Existence of solutions for a third order non-local equation appearing in crack dynamics

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Abstract

In this paper, we prove the existenc e of non-negative solutions for a non-local third order degenerate parabolic equation arising in the modeling of hydraulic fractures. The equation is similar to the well-known thin film equation, but the Laplace operator is replaced by a Dirichlet-to-Neumann type operator (which can be defined using the periodic Hilbert transform). The main difficulties are due to the fact that this equation is non-local, and that the natural energy estimates are not as good as in the case of the thin film equation.

Keywords: Hydraulic fractures, Higher order equation, Non-local equation, Thin film equation, Non-negative solutions, periodic Hilbert transform

MSC: 35G25, 35K25, 35A01, 35B09

1 Introduction

The aim of this paper is to show the existence of non-negative solutions for the following problem, which arises in the modeling of hydraulic fractures:

$$\begin{cases} u_t + (u^3 I(u)_x)_x = 0 & \text{for } x \in \Omega, \quad t > 0 \\ u_x = 0,\ u^3 I(u)_x = 0 & \text{for } x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x) & \text{for } x \in \Omega \end{cases} \tag{1}$$

where Ω is a bounded interval in $\mathbb{R}$ and I is a non-local elliptic operator of order 1 satisfying $I \circ I = -\Delta$; the operator I will be defined precisely in Section 3 as the square root of the Laplace operator with Neumann boundary conditions. In the sequel, we will take $\Omega = (0,1)$ for simplicity.

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The equation

$$u_t + (u^3 I(u)_x)_x = 0 \tag{2}$$

is a non-local parabolic equation of order 3 which is somewhat in between the well known porous media equation

$$u_t - (u^m u_x)_x = 0 \tag{3}$$

and the thin film equation

$$u_t + (u^m u_{xxx})_x = 0. \tag{4}$$

(this equation models the evolution of a thin viscous droplet of liquid on a plane surface ($m = 3$) or in a Hele-Shaw cell ($m = 1$)).

A weak formulation of (2) is given by

$$\iint_Q u\,\partial_t\varphi\,dx\,dt + \iint_Q u^3\partial_x I(u)\,\partial_x\varphi\,dx\,dt = -\iint_Q u_0\varphi(0,\cdot)\,dx\,dt$$

for all $\varphi \in \mathcal{D}(\overline{Q})$ where Q denotes $\Omega \times (0,T)$. However, we will see that because of the degeneracy of the coefficient u^3, it is difficult to give a meaning to the term $u^3\partial_x I(u)$. We thus perform an additional integration by parts to get the following weak formulation of (2):

$$\iint_Q u\,\partial_t\varphi\,dx\,dt - \iint_Q 3u^2\partial_x u\,I(u)\,\partial_x\varphi\,dx\,dt - \iint_Q u^3\,I(u)\,\partial_{xx}\varphi\,dx\,dt = -\iint_Q u_0\varphi(0,\cdot)\,dx\,dt \tag{5}$$

for all $\varphi \in \mathcal{D}(\overline{Q})$ satisfying $\partial_x\varphi|_{\partial\Omega} = 0$.

We are going to prove the following existence theorem:

Theorem 1. *Consider a positive initial condition $u_0 \in H^{\frac12}(\Omega)$ such that*

$$\int_\Omega \frac{1}{u_0(x)}\,dx < \infty. \tag{6}$$

There exists a non-negative function $u \in L^\infty(0,T,H^{\frac12}(\Omega))$ such that

$$u \in L^2(0,T,H_N^{\frac32}(\Omega)) \tag{7}$$

which satisfies (5) for all $\varphi \in \mathcal{D}(\overline{Q})$ satisfying $\partial_x\varphi|_{\partial\Omega} = 0$.
Furthermore u satisfies, for almost every $t \in (0,T)$

$$\int_\Omega u(t,\cdot)\,dx = \int_\Omega u_0\,dx, \tag{8}$$

$$\|u(t,\cdot)\|^2_{H^{\frac12}(\Omega)} + 2\int_0^t\int_\Omega g^2\,dx\,ds \le \|u_0\|^2_{H^{\frac12}(\Omega)} \tag{9}$$

where the function $g \in L^2(Q)$ *satisfies* $g = \partial_x(u^{\frac{3}{2}} I(u)) - \frac{3}{2} u^{\frac{1}{2}} \partial_x u \, I(u)$ *in* $\mathcal{D}'(\Omega)$, *and*

$$\int_\Omega \frac{1}{u(t,x)}\,dx + ||u||^2_{L^2(0,t;\dot{H}^{\frac{3}{2}}(\Omega))} \leq \int_\Omega \frac{1}{u_0(x)}\,dx. \tag{10}$$

Remark 1. Remark that, at least formally, we have $g = u^{\frac{3}{2}} \partial_x I(u)$ (though we do not have enough regularity on u to give a meaning to this product).

Remark 2. Condition (6) requires in particular that $\mathrm{Supp}(u_0) = \Omega$ and inequality (10) implies that this remains true for all time. This is a serious restriction since the case of compactly supported initial data is physically the most relevant and most interesting (see Section 2). We hope to be able to get ride of this assumption in a further work.

The space $H^{\frac{3}{2}}_N(\Omega)$ will be defined precisely in Section 3. In particular, the following characterization will be given:

$$H^{\frac{3}{2}}_N(\Omega) = \left\{ u \in H^{\frac{3}{2}}(\Omega)\,;\, \int_\Omega \frac{u_x^2}{d(x)}\,dx < \infty \right\}$$

where $d(x)$ denotes the distance to $\partial\Omega$. Condition (7) thus implies that u satisfies $u_x = 0$ on $\partial\Omega$ in some very weak sense.

Remark 3. Note that we have $H^{\frac{3}{2}}_N(\Omega) \subset W^{1,p}(\Omega)$ for all $p < \infty$ and so every term in (5) makes sense.

The porous media equation, the thin film equation and ours. Existence and uniqueness of weak solutions for the porous media equation (3) is well known (see for instance [25] or A. Friedman [20] and references therein), and these solutions enjoy many properties such as Hölder regularity [17], finite speed expansion of the support (if the initial data has compact support, then the solution has compact support for all time) and the fact that the support is always expanding (in fact strictly expanding, except maybe for a waiting initial time). Other properties of the porous media equation include the existence of self similar source type solutions and traveling wave solutions.

The main difference between (2) and the porous media equation, however, is the lack of maximum principle for the former, as for any equation of order greater than 2 (in particular, it is well known that non-negative initial data may generate changing sign solutions of the fourth order equation $\partial_t h + \partial_{xxxx} h = 0$).

In that sense, we expect the analysis of (2) to be somewhat closer to that of the thin film equation (4). The existence of non-negative weak solutions of (4) was first addressed by F. Bernis and A. Friedman [10] for $n > 1$. Further results were later obtained, by similar technics, by E. Beretta, M. Bertsch and R. Dal Passo [5] and A. Bertozzi and M. Pugh [12, 13]. A remarkable feature of (4) is that the degeneracy of the diffusion coefficient permits the existence of non-negative solutions (even though no maximum principle hold).

