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Hölder regularity for viscosity solutions of fully nonlinear, local or nonlocal, Hamilton-Jacobi equations with super-quadratic growth in the gradient *

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Abstract

Viscosity solutions of fully nonlinear, local or non local, Hamilton-Jacobi equations with a super-quadratic growth in the gradient variable are proved to be Hölder continuous, with a modulus depending only on the growth of the Hamiltonian. The proof involves some representation formula for nonlocal Hamilton-Jacobi equations in terms of controlled jump processes and a weak reverse inequality.

Key words: Integro-Differential Hamilton-Jacobi equations, viscosity solutions, Hölder continuity, degenerate parabolic equations, reverse Hölder inequalities.

MSC Subject classifications: 49L25, 35K55, 93E20, 26D15

1 Introduction

In a previous paper [9], the first author investigated the regularity of solutions to the Hamilton-Jacobi equation

$$u_t(x,t) - \mathrm{Tr}\left(a(x,t)D^2u(x,t)\right) + H(x,t,Du(x,t)) = 0 \qquad \text{in } \mathbb{R}^N \times (0,T) \qquad (1)$$

under a super-quadratic growth condition on the Hamiltonian H with respect to the gradient variable:

$$\frac{1}{\delta}|z|^q - \delta \leq H(x,t,z) \leq \delta|z|^q + \delta \qquad \forall (x,t,z) \in \mathbb{R}^N \times (0,T) \times \mathbb{R}^N ,$$

for some $\delta \geq 1$, $q \geq 2$. Under this assumption, it is proved in [9] that any continuous, bounded solution u of (1) is Hölder continuous on $\mathbb{R}^N \times [\tau, T]$ (for any $\tau \in (0,T)$), with

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Hölder exponent and constant only depending on N, p, δ, q, τ and $\|u\|_\infty$. In particular, such a modulus of continuity is independent of the regularity of a and of H with respect to the variables (x,t). The result is somewhat surprizing since no uniform ellipticity on a is required.

The aim of this paper is to extend this regularity result to solutions of fully nonlinear, local or nonlocal, Hamilton-Jacobi equations which have a super-quadratic growth with respect to the gradient variable. Beside its own interest, such a uniform estimate is important in homogenization theory where, for instance, it is used to prove the existence of correctors.

Let us consider a fully nonlinear, nonlocal Hamilton-Jacobi equation of the form

$$u_t + F(x,t,Du,[u]) = 0 \qquad \text{in } \mathbb{R}^N \times (0,T)\ . \tag{2}$$

In the above equation we assume that the mapping $F : \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \times \mathcal{C}^2_b(\mathbb{R}^N) \to \mathbb{R}$ is nonincreasing with respect to the nonlocal variable, i.e.,

$$[\phi \le \psi \text{ and } \phi(x) = \psi(x)] \ \Rightarrow\ F(x,t,\xi,[\phi]) \ge F(x,t,\xi,[\psi])$$

for any function $\phi, \psi \in \mathcal{C}^2_b(\mathbb{R}^N)$. Let us recall (see [1, 2, 4, 5, 16, 17, 18, 20] for instance) that a subsolution (resp. a supersolution) of equation (2) is a continuous map $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ such that, for any continuous, bounded test function $\phi : \mathbb{R}^N \times (0,T) \to \mathbb{R}$, which has continuous second order derivatives and such that $u - \phi$ has a global maximum (resp. global minimum) at some point $(\bar{x},\bar{t})$, one has

$$\phi_t(\bar{x},\bar{t}) + F(x,t,D\phi(\bar{x},\bar{t}),[\phi(\cdot,\bar{t})]) \le 0 \qquad (\text{resp. } \ge 0)\ .$$

Our main assumption on F is the following structure condition, which roughly says that F is super-quadratic with respect to the gradient variable:

$$-\delta M^+[\phi](x) + \frac{1}{\delta}|\xi|^q - \delta \ \le\ F(x,t,\xi,[\phi]) \ \le\ -\delta M^-[\phi](x) + \delta|\xi|^q + \delta \tag{3}$$

for any $(x,t,\xi,\phi) \in \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \times \mathcal{C}^2_b(\mathbb{R}^N)$, for some constants $q > 2$ and $\delta \ge 1$, where M^- and M^+ are defined by

$$M^-[\phi](x) = \inf_{\lambda\in(0,1],\ b\in\mathbf{B}\setminus\{0\}} \left\{ \frac{\phi(x+\lambda b) - \phi(x) - \langle D\phi(x), \lambda b\rangle}{|b|^2} \right\}$$

and

$$M^+[\phi](x) = \sup_{\lambda\in(0,1],\ b\in\mathbf{B}\setminus\{0\}} \left\{ \frac{\phi(x+\lambda b) - \phi(x) - \langle D\phi(x), \lambda b\rangle}{|b|^2} \right\}$$

and where $\mathbf{B}$ is the unit ball of $\mathbb{R}^N$. Let us note that, under the above assumption, a solution of (2) is a supersolution of

$$u_t - \delta M^-[u(\cdot,t)](x) + \delta|Du|^q + \delta = 0 \qquad \text{in } \mathbb{R}^N \times (0,T) \tag{4}$$

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and a subsolution of

$$u_t - \delta M^+[u(\cdot,t](x) + \frac{1}{\delta}|Du|^q - \delta = 0 \qquad \text{in } \mathbb{R}^N \times (0,T)\ . \tag{5}$$

We denote by p the conjugate exponent of q. We are interested in solutions which are bounded by some constant M:

$$|u(x,t)| \leq M \qquad \forall (x,t) \in \mathbb{R}^N \times [0,T]\ . \tag{6}$$

In what follows, a *(universal) constant* is a positive number depending on the given data q, δ, N and M only. Universal constants will be typically labeled with C, but also with different letters (e.g., $\theta, A, \ldots$). Dependence on extra quantities will be accounted for by using parentheses (e.g., $C(r)$ denotes a constant depending also on r). The constant C appearing in the proofs may change from line to line.

Theorem 1.1 *Let $u \in \mathcal{C}(\mathbb{R}^N \times [0,T])$ be a viscosity supersolution of* (4) *and a subsolution of (5), such that $|u| \leq M$ in $\mathbb{R}^N \times [0,T]$. For any $\tau \in (0,T)$, there are constants $\theta = \theta(\delta, M, N, q) > p$ and $C(\tau) = (\tau, \delta, M, N, q) > 0$ such that*

$$|u(x_1,t_1) - u(x_2,t_2)| \leq C(\tau)\left[|x_1 - x_2|^{(\theta-p)/(\theta-1)} + |t_1 - t_2|^{(\theta-p)/\theta}\right] \tag{7}$$

for any $(x_1,t_1),(x_2,t_2) \in \mathbb{R}^N \times [\tau,T]$.

The main point of the above result is that (7) holds true uniformly with respect to F, as long as conditions (9) and the bound $|u| \leq M$ are satisfied. In particular, θ and $C(\tau)$ are independent of the continuity modulus of F. In contrast to [7, 19], F can also be degenerate parabolic.

