(t) := \int_{\mathbb{R}} \chi_{f(t)}(\xi) d\xi, \quad (15)$$

with initial datum $f_0(\xi)$ a kinetic function. Integrating (15) with respect to ξ shows that $u = u_0$ is constant and gives

$$f(t) = e^{-t} f_0 + (1 - e^{-t}) \mathbf{1}_{u_0 > \xi},$$

i.e. $f(t)$ is describing the progressive and continuous ‘‘collapse’’ from f_0 to $\mathbf{1}_{u_0 > \xi}$. It is also simple to show that

$$m(t, \xi) := \int_{-\infty}^{\xi} (\mathbf{1}_{u > \zeta} - f(t, \zeta)) d\zeta \geq 0$$

for all t, ξ , and, more generally,

$$\int_{-\infty}^{\xi} (f(\tau, \zeta) - f(t, \zeta)) d\zeta \geq 0, \quad \forall \tau > t, \forall \xi \in \mathbb{R}, \quad (16)$$

so that f satisfies $\partial_t f = \partial_\xi m$, $m \geq 0$. Now erase an interval $[t_1, t_2]$ in the evolution of f . Then

$$g(t) = \hat{f}(t) := f(t)\mathbf{1}_{[0, t_1]}(t) + f(t + t_2 - t_1)\mathbf{1}_{(t_1, +\infty)}(t)$$

satisfies

$$\begin{aligned} \partial_t g &= \partial_\xi \hat{m} + (f(t_2) - f(t_1))\delta(t - t_1) \\ &= \partial_\xi n, \quad n(t, \xi) := \hat{m}(t, \xi) + \int_{-\infty}^{\xi} (f(t_2, \zeta) - f(t_1, \zeta))d\zeta\delta(t - t_1). \end{aligned}$$

By (16), n is non-negative, but, unless $f_0 = \mathbf{1}_{u_0 > \xi}$, g is discontinuous at $t = t_1$. In the analysis of a generalized solution f , we thus first show the existence of modifications f^+ and f^- of f being respectively right- and left-continuous everywhere and we work on f^\pm in most of the proof of uniqueness and reduction (Theorem 11). Finally, we obtain the time-continuity of solutions in Corollary 12.

2.3 Left and right limits of generalized solution

We show in the following proposition that, almost surely, any generalized solution admits possibly different left and right weak limits at any point $t \in [0, T]$. This property is important to prove a comparison principle which allows to prove uniqueness. Also, it allows us to see that the weak form (14) of the equation satisfied by a generalized solution can be strengthened. We write below a formulation which is weak only with respect to x and ξ . Note that we obtain continuity with respect to time of solutions in Corollary 12 below.

Proposition 8 (Left and right weak limits) *Let f_0 be a kinetic initial datum. Let f be a generalized solution to (1) with initial datum f_0 . Then f admits almost surely left and right limits at all point $t_* \in [0, T]$. More precisely, for all $t_* \in [0, T]$ there exists some kinetic functions $f^{*,\pm}$ on $\Omega \times \mathbb{T}^N \times \mathbb{R}$ such that \mathbb{P} -a.s.*

$$\langle f(t_* - \varepsilon), \varphi \rangle \rightarrow \langle f^{*, -}, \varphi \rangle$$

and

$$\langle f(t_* + \varepsilon), \varphi \rangle \rightarrow \langle f^{*, +}, \varphi \rangle$$

as $\varepsilon \rightarrow 0$ for all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$. Moreover, almost surely,

$$\langle f^{*, +} - f^{*, -}, \varphi \rangle = - \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{1}_{\{t_*\}}(t) dm(x, t, \xi). \quad (17)$$

In particular, almost surely, the set of $t_* \in [0, T]$ such that $f^{*,-} \neq f^{*,+}$ is countable.

In the following, for a generalized solution f , we define f^\pm by $f^\pm(t_*) = f^{*\pm}$, $t_* \in [0, T]$. Note that, since we are dealing with a filtration associated to brownian motions, f^\pm are also predictable. Also $f = f^+ = f^-$ almost everywhere in time and we can take any of them in an integral with respect to the Lebesgue measure or in a stochastic integral. On the contrary, if the integration is with respect to a measure - typically a kinetic measure in this article -, the integral is not well defined for f and may differ if one chooses f^+ or f^- .

Proof of Proposition 8. The set of test functions $C_c^1(\mathbb{T}^N \times \mathbb{R})$ (endowed with the topology of the uniform convergence on any compact of the functions and their first derivatives) is separable and we fix a dense countable subset \mathcal{D}_1 . For all $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$, a.s., the map

$$\begin{aligned} J_\varphi : t \mapsto & \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds \\ & - \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) \\ & + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{G}^2(x, \xi) d\nu_{x,s}(\xi) dx ds \end{aligned} \quad (18)$$

is continuous on $[0, T]$. Consequently: a.s., for all $\varphi \in \mathcal{D}_1$, J_φ is continuous on $[0, T]$.

For test functions of the form $(x, t, \xi) \mapsto \varphi(x, \xi) \alpha(t)$, $\alpha \in C_c^1([0, T])$, $\varphi \in \mathcal{D}_1$, Fubini Theorem and the weak formulation (14) give

$$\int_0^T g_\varphi(t) \alpha'(t) dt + \langle f_0, \varphi \rangle \alpha(0) = \langle m, \partial_\xi \varphi \rangle (\alpha), \quad (19)$$

where $g_\varphi(t) := \langle f(t), \varphi \rangle - J_\varphi(t)$. This shows that $\partial_t g_\varphi$ is a Radon measure on $(0, T)$, *i.e.* the function $g_\varphi \in BV(0, T)$. In particular it admits left and right limits at all points $t_* \in [0, T]$. Since J_φ is continuous, this also holds for $\langle f, \varphi \rangle$: for all $t_* \in [0, T]$, the limits

$$\langle f, \varphi \rangle(t_*+) := \lim_{t \downarrow t_*} \langle f, \varphi \rangle(t) \quad \text{and} \quad \langle f, \varphi \rangle(t_*-) := \lim_{t \uparrow t_*} \langle f, \varphi \rangle(t)$$

exist. Note that:

$$\langle f, \varphi \rangle(t_*+) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{t_*}^{t_*+\varepsilon} \langle f, \varphi \rangle(t) dt, \quad \langle f, \varphi \rangle(t_*-) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_{t_*-\varepsilon}^{t_*} \langle f, \varphi \rangle(t) dt.$$

Let $(\varepsilon_n) \downarrow 0$. By the uniform in time integrability condition (13) and Corollary 6, there exist a kinetic functions $f^{*,\pm}$ on $\Omega \times \mathbb{T}^N \times \mathbb{R}$ and subsequences $(\varepsilon_{n_k^\pm})$ such that

$$\frac{1}{\varepsilon_{n_k^-}} \int_{t_* - \varepsilon_{n_k^-}}^{t_*} f(t) dt \rightharpoonup f^{*,-}, \quad \frac{1}{\varepsilon_{n_k^+}} \int_{t_*}^{t_* + \varepsilon_{n_k^+}} f(t) dt \rightharpoonup f^{*,+}$$

weakly-* in $L^\infty(\Omega \times \mathbb{T}^N \times \mathbb{R})$ as $k \rightarrow +\infty$. We deduce:

$$\langle f, \varphi \rangle(t_*+) = \langle f^{*,+}, \varphi \rangle \text{ and } \langle f, \varphi \rangle(t_*-) = \langle f^{*,-}, \varphi \rangle.$$

Taking for α the hat function $\alpha(t) = \frac{1}{\varepsilon} \min((t - t_* + \varepsilon)^+, (t - t_* - \varepsilon)^-)$ in (19), we obtain (17) at the limit $[\varepsilon \rightarrow 0]$. In particular, almost surely, $f^{*,+} = f^{*,-}$ whenever m has no atom at t_* .

We thus have proved the result for $\varphi \in \mathcal{D}_1$. Since \mathcal{D}_1 is dense in $C_c^1(\mathbb{T}^N \times \mathbb{R})$, it is easy to see that in fact everything holds a.s. for every $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$.