Equation (2) arises in the modeling of hydraulic fracture, i.e. when a fracture (with opening u) is propagated in an elastic material due to the pressure exerted by a viscous fluid which fills the fracture (see Section 2 for details). Such fractures occur naturally, for instance in volcanic dikes where magma causes fracture propagation below the surface of the earth, or can be deliberately propagated in oil or gas reservoirs to increase production. From a mathematical point of view, Equation (2) is somehow in between the porous media equation and the thin film equation. It is however worse than both of them: it is still of order greater than 2, and thus lacks a maximum principle, but, unlike the thin-flim equation, it is a non-local equation and the natural regularity given by the energy inequality ($u \in H^{\frac{1}{2}}$ rather than $u \in H^1$) does not give the continuity of weak solutions even in dimension 1.

There is a significant amount of work involving the mathematical modeling of hydraulic fractures, which is beyond the scope of this article. The model that we consider in our paper, which corresponds to very simple fracture geometry, was developed independently by Geertsma and De klerk [21] and Khristianovic and Zheltov [32]. Spence and Sharp [30] initiated the work on self-similar solutions and asymptotic analyses of near tip processes for (2). There is now an abundant literature that has extended this analysis to various regimes (see for instance [1], [2], [24] and reference therein). Several numerical methods have also been developed (see in particular Peirce et al. [26], [28] and [27]). However, to our knowledge, there are no rigorous existence results for this problem for general initial data. This paper is thus a first step toward a rigorous analysis of (1).

Organization of the article. The paper is organized as follows: In Section 2, we recall the main steps of the derivation of (1). In Section 3, we introduce the functional analysis tools that will be needed to prove Theorem 1. In particular, the non-local operator $I(u)$ is defined, first using a spectral decomposition, then as a Dirichlet-to-Neuman map. An integral representation for I, using the periodic Hilbert transform is also given. Section 4 is devoted to the study of a regularized equation while the proof of Theorem 1 is given in Section 5.

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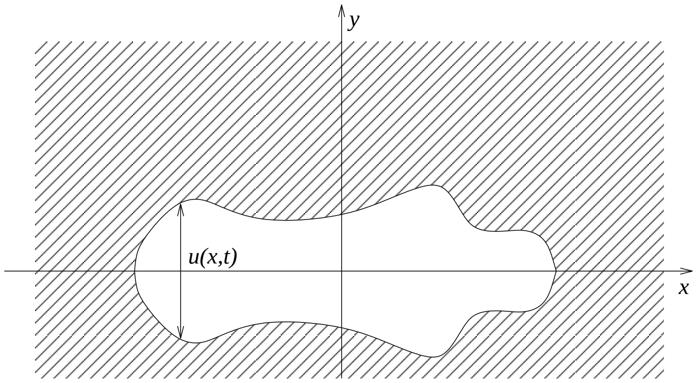
2 The physical model

The model under consideration describes the propagation of an impermeable KGD fracture driven by a viscous fluid in a uniform elastic medium under condition of plane strain. More precisely, denoting by (x, y, z) the standard coordinates in $\mathbb{R}^3$, we consider a fracture which is invariant with respect to one variable (say z) and symmetric with respect to another direction (say y). The fracture can then be entirely described by its opening $u(t, x)$ in the y direction (see Figure 2). Since it assumes that the fracture is an infinite strip whose cross-sections are in a state of plane strain, it is only applicable to rectangular planar fracture with large aspect ratio.



We now briefly describe the main steps of the derivation of (1).

2.1 Conservation of mass and Poiseuille law

The conservation of mass for the fluid inside the fracture, averaged with respect to y yields:

$$\partial_t(\rho u) + \partial_x q = 0 \quad \text{in } \mathbb{R} \tag{11}$$

where ρ is the density of the fluid (which is assumed to be constant) and $q = q(t,x)$ denotes the fluid flux. This flux is given by

$$q = \rho u \overline{v} \tag{12}$$

where $\overline{v}$ is the y-averaged horizontal component of the velocity of the fluid

$$\overline{v} = \frac{1}{u}\int_{-u/2}^{u/2} v_H(t,x,y)\,dy.$$

Under the lubrification approximation, Navier-Stokes equations reduce to

$$\mu\frac{\partial^2 v_H}{\partial y^2}(t,x,y) = \partial_x p(t,x)$$

where p denotes the pression of the fluid at a point x and μ denotes the viscosity coefficient. Assuming a no-slip boundary condition $v = 0$ at $y = \pm u/2$, we deduce

$$v_H(t,x,y) = \frac{1}{\mu}\partial_x p\left[\frac{1}{2}y^2 - \frac{1}{8}u^2\right] \qquad \text{for } -u \le y \le u$$

and so

$$\overline{v}(t,x) = -\frac{u^2}{12\mu}\partial_x p(t,x)$$

Using (12), we deduce *Poiseuille law*

$$q = -\frac{u^3}{12\mu}\partial_x p\,. \tag{13}$$

Together with (11), this implies

$$\rho \partial_t u - \partial_x \left(\frac{u^3}{12\mu} \partial_x p \right) = 0,$$

and we obtain (2) (up to some constant) if the pressure p can be express as a function of the displacement u ($p = I(u)$).

2.2 The pressure law

For a state of plane strain, the pressure is given by

$$p(x) = E' \partial_y U(x, 0) \tag{14}$$

where E' denotes Young's modulus and $U(x, y)$ denotes the displacement of the rock. The function U is computed through the resolution of equations from linear elasticity. By taking advantage of the symmetry of the problem, we get

$$\begin{cases} -\Delta U = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ U(x, 0) = u(t, x), & \text{on } \mathbb{R}\,. \end{cases} \tag{15}$$

By combining (14) and (15), we deduce that the pressure p, seen as a function of the displacement u, is a Dirichlet-to-Neumann operator for the laplace operator. Denoting this operator by $I(u)$, we deduce (2).

A technical assumption. In order to reduce the technicality of the analysis, we will assume that the crack is periodic with respect to x. Since we expect compactly supported initial data to give rise to compactly supported solutions whose supports extend with finite speed, this is a reasonnable physical assumption. We also assume that the initial crack is even with respect to $x = 0$ and we look for solutions that are also even.

By making such an assumption (periodicity and evenness), it is now equivalent to replace (15) with the following boundary value problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \times (0, \infty) \\ v_\nu = 0 & \text{on } \partial\Omega \times (0, \infty) \\ v(x, 0) = u(x) & \text{on } \Omega \end{cases}$$

if the period of the initial crack is 2, say. The cylinder $\Omega \times (0, +\infty)$ is denoted by C in the remaining of the paper.