Let us note that the above result also applies to the solutions of the fully nonlinear, local equation

$$u_t + F(x,t,Du,D^2u) = 0 \qquad \text{in } \mathbb{R}^N \times (0,T) \tag{8}$$

provided $F : \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \times \mathcal{S}^N \to \mathbb{R}$ is nonincreasing with respect to the matrix variable and satisfies the following structure condition:

$$-\delta\Lambda(X) + \frac{1}{\delta}|\xi|^q - \delta \leq F(x,t,\xi,X) \leq -\delta\lambda(X) + \delta|\xi|^q + \delta \tag{9}$$

for any $(x,t,\xi,X) \in \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \times \mathcal{S}^N$, for some constants $q > 2$ and $\delta \geq 1$, where

$$\Lambda(X) = \max_{|z|\leq 1}\langle Xz,z\rangle \text{ and } \lambda(X) = \min_{|z|\leq 1}\langle Xz,z\rangle \qquad \forall X \in \mathcal{S}^N\ ,$$

($\mathcal{S}^N$ being the set of $N \times N$ symmetric matrices). Indeed, since

$$M^-[\phi](x) \leq \lambda(D^2\phi(x)) \qquad \text{and} \qquad M^+[\phi](x) \geq \Lambda(D^2\phi(x))\ ,$$

any solution of (8) is a supersolution of (4) and a subsolution of (5). Note that F is neither required to be concave nor convex with respect to the matrix variable.

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Here are some examples of nonlinear, nonlocal Hamilton-Jacobi equations satisfying the structure condition (3) (see [4] or [5] for instance): let us assume that

$$F(x,t,\xi,[\phi]) = \mathcal{I}[\phi](x,t) + H(x,t,\xi)$$

where H is a first order term with superquadratic growth:

$$\frac{1}{\delta}|\xi|^q - \delta \leq H(x,t,\xi) \leq \delta|\xi|^q + \delta$$

and where the nonlocal term $\mathcal{I}$ can be of the form

$$\mathcal{I}[\phi](x,t) = \inf_{\alpha\in A}\sup_{\beta\in B}\int_{\mathbb{R}^N} \phi(x+j_{\alpha,\beta}(x,t,e)) - \phi(x) - \langle D\phi(x), j_{\alpha,\beta}(x,t,e)\rangle \; d\nu(e)$$

where A, B are some sets, $j_{\alpha,\beta} : \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is such that

$$|j_{\alpha,\beta}(x,t,e)| \leq C(|e| \wedge 1) \qquad \forall (x,t,e,\alpha,\beta) \in \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \times A \times B$$

and the measure ν satisfies

$$\int_{\mathbb{R}^N} |e|^2 \wedge 1 \; d\nu(e) \leq C \; , \tag{10}$$

or of the form

$$\mathcal{I}[\phi](x,t) = \inf_{\alpha\in A}\sup_{\beta\in B}\int_{\mathbb{R}^N} \phi(x+j_{\alpha,\beta}(x,t,e)) - \phi(x) - \langle D\phi(x), j_{\alpha,\beta}(x,t,e)\rangle \mathbf{1}_{\mathbf{B}}(e) \; d\nu(e)$$

where $j_{\alpha,\beta} : \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \to \mathbb{R}^N$ is now of linear growth

$$|j_{\alpha,\beta}(x,t,e)| \leq C|e| \qquad \forall (x,t,e,\alpha,\beta) \in \mathbb{R}^N \times (0,T) \times \mathbb{R}^N \times A \times B$$

and the measure ν again satisfies the integrability condition (10). In this later case, the part

$$\int_{\mathbf{B}} \phi(x+j_{\alpha,\beta}(x,t,e)) - \phi(x) - \langle D\phi(x), j_{\alpha,\beta}(x,t,e)\rangle \mathbf{1}_{\mathbf{B}}(e) \; d\nu(e)$$

can be estimated from above and below by $M^+[\phi]$ and $M^-[\phi]$, while the part

$$\int_{\mathbb{R}^N\setminus\mathbf{B}} \phi(x+j_{\alpha,\beta}(x,t,e)) - \phi(x) \; d\nu(e)$$

is can be bounded—in equation (2)—by CM, where $M = \|u\|_\infty$ and C is universal.

Some comments on the proof of Theorm 1.1 are now in order. As in [9], the main ingredients are representation formulae for simplified Hamilton-Jacobi equations, existence of "nearly optimal trajectories", use of Brownian bridges and, finally, application of a reverse Hölder inequality. However, since we work with fully nonlinear, nonlocal equations, each step is technically more involved: the representation formulae (see Proposition 2.1 or the proof of Proposition 3.1) are inspired by a work on controlled structure equations by the second author [6]. They involve controlled jump processes in a particular form. The

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estimates of the subsolutions in Proposition 3.1, which, in [9], are obtained by controls issued from Brownian bridge techniques, have to be built here in a much more subtle way: indeed the Hamiltonian of equation (4) being non convex, the naturally associated control problem should actually be a differential game, which has never been investigated in this framework. We overcome this difficulty by building explicit feedbacks. Finally, the construction of optimal trajectories in Lemma 2.4, requires careful estimates because we are dealing with jump processes.

Notations : Throughout the paper, $\mathbf{B}$ denotes the closed unit ball of $\mathbb{R}^N$, $\mathcal{B}(\mathbf{B}\backslash\{0\})$ the set of Borel measurable subsets of $\mathbf{B}\backslash\{0\}$, $\mathcal{C}(\mathbb{R}^N\times[0,T])$ the set of continuous functions on $\mathbb{R}^N\times[0,T]$ and $\mathcal{C}^2_b(\mathbb{R}^N)$ the set of bounded continuous functions on $\mathbb{R}^N$ with continuous second order derivatives.

2 Analysis of supersolutions

Let u be as in Theorem 1.1. Throughout the proof of Theorem 1.1 it will be more convenient to work with $u(x,T-t)$ instead of $u(x,t)$. We note that $u(x,T-t)$ is a supersolution of

$$-v_t - \delta M^-[v(\cdot,t)](x) + \delta|Dv|^q + \delta = 0 \qquad \text{in } \mathbb{R}^N\times(0,T) \tag{11}$$

and a subsolution of

$$-v_t - \delta M^+[v(\cdot,t)](x) + \frac{1}{\delta}|Dv|^q - \delta = 0 \qquad \text{in } \mathbb{R}^N\times(0,T)\ . \tag{12}$$

To simplify the notation we will write $u(x,t)$ instead of $u(x,T-t)$.

In this part we are concerned with some monotonity property along particular trajectories of supersolution of equation (11). For this we have to give a representation formula for solutions of this equation in terms of controlled jump processes.

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space on which is defined a $N-$dimensional Poisson random measure μ. We assume that the Levy measure of μ, denoted by ν, is supported in $\mathbf{B}$, has no atom and satisfies the conditions

$$\int_{\mathbf{B}} |e|^2\, d\nu(e) < +\infty \qquad \text{and} \qquad \nu(\mathbf{B}) = +\infty\ . \tag{13}$$

We denote by $\tilde\mu(de,dt) = \mu(de,dt) - \nu(de)dt$ the compensated Poisson measure. For all $t\in[0,T]$, $(\mathcal{F}_{t,s}, s\in[t,T])$ will denote the filtration generated by μ on the interval $[t,T]$, i.e. $\mathcal{F}_{t,s} = \sigma\{\mu([t,r]\times A), t\le r\le s, A\in\mathcal{B}(\mathbb{R}^N)\}$ completed by all null sets of P.

Let $\mathcal{A}(t)$ be the set of $(\mathcal{F}_{t,s})$-adapted controls $(a_s) = (\lambda_s, b_s) : [t,T] \to (0,1]\times\mathbf{B}\backslash\{0\}$ and let $L^p_{\rm ad}([t,T])$ be the set of $(\mathcal{F}_{t,s})$-adapted controls $(\zeta_s) : [t,T]\to\mathbb{R}^N$ such that $\mathbb{E}\left[\int_t^T |\zeta_s|^p ds\right] < +\infty$.

