

Gabor frames with reduced redundancy

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Abstract:

Considering previous constructions of pairs of dual Gabor frames, we discuss ways to reduce the redundancy. The focus is on B-spline type windows.

1. Introduction

We will consider Gabor systems in $L^2(\mathbb{R})$, i.e., families of functions $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$, where

$$E_{mb}T_n g(x) := e^{2\pi i m b x} g(x - na).$$

If there exists a constant $B > 0$ such that

$$\sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_n g \rangle|^2 \leq B \|f\|^2, \quad \forall f \in L^2(\mathbb{R}),$$

then $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ is called a Bessel sequence. If there exist two constants $A, B > 0$ such that

$$A \|f\|^2 \leq \sum_{m,n \in \mathbb{Z}} |\langle f, E_{mb}T_n g \rangle|^2 \leq B \|f\|^2, \quad \forall f \in L^2(\mathbb{R}),$$

then $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ is called a frame. If $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ is a frame with dual frame $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$, then

$$f = \sum_{m,n \in \mathbb{Z}} \langle f, E_{mb}T_n h \rangle E_{mb}T_n g, \quad f \in L^2(\mathbb{R}),$$

where the series expansion converges unconditionally in $L^2(\mathbb{R})$.

Our starting point is the duality condition for Gabor frames, originally due to Ron and Shen [4]. We use the version due to Janssen [3]:

Lemma 1.1 *Two Bessel sequences $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$ form dual Gabor frames for $L^2(\mathbb{R})$ if and only if*

$$\sum_{k \in \mathbb{Z}} \overline{g(x - n/b + k)} h(x + k) = b \delta_{n,0} \quad (1..1)$$

for a.e. $x \in [0, 1]$.

The Bessel condition in Lemma 1.1 is always satisfied for bounded windows with compact support, see [1]. Note that if g and h have compact support, we only need to check a finite number of conditions in (1..1). In this paper we will usually choose b so small that only the condition for $n = 0$ has to be verified.

2. The range $\frac{1}{2N-1} < b < \frac{1}{N}$

We first cite a result from [2]. It yields an explicit construction of dual Gabor frames:

Theorem 2.1 *Let $N \in \mathbb{N}$. Let $g \in L^2(\mathbb{R})$ be a real-valued bounded function with $\text{supp } g \subset [0, N]$, for which*

$$\sum_{n \in \mathbb{Z}} g(x - n) = 1. \quad (2..1)$$

Let $b \in]0, \frac{1}{2N-1}]$. Consider any scalar sequence $\{a_n\}_{n=-N+1}^{N-1}$ for which

$$a_0 = b \text{ and } a_n + a_{-n} = 2b, \quad n = 1, 2, \dots, N-1, \quad (2..2)$$

and define $h \in L^2(\mathbb{R})$ by

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x + n). \quad (2..3)$$

Then g and h generate dual frames $\{E_{mb}T_n g\}_{m,n \in \mathbb{Z}}$ and $\{E_{mb}T_n h\}_{m,n \in \mathbb{Z}}$ for $L^2(\mathbb{R})$.

The above result can be extended:

Corollary 2.2 *Consider any $b \leq 1/N$. With g and a_n as in Theorem 2.1, the function*

$$h(x) = \left(\sum_{n=-N+1}^{N-1} a_n g(x + n) \right) \chi_{[0,N]}(x) \quad (2..4)$$

is a dual frame generator of g .

Proof. Consider the condition (1..1) for $n = 0$; only the values of $h(x)$ for $x \in [0, N]$ play a role, so since the condition holds for the function in (2..3), it also holds for the function in (2..4). \square

The cut-off in (2..4) yields a non-smooth function. However, for any $b < 1/N$, we might modify h slightly and obtain a smooth dual generator:

In particular, we obtain the following:

Corollary 2.3 *Consider any $b < 1/N$, and take $\epsilon < 1/b - N$. With g as in Theorem 2.1, the function $h(x) = b, x \in [0, N]$ has an extension to a function of desired smoothness, supported on $[-\epsilon, N + \epsilon]$, which is a dual frame generator of g .*