■

Taking in (14) a test function of the form $(x, s, \xi) \mapsto \varphi(x, \xi)\alpha(s)$ where α is the function

$$\alpha(s) = \begin{cases} 1, & s \leq t, \\ 1 - \frac{s-t}{\varepsilon}, & t \leq s \leq t + \varepsilon, \\ 0, & t + \varepsilon \leq s, \end{cases}$$

we obtain at the limit $[\varepsilon \rightarrow 0]$: for all $t \in [0, T]$ and $\varphi \in C_c^1(\mathbb{T}^N \times \mathbb{R})$,

$$\begin{aligned} & -\langle f^+(t), \varphi \rangle dt + \langle f_0, \varphi \rangle + \int_0^t \langle f(s), a(\xi) \cdot \nabla \varphi \rangle ds \\ & = - \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, \xi) d\nu_{x,s}(\xi) dx d\beta_k(s) \\ & - \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, \xi) \mathbf{G}^2(x, \xi) d\nu_{(x,s)}(\xi) dx ds + \langle m, \partial_\xi \varphi \rangle([0, t]), \text{ a.s.}, \end{aligned} \tag{20}$$

where $\langle m, \partial_\xi \varphi \rangle([0, t]) = \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \partial_\xi \varphi(x, \xi) dm(x, s, \xi)$.

3 Comparison, uniqueness, entropy solution and regularity

3.1 Doubling of variables

In this paragraph, we prove a technical proposition relating two generalized solutions f_i , $i = 1, 2$ of the equation

$$du_i + \operatorname{div}(A(u_i))dt = \Phi(u_i)dW. \quad (21)$$

Proposition 9 *Let f_i , $i = 1, 2$, be generalized solution to (21). Then, for $0 \leq t \leq T$, and non-negative test functions $\rho \in C^\infty(\mathbb{T}^N)$, $\psi \in C_c^\infty(\mathbb{R})$, we have*

$$\begin{aligned} & \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y) \psi(\xi-\zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(y, t, \zeta) d\xi d\zeta dx dy \\ & \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho(x-y) \psi(\xi-\zeta) f_{1,0}(x, \xi) \bar{f}_{2,0}(y, \zeta) d\xi d\zeta dx dy + I_\rho + I_\psi, \end{aligned} \quad (22)$$

where

$$\begin{aligned} I_\rho = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1(x, s, \xi) \bar{f}_2(y, s, \zeta) (a(\xi) - a(\zeta)) \psi(\xi - \zeta) d\xi d\zeta \\ \cdot \nabla_x \rho(x-y) dx dy ds \end{aligned}$$

and

$$\begin{aligned} I_\psi = \frac{1}{2} \int_{(\mathbb{T}^N)^2} \rho(x-y) \mathbb{E} \int_0^t \int_{\mathbb{R}^2} \psi(\xi-\zeta) \\ \times \sum_{k \geq 1} |g_k(x, \xi) - g_k(y, \zeta)|^2 d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi, \zeta) dx dy ds. \end{aligned}$$

Remark 10 *Each term in (22) is finite. Let us for instance consider the first one on the right hand side. Let us introduce the auxiliary functions*

$$\psi_1(\xi) = \int_{-\infty}^{\xi} \psi(s) ds, \quad \psi_2(\zeta) = \int_{-\infty}^{\zeta} \psi_1(\xi) d\xi,$$

which are well-defined since ψ is compactly supported. Note that both ψ_1 and ψ_2 vanish at $-\infty$. When $\xi \rightarrow +\infty$, ψ_1 remains bounded while ψ_2 has linear growth. To lighten notations, we omit the index 0. Let us set $\bar{f}_2 = 1 - f_2$. In

the case where f_1 and f_2 correspond to kinetic solutions, i.e. $f_i = \mathbf{1}_{u_i > \xi}$, we compute (forgetting the dependence upon t and x): $\bar{f}_2(\zeta) = \mathbf{1}_{u_2 \leq \zeta}$ and

$$\int_{\mathbb{R}^2} \psi(\xi - \zeta) f_1(\xi) \bar{f}_2(\zeta) d\xi d\zeta = \psi_2(u_1 - u_2).$$

In the case of generalized solutions, we introduce the integrable modifications χ_{f_i} of f_i , $i = 1, 2$:

$$f_1(\xi) = \chi_{f_1}(\xi) + \mathbf{1}_{0 > \xi}, \quad \bar{f}_2(\zeta) = \mathbf{1}_{0 \leq \zeta} - \chi_{f_2}(\zeta).$$

Accordingly, we have, by explicit integration:

$$\begin{aligned} \int_{\mathbb{R}^2} \psi(\xi - \zeta) f_1(\xi) \bar{f}_2(\zeta) d\xi d\zeta &= - \int_{\mathbb{R}^2} \psi(\xi - \zeta) \chi_{f_1}(\xi) \chi_{f_2}(\zeta) d\xi d\zeta \\ &+ \int_{\mathbb{R}} \psi_1(\xi) \chi_{f_1}(\xi) d\xi - \int_{\mathbb{R}} \psi_1(\zeta) \chi_{f_2}(-\zeta) d\zeta + \psi_2(0) \end{aligned} \quad (23)$$

Each term in the right hand-side of (23) is indeed finite by (10).

Proof of Proposition 9: Set $G_1^2(x, \xi) = \sum_{k=1}^{\infty} |g_k(x, \xi)|$ and $G_2^2(y, \zeta) = \sum_{k=1}^{\infty} |g_k(y, \zeta)|$. Let $\varphi_1 \in C_c^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi)$ and $\varphi_2 \in C_c^\infty(\mathbb{T}_y^N \times \mathbb{R}_\zeta)$. By (20), we have

$$\langle f_1^+(t), \varphi_1 \rangle = \langle f_{1,0}, \varphi_1 \rangle + F_1(t) + \langle \tilde{m}_1, \partial_\xi \varphi_1 \rangle([0, t])$$

with

$$F_1(t) = - \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_{k,1} \varphi_1 d\nu_{x,s}^1(\xi) dx d\beta_k(s)$$

and

$$\begin{aligned} \langle \tilde{m}_1, \partial_\xi \varphi_1 \rangle([0, t]) &= \int_0^t \langle f_1, a \cdot \nabla \varphi_1 \rangle ds \\ &+ \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi_1 \mathbf{G}_1^2 d\nu_{x,s}^1(\xi) dx ds - \langle m_1, \partial_\xi \varphi_1 \rangle([0, t]). \end{aligned}$$

Similarly

$$\langle \bar{f}_2^+(t), \varphi_2 \rangle = \langle \bar{f}_{2,0}, \varphi_2 \rangle + \bar{F}_2(t) - \langle \tilde{m}_2, \partial_\xi \varphi_2 \rangle([0, t])$$

with

$$\bar{F}_2(t) = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_{k,2} \varphi_2 d\nu_{x,s}^2(\xi) dx d\beta_k(s)$$

and

$$\begin{aligned} \langle \tilde{m}_2, \partial_\xi \varphi_2 \rangle([0, t]) &= - \int_0^t \langle \bar{f}_2, a \cdot \nabla \varphi_2 \rangle ds \\ &\quad + \frac{1}{2} \int_0^t \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi_2 \mathbf{G}_1^2 d\nu_{(x,s)}^2(\xi) dx ds - \langle m_2, \partial_\xi \varphi_2 \rangle([0, t]). \end{aligned}$$

Set $\alpha(x, \xi, y, \zeta) = \varphi_1(x, \xi) \varphi_2(y, \zeta)$. Using Itô formula for $F_1(t) \bar{F}_2(t)$, integration by parts for functions of finite variation (see for instance [RY99], chapter 0) for $\langle \tilde{m}_1, \partial_\xi \varphi_1 \rangle([0, t]) \langle \tilde{m}_2, \partial_\xi \varphi_2 \rangle([0, t])$, the following formula

$$\langle \tilde{m}_1, \partial_\xi \varphi_1 \rangle([0, t]) \bar{F}_2(t) = \int_0^t \langle \tilde{m}_1, \partial_\xi \varphi_1 \rangle([0, s]) d\bar{F}_2(s) + \int_0^t \bar{F}_2(s) \langle \tilde{m}_1, \partial_\xi \varphi_1 \rangle(ds),$$

which is easy to obtain since \bar{F}_2 is continuous, and a similar formula for $\langle \tilde{m}_2, \partial_\xi \varphi_2 \rangle([0, t]) \bar{F}_1(t)$, we obtain that

$$\langle f_1^+(t), \varphi_1 \rangle \langle \bar{f}_2^+(t), \varphi_2 \rangle = \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle$$