Mathematical definition of the pressure. It turns out that it is easier to define first the operator I by using the spectral decomposition of the Laplace operator: We take $\{\lambda_k, \varphi_k\}$ the eigenvalues and corresponding eigenvectors of the Laplacian operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta \varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\ \partial_\nu \varphi_k = 0 & \text{on } \partial\Omega. \end{cases}$$

We then define the operator I by

$$Iu = \sum_{k=0}^{\infty} c_k \varphi_k(x) \mapsto -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k(x)$$

which maps $H^1(\Omega)$ onto $L^2(\Omega)$. We will prove that this operator can be characterized as a Dirichlet-to-Neumann map (see Proposition 2) and has an integral representation too (see Proposition 3).

2.3 Boundary conditions

Equation (2) must be supplemented with *boundary conditions* at the tip of the fracture. This is not really relevant to our analysis since we will be considering fracture with support equal to Ω, but we include this discussion here for the sake of completeness.

Assuming that $\mathrm{Supp}(u(t,\cdot)) = [-\ell(t), \ell(t)]$, it is usually assumed that

$$u = 0, \qquad u^3 \partial_x p = 0 \qquad \text{at } x = \pm\ell(t)$$

which ensures zero width and zero fluid loss at the tip.

Here, we want to point out that the problem is really a free boundary problem, since the support $[-\ell(t), \ell(t)]$ of the fracture is not known a priori. Since the equation is of order 3, those two conditions are thus not enough to fully determine a solution. In fact, there should be an additional condition which takes into account the energy required to break the rock at the tip of the crack. Consistent with linear elastic fracture propagation, we assume that the rock toughness K_{Ic} equals the stress intensity factor K_I, which for this simple geometry is given by

$$K_I = \frac{\ell}{\pi} \int_{-\ell}^{\ell} \frac{p}{\sqrt{\ell^2 - x^2}}\, dx.$$

When the crack propagation is determined by the toughness of the rock, a formal asymptotic analysis of fracture profile at the tip (see [1, 18]) shows that

$$u(t,x) \sim \frac{K'}{E'} \sqrt{\ell(t) - x} \qquad \text{as } x \to \ell(t) \tag{16}$$

with $K' = 4\sqrt{\frac{2}{\pi}} K_{Ic}$ (and a similar condition for $x \to -\ell$). One can now take this condition on the profile of u at the tip as the missing free boundary condition. The resulting free boundary problem is clearly very delicate to study (remember that (1) is a third order degenerate non-local parabolic equation).

A particular case which is simpler and interesting is the case of zero toughness ($K_{Ic} = 0$). This is relevant mainly if there is a pre-fracture (i.e. the rock is already cracked, even though $u = 0$ outside the initial fracture). Mathematically speaking, this means that Equation (1) is now satisfied everywhere in $\mathbb{R}$ even though u is expected to have compact support. No free boundary conditions are necessary. One should then have $\lim_{x\to\ell}(\ell(t) - x)^{-1/2} u(t,x) = 0$ at the tip of the crack. Furthermore, formal arguments show that the asymptotic behavior of the fracture opening near the fracture tip should be proportional to $(\ell(t)-x)^{2/3}$ (see [1, 18]).

Such a problem could easily be investigated in our framework (bounded Ω with Neumann boundary conditions), if we were able to work with compactly supported initial data in Ω. Note that this approach is very similar to what is usually done with the porous media equation, and it has been used successfully in the case of the thin film equation to prove the existence of solutions with zero contact angle (in that case, we speak of precursor film, or pre-wetting).

2.4 Conservation of mass and energy inequalities

In this section, we briefly describe the main inequalities satisfied by the (smooth) solutions of (1).

Conservation of mass. Integrating (2) on Ω and using the boundary conditions, we get

$$\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx \qquad \text{for all } t>0.$$

Energy equality. Multiplying (2) by $-I(u)$ and integrating on $[0,T]\times\Omega$, we get:

$$-\frac{1}{2}\int_\Omega u(T)I(u(T))dx + \int_0^T\int_\Omega u^3(\partial_x I(u))^2 dx\,dt = -\frac{1}{2}\int_\Omega u_0 I(u_0)dx$$

which can be rewritten as (see Section 3)

$$\|u(T,\cdot)\|^2_{\dot H^{\frac12}(\Omega)} + 2\int_0^T\int_\Omega u^3(\partial_x I(u))^2 dx\,dt = \|u_0\|^2_{\dot H^{\frac12}(\Omega)} \tag{17}$$

where $\dot H^{\frac12}(\Omega)$ denotes the homogeneous fractional Sobolev space. This equality will play a central role in proving the existence of a solution.

Entropy equality. As in the case of the thin film equation, positive solutions of (2) satisfy another surprising equality: Multiplying (2) by u^{-2} we get:

$$\int_\Omega \frac{1}{u(T,x)}dx - 2\int_0^T\int_\Omega \partial_x u\,\partial_x I(u)\,dx\,dt = \int_\Omega \frac{1}{u_0(x)}dx. \tag{18}$$

When u satisfies Neumann boundary conditions we will show that

$$-\int_\Omega \partial_x u\,\partial_x I(u)\,dx = ||\partial_x u||^2_{\dot H^{\frac12}(\Omega)} \geq 0$$

and so that the quantity $\int_\Omega u^{-1}(t,x)\,dx$ is monotone decreasing in time. This inequality suggests that an open fracture cannot close, and it will play a crucial role in showing the existence of *non-negative* solutions.

3 Preliminaries

In this section, we define the operator I and give the functional analysis results that will play an important role in the proof of the main theorem. A very similar operator, with Dirichlet boundary conditions rather than Neumann boundary conditions, was studied by Cabré and Tan [15]. This section follows their analysis very closely.

3.1 Functional spaces

The space $H^s_N(\Omega)$. We denote by $\{\lambda_k, \varphi_k\}_{k=0,1,2...}$ the eigenvalues and corresponding eigenfunctions of the Laplace operator in Ω with Neumann boundary conditions on $\partial\Omega$:

$$\begin{cases} -\Delta\varphi_k = \lambda_k\varphi_k & \text{in } \Omega \\ \partial_\nu\varphi_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{19}$$

normalized so that $\int_\Omega \varphi_k^2\,dx = 1$. When $\Omega = (0,1)$, we have

$$\lambda_0 = 0\,, \qquad \varphi_0(x) = 1$$

and

$$\lambda_k = (k\pi)^2\,, \qquad \varphi_k(x) = \sqrt{2}\cos(k\pi x) \qquad k = 1,\, 2,\, 3,\, \ldots$$

The φ_k's clearly form an orthonormal basis of $L^2(\Omega)$. Furthermore, the φ_k's also form an orthogonal basis of the space $H^s_N(\Omega)$ defined by

$$H^s_N(\Omega) = \{u = \sum_{k=0}^{\infty} c_k\varphi_k\,;\, \sum_{k=0}^{\infty} c_k^2(1+\lambda_k^s)\}$$

equipped with the norm

$$||u||^2_{H^s_N(\Omega)} = \sum_{k=0}^{\infty} c_k^2(1+\lambda_k^s)$$

or equivalently (noting that $c_0 = \|u\|_{L^1(\Omega)}$ and $\lambda_k \geq 1$ for $k \geq 1$):

$$||u||^2_{H^s_N(\Omega)} = \|u\|^2_{L^1(\Omega)} + \|u\|^2_{\dot{H}^s_N(\Omega)}$$

where

$$\|u\|^2_{\dot{H}^s_N(\Omega)} = \sum_{k=1}^{\infty} c_k^2\lambda_k^s.$$

A characterisation of $H^s_N(\Omega)$. The precise description of the $H^s_N(\Omega)$ is a classical problem.