Proof. The choice $a_n = b$, $n = -N + 1, \dots, N - 1$, leads to

$$\sum_{n=-N+1}^{N-1} a_n g(x+n) = b, \quad x \in [0, N].$$

Given $\epsilon < 1/b - N$ and any functions $\phi_1 : [-\epsilon, 0[\rightarrow \mathbb{R}$ and $\phi_2 :]N, N + \epsilon] \rightarrow \mathbb{R}$, the function

$$h(x) = \begin{cases} \phi_1(x), & x \in [-\epsilon, 0[, \\ \sum_{n=-N+1}^{N-1} a_n g(x+n) = b, & x \in [0, N], \\ \phi_2, & x \in]N, N + \epsilon], \\ 0, & x \notin [-\epsilon, N + \epsilon], \end{cases}$$

will satisfy (1..1); in fact, for $n \neq 0$, the support of the functions $g(\cdot \pm n/b)$ and h are disjoint, and for $n = 0$ we are (for all relevant values of x) back at the function in (2..4). The functions ϕ_1 and ϕ_2 can be chosen such that the function h has the desired smoothness. \square

The assumptions in Theorem 2..1 are tailored to B-splines, defined inductively by

$$B_1 := \chi_{[0,1]}, \quad B_{N+1} := B_N * B_1.$$

Direct calculations shows that

$$B_2(x) = \begin{cases} x & \text{if } x \in [0, 1], \\ 2 - x & \text{if } x \in [1, 2], \\ 0 & \text{otherwise,} \end{cases}$$

and

$$B_3(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \in [0, 1], \\ -x^2 + 3x - \frac{3}{2} & \text{if } x \in [1, 2], \\ \frac{1}{2}x^2 - 3x + \frac{9}{2} & \text{if } x \in [2, 3], \\ 0 & \text{otherwise.} \end{cases}$$

In general, the functions B_N are $(N - 2)$ -times differentiable piecewise polynomials (explicit expressions are known). Furthermore, $\text{supp } B_N = [0, N]$, and the partition of unity condition (2..1) is satisfied.

In case $g = B_N$, the dual generators in Theorem 2..1 are splines, of the same smoothness as B_N itself. By compressing the function $\sum_{n=-N+1}^{N-1} a_n g(x+n)$ from the interval $[-N + 1, 0]$ to $[-\epsilon, 0]$ and from $[N, 2N - 1]$ to $[N, N + \epsilon]$ we obtain a dual in (2..3) with the same features:

Example 2..4 For the B-spline $B_3(x)$ and $b = 1/5$, Theorem 2..1 yields the symmetric dual

$$h_3(x) = \frac{1}{5} \begin{cases} 1/2 x^2 + 2x + 2, & x \in [-2, -1[, \\ -1/2 x^2 + 1, & x \in [-1, 0[, \\ 1, & x \in [0, 3[, \\ -1/2 x^2 + 3x - 7/2, & x \in [3, 4[, \\ 1/2 x^2 - 5x + 25/2, & x \in [4, 5[, \\ 0, & x \notin [0, 5]. \end{cases} \quad (2..5)$$

See Figure 1.

Now, for $b = 1/4$, we can use Corollary 2..3 for $\epsilon < 4 - 3 = 1$. Taking $\epsilon = 1/2$, we compress the function

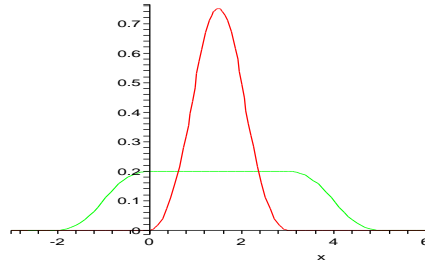


Figure 1: B_3 and the dual generator h_3 in (2..5).