satisfies

$$\begin{aligned} \mathbb{E} \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle &= \langle \langle f_{1,0} \bar{f}_{2,0}, \alpha \rangle \rangle \\ &\quad - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) \cdot \nabla_x + a(\zeta) \cdot \nabla_y) \alpha d\xi d\zeta dx dy ds \\ &\quad + \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\xi \alpha \bar{f}_2(s) \mathbf{G}_1^2 d\nu_{(x,s)}^1(\xi) d\zeta dx dy ds \\ &\quad - \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \partial_\zeta \alpha f_1(s) \mathbf{G}_2^2 d\nu_{(x,s)}^2(\zeta) d\xi dy dx ds \\ &\quad - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathbf{G}_{1,2} \alpha d\nu_{x,s}^1(\xi) d\nu_{y,s}^2(\zeta) dx dy \\ &\quad - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s) \partial_\xi \alpha dm_1(x, s, \xi) d\zeta dy \\ &\quad + \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\zeta \alpha dm_2(y, s, \zeta) d\xi dx \quad (24) \end{aligned}$$

where $\mathbf{G}_{1,2}(x, y; \xi, \zeta) := \sum_{k \geq 1} g_k(x, \xi) g_k(y, \zeta)$ and $\langle \langle \cdot, \cdot \rangle \rangle$ denotes the duality distribution over $\mathbb{T}_x^N \times \mathbb{R}_\xi \times \mathbb{T}_y^N \times \mathbb{R}_\zeta$. By a density argument, (24) remains true for any test-function $\alpha \in C_c^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi \times \mathbb{T}_y^N \times \mathbb{R}_\zeta)$. Using similar arguments as in Remark 10, the assumption that α is compactly supported can be relaxed thanks to the condition at infinity (5) on m_i and (9) on ν^i , $i = 1, 2$. Using

truncates of α , we obtain that (24) remains true if $\alpha \in C_b^\infty(\mathbb{T}_x^N \times \mathbb{R}_\xi \times \mathbb{T}_y^N \times \mathbb{R}_\zeta)$ is compactly supported in a neighborhood of the diagonal

$$\{(x, \xi, x, \xi); x \in \mathbb{T}^N, \xi \in \mathbb{R}\}.$$

We then take $\alpha = \rho\psi$ where $\rho = \rho(x-y)$, $\psi = \psi(\xi-\zeta)$. Note the remarkable identities

$$(\nabla_x + \nabla_y)\alpha = 0, \quad (\partial_\xi + \partial_\zeta)\alpha = 0. \quad (25)$$

In particular, the last term in (24) is

$$\begin{aligned} & \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\zeta \alpha d\xi dx dm_2(y, s, \zeta) \\ &= - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1^-(s) \partial_\xi \alpha d\xi dx dm_2(y, s, \zeta) \\ &= - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu_{x,s}^{1,-}(\xi) dx dm_2(y, s, \zeta) \leq 0 \end{aligned}$$

since $\alpha \geq 0$. The symmetric term

$$\begin{aligned} & - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \bar{f}_2^+(s) \partial_\xi \alpha dm_1(x, s, \xi) d\zeta dy \\ &= - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha d\nu_{y,s}^{2,+}(\zeta) dy dm_1(x, s, \xi) \end{aligned}$$

is, similarly, non-positive. Consequently, we have

$$\mathbb{E} \langle \langle f_1^+(t) \bar{f}_2^+(t), \alpha \rangle \rangle \leq \langle \langle f_{1,0} \bar{f}_{2,0}, \alpha \rangle \rangle + I_\rho + I_\psi, \quad (26)$$

where

$$I_\rho := - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) \cdot \nabla_x + a(\zeta) \cdot \nabla_y) \alpha d\xi d\zeta dx dy ds$$

and

$$\begin{aligned} I_\psi &= \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \partial_\xi \alpha \bar{f}_2(s) \mathbf{G}_1^2 d\nu_{(x,s)}^1(\xi) d\zeta dx dy ds \\ &\quad - \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \partial_\zeta \alpha f_1(s) \mathbf{G}_2^2 d\nu_{(x,s)}^2(\zeta) d\xi dy dx ds \\ &\quad - \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}} \mathbf{G}_{1,2} \alpha d\nu_{x,s}^1(\xi) d\nu_{y,s}^2(\zeta) dx dy. \end{aligned}$$

Equation (26) is indeed equation (22) for f_i^+ since, by (25),

$$I_\rho = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 (a(\xi) - a(\zeta)) \cdot \nabla_x \alpha d\xi d\zeta dx dy ds$$

and, by (25) also and integration by parts,

$$\begin{aligned} I_\psi &= \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha (\mathbf{G}_1^2 + \mathbf{G}_2^2 - 2\mathbf{G}_{1,2}) d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi, \zeta) dx dy ds \\ &= \frac{1}{2} \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \alpha \sum_{k \geq 0} |g_k(x, \xi) - g_k(y, \zeta)|^2 d\nu_{x,s}^1 \otimes \nu_{y,s}^2(\xi, \zeta) dx dy ds. \end{aligned}$$

To obtain the result for f_i^- , we take $t_n \uparrow t$, write (22) for $f_i^+(t_n)$ and let $n \rightarrow \infty$.

3.2 Uniqueness, reduction of generalized solution

In this section we use Proposition 9 above to deduce the uniqueness of solutions and the reduction of generalized solutions to solutions.

Theorem 11 (Uniqueness, Reduction) *Let $u_0 \in L^\infty(\Omega)$. Assume (2)-(3). Then, there is at most one solution with initial datum u_0 to (1). Besides, any generalized solution f is actually a solution, i.e. if f is a generalized solution to (1) with initial datum $\mathbf{1}_{u_0 > \xi}$, then there exists a solution u to (1) with initial datum u_0 such that $f(x, t, \xi) = \mathbf{1}_{u(x,t) > \xi}$ a.s., for a.e. (x, t, ξ) . Moreover u has left and right limit at any point in the sense of $L^p(\mathbb{T}^N)$ for any $p \geq 1$ and, for any $t \in [0, T]$, $f^\pm(x, t, \xi) = \mathbf{1}_{u(x,t^\pm) > \xi}$ a.s., for a.e. (x, ξ) .*

Corollary 12 (Continuity in time) *Let $u_0 \in L^\infty(\Omega)$. Assume (2)-(3). Then, for every $p \in [1, +\infty)$, the solution u to (1) with initial datum u_0 has almost surely continuous trajectories in $L^p(\mathbb{T}^N)$.*

Proof of Theorem 11: Consider first the additive case: $\Phi(u)$ independent on u . Let f_i , $i = 1, 2$ be two generalized solutions to (1). Then, we use (22) with g_k independent on ξ and ζ . By (3), the last term I_ψ is bounded by

$$\frac{tD_1}{2} \|\psi\|_{L^\infty} \int_{(\mathbb{T}^N)^2} |x - y|^2 \rho(x - y) dx dy.$$

We then take $\psi := \psi_\delta$ and $\rho = \rho_\varepsilon$ where (ψ_δ) and (ρ_ε) are approximations to the identity on \mathbb{R} and \mathbb{T}^N respectively to obtain

$$I_\psi \leq \frac{tD_1}{2} \varepsilon^2 \delta^{-1}. \quad (27)$$

Let $t \in [0, T]$ and let $(t_n) \downarrow t$ be such that (13) is satisfied for $\nu = \nu^{1,+}$ at t_n . Then $\nu_{x,t}^{i,+}$, being the weak-limit (in the sense of (12)) of $\nu_{x,t_n}^{i,+}$ satisfies (13). Similarly for $\nu^{i,-}$. In particular, by (10), $\chi_{f_i^\pm(t)}$ is integrable on $\mathbb{T}^N \times \mathbb{R}$ and

$$\begin{aligned} & \mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) dx d\xi \\ &= \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x - y) \psi_\delta(\xi - \zeta) f_1^\pm(x, t, \xi) \bar{f}_2^\pm(x, t, \xi) d\xi d\zeta dx dy + \eta_t(\varepsilon, \delta), \end{aligned}$$

where $\lim_{\varepsilon, \delta \rightarrow 0} \eta_t(\varepsilon, \delta) = 0$. To conclude, we need a bound on the term I_ρ . Since a has at most polynomial growth, there exists $C \geq 0$, $p > 1$, such that

$$|a(\xi) - a(\zeta)| \leq \Gamma(\xi, \zeta) |\xi - \zeta|, \quad \Gamma(\xi, \zeta) = C(1 + |\xi|^{p-1} + |\zeta|^{p-1}).$$