Intuitively, for $s < 3/2$, the boundary condition $u_\nu = 0$ does not make sense, and one can show that (see Agranovich and Amosov [4] and references therein):

$$H^s_N(\Omega) = H^s(\Omega) \quad \text{for all } 0 \leq s < \frac{3}{2}.$$

In particular, we have $H_N^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$ and we will see later that

$$\|u\|^2_{\dot{H}^{\frac{1}{2}}(\Omega)} = \int_\Omega \int_\Omega (u(y) - u(x))^2 \nu(x,y) dx dy$$

where $\nu(x,y)$ is a given positive function; see (23) below.

For $s > 3/2$, the Neumann condition has to be taken into account, and we have in particular

$$H_N^2(\Omega) = \{u \in H^2(\Omega)\,;\, u_\nu = 0 \text{ on } \partial\Omega\}$$

which will play a particular role in the sequel. More generally, a similar characterization holds for $3/2 < s < 7/2$. For $s > 7/2$, additional boundary conditions have to be taken into account.

The case $s = 3/2$ is critical (note that $u_\nu|_{\partial\Omega}$ is not well defined in that space) and one can show that

$$H_N^{\frac{3}{2}}(\Omega) \quad = \quad \left\{ u \in H^{\frac{3}{2}}(\Omega)\,;\, \int_\Omega \frac{u_x^2}{d(x)}\,dx < \infty \right\}$$

where $d(x)$ denotes the distance to $\partial\Omega$. A similar result appears in [15]; more precisely, such a characterization of $H_N^{\frac{3}{2}}(\Omega)$ can be obtained by considering functions u such that $u_x \in \mathcal{V}_0(\Omega)$ where $\mathcal{V}_0(\Omega)$ is defined in [15] as the equivalent of our space $H_N^{1/2}(\Omega)$ with Dirichlet rather than Neumann boundary conditions. We do not dwell on this issue since we will not need this result in our proof.

3.2 The operator I

As it is explained in the Introduction, the operator I is related to the computation of the pression as a function of the displacement. ¿From this point of view, the operator I is a Dirichlet-to-Neumann operator associated with the Laplacian. Since we study the problem in a periodic setting we explained that this yields to consider Neumann boundary conditions on a cylinder C.

Spectral definition. It is convenient to begin with the spectral definition of the operator I: With λ_k and φ_k defined by (19), we define the operator

$$I : \sum_{k=0}^{\infty} c_k \varphi_k \;\longmapsto\; -\sum_{k=0}^{\infty} c_k \lambda_k^{\frac{1}{2}} \varphi_k \tag{20}$$

which clearly maps $H^1(\Omega)$ onto $L^2(\Omega)$ and $H_N^2(\Omega)$ onto $H^1(\Omega)$.

Dirichlet-to-Neuman map. With the spectral definition in hand, we are now going to show that I can also be defined as the Dirichlet-to-Neumann operator associated with the Laplace operator supplemented with Neumann boundary conditions.

To be more precise, we consider the following boundary problem in the cylinder $C = \Omega \times (0, +\infty)$"

$$\begin{cases} -\Delta v = 0 & \text{in } C, \\ v(x,0) = u(x) & \text{on } \Omega, \\ v_\nu = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{21}$$

We will show that we have

$$I(u) = \partial_y v(\cdot, 0).$$

We start with the following result which show the existence of a unique harmonic extension v:

Proposition 1 (Existence and uniqueness for (21))**.** *For all $u \in H_N^{\frac{1}{2}}(\Omega)$, there exists a unique extension $v \in H^1(C)$ solution of (21).*
Furthermore, if $u(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x)$, then

$$v(x,y) = \sum_{k=1}^{\infty} c_k \varphi_k(x) \exp(-\lambda_k^{\frac{1}{2}} y). \tag{22}$$

Proof. We recall that $H_N^{\frac{1}{2}}(\Omega) = H^{\frac{1}{2}}(\Omega)$, and for a given $u \in H^{\frac{1}{2}}(\Omega)$ we consider the following minimization problem:

$$\inf \left\{ \int_C |\nabla w|^2 \, dx \, dy \, ; \, w \in H^1(C) \, , \; w(\cdot, 0) = u \text{ on } \Omega \right\}.$$

Using classical arguments, one can show that this problem has a unique minimizer v (note that the set of functions on which we minimize the functional is not empty). This minimizer is a weak solution of (21). In particular, it satisfies

$$\int_C \nabla v \cdot \nabla w \, dx \, dy = 0$$

for all $w \in H^1(\Omega)$ such that $w(\cdot, 0) = 0$ on Ω, which includes a weak formulation of the Neumann condition.

Finally, the representation formula (22) follows from a straightforward computation. Indeed, we have

$$\begin{aligned} \int_0^\infty \int_\Omega |\nabla v|^2 \, dx \, dy &= \int_0^\infty |\partial_x v|^2 + |\partial_y v|^2 \, dx \, dy \\ &= 2 \sum_{k=1}^{\infty} b_k^2 \lambda_k \int_0^\infty \exp(-2\lambda_k^{1/2} y) \, dy \\ &= 2 \sum_{k=1}^{\infty} b_k^2 \lambda_k \frac{1}{2\lambda_k^{1/2}} \\ &= \sum_{k=1}^{\infty} b_k^2 \lambda_k^{1/2} = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)} \end{aligned}$$

which shows that v belongs to $H^1(C)$. The fact that v satisfies (21) is now easy to check. $\square$

We can now show:

Proposition 2 (The operator I is of Dirichlet-to-Neumann type)**.** *For all $u \in H^2_N(\Omega)$, we have*

$$I(u)(x) = -\frac{\partial v}{\partial \nu}(x,0) = \partial_y v(x,0) \quad \textit{for all } x \in \Omega,$$

where v is the unique harmonic extension solution of (21).

Furthermore $I \circ I(u) = -\Delta u$.