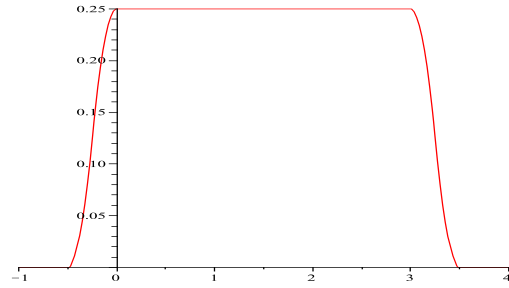


Figure 2: The function h in (3..13)..

h_3 in (2..5) from $[-2, 0]$ to $[-1/2, 0]$ and from $[3, 5]$ to $[3, 31/2]$ and obtain the dual

$$h(x) = \begin{cases} 1/2 (4x)^2 + 2(4x) + 2, & x \in [-1/2, -1/4[, \\ -1/2 (4x)^2 + 1, & x \in [-1/4, 0[, \\ 1, & x \in [0, 3[, \\ -1/2 (4(x-3) + 3)^2 + 3(4(x-3) + 3) - 7/2, & x \in [3, 3 + 1/4[, \\ 1/2 (4(x-3) + 3)^2 - 5(4(x-3) + 3) + 25/2, & x \in [3 + 1/4, 3 + 1/2[, \\ 0, & x \notin [-1/2, 3 + 1/2]. \end{cases}$$

$$= \frac{1}{4} \begin{cases} 8x^2 + 8x + 2, & x \in [-1/2, -1/4[, \\ -8x^2 + 1, & x \in [-1/4, 0[, \\ 1, & x \in [0, 3[, \\ -8x^2 + 48x - 71, & x \in [3, 3 + 1/4[, \\ 8x^2 - 56x + 98, & x \in [3 + 1/4, 3 + 1/2[, \\ 0, & x \notin [-1/2, 3 + 1/2]. \end{cases}$$

See Figure 2. \square

3. B_2 and $1/2 < b < 1$

In the following discussion, we consider dual windows associated with a Gabor frame $\{E_{mb}T_n B_2\}_{m,n \in \mathbb{Z}}$ generated by the B-spline B_2 . The arguments can be extended to general functions supported on $[0, 2]$. Take any function h with values specified only on $[0, 2]$ and such that

$$\sum_{k \in \mathbb{Z}} B_2(x+k)h(x+k) = 1, \quad x \in [0, 1]. \quad (3..1)$$

In fact, due to the support of B_2 , only the values for $h(x)$ for $x \in [0, 2]$ play a role for that condition. We know that

for any $b \leq 1/2$ the function generates – up to a certain scalar multiple – a dual of g .
 Now consider any $1/2 < b < 1$; that is, we have $1 < 1/b < 2$.

Lemma 3..1 Assume that $h(x)$, $x \in [0, 2]$ is chosen such that (3..1) is satisfied. The following hold:

(i) If

$$\sum_{k \in \mathbb{Z}} B_2(x - 1/b + k)h(x + k) = 0, \quad x \in \mathbb{R}, \quad (3..2)$$

and

$$\sum_{k \in \mathbb{Z}} B_2(x + 1/b + k)h(x + k) = 0, \quad x \in \mathbb{R}, \quad (3..3)$$

then

$$B_2(x - 1/b)h(x) + B_2(x - 1/b + 1)h(x + 1) = 0,$$

$$x \in [1/b, 2], \quad (3..4)$$

$$B_2(x + 1/b - 1)h(x - 1) + B_2(x + 1/b)h(x) = 0$$

$$x \in [0, 2 - 1/b]. \quad (3..5)$$

These equations determine $h(x)$ for

$$x \in [-1, 1 - 1/b] \cup [1 + 1/b, 3].$$

(ii) If $h(x)$ for $x \in [-1, 1 - 1/b] \cup [1 + 1/b, 3]$ is chosen such that (3..4) and (3..5) are satisfied, and

$$h(x) = 0, \quad x \notin [0, 2] \cup [-1, 1 - 1/b] \cup [1 + 1/b, 3],$$

then (3..2) and (3..3) hold.