Supposing additionally that $\psi_\delta(\xi) = \delta^{-1} \psi_1(\delta^{-1} \xi)$ where ψ_1 is supported in $(-1, 1)$, this gives

$$|I_\rho| \leq \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} f_1 \bar{f}_2 \Gamma(\xi, \zeta) |\xi - \zeta| \psi_\delta(\xi - \zeta) |\nabla_x \rho_\varepsilon(x - y)| d\xi d\zeta dx dy d\sigma.$$

By integration by parts with respect to (ξ, ζ) , we deduce

$$|I_\rho| \leq \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \Upsilon(\xi, \zeta) d\nu_{x,\sigma}^1 \otimes \nu_{y,\sigma}^2(\xi, \zeta) |\nabla_x \rho_\varepsilon(x - y)| dx dy d\sigma,$$

where

$$\Upsilon(\xi, \zeta) = \int_\zeta^{+\infty} \int_{-\infty}^\xi \Gamma(\xi', \zeta') |\xi' - \zeta'| \psi_\delta(\xi' - \zeta') d\xi' d\zeta'.$$

It is shown below that Υ admits the bound

$$\Upsilon(\xi, \zeta) \leq C(1 + |\xi|^p + |\zeta|^p) \delta. \quad (28)$$

Since ν^1 and ν^2 vanish at infinity, we then obtain, for a given constant C_p ,

$$|I_\rho| \leq t C_p \delta \left(\int_{\mathbb{T}^N} |\nabla_x \rho_\varepsilon(x)| dx \right).$$

It follows that, for possibly a different C_p ,

$$|I_\rho| \leq t C_p \delta \varepsilon^{-1}. \quad (29)$$

We then gather (27), (29) and (22) to deduce for $t \in [0, T]$

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(t) \bar{f}_2^\pm(t) dx d\xi \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_{1,0} \bar{f}_{2,0} dx d\xi + r(\varepsilon, \delta), \quad (30)$$

where the remainder $r(\varepsilon, \delta)$ is $r(\varepsilon, \delta) = TC_p \delta \varepsilon^{-1} + \frac{TD_1}{2} \varepsilon^2 \delta^{-1} + \eta_t(\varepsilon, \delta) + \eta_0(\varepsilon, \delta)$. Taking $\delta = \varepsilon^{4/3}$ and letting $\varepsilon \rightarrow 0$ gives

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_1^\pm(t) \bar{f}_2^\pm(t) dx d\xi \leq \int_{\mathbb{T}^N} \int_{\mathbb{R}} f_{1,0} \bar{f}_{2,0} dx d\xi. \quad (31)$$

Assume that f is a generalized solution to (1) with initial datum $\mathbf{1}_{u_0 > \xi}$. Since f_0 is the (translated) Heavyside function $\mathbf{1}_{u_0 > \xi}$, we have the identity $f_0 \bar{f}_0 = 0$. Taking $f_1 = f_2 = f$ in (31), we deduce $f^\pm(1 - f^\pm) = 0$ a.e., *i.e.* $f^\pm \in \{0, 1\}$ a.e. The fact that $-\partial_\xi f$ is a Young measure then gives the conclusion: indeed, by Fubini Theorem, for any $t \in [0, T]$, there is a set E_t of full measure in $\mathbb{T}^N \times \Omega$ such that, for $(x, \omega) \in E_t$, $f^\pm(x, t, \xi, \omega) \in \{0, 1\}$ for a.e. $\xi \in \mathbb{R}$. Recall that $-\partial_\xi f^\pm(x, t, \cdot, \omega)$ is a probability measure on \mathbb{R} so that, necessarily, there exists $u^\pm(x, t, \omega) \in \mathbb{R}$ such that $f^\pm(t, x, \xi, \omega) = \mathbf{1}_{u^\pm(x, t, \omega) > \xi}$ for almost every (x, ξ, ω) . In particular, $u^\pm = \int_{\mathbb{R}} (f^\pm - \mathbf{1}_{\xi > 0}) d\xi$ for almost every (x, ω) . The discussion after Definition 7 tells us that f^\pm is solution to (1) implies that u^\pm is a solution to (1). Since $f = f^+$ a.e., this shows the reduction of generalized solutions to solutions.

Write for $p \geq 1$, $t, s \in [0, T]$,

$$\begin{aligned} \int_{\mathbb{T}^N} [(u^\pm(x, t) - u^\pm(x, s))^+]^p dx &= \int_{\mathbb{T}^N} \int_{\mathbb{R}} \frac{d|\xi|^p}{d\xi} (\mathbf{1}_{u^\pm(x, t) > \xi} - \mathbf{1}_{u^\pm(x, s) > \xi}) d\xi dx \\ &= \int_{\mathbb{T}^N} \int_{\mathbb{R}} \frac{d|\xi|^p}{d\xi} (f^\pm(x, t, \xi) - f^\pm(x, s, \xi)) d\xi dx. \end{aligned}$$

By (10) and Proposition 8, we deduce that $u^+ = u^-$ except at a countable set of t and at these points u^\pm have left and right limits. Setting $u = u^+$ yield the second part of the result.

If now u_1 and u_2 are two solutions to (1), we deduce from (31) with $f_i = \mathbf{1}_{u_i > \xi}$ and from the identity

$$\int_{\mathbb{R}} \mathbf{1}_{u_1 > \xi} \overline{\mathbf{1}_{u_2 > \xi}} d\xi = (u_1 - u_2)^+$$

the contraction property

$$\mathbb{E} \|(u_1^\pm(t) - u_2^\pm(t))^+\|_{L^1(\mathbb{T}^N)} \leq \mathbb{E} \|(u_{1,0} - u_{2,0})^+\|_{L^1(\mathbb{T}^N)}. \quad (32)$$

This implies the L^1 -contraction property, comparison and uniqueness of solutions.

In the multiplicative case (Φ depending on u), the reasoning is similar, except that there is an additional term in the bound on I_ψ . More precisely, by

Hypothesis (3) we obtain in place of (27) the estimate

$$I_\psi \leq \frac{TD_1}{2}\varepsilon^2\delta^{-1} + \frac{D_1}{2}I_\psi^h,$$

where

$$I_\psi^h = \mathbb{E} \int_0^t \int_{(\mathbb{T}^N)^2} \rho_\varepsilon \int_{\mathbb{R}^2} \psi_\delta(\xi - \zeta) |\xi - \zeta| h(|\xi - \zeta|) d\nu_{x,\sigma}^1 \otimes \nu_{y,\sigma}^2(\xi, \zeta) dx dy d\sigma.$$

Choosing $\psi_\delta(\xi) = \delta^{-1}\psi_1(\delta^{-1}\xi)$ with ψ_1 compactly supported gives

$$I_\psi \leq \frac{TD_1}{2}\varepsilon^2\delta^{-1} + \frac{TD_1 C_\psi h(\delta)}{2}, \quad C_\psi := \sup_{\xi \in \mathbb{R}} \|\xi \psi_1(\xi)\|. \quad (33)$$

We deduce (30) with a remainder term $r'(\varepsilon, \delta) := r(\varepsilon, \delta) + \frac{TD_1 C_\psi h(\delta)}{2}$ and conclude the proof as in the additive case.

There remains to prove (28): setting $\xi'' = \xi' - \zeta'$, we have

$$\begin{aligned} \Upsilon(\xi, \zeta) &= \int_\zeta^{+\infty} \int_{|\xi''| < \delta, \xi'' < \xi - \zeta'} \Gamma(\xi'' + \zeta', \zeta') |\xi''| \psi_\delta(\xi'') d\xi'' d\zeta' \\ &\leq C \int_\zeta^{\xi + \delta} \max_{|\xi''| < \delta, \xi'' < \xi - \zeta'} \Gamma(\xi'' + \zeta', \zeta') d\zeta' \delta \\ &\leq C \int_\zeta^{\xi + \delta} (1 + |\xi|^{p-1} + |\zeta'|^{p-1}) d\zeta' \delta, \end{aligned}$$

which gives (28). ■

Proof of Corollary 12: set $f = \mathbf{1}_{u > \xi}$. Let us first show the continuity at $t = 0$. We apply (17), which reads, at $t = 0$,

$$\langle f^+(0) - \mathbf{1}_{u_0 > \xi}, \varphi \rangle = \langle \partial_\xi m_0, \varphi \rangle,$$

where m_0 is the restriction of m to $\mathbb{T}^N \times \{0\} \times \mathbb{R}$. In particular, by the condition at infinity (5) on m , we have: almost surely, for a.e. $x \in \mathbb{T}^N$,

$$\int_{\mathbb{R}} f^{0,+}(x, \xi) - \mathbf{1}_{u_0 > \xi} d\xi = 0.$$

We then write

$$u^+(x, 0) = \int_{\mathbb{R}} (f^{0,+}(x, \xi) - \mathbf{1}_{0 > \xi}) d\xi = \int_{\mathbb{R}} (\mathbf{1}_{u_0(x) > \xi} - \mathbf{1}_{0 > \xi}) d\xi = u_0(x).$$

This gives continuity at $t = 0$, indeed we already know that u converges to $u^+(0)$ in $L^p(\mathbb{T}^N)$.