Proof. This follows from a direct computation using (22). Furthermore, if u is in $H^2_N(\Omega)$, we know that $\sum_{k=0}^{\infty} c_k^2 \lambda_k^2 < \infty$. It is now easy to derive the following equality

$$I(I(u)) = \sum_{k=0}^{\infty} c_k \lambda_k \varphi_k(x) = -\Delta u.$$

□

Integral representation. The operator I can also be represented as a singular integral operator. Indeed, we will prove the following

Proposition 3. *Consider a smooth function $u : \Omega \to \mathbb{R}$. Then for all $x \in \Omega$,*

$$I(u)(x) = \int_\Omega (u(y) - u(x))\nu(x,y)dy$$

where $\nu(x,y)$ is defined as follows: for all $x, y \in \Omega$,

$$\nu(x,y) = \frac{\pi}{2}\left(\frac{1}{1-\cos(\pi(x-y))} + \frac{1}{1-\cos(\pi(x+y))}\right). \tag{23}$$

Proof. We use the Dirichlet-to-Neumann definition of I. Let v denote the solution of (21). Then v is the restriction to $(0,1)$ of the unique solution w of (21) where Ω is replaced with $(-1,1)$ and u is replaced by its even extension to $(-1,1)$. In particular, w is even with respect to x. Then there exists an holomorphic function W defined in the cylinder $(-1,1)\times(0,+\infty)$ such that $w = \mathrm{Re}(W)$. Next, we consider the holomorphic function $z \mapsto e^{i\pi z} = e^{-\pi y}e^{i\pi x}$ defined on the cylinder $(-1,1)\times(0,+\infty)$ with values into the unit disk $D_1 = \{(x,y) : x^2+y^2 < 1\}$. If z denotes the complex variable $x+iy$, then a new holomorphic function W_0 is obtained by the following formula

$$W(z) = W_0(e^{i\pi z}).$$

In particular, W_0 is defined and harmonic in D_1. This implies that the function W_0 can be represented by the Poisson integral. Precisely,

$$W_0(Z) = \frac{1-|Z|^2}{2\pi}\int_{\partial D_1} \frac{W_0(Y)}{|Y-Z|^2} d\sigma(Y).$$

This implies that for all $z \in C$,

$$W(z) = \frac{1-e^{-2\pi y}}{2\pi} \int_{-1}^{1} \frac{W(\theta)}{|e^{i\pi\theta} - e^{-\pi y} e^{i\pi x}|^2} \pi d\theta$$

and we finally obtain

$$w(x,y) = \frac{1-e^{-2\pi y}}{2} \int_{-1}^{1} \frac{w(\theta,0)}{|e^{i\pi\theta} - e^{-\pi y} e^{i\pi x}|^2} d\theta.$$

Taking $w = 1$, we get in particular the following equality:

$$1 = \frac{1-e^{-2\pi y}}{2} \int_{-1}^{1} \frac{1}{|e^{i\pi\theta} - e^{-\pi y} e^{i\pi x}|^2} d\theta.$$

We deduce:

$$\frac{w(x,y) - w(x,0)}{y} = \frac{1-e^{-2\pi y}}{2y} \int_{-1}^{1} \frac{w(\theta,0) - w(x,0)}{|e^{i\pi\theta} - e^{-\pi y} e^{i\pi x}|^2} d\theta$$

which implies (letting y go to zero):

$$\partial_y w(x,0) = \pi \int_{-1}^{1} \frac{w(\theta,0) - w(x,0)}{|e^{i\pi\theta} - e^{i\pi x}|^2} d\theta.$$

The integral in the right hand side of the previous equality is understood in the sense of the principal value of the associated distribution. We finally use the fact that w is even in x and equal to u on Ω to obtain the following singular integral representation of $I(u)$:

$$I(u)(x) = \pi \int_{0}^{1} (u(\theta,0) - u(x,0)) \left(\frac{1}{|1 - e^{i\pi(x-\theta)}|^2} + \frac{1}{|1 - e^{i\pi(x+\theta)}|^2} \right) d\theta.$$

□

The space $H^{-\frac{1}{2}}(\Omega)$. The space $H^{-\frac{1}{2}}(\Omega)$ is defined as the topological dual space of $H^{\frac{1}{2}}(\Omega)$. It is classical that for any $u \in H^{-\frac{1}{2}}(\Omega)$, there exists $u_1 \in L^2(\Omega)$ and $u_2 \in H^{\frac{1}{2}}(\Omega)$ such that $u = u_1 + \partial_x u_2$ (in the sense of distributions). We will also use repeatedly the following elementary lemma

Lemma 1. *If $u \in H^{\frac{1}{2}}(\Omega)$, then the distribution $I(u)$ is in $H^{-\frac{1}{2}}(\Omega)$ and for all $v \in H^{\frac{1}{2}}(\Omega)$,*

$$\langle I(u), v \rangle_{H^{-\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega)} = -\sum_{k=0}^{+\infty} \lambda_k^{\frac{1}{2}} c_k d_k$$

where $u = \sum_{k=0}^{+\infty} c_k \varphi_k$ and $v = \sum_{k=0}^{+\infty} d_k \varphi_k$. In particular,

$$-\langle I(u), u \rangle_{H^{-\frac{1}{2}}(\Omega), H^{\frac{1}{2}}(\Omega)} = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}.$$

Important equalities. The semi-norms $||\cdot||_{\dot{H}^{\frac{1}{2}}(\Omega)}$, $||\cdot||_{\dot{H}^1(\Omega)}$, $||\cdot||_{\dot{H}^{\frac{3}{2}}(\Omega)}$ and $||\cdot||_{\dot{H}^2_N(\Omega)}$ are related to the operator I by equalities which will be used repeatedly.

Proposition 4 (The operator I and several semi-norms)**.**
For all $u \in H^{\frac{1}{2}}(\Omega)$, we have

$$\frac{1}{2}\int_\Omega\int_\Omega (u(x)-u(y))^2\nu(x,y)dxdy = ||u||^2_{\dot{H}^{\frac{1}{2}}(\Omega)}.$$

For all $u \in H^1(\Omega)$, we have

$$\int_\Omega (I(u))^2 dx = \|u\|^2_{\dot{H}^1(\Omega)}.$$

For all $u \in H^2_N(\Omega)$, we have

$$-\int_\Omega I(u)_x u_x\, dx = ||u||^2_{\dot{H}^{\frac{3}{2}}_N(\Omega)}.$$

For all $u \in H^2_N(\Omega)$, we have

$$\int_\Omega (\partial_x I(u))^2\, dx = \|u\|^2_{\dot{H}^2_N(\Omega)}.$$

Remark 4. Note that $I(u)_x \neq I(u_x)$.

Proof. The two first equalities are easily derived form the definition of I, definitions of the semi-norms, the integral representation of I and the fact that $\nu(x,y) = \nu(y,x)$.

In order to prove the third and fourth equalities, we first remark that $\partial_x\varphi_k = -\lambda_k^{\frac{1}{2}}\sin(k\pi x)$ form an orthogonal basis of $L^2(\Omega)$.