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To all control $(a_s) = (\lambda_s, b_s) \in \mathcal{A}(t)$ we associate a $(\mathcal{F}_{t,s})$-martingale M^a in the following way:
First we set

$$\rho_s = \inf \left\{ r > 0 \ ; \ \nu(\mathbf{B} \backslash B(0,r)) \leq \delta \frac{1}{|b_s|^2} \right\} \ , \ s \in [t,T].$$

By the assumptions (13) on the measure ν, the process (ρ_s) is well defined. It takes its values in $[0,1]$ and is adapted to the filtration $(\mathcal{F}_{t,s})$.
Then we introduce $A_s = A_s(a) = \mathbf{B} \backslash B(0, \rho_s)$. For all $s \in [0,T]$, the set A_s belongs to $\mathcal{B}(\mathbf{B} \backslash \{0\}) \otimes \mathcal{F}_{t,s}$ and, since ν has no atoms, it satisfies

$$\nu(A_s) = \frac{\delta}{|b_s|^2} \qquad \text{for almost all } s \in [t,T], \ \mathbb{P}-\text{a.s.} \tag{14}$$

We finally denote by M^a the controlled martingale

$$M^a_s = \int_t^s \int_{\mathbf{B}} \lambda_r b_r \mathbf{1}_{A_r}(e) \tilde{\mu}(de, dr) \ . \tag{15}$$

Let us precise Itô's formula satisfied by M^a: for any smooth function $\phi \in \mathcal{C}^2_b$,

$$\begin{aligned}
&\mathbb{E}[\phi(M^a_s)] \\
&\quad = \phi(0) + \mathbb{E}\left[\int_t^s \int_{\mathbf{B}} (\phi(M^a_{r^-} + \lambda_r b_r \mathbf{1}_{A_r}(e)) - \phi(M^a_{r^-}) - \langle D\phi(M^a_{r^-}), \lambda_r b_r \mathbf{1}_{A_r}(e) \rangle) \, d\nu(e) dr \right] \\
&\quad = \phi(0) + \delta \mathbb{E}\left[\int_t^s (\phi(M^a_{r^-} + \lambda_r b_r) - \phi(M^a_{r^-}) - \langle D\phi(M^a_{r^-}), \lambda_r b_r \rangle) \frac{dr}{|b_r|^2} \right]
\end{aligned}$$

Now we consider the controlled system

$$\left\{ \begin{array}{l} dY_s = \zeta_s ds + dM^a_s \ , \ s \in [t,T], \\ Y_t = x \ , \end{array} \right. \tag{16}$$

where $\zeta \in L^p_{\text{ad}}([t,T])$ and $a \in \mathcal{A}(t)$. The system (16) is related to equation (11) by the following proposition:

Proposition 2.1 *Let $v : \mathbb{R}^N \times [0,T]$ be a continuous viscosity solution to (11). Then*

$$v(x,t) = \inf_{(\zeta,a) \in L^p_{\text{ad}}([t,T]) \times \mathcal{A}(t)} \mathbb{E}\left[v(Y^{x,t,\zeta,a}_T, T) + C_+ \int_t^T |\zeta_s|^p ds - \delta(T-t) \right]$$

where $(Y^{x,t,\zeta,a}_s)$ is the solution to (16) and $C_+ > 0$ is the universal constant given by

$$C_+ = \frac{\delta^{-p/q}}{p q^{p/q}} \ . \tag{17}$$

Proof : The proof relies on two arguments: first the map

$$w(x,t) = \inf_{(\zeta,a) \in L^p_{\text{ad}}([t,T]) \times \mathcal{A}(t)} \mathbb{E}\left[v(Y^{x,t,\zeta,a}_T, T) + C_+ \int_t^T |\zeta_s|^p ds - \delta(T-t) \right]$$

is a viscosity solution of (11). This result has been proved—in a slightly different framework—in [6], and we omit the proof, which is very close to that of [6]. Second, in order to conclude that $w = v$, we need a uniqueness argument for the solution of (11) with terminal condition $v(\cdot, T)$. This is a direct consequence of the following comparison principle. □

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Lemma 2.2 (Comparison) *Let u be a continuous, bounded subsolution of (11) on $\mathbb{R}^N\times[0,T]$ and v be a continuous, bounded supersolution of (11). If $u(\cdot,T)\leq v(\cdot,T)$, then $u\leq v$ on $\mathbb{R}^N\times[0,T]$.*

Proof : We use ideas of [5] for the treatment of the nonlocal term and of [11] for the treatment of the super-linear growth with respect to the gradient variable. Let $M=\max\{\|u\|_\infty,\|v\|_\infty\}$. Our aim is to show that, for any $\mu\in(0,1)$,

$$\tilde u(x,t):=\mu u(x,t)-(1-\mu)M-(1-\mu)\delta(T-t)\ \leq\ v(x,t) \text{ in } \mathbb{R}^N\times[0,T]\,. \tag{18}$$

Note that this inequality holds at $t=T$. One also easily checks that $\tilde u$ is a subsolution of equation

$$-w_t-\delta M^-[w(\cdot,t)](x)+\delta\mu^{1-q}|Dw|^q+\delta=0 \qquad \text{in } \mathbb{R}^N\times(0,T)\,. \tag{19}$$

For any $\epsilon>0$, let $\tilde u^\epsilon$ be the space-time sup-convolution of $\tilde u$ and v_ϵ be the space-time inf-convolution of v (see [10]):

$$\tilde u^\epsilon(x,t)=\sup_{(y,s)\in\mathbb{R}^N\times[0,T]}\left\{\tilde u(y,s)-\frac{1}{\epsilon}|(x,t)-(y,s)|^2\right\}$$

and

$$v_\epsilon(x,t)=\inf_{(y,s)\in\mathbb{R}^N\times[0,T]}\left\{v(y,s)+\frac{1}{\epsilon}|(x,t)-(y,s)|^2\right\}\,.$$

It is known that $\tilde u^\epsilon$ is still a subsolution of (19) in $\mathbb{R}^N\times((2M\epsilon)^{\frac12},T-(2M\epsilon)^{\frac12})$ while v_ϵ a supersolution of (11) and that u^ϵ is semiconvex while v^ϵ is semiconcave $\mathbb{R}^N\times((2M\epsilon)^{\frac12},T-(2M\epsilon)^{\frac12})$. In particular, u^ϵ and v_ϵ have almost everywhere a second order expansion and at such a point $(x,t)\in\mathbb{R}^N\times((2M\epsilon)^{\frac12},T-(2M\epsilon)^{\frac12})$ one has

$$-\tilde u^\epsilon_t(x,t)-\delta M^-[\tilde u^\epsilon(\cdot,t)](x)+\delta\mu^{1-q}|D\tilde u^\epsilon(x,t)|^q+\delta\leq 0$$

and

$$-v^\epsilon_t(x,t)-\delta M^-[v^\epsilon(\cdot,t)](x)+\delta|Dv^\epsilon(x,t)|^q+\delta\geq 0\,.$$