To prove similar results at time $t_* \in (0, T)$, we consider t_* as the origin of time: indeed it follows from (14) and Proposition. 8 that

$$\begin{aligned} & \int_{t_*}^T \langle f(t), \partial_t \varphi(t) \rangle dt + \langle f^-(t_*), \varphi(t_*) \rangle + \int_{t_*}^T \langle f(t), a(\xi) \cdot \nabla \varphi(t) \rangle dt \\ &= - \sum_{k \geq 1} \int_{t_*}^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}(\xi) dx d\beta_k(t) \\ & \quad - \frac{1}{2} \int_{t_*}^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, t, \xi) \mathbf{G}^2(x, \xi) d\nu_{(x,t)}(\xi) dx dt + m(\mathbf{1}_{[t_*, T]} \partial_\xi \varphi). \end{aligned}$$

In other words, $t \mapsto f(t_* + t)$ is a generalized solution to (1) on $[0, T - t_*]$ with initial datum $f^-(t_*) = \mathbf{1}_{u^-(t_*) > \xi}$. We obtain $u^+(t_*) = u^-(t_*)$ and the result follows. ■

3.3 Entropy solutions

For deterministic first-order scalar conservation laws, the notion of entropy solution was introduced by Kruzhkov [Kru70] prior to the notion of kinetic solution [LPT94]. For the first-order scalar conservation law with stochastic forcing, a corresponding notion of weak entropy solution has been introduced by Feng and Nualart [FN08]:

Definition 13 (Weak entropy solution) *A measurable function $u: \mathbb{T}^N \times [0, T] \times \Omega \rightarrow \mathbb{R}$ is said to be a weak entropy solution to (1) if $(u(t))$ is an adapted $L^2(\mathbb{T}^N)$ -valued process, for all $p \geq 1$, there exists $C_p \geq 0$ such that for a.e. $t \in [0, T]$,*

$$\|u(t)\|_{L^p(\Omega \times \mathbb{T}^N)} \leq C_p$$

and for all convex $\eta \in C^2(\mathbb{R})$, for all non-negative $\theta \in C^1(\mathbb{T}^N)$, for all $0 \leq s \leq t \leq T$,

$$\begin{aligned} & \langle \eta(u(t)), \theta \rangle - \langle \eta(u(s)), \theta \rangle \leq \int_s^t \langle q(u(r)), \nabla \theta \rangle dr \\ & \quad + \sum_{k \geq 1} \int_s^t \langle g_k(\cdot, u(r)) \eta'(u(r)), \theta \rangle d\beta_k(r) + \frac{1}{2} \int_s^t \langle \mathbf{G}^2(\cdot, u(r)) \eta''(u(r)), \theta \rangle dr, \end{aligned}$$

a.s., where $q(u) = \int_0^u a(\xi) \eta'(\xi) d\xi$.

An entropy solution is a kinetic solution and vice versa. Let us introduce an auxiliary definition:

Definition 14 (Time-weak weak entropy solution) *Let $u_0 \in L^\infty(\Omega)$. A measurable function $u: \mathbb{T}^N \times [0, T] \times \Omega \rightarrow \mathbb{R}$ is said to be a time-weak weak entropy solution to (1) with initial datum u_0 if $(u(t))$ is an adapted $L^2(\mathbb{T}^N)$ -valued process, for all $p \geq 1$, there exists $C_p \geq 0$ such that for a.e. $t \in [0, T]$,*

$$\|u(t)\|_{L^p(\Omega \times \mathbb{T}^N)} \leq C_p$$

and for all convex $\eta \in C^2(\mathbb{R})$, for all non-negative $\rho \in C_c^1(\mathbb{T}^N \times [0, T])$,

$$\begin{aligned} & \int_0^T \langle \eta(u), \partial_t \rho \rangle dr + \langle \eta(u_0), \rho(0) \rangle + \int_0^T \langle q(u), \nabla \rho \rangle dr \\ & \geq - \sum_{k \geq 1} \int_0^T \langle g_k(\cdot, u(r)) \eta'(u(r)), \rho \rangle d\beta_k(r) - \frac{1}{2} \int_0^T \langle \mathbf{G}^2(\cdot, u(r)) \eta''(u(r)), \rho \rangle dr, \end{aligned} \quad (34)$$

a.s., where $q(u) = \int_0^u a(\xi) \eta'(\xi) d\xi$.

Proposition 15 (Entropy and kinetic solutions) *Let $u_0 \in L^\infty(\Omega)$. For a measurable function $u: \mathbb{T}^N \times [0, T] \times \Omega \rightarrow \mathbb{R}$, it is equivalent to be a kinetic solution to (1), i.e. a solution in the sense of Definition 2, and a time-weak weak solution.*

The proof of the proposition is classical. Choosing test functions $\varphi(x, t, \xi) = \rho(x, t) \eta'(\xi)$ in (7) and using the inequality $m \eta'' \geq 0$ gives (34). Conversely, starting from (34), one defines the measure m (actually $\partial_\xi^2 m$) by

$$\begin{aligned} m(\rho \otimes \eta'') &= \int_0^T \langle \eta(u), \partial_t \rho \rangle dr + \langle \eta(u_0), \rho(0) \rangle + \int_0^T \langle q(u), \nabla \rho \rangle dr \\ &+ \sum_{k \geq 1} \int_0^T \langle g_k(\cdot, u(r)) \eta'(u(r)), \rho \rangle d\beta_k(r) + \frac{1}{2} \int_0^T \langle \mathbf{G}^2(\cdot, u(r)) \eta''(u(r)), \rho \rangle dr, \end{aligned}$$

and then derives (7). See [Per02] for precise references.

It is clear also that a weak entropy solution, satisfying $u(0) = u_0$, is a time-weak entropy solution, while, for the converse assertion, time-continuity of the solution is required. We have seen that a kinetic solution is continuous in time, it follows that it is indeed a weak entropy solution.

3.4 Spatial regularity

To conclude this paragraph and our applications of Proposition 9, we give a result on the spatial regularity of the solution. To that purpose, we introduce two semi-norms that measure the $W^{\sigma,1}$ -regularity of a function $u \in L^1(\mathbb{T}^N)$ ($\sigma \in (0, 1)$): we set

$$p^\sigma(u) := \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \frac{|u(x) - u(y)|}{|x - y|^{N+\sigma}} dx dy,$$

and

$$p_\rho^\sigma(u) = \sup_{0 < \varepsilon < 2D_N} \frac{1}{\varepsilon^\sigma} \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} |u(x) - u(y)| \rho_\varepsilon(x - y) dx dy,$$

where (ρ_ε) is a fixed regularizing kernel: $\rho_\varepsilon(x) = \varepsilon^{-N} \rho(\varepsilon^{-1}|x|)$ where ρ is supported in the ball $B(0, 1)$ of \mathbb{R}^N . We define $W^{\sigma,1}(\mathbb{T}^N)$ as the subspace of $u \in L^1(\mathbb{T}^N)$ with finite norm

$$\|u\|_{W^{\sigma,1}(\mathbb{T}^N)} = \|u\|_{L^1(\mathbb{T}^N)} + p^\sigma(u).$$

Lemma 16 (Comparison of the $W^{\sigma,1}$ semi-norms) *Let $\sigma \in (0, 1)$. There exists C depending on σ, ρ, N such that, for all $0 < s < \sigma$, for all $u \in L^1(\mathbb{T}^N)$,*

$$p_\rho^\sigma(u) \leq C p^\sigma(u), \quad p^s(u) \leq \frac{C}{\sigma - s} p_\rho^\sigma(u).$$