In order to prove the fourth equality, we first write

$$\partial_x(I(u)) = -\sum_{k=1}^{\infty} c_k\lambda_k^{\frac{1}{2}}\partial_x\varphi_k \quad \text{in } L^2(\Omega)$$

from which we deduce

$$\begin{aligned}\int_\Omega (I(u)_x)^2\, dx &= \sum_{k=1}^{\infty} c_k^2\lambda_k \int_\Omega (\partial_x\varphi_k)^2\, dx \\ &= \sum_{k=1}^{\infty} c_k^2\lambda_k \int_\Omega \varphi_k(-\partial_{xx}\varphi_k)\, dx \\ &= \sum_{k=0}^{\infty} c_k^2\lambda_k^2.\end{aligned}$$

As far as the third equality is concerned, we note that

$$u_x = \sum_{k=0}^{\infty} c_k \partial_x \varphi_k \qquad \text{in } L^2(\Omega).$$

We then have

$$\begin{aligned}
-\int_\Omega I(u)_x u_x \, dx &= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{1}{2}} \int_\Omega (\partial_x \varphi_k)^2 \, dx \\
&= -\sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{1}{2}} \int_\Omega \varphi_k \partial_{xx} \varphi_k \, dx \\
&= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{1}{2}} \int_\Omega \lambda_k \varphi_k^2 \, dx \\
&= \sum_{k=0}^{\infty} c_k^2 \lambda_k^{\frac{3}{2}} = ||u||^2_{\dot{H}^{\frac{3}{2}}(\Omega)}.
\end{aligned}$$

□

3.3 The problem $-I(u) = g$

We conclude this preliminary section by giving a few results about the following problem:

$$\begin{aligned} &\text{For a given } g \in L^2(\Omega), \text{ find } u \in H^1(\Omega) \text{ such that} \\ &-I(u) = g. \end{aligned} \tag{24}$$

Note that $\int_\Omega I(u)\, dx = 0$ for all $u \in H^1(\Omega)$ and so a necessary condition for the existence of a solution to (24) is

$$\int_\Omega g(x)\, dx = 0.$$

Note also that there is no uniqueness since if u is a solution then $u + C$ is also a solution for any constant C. We may however expect a unique solution if we add the further constraint $\int u\, dx = 0$. Indeed, a weak solution $u \in H^{\frac{1}{2}}(\Omega)$ for $g \in H^{-\frac{1}{2}}(\Omega)$ can be found using Lax-Milgram theorem in $\{u \in H^{\frac{1}{2}}(\Omega)\,;\, \int_\Omega u\, dx = 0\}$ equipped with the norm $||u||_{\dot{H}^{\frac{1}{2}}(\Omega)}$. Alternatively, we can use the spectral framework: For $g \in L^2(\Omega)$ such that $\int_\Omega g(x)\, dx = 0$, we have

$$g = \sum_{k=1}^{\infty} d_k \varphi_k \qquad \text{with } \sum_{k=1}^{\infty} d_k^2 < \infty.$$

We can then write:

$$u = I^{-1}(g) := \sum_{k=1}^{\infty} \frac{d_k}{\lambda_k^{\frac{1}{2}}} \varphi_k \tag{25}$$

which clearly lies in $H^1(\Omega)$ and satisfies $\int_\Omega u\,dx = 0$. The fact that the φ_k's form an orthogonal basis of $L^2(\Omega)$ implies that there is only one solution of (24) such that $\int_\Omega u\,dx = 0$. Finally it is clear from (25) that further regularity on g will imply further regularity on u. We sum up this discussion in the following statement

Theorem 2. *For all $g \in L^2(\Omega)$ such that $\int_\Omega g\,dx = 0$, there exists a unique function $u \in H^1(\Omega)$ such that $-I(u) = g$ in $L^2(\Omega)$ and $\int_\Omega u\,dx = 0$. Furthermore, if g is in $H^1(\Omega)$, then $u \in H^2_N(\Omega)$.*

We will also use the following corollary of the previous theorem

Corollary 1. *For all $g \in L^2(\Omega)$, there exists a unique solution $u \in H^1(\Omega)$ of*

$$-I(v) + \int_\Omega v\,dx = g.$$

Proof. Set $m = \int_\Omega g(x)\,dx$ and consider $\tilde{g} = g - m$. Then $\tilde{g} \in L^2(\Omega)$ and $\int_\Omega \tilde{g}dx = 0$. There is a (unique) $u \in H^1(\Omega)$ such that

$$-I(u) = g - m, \qquad \int_\Omega u(x)\,dx = 0.$$

We then set $v = u + m$. Then $\int_\Omega v\,dx = m$ and

$$-I(v) = -I(u) = g - m = g - \int_\Omega v\,dx.$$

As far as uniqueness is concerned, if we consider two solutions v_1 and v_2 then we have

$$\int_\Omega v_1 dx = \int_\Omega v_2 dx = \int_\Omega g$$

and this implies that $w = v_1 - v_2$ satisfies $-I(w) = 0$. The uniqueness of the solution given by Theorem 2 implies that $w = 0$ and the proof is complete. □

4 A regularized problem

We now turn to the proof of Theorem 1. The degeneracy of the diffusion coefficient is a major obstacle to the development of a variational argument. As in [10], the existence of solution for (2) is thus obtained via a regularization approach: Given $\varepsilon > 0$, we consider

$$\partial_t u + \partial_x(f_\varepsilon(u)\partial_x I(u)) = 0, \quad t \in (0,T), x \in \Omega \tag{26}$$

where

$$f_\varepsilon(u) = u_+^3 + \varepsilon$$

with the initial condition

$$u(x,0) = u_0(x). \tag{27}$$

The first step in the proof of Theorem 1, is to prove the following proposition:

Proposition 5 (Existence for the regularized problem)**.** *For all $u_0 \in H^{\frac{1}{2}}(\Omega)$, there exists a unique function u^ε such that*

$$u^\varepsilon \in L^\infty(0,T;H^{\frac{1}{2}}(\Omega)) \cap L^2(0,T;H^2_N(\Omega))$$

solution of

$$\iint_Q u^\varepsilon \partial_t \varphi \, dx\, dt + \iint_Q f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon)\partial_x \varphi \, dx\, dt = -\int_\Omega u_0 \varphi(0,\cdot)\, dx \qquad (28)$$

for all $\varphi \in \mathcal{C}^1_c([0,T), H^1(\Omega))$ with $Q = \Omega \times (0,T)$.

Moreover, the function u^ε satisfies

$$\int_\Omega u^\varepsilon(t,x)\, dx = \int_\Omega u_0(x)\, dx \qquad a.e.\ t \in (0,T) \qquad (29)$$

and

$$\|u^\varepsilon(t,\cdot)\|^2_{\dot H^{\frac{1}{2}}(\Omega)} + 2\int_0^t \int_\Omega f_\varepsilon(u^\varepsilon)(\partial_x I(u^\varepsilon))^2\, dx\, ds \le \|u_0\|^2_{\dot H^{\frac{1}{2}}(\Omega)} \quad a.e.\ t \in (0,T). \qquad (30)$$

Finally, if $u_0 \geq 0$ and F_ε is a non-negative function such that $F''_\varepsilon(s) = \frac{1}{f_\varepsilon(s)}$, then u^ε satisfies for almost every $t \in (0,T)$

$$\int_\Omega F_\varepsilon(u^\varepsilon)(t,x)\, dx + \int_0^t \| u^\varepsilon(s)\|^2_{\dot H^{\frac{3}{2}}_N(\Omega)}\, ds \le \int_\Omega F_\varepsilon(u_0)\, dx. \qquad (31)$$

Remark 5. Note that this result does not require condition (6) to be satisfied and is thus valid with compactly supported initial data. However, we will need condition (6) to get enough compactness on u^ε to pass to the limit $\varepsilon \to 0$ and complete the proof of Theorem 1.