As usual we prove (18) by contradiction and assume that $\sup_{x,t}\tilde u(x,t)-v(x,t)>0$. Then, for any $\alpha>0$, $\sigma>0$ sufficiently small, one can choose $\epsilon>0$ sufficiently small such that the map $(x,t)\to\tilde u^\epsilon(x,t)-v_\epsilon(x,t)-\alpha|x|^2+\sigma t$ reaches its maximum on $\mathbb{R}^N\times[0,T]$ at some point $(\bar x,\bar t)\in\mathbb{R}^N\times((2M\epsilon)^{\frac12},T-(2M\epsilon)^{\frac12})$. For any $\eta>0$, the point $(\bar x,\bar t)$ is a strict maximum of the map $(x,t)\to\tilde u^\epsilon(x,t)-v_\epsilon(x,t)-\alpha|x|^2+\sigma t-\eta(|x-\bar x|^2+(t-\bar t)^2)$. Jensen's Lemma (see [10]) then states that one can find $p^n=(p^n_x,p^n_t)\in\mathbb{R}^{N+1}$ such that $p^n\to 0$ and the map

$$(x,t)\to\tilde u^\epsilon(x,t)-v_\epsilon(x,t)-\alpha|x|^2+\sigma t-\eta(|x-\bar x|^2+(t-\bar t)^2)-\langle p^n_x,x\rangle-p^n_t t$$

has a maximum at some point (x_n,t_n) where $\tilde u^\epsilon$ and v_ϵ have a second order expansion. Note that $(x_n,t_n)\to(\bar x,\bar t)$. At the point (x_n,t_n) we have

$$\tilde u^\epsilon_t(x_n,t_n)=v^\epsilon_t(x_n,t_n)-\sigma+2\eta(t_n-\bar t)+p^n_t\ ,$$

$$D\tilde{u}^\epsilon(x_n,t_n) = Dv^\epsilon(x_n,t_n) + 2\alpha x_n + 2\eta(x_n-\bar{x}) + p_x^n \ ,$$

$$-\tilde{u}^\epsilon_t(x_n,t_n) - \delta M^-[\tilde{u}^\epsilon(\cdot,t_n)](x_n) + \delta\mu^{1-q}|D\tilde{u}^\epsilon(x_n,t_n)|^q + \delta \le 0 \ , \qquad (20)$$

and

$$-v^\epsilon_t(x_n,t_n) - \delta M^-[v^\epsilon(\cdot,t_n)](x_n) + \delta|Dv^\epsilon(x_n,t_n)|^q + \delta \ge 0 \ . \qquad (21)$$

For the optimality conditions, we have, for any $x \in \mathbb{R}^N$,

$$\tilde{u}^\epsilon(x,t_n) \le v_\epsilon(x,t_n) + \tilde{u}^\epsilon(x_n,t_n) - v_\epsilon(x_n,t_n) + \alpha(|x|^2-|x_n|^2) + \eta(|x-\bar{x}|^2-|x_n-\bar{x}|^2) - \langle p_x^n, x-x_n\rangle$$

so that

$$M^-[\tilde{u}^\epsilon(\cdot,t_n)](x_n) \le M^-[v^\epsilon(\cdot,t_n)](x_n) + 2\alpha + 2\eta \ .$$

Let us set $\xi_n = Dv^\epsilon(x_n,t_n)$ and estimate the difference between (20) and (21): we get

$$\sigma - 2\eta(t_n-\bar{t}) - p_t^n - 2\alpha - 2\eta + \delta\left(\mu^{1-q}|\xi_n + 2\alpha x_n + 2\eta(x_n-\bar{x}) + p_x^n|^q - |\xi_n|^q\right) \le 0 \ .$$

When $n \to +\infty$, ξ_n remains bounded since v^ϵ is semi-concave. So we can assume that $\xi_n \to \xi_{\alpha,\epsilon}$ with

$$\sigma - 2\alpha - 2\eta + \delta\left(\mu^{1-q}|\xi_{\alpha,\epsilon} + 2\alpha\bar{x}|^q - |\xi_{\alpha,\epsilon}|^q\right) \le 0 \ .$$

Since, $\tilde{u}$ and v are bounded, so is $\alpha|\bar{x}|^2$. So $\alpha\bar{x}$ is bounded (in fact $\alpha\bar{x} \to 0$ as $\alpha \to 0$). Then the above inequality implies that $\xi_{\alpha,\epsilon}$ is bounded, because since $\mu < 1$ and $q > 1$. So, letting $\eta, \epsilon \to 0$ and then $\alpha \to 0$, we get that $\sigma \le 0$, which contradicts our assumption on σ. □

Lemma 2.3 *Let $u \in \mathcal{C}(\mathbb{R}^N \times [0,T])$ be a supersolution of (11) satisfying $|u| \le M$ in $\mathbb{R}^N \times (0,T)$. Then, for all $(\bar{x},\bar{t}) \in \mathbb{R}^N \times [0,T]$, $n \in \mathbb{N}$, $R > 0$ large and $\sigma > 0$ small, there exist a $(\mathcal{F}_{\bar{t},s})$-adapted càdlàg process Y^n and a control $\zeta^n \in L^p_{\mathrm{ad}}([\bar{t},T])$ such that*

$$u(\bar{x},\bar{t}) \ge \mathbb{E}\left[u(Y^n_t,t) + C_+\int_{\bar{t}}^t |\zeta^n_s|^p\,ds - (\delta+\tau)(t-\bar{t}) - c_n(\sigma,R)\right] \qquad \forall t \in [\bar{t},T] \ , \quad (22)$$

where

$$c_n(\sigma,R) = C\left(R^{-p} + \frac{\tau^{p-1}}{\sigma^p} + \omega(\sigma) + (\delta+\tau)\tau\right) \ ,$$

with ω the modulus of continuity of u in $B_R(\bar{x}) \times [0,T]$, and C an universal constant.

Proof: For any $(y,t) \in \mathbb{R}^N \times [0,T)$, $\zeta \in L^p_{\mathrm{ad}}([t,T])$, $a \in \mathcal{A}(t)$, let us denote by $Y^{x,t,\zeta,a}$ the solution to

$$\begin{cases} dY_s = \zeta_s ds + dM^a_s, \ t \le s \le T, \\ Y_t = x, \end{cases}$$

where the martingale M^a is defined by (15).
Let us now fix an initial condition $(\bar{x},\bar{t}) \in \mathbb{R}^N \times [0,T]$. For a large $n \in \mathbb{N}$, we set

$$\tau = (T-\bar{t})/n \qquad \text{and} \qquad t_k = \bar{t} + k\tau \qquad \text{for } k \in \{0,\dots,n\} \ .$$

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We will build some controls $\zeta^n \in L^p_{\rm ad}([\bar t,T])$ and $a^n \in \mathcal{A}(\bar t)$ such that the process $Y^n = Y^{\bar x,\bar t,\zeta^n,a^n}$ satisfies the relation

$$u(\bar x,\bar t) \geq \mathbb{E}\left[u(Y^n_{t_k},t_k) + C_+\int_{\bar t}^{t_k} |\zeta^n_s|^p\,ds - (\delta+\tau)(t_k-\bar t)\right] \qquad \forall k\in\{1,\dots,n\}\ , \quad (23)$$

and then deduce from (23) that (Y^n,ζ^n) also satisfy (22). We follow closely the construction in [9].
For any $k\in\{1,\dots,n\}$, let v^k be the solution of (11), defined on the time interval $[0,t_k]$, with terminal condition $u(\cdot,t_k)$. From the representation formula given in Proposition 2.1 we have, for all $x\in\mathbb{R}^N$,

$$v^k(x,t_{k-1}) = \inf_{(\zeta,a)\in L^p_{\rm ad}([t_{k-1},t_k])\times\mathcal{A}(t_{k-1})} \mathbb{E}\left[\ u(Y^{x,t_{k-1},\zeta,a}_{t_k},t_k) + C_+\int_{t_{k-1}}^{t_k}|\zeta_s|^p ds - \delta\tau\right]\ .$$

Since the filtration $(\mathcal{F}_{t_{k-1},s})$ is generated by a random Poisson measure, the set $L^p_{\rm ad}([t_{k-1},t_k])\times\mathcal{A}(t_{k-1})$ is a complete separable space. Moreover $u(\cdot,t_k)$ is continuous. Therefore, thanks to the measurable selection theorem (see [3]), one can build Borel measurable maps $x\to Z^{x,k}$ and $x\to A^{x,k}$ from $\mathbb{R}^N$ to $L^p_{\rm ad}([t_{k-1},t_k])$ and $\mathcal{A}(t_{k-1})$ respectively, such that

$$v^k(x,t_{k-1}) \geq \mathbb{E}\left[u(Y^{x,t_{k-1},Z^{x,k},A^{x,k}}_{t_k},t_k) + C_+\int_{t_{k-1}}^{t_k}|Z^{x,k}_s|^p ds - (\delta+\tau)\tau\right] \qquad \forall x\in\mathbb{R}^N\ . \quad (24)$$