Proof: we have

$$\frac{1}{\varepsilon^\sigma} \rho_\varepsilon(x - y) \leq \frac{\|\rho\|_{L^\infty}}{\varepsilon^{N+\sigma}} \mathbf{1}_{|x-y| < \varepsilon} \leq \frac{\|\rho\|_{L^\infty}}{|x - y|^{N+\sigma}},$$

hence $p_\rho^\sigma(u) \leq C p^\sigma(u)$. Multiplying the inequality

$$\frac{1}{\varepsilon^\sigma} \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} |u(x) - u(y)| \frac{1}{\varepsilon^N} \rho\left(\frac{|x - y|}{\varepsilon}\right) dx dy \leq p_\rho^\sigma(u)$$

by $\varepsilon^{-1+(\sigma-s)}$ and integrating over $\varepsilon \in (|x - y|, 2D_N)$ where $D_N = \sqrt{N}$ is the diameter of $[0, 1]^N$, we obtain the second inequality. ■

Theorem 17 ($W^{\sigma,1}$ -regularity) *Let $u_0 \in L^\infty(\mathbb{T}^N)$, let $u: \mathbb{T}^N \times (0, +\infty) \times \Omega \rightarrow \mathbb{R}$ be the solution to (1) with initial datum u_0 . Assume that h satisfies*

$$h(\delta) \leq C\delta^\alpha, \quad \delta < 1, \quad 0 < \alpha. \quad (35)$$

Set $\sigma = \min\left(\frac{2\alpha}{1+\alpha}, \frac{1}{2}\right)$. Then, there exists a constant C such that, for all $t \geq 0$, we have

$$\mathbb{E}p_\rho^\sigma(u(t)) \leq C(\mathbb{E}p_\rho^\sigma(u_0) + t). \quad (36)$$

In particular, for all $0 < s < \sigma$, there exists a constant $C_s > 0$ such that for $t \geq 0$,

$$\mathbb{E}\|u(t)\|_{W^{s,1}(\mathbb{T}^N)} \leq C_s(\|u_0\|_{W^{\sigma,1}(\mathbb{T}^N)} + t).$$

Proof: the last assertion is proved as follows: by Lemma 16, (36) implies $\mathbb{E}p^s(u(t)) \leq C_s(p^\sigma(u_0) + t)$. Poincaré Inequality gives

$$\left\|u(t) - \int_{\mathbb{T}^N} u(t)dx\right\|_{L^1(\mathbb{T}^N)} \leq C_s p^s(u(t)).$$

Since $\mathbb{E} \int_{\mathbb{T}^N} u(t)dx = \mathbb{E} \int_{\mathbb{T}^N} u_0 dx$, we obtain a bound on the L^1 -norm of u :

$$\|u(t)\|_{L^1(\mathbb{T}^N)} \leq C_s(p^\sigma(u_0) + t + \|u_0\|_{L^1(\mathbb{T}^N)}),$$

hence $\mathbb{E}\|u(t)\|_{W^{s,1}(\mathbb{T}^N)} \leq C_s(\|u_0\|_{W^{\sigma,1}(\mathbb{T}^N)} + t)$. To prove (36), we apply Prop. 9 with $f_1 = f_2 = \mathbf{1}_{u>\xi}$, $\rho = \rho^\varepsilon$, $\psi = \psi_\delta$. Since $\partial_\xi \mathbf{1}_{u>\xi} = -\delta_{u=\xi}$ is a Radon measure with mass 1, we have

$$\begin{aligned} & \mathbb{E} \int_{(\mathbb{T}^N)^2} \rho_\varepsilon(x-y)(u(x,t) - u(y,t))^+ dx dy \\ & \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x-y) \psi_\delta(\xi - \zeta) \mathbf{1}_{u(x,t)>\xi} (1 - \mathbf{1}_{u(y,t)>\zeta}) dx dy d\xi d\zeta + \delta \end{aligned}$$

and

$$\begin{aligned} & \mathbb{E} \int_{(\mathbb{T}^N)^2} \int_{\mathbb{R}^2} \rho_\varepsilon(x-y) \psi_\delta(\xi - \zeta) \mathbf{1}_{u_0(x)>\xi} (1 - \mathbf{1}_{u_0(y)>\zeta}) dx dy d\xi d\zeta \\ & \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \rho_\varepsilon(x-y)(u_0(x) - u_0(y))^+ dx dy + \delta. \end{aligned}$$

We deduce that

$$\begin{aligned} & \mathbb{E} \int_{(\mathbb{T}^N)^2} \rho_\varepsilon(x-y)(u(x,t) - u(y,t))^+ dx dy \\ & \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \rho_\varepsilon(x-y)(u_0(x) - u_0(y))^+ dx dy + I_\rho + I_\psi + 2\delta. \end{aligned}$$

As in (33)-(29), we have

$$I_\psi \leq tC(\varepsilon^2\delta^{-1} + h(\delta)), \quad I_\rho \leq tC\delta\varepsilon^{-1},$$

hence

$$\begin{aligned} & \mathbb{E} \int_{(\mathbb{T}^N)^2} \rho_\varepsilon(x-y)(u(x,t) - u(y,t))^+ dx dy \\ & \leq \mathbb{E} \int_{(\mathbb{T}^N)^2} \rho_\varepsilon(x-y)(u_0(x) - u_0(y))^+ dx dy + tC(\varepsilon^2\delta^{-1} + h(\delta) + \delta\varepsilon^{-1}) + 2\delta. \end{aligned}$$

By optimization in δ , using (35), we obtain (36). ■

4 Existence

4.1 The parabolic approximation, kinetic formulation

Let $u_0 \in L^\infty(\mathbb{T}^N)$. To prove the existence of a solution to (1) with initial datum u_0 , we show the convergence of the parabolic approximation

$$\begin{cases} du^\eta + \operatorname{div}(A(u^\eta))dt - \eta\Delta u^\eta dt & = \Phi_\eta(u^\eta)dW(t), \quad t > 0, x \in \mathbb{T}^N, \\ u^\eta(x, 0) & = u_0(x), \quad x \in \mathbb{T}^N. \end{cases} \quad (37)$$

Where Φ_η is a suitable Lipschitz approximation of Φ satisfying (2), (3) uniformly. We define g_k^η and \mathbf{G}^η as in the case $\eta = 0$.

It is shown in [GR00] that equation (37) has a unique $L^\rho(\mathbb{T}^N)$ valued continuous solution provided ρ is large enough and $u_0 \in L^\rho(\mathbb{T}^N)$, hence in particular for $u_0 \in L^\infty(\mathbb{T}^N)$. Moreover, it is also shown in [GR00] that using Itô Formula one can prove that u^η satisfies the energy inequality

$$\begin{aligned} \mathbb{E}\|u^\eta(t)\|_{L^2(\mathbb{T}^N)}^2 + 2\eta\mathbb{E} \int_0^t \|\nabla u^\eta\|_{L^2(\mathbb{T}^N)}^2 ds \\ \leq \mathbb{E}\|u_0\|_{L^2(\mathbb{T}^N)}^2 + \mathbb{E} \int_0^t \|\mathbf{G}_\eta(u^\eta)\|_{L^2(\mathbb{T}^N)}^2 ds. \end{aligned} \quad (38)$$

By (2) and Gronwall Lemma, we easily derive

$$\mathbb{E}\|u^\eta(t)\|_{L^2(\mathbb{T}^N)}^2 + \eta\mathbb{E} \int_0^t \|\nabla u^\eta\|_{L^2(\mathbb{T}^N)}^2 ds \leq C(T)(\mathbb{E}\|u_0\|_{L^2(\mathbb{T}^N)}^2 + 1). \quad (39)$$

Also, for $p \geq 2$, by Itô Formula applied to $|u|^p$ and a martingale inequality

$$\mathbb{E} \left(\sup_{t \in [0, T]} \|u^\eta(t)\|_{L^p(\mathbb{T}^N)}^p \right) + \eta\mathbb{E} \int_0^T \int_{\mathbb{T}^N} |u^\eta(t, x)|^{p-2} |\nabla u^\eta(t)|^2 dx dt \leq C(p, u_0, T). \quad (40)$$