4.1 Stationary problem

In order to prove Proposition 5, we first consider the following stationary problem:

For a given $g \in H^{\frac{1}{2}}(\Omega)$, find $u \in H^2_N(\Omega)$ such that

$$\begin{cases} u + \tau \partial_x(f_\varepsilon(u)\partial_x I(u)) &= g \text{ in } \Omega \\ \partial_x u = 0 \text{ and } \partial_x I(u) &= 0 \text{ on } \partial\Omega. \end{cases} \qquad (32)$$

Once we prove the existence of a solution for (32), a simple time discretization method will provide the existence of a solution to (28). We are going to prove:

Proposition 6 (The stationary problem)**.** *For all $g \in H^{\frac{1}{2}}(\Omega)$, there exists $u \in H^2_N(\Omega)$ such that for all $\varphi \in H^1(\Omega)$,*

$$\frac{1}{\tau}\int_\Omega (u-g)\varphi\, dx - \int_\Omega f_\varepsilon(u)\, \partial_x I(u)\, \partial_x \varphi\, dx = 0\,. \qquad (33)$$

Furthermore,

$$\int_\Omega u(x)\,dx = \int_\Omega g(x)\,dx\,, \tag{34}$$

$$\|u\|^2_{\dot H^{\frac12}(\Omega)} + 2\tau \int f_\varepsilon(u)(\partial_x I u)^2\,dx \leq \|g\|^2_{\dot H^{\frac12}(\Omega)}\,, \tag{35}$$

and if $\int_\Omega F_\varepsilon(g)\,dx < \infty$ *then*

$$\int_\Omega F_\varepsilon(u)\,dx + \tau\|u\|^2_{\dot H_N^{\frac32}(\Omega)} \leq \int_\Omega F_\varepsilon(g)\,dx. \tag{36}$$

In order to prove such a result, we have to reformulate the problem:

New formulation of (33)**.** We are going to use classical variational methods to show the existence of a solution to (33). In order to work with a coercive non-linear operator, we need to take $\varphi = -I(v)$ as a test function. We note, however, that by doing that we would restrict ourself to test functions with zero mean value. In order to recover all test functions from $H^1(\Omega)$, we use Corollary 1 and consider

$$\varphi = -I(v) + \int_\Omega v\,dx \tag{37}$$

for some function $v \in H^2_N(\Omega)$. Let us emphasize the fact that Corollary 1 implies in particular that there is a one-to-one correspondence between $\varphi \in H^1(\Omega)$ and $v \in H^2_N(\Omega)$ satisfying (37).

Using (37), Equation (33) becomes:

$$\begin{aligned} -\int_\Omega u\,I(v)\,dx + \left(\int_\Omega u\,dx\right)\left(\int_\Omega v\,dx\right) + \tau\int_\Omega f_\varepsilon(u)\partial_x I(u)\partial_x I(v)\,dx \\ = -\int_\Omega g\,I(v)\,dx + \left(\int_\Omega g\,dx\right)\left(\int_\Omega v\,dx\right). \end{aligned} \tag{38}$$

We can now introduce the non-linear operator we are going to work with.

A non-linear operator. We define for all u and $v \in H^2_N(\Omega)$

$$A(u)(v) = -\int_\Omega u\,I(v)\,dx + \left(\int_\Omega u\,dx\right)\left(\int_\Omega v\,dx\right) + \tau\int_\Omega f_\varepsilon(u)\partial_x I(u)\partial_x I(v)\,dx.$$

We will show that this non-linear operator is continuous, coercive and pseudo-monotone. Classical theorems will permit us to conclude that we can solve the equation $A(u) = g$ for proper g's. Precisely, we have the following existence theorem

Proposition 7 (Existence for the new problem)**.** *For all* $g \in H^{\frac12}(\Omega)$ *there exists* $u \in H^2_N(\Omega)$ *such that*

$$A(u)(v) = -\int_\Omega g\,I(v)\,dx + \left(\int_\Omega g\,dx\right)\left(\int_\Omega v\,dx\right) \qquad \textit{for all } v \in H^2_N(\Omega). \tag{39}$$

Proof. We denote

$$V = H^2_N(\Omega).$$

For any $u \in V$ the functional $A(u)$ is clearly linear on V and since V is continuously embedded in $L^\infty(\Omega)$, we have

$$|A(u)(v)| \le \left[||u||_{H^{\frac{1}{2}}(\Omega)} + \tau(\varepsilon + ||u||_V^3)||u||_V\right] ||v||_V \, . \tag{40}$$

(Note that we used Proposition 4 to get this inequality). The non-linear operator A is thus well-defined as a map from V to V'. Moreover, it is bounded.

Next, we remark that we have

$$A(u)(u) \ge -\int_\Omega u\, I(u)\, dx + \left(\int_\Omega u\, dx\right)^2 + \varepsilon \int_\Omega |\partial_x I(u)|^2\, dx.$$

We deduce from Proposition 4 that

$$A(u)(u) \ge \tau\varepsilon ||u|^2_{H^2_N(\Omega)}. \tag{41}$$

In particular, we have

$$\frac{A(u)(u)}{\|u\|_V} \to +\infty \quad \text{ as } \|u\|_V \to +\infty.$$

The operator A is thus coercive. Proposition 7 will now be a consequence of classical theorems if we prove that A is a pseudo-monotone operator. Since we already know that A is bounded, it remains to prove the following lemma:

Lemma 2 (A is pseudo-monotone)**.** *Let u_j be a sequence of functions in V such that $u_j \rightharpoonup u$ weakly in V. Then*

$$\liminf_j \; A(u_j)(u_j - v) \ge A(u)(u - v).$$

Before we prove this lemma, let us notice that for $g \in H^{\frac{1}{2}}(\Omega)$, the application

$$T_g : v \mapsto -\int_\Omega g I(v)\, dx + \left(\int_\Omega g\, dx\right)\left(\int_\Omega v\, dx\right)$$

belongs to V'. Hence, using Theorem 2.7 (page 180) of [22], we deduce that for all $g \in H^{\frac{1}{2}}(\Omega)$, there exists a function $u \in V$ such that $A(u) = T_g$ in V', which completes the proof of Proposition 7. □

It remains to prove Lemma 2.