We now construct ζ^n, a^n and Y^n by induction on the time intervals $[t_{k-1},t_k)$:
On $[\bar t,t_1)$ we set $\zeta^n_t = Z^{\bar x,1}_t$, $a^n = A^{\bar t,1}$ and $Y^n = Y^{\bar x,\bar t,\zeta^n,a^n}$. Assume that ζ^n, Y^n and a^n have been built on $[\bar t,t_{k-1})$. Then we set

$$\zeta^n = Z^{Y^n_{t_{k-1}},k},\ a^n = A^{Y^n_{t_{k-1}},k}, \qquad \text{and} \qquad Y^n = Y^{\bar x,\bar t,\zeta^n,a^n} \qquad \text{on } [t_{k-1},t_k)\ .$$

(The process $Y^{\bar x,\bar t,\zeta^n,a^n}$ is P-a.s. continuous on each fixed t, so we have $Y^n_{t_{k-1}-} = Y^n_{t_{k-1}}$ P-a.s., which means that ζ^n and A^n are defined P-as surely.)
We remark that, on $[t_{k-1},t_k)$, we have $Y^n = Y^{Y^n_{t_{k-1}},t_{k-1},\zeta^n,a^n}$.
Let us fix now some $k\in\{1,\dots,n\}$. Since the processes $A^{x,k}$ and $Z^{x,k}$ are $(\mathcal{F}_{t_{k-1},s})$-adapted and therefore independent of $\mathcal{F}_{\bar t,t_{k-1}}$, the same holds also for $M^{A^{x,k}}_{t_k}$ and finally for $Y^{x,t_{k-1},Z^{x,k},A^{x,k}}_{t_k}$, while $Y^n_{t_{k-1}}$ is $\mathcal{F}_{\bar t,t_{k-1}}$-measurable. It follows that

$$\mathbb{E}\left[u(Y^{x,t_{k-1},Z^{x,k},A^{x,k}}_{t_k},t_k) + C_+\int_{t_{k-1}}^{t_k}|Z^{x,k}_s|^p ds\right]_{x=Y^n_{t_{k-1}}} = \mathbb{E}\left[u(Y^n_{t_k},t_k) + C_+\int_{t_{k-1}}^{t_k}|\zeta^n_s|^p ds\ \Big|\ \mathcal{F}_{\bar t,t_{k-1}}\right].$$

Using (24), the fact that u is a supersolution of (11) and the comparison Lemma 2.2, this leads to the relation

$$u(Y^n_{t_{k-1}},t_{k-1}) \geq \mathbb{E}\left[u(Y^n_{t_k},t_k) + C_+\int_{t_{k-1}}^{t_k}|\zeta^n_s|^p\,ds - (\delta+\tau)\tau\ \Big|\ \mathcal{F}_{\bar t,t_{k-1}}\right] \qquad \mathbb{P}-\text{a.s.}\ .$$

Taking the expectation on both sides of the above inequality and summing up gives (23).

We now extend this inequality to the full interval $[\bar{t}, T]$ and prove (22). Let $t \in [\bar{t}, T]$ and k be such that $t \in [t_{k-1}, t_k)$. From (23), we have

$$u(\bar{x}, \bar{t}) \geq \mathbb{E}\left[u(Y_t^n, t) + C_+ \int_{\bar{t}}^t |\zeta_s^n|^p \, ds - (\delta + \tau)(t - \bar{t})\right] \\ + \mathbb{E}\left[u(Y_{t_{k-1}}^n) - u(Y_t^n, t)\right] - (\delta + \tau)\tau \tag{25}$$

Let us fix $R > 0$ and $\sigma > 0$. Since u is bounded by M and from the definition of the modulus ω, we have

$$\mathbb{E}\left[u(Y_{t_{k-1}}^n) - u(Y_t^n, t)\right] \leq \omega(\sigma)\mathbb{P}\left[|Y_{t_k}^n - \bar{x}| \leq R, \ |Y_t^n - \bar{x}| \leq R, \ |Y_{t_k}^n - Y_t^n| \leq \sigma\right] \\ + 2M(\mathbb{P}\left[|Y_{t_k}^n - \bar{x}| > R\right] + \mathbb{P}\left[|Y_t^n - \bar{x}| > R\right] + \mathbb{P}\left[|Y_{t_k}^n - Y_t^n| > \sigma\right]) \tag{26}$$

To estimate the right hand side term of (26), we first note that, for any $0 < s < t$, it holds that

$$\mathbb{E}\left[|Y_t^n - Y_s^n|^p\right] \leq 2^{p-1}\left\{\mathbb{E}\left[\left|\int_s^t \zeta_\tau^n d\tau\right|^p\right] + \mathbb{E}\left[|M_t^{a_n} - M_s^{a_n}|^p\right]\right\}$$

But, thanks to (23) again, we have

$$\mathbb{E}\left[\int_{\bar{t}}^T |\zeta_\tau^n|^p \, d\tau\right] \ \leq \ 2M + (\delta + \tau)T \ \leq \ C \tag{27}$$

so that, by Hölder's inequality,

$$\mathbb{E}\left[\left|\int_s^t \zeta_\tau^n d\tau\right|^p\right] \leq C(t - s)^{p-1} \, .$$

Also by Hölder we have $\mathbb{E}\left[|M_t^{a_n} - M_s^{a_n}|^p\right] \leq \left(\mathbb{E}\left[|M_t^{a_n} - M_s^{a_n}|^2\right]\right)^{p/2}$ where, by Itô,

$$\mathbb{E}\left[|M_t^{a_n} - M_s^{a_n}|^2\right] \ = \ \mathbb{E}\left[\int_s^t \int_{\mathbf{B}} \lambda_s^2 |b_s|^2 \mathbf{1}_{A_s}(e) d\nu(e) ds\right] \\ = \delta\mathbb{E}\left[\int_s^t \lambda_s^2 ds\right] \ \leq \ \delta(t - s) \tag{28}$$

To summarize

$$\mathbb{E}\left[|Y_t^n - Y_s^n|^p\right] \ \leq \ C((t - s)^{p-1} + (t - s)^{p/2}) \ \leq \ C(t - s)^{p-1}$$

since $p < 2$. Therefore we get

$$\mathbb{P}[|Y_{t_k}^n - \bar{x}| > R] + \mathbb{P}[|Y_t^n - \bar{x}| > R] + \mathbb{P}[|Y_t^n - Y_{t_k}^n| > \sigma] \leq C(R^{-p} + \frac{|t - t_k|^{p-1}}{\sigma^p})$$

which, coming back to (25) and (26), proves claim (22). □

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Lemma 2.4 *Let $u \in \mathcal{C}(\mathbb{R}^N \times [0,T])$ be a supersolution of (11) satisfying $|u| \le M$ in $\mathbb{R}^N \times (0,T)$. Then, for any $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0,T)$ there is a stochastic basis $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$, a filtration $(\bar{\mathcal{F}}_t)_{t\ge\bar{t}}$, a càdlàg process $(\bar{Y}_t)$ adapted to $(\bar{\mathcal{F}}_t)_{t\ge\bar{t}}$ and a process $\bar{\zeta} \in L^p_{\rm ad}([\bar{t},T])$ such that*

$$u(\bar{x},\bar{t}) \ge \mathbb{E}\left[u(\bar{Y}_t,t) + C_+ \int_{\bar{t}}^t |\bar{\zeta}_s|^p ds\right] - \delta(t-\bar{t}) \qquad \forall t \in (\bar{t},T)\ , \tag{29}$$

where $C_+ > 0$ is the universal constant given by (17), and

$$\mathbb{E}\left[\left|\bar{Y}_t - \bar{x} - \int_{\bar{t}}^t \bar{\zeta}_s ds\right|^r\right] \le \delta^{\frac{r}{2}} |t-\bar{t}|^{r/2} \qquad \forall t \in [\bar{t},T) \tag{30}$$

for any $r \in (0,2]$.