Proof. We consider (3..2) for $x \in [1, 2]$, and split into two cases:

For $x \in [1, 1/b]$, (3..2) yields that

$$0 = B_2(x - 1/b + 1)h(x + 1) + B_2(x - 1/b + 2)h(x + 2); \quad (3..6)$$

the equation only involve $h(x)$ for

$$x \in [2, 1 + 1/b] \cup [3, 2 + 1/b].$$

For $x \in [1/b, 2]$, (3..2) yields that

$$0 = B_2(x - 1/b)h(x) + B_2(x - 1/b + 1)h(x + 1);$$

since $h(x)$ is known, this implies that

$$h(x + 1) = \frac{-B_2(x - 1/b)h(x)}{B_2(x - 1/b + 1)}, \quad x \in [1/b, 2],$$

that is,

$$h(x) = \frac{-B_2(x - 1/b - 1)h(x - 1)}{B_2(x - 1/b)}, \quad x \in [1/b + 1, 3].$$

Similarly, considering (3..3) for

$$x \in [0, 1] = [0, 2 - 1/b] \cup [2 - 1/b, 1]$$

leads to (3..5) and

$$B_2(x + 1/b - 2)h(x - 2) + B_2(x + 1/b - 1)h(x - 1) = 0, \quad x \in [2 - 1/b, 1]; \quad (3..7)$$

the equation (3..7) only involves $h(x)$ for

$$x \in [-1/b, -1] \cup [1 - 1/b, 0],$$

and (3..5) implies that

$$h(x - 1) = \frac{-B_2(x + 1/b)h(x)}{B_2(x + 1/b - 1)}, \quad x \in [0, 2 - 1/b],$$

i.e.,

$$h(x) = \frac{-B_2(x + 1/b + 1)h(x + 1)}{B_2(x + 1/b)}, \quad x \in [-1, 1 - 1/b].$$

For the proof of (ii), the condition

$$h(x) = 0, \quad x \notin [0, 2] \cup [-1, 1 - 1/b] \cup [1 + 1/b, 3],$$

implies that (3..6) and (3..7) are satisfied. By construction, (3..2) and (3..3) are satisfied. \square

Lemma 3..1 shows that if we want that (3..1), (3..2), and (3..3) hold for some $b \in]1/2, 1]$, then h in general will take values outside $[0, 2]$. However, the proof shows that we under certain circumstances can find a solution h having support in $[0, 2]$. In that case, the support will actually be a subset of $[0, 2]$:

Corollary 3..2 Let $b \in]1/2, 1]$. Assume that $\text{supp } h \subseteq [0, 2]$ and that (3..1) and (3..2) holds. Then

$$h(x) = 0, \quad x \in [0, 2 - 1/b] \cup [1/b, 2]. \quad (3..8)$$

Proof. According to the proof of Lemma 3..1, we obtain that $h(x) = 0$ on $[1/b + 1, 3]$ by requiring that $h(x) = 0$ for $x \in [1/b, 2]$; and we obtain that $h(x) = 0$ on $[-1, 1 - 1/b]$ by requiring that $h(x) = 0$ for $x \in [0, 2 - 1/b]$. \square

If $\text{supp } h \subseteq [0, 2]$, the condition (3..8) implies that h at most can be nonzero on the interval $[2 - 1/b, 1/b]$ having length $2/b - 2$. In order for (3..1) to hold, this interval must have length at least 1; thus, we need to consider b such that $2/b - 2 \geq 1$, i.e., $b \leq 2/3$. Note that if $b \leq 2/3$, then $2/b \geq 3$: that is, because B_2 and h are supported on $[0, 2]$, Janssen's duality conditions in (1..1) are automatically satisfied for $n = \pm 2, \pm 3, \dots$

Corollary 3..3 Consider $b \in]1/2, 2/3]$. Then there exists a function h with $\text{supp } h \subseteq [0, 2]$ such that (3..1) and (3..2) hold; and $bh(x)$ is a dual generator of B_2 for these values of b .

Proof. For $x \in [0, 2 - 1/b] \cup [1/b, 2]$, let $h(x) = 0$. For $x \in [0, 1]$, the equation (3..1) means that

$$xh(x) + (1 - x)h(x + 1) = 1.$$

This implies that

$$\begin{aligned} xh(x) &= 1, & x \in [1/b - 1, 1], \\ (1 - x)h(x + 1) &= 1, & x \in [0, 2 - 1/b]; \end{aligned}$$

that is,

$$h(x) = \frac{1}{x}, \quad x \in [1/b - 1, 1], \quad (3..9)$$

and

$$h(x) = \frac{1}{2 - x}, \quad x \in [1, 3 - 1/b]. \quad (3..10)$$

Finally, for $x \in [2 - 1/b, 1/b - 1]$ and $x \in [3 - 1/b, 1/b]$, choose $h(x)$ such that

$$xh(x) + (1 - x)h(x + 1) = 1.$$

By construction, $bh(x)$ is a dual generator. \square

For $b = 3/5$ we will now explicitly construct a continuous dual generator h of B_2 with support in $[0, 2]$. Putting Corollary 3..2, (3..9), and (3..10) together, we can state a result about how a dual window supported on $[0, 2]$ must look like on parts of $[0, 2]$:

Lemma 3..4 For $b = 3/5$, every dual generator of B_2 with support in $[0, 2]$ has the form

$$h(x) = \begin{cases} 0 & \text{if } x \leq 1/3; \\ \frac{1}{x} & \text{if } x \in [2/3, 1]; \\ \frac{1}{2-x} & \text{if } x \in [1, 4/3]; \\ 0 & \text{if } x \geq 5/3. \end{cases}$$

That is, we only have freedom on the definition of h on $]1/3, 2/3[\cup]4/3, 5/3[$.

Note that on $[2/3, 4/3]$, the function h is symmetric around $x = 1$. We will now show that it is possible to define h on $]1/3, 2/3[\cup]4/3, 5/3[$ in such a way that h becomes symmetric around $x = 1$.

First, we note that this form of symmetry means that

$$h(1 - x) = h(1 + x), \quad x \in]1/3, 2/3[. \quad (3..11)$$

Put together with the duality condition, we thus require that

$$xh(x) = 1 - (1 - x)h(1 - x), \quad x \in]1/3, 2/3[. \quad (3..12)$$

The condition (3..12) shows that must define $h(1/2) = 1$. Now, taking any continuous function h defined on $]1/3, 1/2[$ with the properties that $h(1/3) = 0$ and $h(1/2) = 1$, the condition (3..12) shows how to define $h(x)$ on $]1/2, 2/3[$; and, finally, the condition (3..11) shows how to define h on $]4/3, 5/3[$ such that the resulting function is a symmetric dual generator.

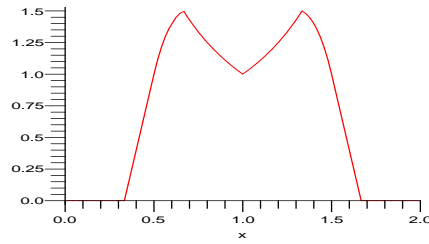


Figure 3: The function h in (3..13)..

Put

$$h(x) = 6x - 2, \quad x \in [1/3, 1/2].$$

Then, for $x \in [1/2, 2/3]$,

$$\begin{aligned} h(x) &= \frac{1 - (1 - x)h(1 - x)}{x} \\ &= \frac{-6x^2 + 10x - 3}{x}. \end{aligned}$$

The condition $h(1 + x) = h(1 - x)$, $x \in]1/3, 2/3[$ can also be expressed as $h(x) = h(2 - x)$, $x \in]4/3, 5/3[$. Thus, for $x \in [4/3, 3/2]$ we arrive at

$$h(x) = h(2 - x) = \frac{-6x^2 + 14x - 7}{2 - x}, \quad x \in [4/3, 3/2];$$

while, for $x \in [3/2, 5/3]$,

$$h(x) = h(2 - x) = 6(2 - x) - 2 = 10 - 6x.$$

We have arrived at the following conclusion:

Lemma 3..5 For $b = 3/5$, the function

$$h(x) = \begin{cases} 0 & \text{if } x \leq 1/3; \\ 6x - 2 & \text{if } x \in [1/3, 1/2]; \\ \frac{-6x^2 + 10x - 3}{x} & \text{if } x \in [1/2, 2/3]; \\ \frac{1}{x} & \text{if } x \in [2/3, 1]; \\ \frac{1}{2-x} & \text{if } x \in [1, 4/3]; \\ \frac{-6x^2 + 14x - 7}{2-x} & \text{if } x \in [4/3, 3/2]; \\ 10 - 6x & \text{if } x \in [3/2, 5/3]; \\ 0 & \text{if } x \geq 5/3 \end{cases} \quad (3..13)$$

is a continuous symmetric dual generator of B_2 .

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