Proposition 18 (Kinetic formulation) *Let $u_0 \in L^\infty(\mathbb{T}^N)$ and let u^η be the solution to (37). Then $f^\eta := \mathbf{1}_{u^\eta > \xi}$ satisfies: for all $\varphi \in C_c^1(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,*

$$\begin{aligned} & \int_0^T \langle f^\eta(t), \partial_t \varphi(t) \rangle dt + \langle f_0, \varphi(0) \rangle + \int_0^T \langle f^\eta(t), a(\xi) \cdot \nabla \varphi(t) - \eta \Delta \varphi(t) \rangle dt \\ &= - \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} g_k^\eta(x, \xi) \varphi(x, t, \xi) d\nu_{x,t}^\eta(\xi) dx d\beta_k(t) \\ & - \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} \partial_\xi \varphi(x, t, \xi) \mathbf{G}_\eta^2(x, \xi) d\nu_{(x,t)}^\eta(\xi) dx dt + m^\eta(\partial_\xi \varphi), \end{aligned} \quad (41)$$

a.s., where $f_0(\xi) = \mathbf{1}_{u_0 > \xi}$ and, for $\phi \in C_b(\mathbb{T}^N \times [0, T] \times \mathbb{R})$,

$$\nu_{(x,t)}^\eta = \delta_{u^\eta(x,t)}, \quad m^\eta(\phi) = \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \phi(x, t, u^\eta(x, t)) \eta |\nabla u^\eta|^2 dx dt.$$

Note that the measure m^η is explicitly known here: $m^\eta = \eta |\nabla u^\eta|^2 \delta_{u^\eta = \xi}$.

Proof: By Itô Formula, we have, for $\theta \in C^2(\mathbb{R})$ with polynomial growth at $\pm\infty$,

$$\begin{aligned} d(\mathbf{1}_{u^\eta > \xi}, \theta') &:= d \int_{\mathbb{R}} \mathbf{1}_{u^\eta > \xi} \theta'(\xi) d\xi = d\theta(u^\eta) \\ &= \theta'(u^\eta) (-a(u^\eta) \cdot \nabla u^\eta dt + \eta \Delta u^\eta dt + \Phi_\eta(u^\eta) dW) + \frac{1}{2} \theta''(u^\eta) \mathbf{G}_\eta^2 dt. \end{aligned}$$

We rewrite the first term as

$$-\theta'(u^\eta) a(u^\eta) \cdot \nabla u^\eta = -\operatorname{div} \left\{ \int_0^{u^\eta} a(\xi) \theta'(\xi) d\xi \right\} = -\operatorname{div}(a \mathbf{1}_{u^\eta > \xi}, \theta'),$$

the second term as

$$\begin{aligned} \theta'(u^\eta) \eta \Delta u^\eta &= \eta \Delta \theta(u^\eta) dt - \eta |\nabla u^\eta|^2 \theta''(u^\eta) \\ &= \eta \Delta(\mathbf{1}_{u^\eta > \xi}, \theta') dt + (\partial_\xi(\eta |\nabla u^\eta|^2 \delta_{u^\eta = \xi}), \theta') \end{aligned}$$

to obtain the kinetic formulation

$$\begin{aligned} d(\mathbf{1}_{u^\eta > \xi}, \theta') &= -\operatorname{div}[(a \mathbf{1}_{u^\eta > \xi}, \theta')] dt + \eta \Delta(\mathbf{1}_{u^\eta > \xi}, \theta') dt \\ &+ (\partial_\xi(\eta |\nabla u^\eta|^2 \delta_{u^\eta = \xi} - \frac{1}{2} \mathbf{G}_\eta^2 \delta_{u^\eta = \xi}), \theta') dt + \sum_{k \geq 1} (\delta_{u^\eta = \xi}, \theta' g_{k,\eta}) d\beta_k. \end{aligned} \quad (42)$$

Taking $\theta(\xi) = \int_{-\infty}^{\xi} \beta$, we then obtain (42) with the test function β in place of θ' . Since the test functions $\varphi(x, \xi) = \alpha(x)\beta(\xi)$ form a dense subset of $C_c^\infty(\mathbb{T}^N \times \mathbb{R})$, (41) follows. ■

Equation (41) is close to the kinetic equation (7) satisfied by the solution to (1). For $\eta \rightarrow 0$, we lose the precise structure of $m^\eta = \eta|\nabla u^\eta|^2 \delta_{u^\eta=\xi}$ and obtain a solution u to (1). More precisely, we will prove the

Theorem 19 (Convergence of the parabolic approximation) *Let $u_0 \in L^\infty(\mathbb{T}^N)$. There exists a unique solution u to (1) with initial datum u_0 which is the strong limit of (u^η) as $\eta \rightarrow 0$: for every $T > 0$, for every $1 \leq p < +\infty$,*

$$\lim_{\eta \rightarrow 0} \mathbb{E} \|u^\eta - u\|_{L^p(\mathbb{T}^N \times (0, T))} = 0. \quad (43)$$

The proof of Theorem 19 is quite a straightforward consequence of both the result of reduction of generalized solution to solution - Theorem 11 - and the a priori estimates derived in the following section.

4.1.1 A priori estimates

We denote indifferently by C_p various constants that may depend on $p \in [1, +\infty)$, on u_0 , on the noise and on the terminal time T , but not on $\eta \in (0, 1)$.

1. Estimate of m^η : we analyze the kinetic measure $m^\eta = \eta|\nabla u^\eta|^2 \delta_{u^\eta=\xi}$. By (39), we have a uniform bound $\mathbb{E} m^\eta(\mathbb{T}^N \times [0, T] \times \mathbb{R}) \leq C$. Furthermore, the second term in the left hand-side of (40) is $\mathbb{E} \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} |\xi|^{p-2} dm^\eta(x, t, \xi)$, so we have

$$\mathbb{E} \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} |\xi|^p dm^\eta(x, t, \xi) \leq C_p. \quad (44)$$

We also have the improved estimate, for $p \geq 0$,

$$\mathbb{E} \left| \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} |\xi|^{2p} dm^\eta(x, t, \xi) \right|^2 \leq C_p. \quad (45)$$

To prove (45), we apply Itô Formula to $\psi(u^\eta)$, $\psi(\xi) := |\xi|^{2p+2}$:

$$d\psi(u^\eta) - \operatorname{div}(\mathcal{F})dt + \eta\psi''(u^\eta)|\nabla u^\eta|^2 dt = \psi'(u^\eta)\Phi_\eta(u^\eta)dW + \frac{1}{2}\psi''(u^\eta)\mathbf{G}_\eta^2 dt,$$

where $\mathcal{F} := \int_0^{u^\eta} a(\xi)\psi'(\xi)d\xi - \eta\nabla\psi(u^\eta)$. It follows

$$\begin{aligned} & \int_0^T \int_{\mathbb{T}^N} \eta\psi''(u^\eta)|\nabla u^\eta|^2 dxdt \\ & \leq \int_{\mathbb{T}^N} \psi(u_0)dx + \sum_{k \geq 1} \int_0^T \int_{\mathbb{T}^N} \psi'(u^\eta)g_{k,\eta}(x, u^\eta) dx d\beta_k(t) \\ & \quad + \frac{1}{2} \int_0^T \int_{\mathbb{T}^N} \mathbf{G}_\eta^2(x, u^\eta)\psi''(u^\eta) dxdt. \end{aligned}$$

Taking the square, then expectation, we deduce by Itô isometry

$$\begin{aligned} & \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^N} \eta\psi''(u^\eta)|\nabla u^\eta|^2 dxdt \right|^2 \leq 2\mathbb{E} \left| \int_{\mathbb{T}^N} \psi(u_0)dx \right|^2 \\ & + 2\mathbb{E} \int_0^T \sum_{k \geq 1} \left| \int_{\mathbb{T}^N} g_k(x, u^\eta)\psi'(u^\eta)dx \right|^2 dt + \mathbb{E} \left| \int_0^T \int_{\mathbb{T}^N} \mathbf{G}^2(x, u^\eta)\psi''(u^\eta) dxdt \right|^2. \end{aligned}$$

By (2), (40) and Cauchy-Schwarz inequality, we obtain (45).