Proof: This Lemma will follow from Lemma 2.3 by passing to the limit as $n \to +\infty$ in (22). For this we set

$$\Lambda^n_t \doteq \int_{\bar{t}}^t \zeta^n_s ds \qquad \forall t \in [\bar{t},T]\ .$$

From (27), the sequence of probability measures $(\mathbb{P}_{\Lambda^n})$ on $\mathcal{C}([\bar{t},T],\mathbb{R}^N)$ is tight. Let $\mathbf{D}(\bar{t})$ be the set of càdlàg functions from $[\bar{t},T]$ to $\mathbb{R}^N$, endowed with the Meyer-Zheng topology (see [15]). Since $\mathbb{E}[|M^{a_n}_T|]$ is uniformly bounded (thanks to (28)), Theorem 4 of [15] states that the sequence of martingale measures $(\mathbb{P}_{M^{a_n}})$ is tight on $\mathbf{D}(\bar{t})$. Then, from Prohorov's Theorem (Theorem 4.7 of [14]), we can find a subsequence of (Y^n,Λ^n), again labeled (Y^n,Λ^n), and a measure m on $\mathcal{C}([\bar{t},T],\mathbb{R}^N) \times \mathbf{D}(\bar{t})$ such that $(\mathbb{P}_{(Y^n,\ \Lambda^n)})$ weakly converges to m. Skorokhod's embedding Theorem (Theorem 2.4 of [12]) implies that we can find random variables $(\bar{Y}^n,\bar{\Lambda}^n)$ and $(\bar{Y},\bar{\Lambda})$ defined on a new probability space $(\bar{\Omega},\bar{\mathcal{A}},\bar{\mathbf{P}})$, such that $(\bar{Y}^n,\bar{\Lambda}^n)$ has the same law as (Y^n,Λ^n) for any n, the law of $(\bar{Y},\bar{\Lambda})$ is m and, $\bar{\mathbb{P}}-$almost surely, the sequence $(\bar{\Lambda}^n)$ converges to $(\bar{\Lambda})$ in $\mathcal{C}([\bar{t},T],\mathbb{R}^N)$ while, for any t belonging to some set $I \subset [\bar{t},T]$ of full measure in $[\bar{t},T]$, the sequence $(\bar{Y}^n_t)$ converges to $\bar{Y}_t$ (Theorem 5 of [15]).

Since $t \to \Lambda^n_t$ is absolutely continuous $\mathbb{P}-$a.s. and since $\bar{\Lambda}^n$ has the same law as Λ^n, $t \to \bar{\Lambda}^n_t$ is absolutely continuous $\bar{\mathbb{P}}-$a.s.. Let us set $\bar{\zeta}^n_s = \frac{d}{ds}\bar{\Lambda}^n_s$. Then, by (27), $\mathbb{E}[\int_{\bar{t}}^T |\bar{\zeta}^n_s|^p\, ds] \le C$ for all $n \ge 0$. Therefore, up to a subsequence again labeled in the same way, $(\bar{\zeta}^n)$ converges weakly in $L^p([\bar{t},T])$ to some limit, $\bar{\zeta}$, which, $\bar{\mathbb{P}}-$a.s., satisfies $\bar{\Lambda}_t = \int_{\bar{t}}^t \bar{\zeta}_s ds$ for all $t \in [\bar{t},T]$.

Note that $\bar{M}^n_t \doteq \bar{Y}^n_t - \bar{x} - \bar{\Lambda}^n_t$ has the same law as $M^{a_n}_t$, so that by Hölder and (28),

$$\bar{\mathbb{E}}\left[\left|\bar{M}^n_t\right|^r\right] \le \delta^{\frac{r}{2}} (t-\bar{t})^{r/2}$$

for all $r \in (0,2]$ and for all $t \in [\bar{t},T]$. Passing to the limit in the above inequality gives

$$\bar{\mathbb{E}}\left[\left|\bar{Y}_t - \bar{x} - \bar{\Lambda}_t\right|^r\right] \le \delta^{\frac{r}{2}} (t-\bar{t})^{r/2} \qquad \forall t \in I,\ \forall r \in (0,2]\ .$$

We get the above inequality for all $t \in [\bar{t},T)$ thanks to the càdàg property of the trajectories of Y. Recalling (22), a classical lower semicontinuity argument yields

$$u(\bar{x},\bar{t}) \ge \bar{\mathbb{E}}\left[u(\bar{Y}_t,t) + C_+ \int_{\bar{t}}^t \left|\bar{\zeta}_s\right|^p ds - \delta(t-\bar{t})\right] \qquad \forall t \in I\ ,$$

and we conclude the proof by using again the càdàg property of the trajectories of Y. □

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3 Analysis of subsolutions

In this section we investigate properties of subsolution of equation (12).

Proposition 3.1 *For any fixed $(x,t)\in\mathbb{R}^N\times(0,T]$, there is a continuous supersolution w of (12) in $\mathbb{R}^N\times[0,t)$ such that*

$$\frac{1}{C}(t-s)^{1-p}|x-y|^p - C(t-s)^{1-p/2} \leq w(y,s) \leq C(t-s)^{1-p}|x-y|^p + C(t-s)^{1-p/2} \quad (31)$$

for any $(y,s)\in\mathbb{R}^N\times[0,t)$ and for some universal constant C.

Proof : It relies on control interpretation of equation (12) as well as the construction of some Brownian bridges (see [13]). Let us assume, without loss of generality, that $x=0$. Having fixed $\alpha\in(1-1/p,1/2)$, $(y,s)\in\mathbb{R}^N\times[0,t)$ and $a\in\mathcal{A}(s)$, let $Y_\tau^{y,s,a}$ be the solution to

$$\begin{cases} dY_\tau = -\alpha\,\frac{Y_\tau}{t-\tau}\,d\tau + dM_\tau^a \\ Y_s = y \end{cases}$$

Then one easily checks that

$$Y_\tau = (t-s)^{-\alpha}(t-\tau)^\alpha y + (t-\tau)^\alpha \int_s^\tau (t-\sigma)^{-\alpha} dM_\sigma^a \,. \quad (32)$$

Let us set

$$Z_\tau^{y,s,a} \doteq -\,\alpha\, Y_\tau/(t-\tau) \qquad \text{and} \qquad J(y,s,a) = \mathbb{E}\left[\int_s^t |Z_\tau^{y,s,a}|^p\,d\tau\right]\,.$$

We claim that there is a universal constant $C>0$ such that

$$\frac{1}{C}(t-s)^{1-p}|x-y|^p - C(t-s)^{1-p/2} \leq J(y,s,a) \leq C(t-s)^{1-p}|x-y|^p + C(t-s)^{1-p/2}\,. \quad (33)$$

Indeed

$$\begin{aligned} J(y,s,a) &= \mathbb{E}\left[\int_s^t |Z_\tau^{y,s,a}|^p d\tau\right] \\ &\leq 2^{p-1}\alpha^p(t-s)^{-\alpha p}|y|^p \int_s^t (t-\tau)^{p(\alpha-1)}d\tau \\ &\qquad +2^{p-1}\alpha^p \int_s^t (t-\tau)^{p(\alpha-1)}\mathbb{E}\left[\left|\int_s^\tau (t-\sigma)^{-\alpha}dM_\sigma^a\right|^p\right] d\tau \end{aligned}$$

where, by Hölder and Itô,

$$\mathbb{E}\left[\left|\int_s^\tau (t-\sigma)^{-\alpha}dM_\sigma^a\right|^p\right] \leq C(t-s)^{\frac{p}{2}(1-2\alpha)}$$

So

$$J(y,s,a) \leq C(t-s)^{1-p}|y|^p + C(t-s)^{1-p/2}\,.$$

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In the same way,

$$\begin{aligned} J(y,s,a) &= \mathbb{E}\left[\int_s^t |Z_\tau^{y,s,a}|^p d\tau\right] \\ &\geq 2^{1-p}\alpha^p(t-s)^{-\alpha p}|y|^p\int_s^t (t-\tau)^{p(\alpha-1)}d\tau \\ &\qquad -\alpha^p\int_s^t (t-\tau)^{p(\alpha-1)}\mathbb{E}\left[\Big|\int_s^\tau (t-\sigma)^{-\alpha}dM_\sigma^a\Big|^p\right]d\tau \\ &\geq (1/C)(t-s)^{1-p}|y|^p - C(t-s)^{1-p/2}\,, \end{aligned}$$

Whence (33).

Next we introduce the value function w of the optimal control problem

$$w(y,s) = C_{\_}\sup_{a\in\mathcal{A}(s)} J(y,s,a) - \delta(t-s)\ ,$$

where $C_{\_} \doteq \frac{\delta^{p/q}}{pq^{p/q}}$. Let us first show that w is continuous on $\mathbb{R}^N\times[0,t)$. The map $y\to J(y,s,a)$ being convex (since the map $y\to Y_\tau^{y,s,a}$ is affine and $p>1$) and locally uniformly bounded (thanks to (33)), it is has a modulus of continuity which is locally uniform with respect to s and a. The map $s\to J(y,s,a)$ being locally Hölder continuous on $[0,t)$, locally uniformly with respect to y and a, this implies that the map $(y,s)\to J(y,s,a)$ has a modulus of continuity which is uniform with respect to a. Therefore w is continuous on $\mathbb{R}^N\times[0,t)$.

Using the fact that w is continuous and arguments similar to the ones in [6] one can prove that w satisfies the Hamilton-Jacobi equation

$$\begin{aligned} -w_t + \inf_{\lambda\in(0,1],\ b\in\mathbf{B}\backslash\{0\}}\Bigg\{ &-\langle -\alpha\frac{y}{t-s}, Dw\rangle + C_{\_}\left|-\alpha\frac{y}{t-s}\right|^p \\ &-\delta\ \frac{w(y+\lambda b,s)-w(y,s)-\langle Dw(y,s),\lambda b\rangle}{|b|^2}\Bigg\} - \delta = 0 \end{aligned}$$

Since

$$\inf_{\lambda\in(0,1],\ b\in\mathbf{B}\backslash\{0\}}\left\{-\ \frac{w(y+\lambda b,s)-w(y,s)-\langle Dw(y,s),\lambda b\rangle}{|b|^2}\right\} = -M^+[w(\cdot,s)](y)$$

while

$$-\langle -\alpha\frac{y}{t-s}, Dw\rangle + C_{\_}\left|-\alpha\frac{y}{t-s}\right|^p \geq \frac{1}{\delta}|Dw|^q\ ,$$

w is a supersolution of (12). We finally note that w satisfies (31) because the inequalities (33) are uniform with respect to a. □

Lemma 3.2 *Let $u\in\mathcal{C}(\mathbb{R}^N\times[0,T])$ be a subsolution of (12) satisfying $|u|\leq M$. Then, for all $(x,t)\in\mathbb{R}^N\times(0,T)$ and all $(y,s)\in\mathbb{R}^N\times[0,t)$,*

$$u(y,s)\leq u(x,t) + C\left\{|y-x|^p(t-s)^{1-p} + (t-s)^{1-p/2}\right\} \qquad (34)$$

for some universal constant $C>0$.

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Remark 3.3 In particular, if $u = u(x)$ is a subsolution of the stationary equation

$$-\delta M^+[u(\cdot,t](x) + \frac{1}{\delta}|Du|^q - \delta = 0 \qquad \text{in} \quad \mathbb{R}^N,$$

then inequality (34) implies that, for any $x, y \in \mathbb{R}^N$ and any $\tau > 0$,

$$u(x) \leq u(y) + C\left\{|y-x|^p\tau^{1-p} + \tau^{1-p/2}\right\},$$

for some universal constant C. Thus, choosing $\tau = |x-y|^2$ yields $u(x) \leq u(y) + C\,|y-x|^{2-p}$, that is, u is Hölder continuous. This extends to nonlocal equations one of the results of [8].

Proof: According to Proposition 3.1 there is a supersolution w of (12) which satisfies

$$\frac{1}{C}(t-s)^{1-p}|x-y|^p - C(t-s)^{1-p/2} \leq w(y,s) \leq C(t-s)^{1-p}|x-y|^p + C(t-s)^{1-p/2}$$

for any $(y,s) \in \mathbb{R}^N \times [0,t)$ and for some universal constant C. Since u is continuous and bounded and $u(y,t) \leq \lim_{s\to t} w(y,s) + u(x,t)$ for any $y \in \mathbb{R}^N$, we get $u \leq w + u(x,t)$ on $\mathbb{R}^N \times [0,t)$ by comparison (Lemma 2.2). Whence the result. □

Lemma 3.4 *Let $u \in \mathcal{C}(\mathbb{R}^N \times [0,T])$ be a subsolution of (12) satisfying $|u| \leq M$. Fix $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0,T)$, $\zeta \in L^p_{\rm ad}([\bar{t},T])$ and let (X_t,ζ_t) be stochastic processes satisfying (30). Then, for any $x \in \mathbb{R}^N$ and $t \in (\bar{t},T)$,*

$$\begin{aligned} &u(x,\bar{t}) - \mathbb{E}[u(X_t,t)] \\ &\quad \leq C\left\{(t-\bar{t})^{1-p}\left(\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s|ds\right)^p\right] + |\bar{x}-x|^p\right) + (t-\bar{t})^{1-p/2}\right\} \end{aligned} \tag{35}$$

for some constant $C > 0$.

Proof: Fix $t \in (\bar{t},T)$ and apply Lemma 3.2 to $(x,\bar{t})$ and $(X_t(\omega),t)$. Then, for almost all $\omega \in \Omega$,

$$u(x,\bar{t}) \;\leq\; u(X_t(\omega),t) + C\left\{|X_t(\omega)-x|^p(t-\bar{t})^{1-p} + (t-\bar{t})^{1-p/2}\right\}.$$

Hence,

$$u(x,\bar{t}) \;\leq\; \mathbb{E}\left[u(X_t,t)\right] + C\left\{(\mathbb{E}\left[|X_t-\bar{x}|^p\right] + |\bar{x}-x|^p)(t-\bar{t})^{1-p} + (t-\bar{t})^{1-p/2}\right\}.$$

Since, on account of (30),

$$\mathbb{E}\left[|X_t-\bar{x}|^p\right] \leq C\left\{\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s|ds\right)^p\right] + (t-\bar{t})^{\frac{p}{2}}\right\},$$

the conclusion follows. □

In order to proceed, we need to recall the following weak reverse Hölder inequality:

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Lemma 3.5 ([9]) *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let $p \in (1,2)$ and assume that the function $\xi \in L^p((a,b);\mathbb{R}^N)$ satisfies the inequality*

$$\mathbb{E}\left[\frac{1}{t-a}\int_a^t |\xi_s|^p ds\right] \leq A\,\mathbb{E}\left[\left(\frac{1}{t-a}\int_a^t |\xi_s| ds\right)^p\right] + \frac{B}{(t-a)^{\frac{p}{2}}} \qquad \forall t \in (a,b] \tag{36}$$

for some positive constants A and B. Then there are constants $\theta \in (p,2)$ and $C > 0$, depending only on p and A, such that

$$\mathbb{E}\left[\left(\int_a^t |\xi_s| ds\right)^p\right] \leq C(t-a)^{p-\frac{p}{\theta}}\left\{(b-a)^{\frac{p}{\theta}-1}\|\xi\|_p^p + B(b-a)^{\frac{p}{\theta}-\frac{p}{2}}\right\} \qquad \forall t \in (a,b]\ .$$

Thanks to this inequality, we can estimate the L^p norm of the process ζ appearing in Lemma 2.4.