2. Estimate on ν^η : By the bound (40) on u^η in L^p , we have, for $t \in (0, T)$,

$$\mathbb{E} \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\eta(\xi) dx \leq C_p \quad (46)$$

and, in particular,

$$\mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} |\xi|^p d\nu_{x,t}^\eta(\xi) dxdt \leq C_p. \quad (47)$$

4.1.2 Generalized solution

Consider a sequence $(\eta_n) \downarrow 0$. We use the a priori bounds derived in the preceding subsection to deduce, up to subsequences:

1. by (47) and Theorem 5 and Corollary 6 respectively, the convergence $\nu^{\eta_n} \rightarrow \nu$ (in the sense of (12)) and the convergence $f^{\eta_n} \rightharpoonup f$ in $L^\infty(\Omega \times \mathbb{T}^N \times (0, T) \times \mathbb{R})$ -weak-*. Besides, the bound (46) is stable: ν satisfies (13).
2. Let \mathcal{M}_b denote the space of bounded Borel measures over $\mathbb{T}^N \times [0, T] \times \mathbb{R}$ (with norm given by the total variation of measures), i.e. the dual space of C_b , the set of continuous bounded functions on $\mathbb{T}^N \times [0, T] \times \mathbb{R}$. Then $L^2(\Omega; \mathcal{M}_b)$ is the dual space of $L^2(\Omega, C_b)$. By (45) with $p = 0$, we have,

up to subsequence, the convergence $m^{\eta_n} \rightharpoonup m$ in $L^2(\Omega; \mathcal{M}_b)$ -weak star. In particular, we have, for any $k > 0$,

$$\begin{aligned} \mathbb{E} \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \min(|\xi|^p, k) dm(x, t, \xi) \\ = \lim_{n \rightarrow +\infty} \mathbb{E} \int_{\mathbb{T}^N \times [0, T] \times \mathbb{R}} \min(|\xi|^p, k) dm^{\eta_n}(x, t, \xi) \leq C_p \end{aligned}$$

by (44), hence m vanishes at infinity, *i.e.* satisfies (5).

Let $\phi \in C_b(\mathbb{T}^N \times \mathbb{R})$ and set $x^n(t) := \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \phi(x, \xi) dm^{\eta_n}(x, s, \xi)$, $\alpha \in L^2(\Omega)$, $\gamma \in L^2([0, T])$, then, by Fubini's Theorem,

$$\mathbb{E} \left(\alpha \int_0^T \gamma(t) x^n(t) dt \right) = \mathbb{E} \left(\alpha \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \phi(x, \xi) \Gamma(s) dm^{\eta_n}(x, s, \xi) \right),$$

where $\Gamma(s) = \int_s^T \gamma(t) dt$. Since Γ is continuous, and by weak convergence of m^{η_n} , we have

$$\mathbb{E} \left(\alpha \int_0^T \gamma(t) x^n(t) dt \right) \rightarrow \mathbb{E} \left(\alpha \int_0^T \gamma(t) x(t) dt \right),$$

where

$$x(t) = \int_{\mathbb{T}^N \times [0, t] \times \mathbb{R}} \phi(x, \xi) dm(x, s, \xi).$$

Since tensor functions are dense in $L^2(\Omega \times [0, T])$, we obtain the weak convergence $x_n \rightarrow x$ in $L^2(\Omega \times [0, T])$. In particular, since the space of predictable process is weakly-closed, x is predictable.

At the limit [$n \rightarrow +\infty$] in (41), we obtain (14), so f is a generalized solution to (1) with initial datum $\mathbf{1}_{u_0 > \xi}$.

4.1.3 Conclusion: proof of Theorem 19

By Theorem 11, there corresponds a solution u to this f : $f = \mathbf{1}_{u > \xi}$. This proves the existence of a solution u to (1), unique by Theorem 11. Besides, owing to the particular structure of f^n and f , we have

$$\|u^{\eta_n}\|_{L^2(\mathbb{T}^N \times (0, T))}^2 - \|u\|_{L^2(\mathbb{T}^N \times (0, T))}^2 = \int_0^T \int_{\mathbb{T}^N} \int_{\mathbb{R}} 2\xi(f^{\eta_n} - f) d\xi dx dt$$

and (using the bound on u^n in $L^3(\mathbb{T}^N)$)

$$\mathbb{E} \int_0^T \int_{\mathbb{T}^N} \int_{|\xi| > R} |2\xi(f^{\eta_n} - f)(\xi)| d\xi dx \leq \frac{C}{1 + R}.$$

It follows that u^{n_n} converges in norm to u in the Hilbert space $L^2(\mathbb{T}^N \times (0, T) \times \Omega)$. Using the weak convergence, we deduce the strong convergence. Since u is unique, the whole sequence actually converges. This gives the result of the theorem for $p = 2$. The case of general p follows from the bound on u^n in L^q for arbitrary q and Hölder Inequality. ■

A Proof of Theorem 5 and Corollary 6

Let $h \in L^1(X)$ be non-negative. By the condition at infinity (11), the sequence of measure (ν_h^n) defined by

$$\int_{\mathbb{R}} \phi(\xi) d\nu_h^n(\xi) = \int_X h(z) \int_{\mathbb{R}} \phi(\xi) d\nu_z^n(\xi) dz, \quad \phi \in C_b(\mathbb{R})$$

is tight. By Prokhorov Theorem, there exists a subsequence still denoted (ν_h^n) that converges weakly in the sense of measure to a measure ν_h on \mathbb{R} having the same mass as the measures (ν_h^n) :

$$\nu_h(\mathbb{R}) = \int_X h(z) dz. \quad (48)$$

Since $L^1(X)$ is separable and $h \mapsto \nu_h^n$ is uniformly continuous in the sense that

$$|\nu_{h^b}^n(\phi) - \nu_{h^\sharp}^n(\phi)| \leq \|h^b - h^\sharp\|_{L^1(X)} \|\phi\|_{C_b(\mathbb{R})}$$

for all $\phi \in C_b(\mathbb{R})$, standard diagonal and limiting arguments give $\nu_h^n \rightharpoonup \nu_h$ along a subsequence independent on the choice of $h \in L^1(X)$. At fixed $\phi \in C_b(\mathbb{R})$, the estimate $0 \leq \nu_h(\phi) \leq \|h\|_{L^1(X)} \|\phi\|_{C_b(\mathbb{R})}$ and the linearity of $h \mapsto \nu_h(\phi)$ show that

$$\nu_h(\phi) = \int_X h(z) g(z, \phi) dz, \quad \|g(\cdot, \phi)\|_{L^\infty(X)} \leq \|\phi\|_{C_b(\mathbb{R})}.$$

Besides, $g(\cdot, \phi) \geq 0$ a.e. since $\nu_h(\phi) \geq 0$ for non-negative h , and $\phi \mapsto g(\cdot, \phi)$ is linear. Consequently, for a.e. $z \in X$, we have

$$g(z, \phi) = \int_{\mathbb{R}} \phi(\xi) d\nu_z(\phi)$$

where ν_z is non-negative finite measure on \mathbb{R} . By (48), $\nu_z(\mathbb{R}) = 1$. At last, ν vanishes at infinity since

$$\int_X \int_{\mathbb{R}} |\xi|^p d\nu_z(\xi) d\lambda(z) \leq \limsup_{n \rightarrow +\infty} \int_X \int_{\mathbb{R}} |\xi|^p d\nu_z^n(\xi) d\lambda(z) < +\infty.$$

This concludes the proof of the Theorem. To prove Corollary 6 we start from the weak convergence $\nu_h^n \rightharpoonup \nu_h$ ($h \in L^1(X)$ being fixed). This implies

$$\nu_h^n(\xi, +\infty) \rightarrow \nu_h(\xi, +\infty)$$

at all points ξ except the atomic points of ν_h , that are at most countable, hence of zero measure. It follows in particular that

$$\int_{\mathbb{R}} \nu_h^n(\xi, +\infty)g(\xi)d\xi \rightarrow \int_{\mathbb{R}} \nu_h(\xi, +\infty)g(\xi)d\xi$$

for all $g \in L^1(\mathbb{R})$. In other words, we have

$$\int_{X \times \mathbb{R}} f_n(z, \xi)H(z, \xi)d\xi d\lambda(z) \rightarrow \int_{X \times \mathbb{R}} f(z, \xi)H(z, \xi)d\xi d\lambda(z) \quad (49)$$

for all $H \in L^1(X \times \mathbb{R})$ of the form $H(z, \xi) = h(z)g(\xi)$. This implies the result since tensor functions are dense in $L^1(X \times \mathbb{R})$. ■

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