Lemma 3.6 *Let $u \in \mathcal{C}(\mathbb{R}^N \times [0,T])$ be a subsolution of (12) such that $|u| \leq M$ and let $\tau \in (0,T)$. Then there is a universal constant $\theta \in (p,2)$ and a constant $C(\tau,\delta) > 0$ such that, for every $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0, T-\tau)$, and every stochastic processes (X_t, ζ_t) satisfying (29) and (30), we have*

$$\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^p\right] \leq C(\tau)(t-\bar{t})^{p-\frac{p}{\theta}} \qquad \forall t \in (\bar{t}, T)\ .$$

Proof: First, observe that, by Lemma 3.4 applied to $x = \bar{x}$,

$$u(\bar{x},\bar{t}) \leq \mathbb{E}[u(X_t,t)] + C\left((t-\bar{t})^{1-p}\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^p\right] + (t-\bar{t})^{1-p/2}\right)$$

for all $t \in [\bar{t}, T)$. Moreover, in view of (29),

$$\mathbb{E}\left[u(X_t,t)\right] \leq u(\bar{x},\bar{t}) - C_+\mathbb{E}\left[\int_{\bar{t}}^t |\zeta_s|^p ds\right] + \delta(t-\bar{t}) \qquad \forall t \in [\bar{t}, T)\ .$$

Hence, taking into account that $t - \bar{t} \leq C(t-\bar{t})^{1-p/2}$,

$$\mathbb{E}\left[\int_{\bar{t}}^t |\zeta_s|^p ds\right] \leq C\,(t-\bar{t})^{1-p}\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^p\right] + C(t-\bar{t})^{1-p/2} \qquad \forall t \in [\bar{t}, T)\ .$$

Then, owing to Lemma 3.5, there are universal constants $\theta \in (p,2)$ and $C > 0$ such that

$$\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^p\right] \leq C\left(\|\zeta\|_p^p + 1\right)\frac{(t-\bar{t})^{p-\frac{p}{\theta}}}{(T-\bar{t})^{1-\frac{p}{\theta}}} \qquad \forall t \in (\bar{t}, T)\ .$$

Since u is bounded by M, assumption (29) implies that $\|\zeta\|_p \leq C$. So, we finally get

$$\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^p\right] \leq C(\tau)(t-\bar{t})^{p-\frac{p}{\theta}} \qquad \forall t \in (\bar{t}, T)\ ,$$

because $\bar{t} \leq T - \tau$. □

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4 Proof of Theorem 1.1

Let $u : \mathbb{R}^N \times [0,T] \to \mathbb{R}$ be a continuous supersolution of (11), a subsolution of (12) and such that $|u| \le M$.

Space regularity: Fix $(\bar{x},\bar{t}) \in \mathbb{R}^N \times (0, T-\tau)$ and let $x \in \mathbb{R}^N$. By Lemma 2.4 there is a control $\zeta \in L^p_{\rm ad}([\bar{t},T])$ and an adapted process X such that (29) and (30) hold. So,

$$u(\bar{x},\bar{t}) \ge \mathbb{E}\left[u(X_t,t)\right] - \delta(t-\bar{t}) \qquad \forall t \in [\bar{t},T)\ . \tag{37}$$

Also, Lemma 3.6 ensures that

$$\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s| ds\right)^p\right] \le C(\tau)(t-\bar{t})^{p-\frac{p}{\theta}} \qquad \forall t \in (\bar{t},T) \tag{38}$$

for some universal constant $\theta \in (p,2)$ and some constant $C(\tau) > 0$. Furthermore, applying Lemma 3.4, for any $t \in (\bar{t},T)$ we have

$$\begin{aligned} &u(x,\bar{t}) - \mathbb{E}[u(X_t,t)] \\ &\quad \le C\left\{(t-\bar{t})^{1-p}\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s|ds\right)^p\right] + |\bar{x}-x|^p(t-\bar{t})^{1-p} + (t-\bar{t})^{1-p/2}\right\}\ . \end{aligned}$$

Plugging (37) and (38) into the above inequality leads to

$$u(x,\bar{t}) \le u(\bar{x},\bar{t}) + \delta(t-\bar{t}) + C(\tau)(t-\bar{t})^{(\theta-p)/\theta} + C|\bar{x}-x|^p(t-\bar{t})^{1-p} + C(t-\bar{t})^{1-p/2}$$

for any $t \in (\bar{t},T)$.
Since $1 > 1-p/2 > (\theta-p)/\theta$ (recall that $\theta < 2$),

$$u(x,\bar{t}) \le u(\bar{x},\bar{t}) + C(\tau)(t-\bar{t})^{(\theta-p)/\theta} + C|\bar{x}-x|^p(t-\bar{t})^{1-p}\ .$$

Then, for $|x-\bar{x}|$ sufficiently small, choose $t = \bar{t} + |x-\bar{x}|^{\theta/(\theta-1)}$ to obtain

$$u(x,\bar{t}) \ \le u(\bar{x},\bar{t}) + C(\tau)|x-\bar{x}|^{(\theta-p)/(\theta-1)}.$$

Time regularity : Let now $t \in (0, T-\tau)$. Then, in light of (37),

$$u(\bar{x},\bar{t}) \ge \mathbb{E}\left[u(X_t,t)\right] - \delta(t-\bar{t}).$$

Now, applying the space regularity result we have just proved, we obtain

$$\mathbb{E}\left[u(X_t,t)\right] \ge u(\bar{x},t) - C(\tau)\mathbb{E}\left[|X_t-\bar{x}|^{\frac{\theta-p}{\theta-1}}\right].$$

Moreover, since $(\theta-p)/(\theta-1) \ < \ 1$, by (30) we get

$$\mathbb{E}\left[|X_t-\bar{x}|^{\frac{\theta-p}{\theta-1}}\right] \le C\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s|ds\right)^{\frac{\theta-p}{\theta-1}}\right] + C(t-\bar{t})^{\frac{\theta-p}{2(\theta-1)}}\ .$$

Also, by Hölder's inequality and (38),

$$\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s|ds\right)^{\frac{\theta-p}{\theta-1}}\right] \le C\ \left\{\mathbb{E}\left[\left(\int_{\bar{t}}^t |\zeta_s|ds\right)^p\right]\right\}^{\frac{\theta-p}{p(\theta-1)}} \le C(\tau)(t-\bar{t})^{\frac{\theta-p}{\theta}}\ .$$

Notice that $(\theta - p)/(2(\theta-1)) > (\theta-p)/\theta$ since $\theta < 2$. So,

$$u(\bar{x}, \bar{t}) \geq u(\bar{x}, t) - C(\tau)(t - \bar{t})^{\frac{\theta - p}{\theta}} .$$

To derive the reverse inequality, one just needs to apply Lemma 3.2 with $y = x = \bar{x}$ to get

$$u(\bar{x}, \bar{t}) \leq u(\bar{x}, t) + C(t - \bar{t})^{1-p/2} .$$

This leads to the desired result since $1 - p/2 > (\theta - p)/\theta$. □

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