Proof of Lemma 2. We first write

$$\begin{aligned} A(u_j)(u_j - v) &= -\int_\Omega u_j I(u_j - v)\, dx + \left(\int_\Omega u_j\, dx\right)\left(\int_\Omega (u_j - v)\, dx\right) \\ &\quad + \tau \int_\Omega f_\varepsilon(u_j)\partial_x I(u_j)\partial_x I(u_j - v)\, dx \\ &= \|u_j\|^2_{H^{\frac{1}{2}}(\Omega)} - \langle u_j, v\rangle_{H^{\frac{1}{2}}} \\ &\quad + \tau \int_\Omega f_\varepsilon(u_j)(\partial_x I(u_j))^2 - \tau \int_\Omega f_\varepsilon(u_j)\partial_x(Iu_j)\partial_x(Iv) \end{aligned} \tag{42}$$

where

$$\langle u, v\rangle_{H^{\frac{1}{2}}} = -\int_\Omega uI(v)\,dx + \left(\int_\Omega u\,dx\right)\left(\int_\Omega v\,dx\right).$$

We need to check that we can pass to the limit in each of those terms.

Since u_j converges weakly in V we immediately get

$$\liminf_{j\to+\infty} \|u_j\|^2_{H^{\frac{1}{2}}(\Omega)} \geq \|u\|^2_{H^{\frac{1}{2}}(\Omega)}$$

and

$$\lim_{j\to+\infty} \langle u_j, v\rangle_{H^{\frac{1}{2}}} = -\langle u, v\rangle_{H^{\frac{1}{2}}}.$$

Since u_j is bounded in $H^2_N(\Omega)$, it is compact in $L^\infty(\Omega)$, and so $f_\varepsilon(u_j)$ converges to $f_\varepsilon(u)$ strongly in $L^\infty(\Omega)$. We thus write

$$\begin{aligned}\int_\Omega f_\varepsilon(u_j)(\partial_x I(u_j))^2 &= \int_\Omega \big(f_\varepsilon(u_j) - f_\varepsilon(u)\big)(\partial_x I(u_j))^2 + \int_\Omega f_\varepsilon(u)(\partial_x I(u_j))^2 \\ &\geq -\|f_\varepsilon(u_j) - f_\varepsilon(u)\|_{L^\infty(\Omega)}\|u_j\|_V^2 + \int_\Omega f_\varepsilon(u)(\partial_x I(u_j))^2.\end{aligned}$$

The first term goes to zero and we have

$$\sqrt{f_\varepsilon(u)}\partial_x I(u_j) \rightharpoonup \sqrt{f_\varepsilon(u)}\partial_x I(u) \text{ in } L^2(\Omega).$$

Again, the lower semicontinuity of the L^2-norm gives

$$\lim_{j\to\infty} \tau \int_\Omega f_\varepsilon(u)(\partial_x I(u_j))^2 \geq \int_\Omega f_\varepsilon(u)(\partial_x I(u))^2.$$

Finally, we have

$$\begin{aligned} f_\varepsilon(u_j)\partial_x I(v) &\to f_\varepsilon(u)\partial_x I(v) && \text{in } L^2(\Omega) \text{ strong,} \\ \partial_x I(u_j) &\rightharpoonup \partial_x I(u) && \text{in } L^2(\Omega) \text{ weak}\end{aligned}$$

which gives the convergence of the last term in (42) and completes the proof of the lemma. □

Finally, we prove Proposition 6.

Proof of Proposition 6. For a given $g \in H^{\frac{1}{2}}(\Omega)$, Proposition 7 yields the existence of a solution $u \in V$ of (38). We recall that for any $\varphi \in H^1(\Omega)$, there exists $v \in V$ such that

$$\varphi = -I(v) + \int_\Omega v\,dx$$

and so equivalently, we have that u satisfies (33) for all $\varphi \in H^1(\Omega)$.

Next, we note that the mass conservation equality (34) is readily obtained by taking $v = 1$ as a test function in (38), while (35) follows by taking $v = u - \int_\Omega u\,dx$:

$$\|u\|^2_{\dot H^{\frac12}(\Omega)} + \tau \int_\Omega f_\varepsilon(u)|\partial_x I(u)|^2 = -\int_\Omega gI(u)\,dx$$
$$\leq \|g\|_{\dot H^{\frac12}(\Omega)}\|u\|_{\dot H^{\frac12}(\Omega)} \leq \frac12 ||g||^2_{\dot H^{\frac12}(\Omega)} + \frac12 ||u||^2_{\dot H^{\frac12}(\Omega)}.$$

Finally since F'_ε is smooth with F'_ε and F''_ε bounded and Ω is bounded, we have $F'_\varepsilon(u) \in H^1(\Omega)$. We can thus find $v \in V$ such that

$$-I(v) + \int_\Omega v(x)\,dx = F'_\varepsilon(u).$$

Equation (38) then implies:

$$-\int_\Omega uF'_\varepsilon(u)\,dx + \tau\int_\Omega f_\varepsilon(u)\,F''_\varepsilon(u)\,\partial_x I(u)\,\partial_x u\,dx = -\int_\Omega gF'_\varepsilon(u)\,dx$$

and so (using the definition of F_ε given in Proposition 5)

$$-\tau\int_\Omega \partial_x I(u)\,\partial_x u\,dx = \int_\Omega F'_\varepsilon(u)(g-u)\,dx$$

Since F_ε is convex ($F''_\varepsilon \geq 0$), we have $F'_\varepsilon(u)(g-u) \leq F_\varepsilon(g) - F_\varepsilon(u)$ and we deduce (36) from Proposition 4. □

4.2 Proof of Proposition 5

In order to construct the solution u^ε of (26), we discretize the problem with respect to t, and construct a piecewise constant function

$$u^\tau(t,x) = u^n(x) \text{ for } t \in (n\tau,(n+1)\tau)), n \in \{0,\dots,N+1\},$$

where $\tau = T/N$ and $(u^n)_{n\in\{0\dots N+1\}}$ is such that

$$\frac1\tau(u^{n+1}-u^n) + \partial_x(f_\varepsilon(u^{n+1})\partial_x I(u^{n+1})) = 0\,.$$

The existence of the u^n follows from Proposition 6 by induction on n. We deduce:

Corollary 2 (Discrete in time approximate solution)**.** *For any $N > 0$ and $u_0^\varepsilon \in H^{\frac12}(\Omega)$, there exists a function $u^\tau \in L^\infty(0,T;H^{\frac12}(\Omega))$ such that*

- *$t \mapsto u^\tau(t,x)$ is constant on $[k\tau,(k+1)\tau)$ for $k \in \{0,\dots,N\}$, $\tau = \frac{T}{N}$*
- *$u^\tau = u_0$ on $[0,\tau)\times\mathbb{R}$,*

- *for all* $\varphi \in \mathcal{C}^1(0,T,H^1(\Omega))$,
$$\iint_{Q_{\tau,T}} \frac{u^\tau - S_\tau u^\tau}{\tau} \varphi \, dx\, dt = \iint_{Q_{\tau,T}} f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi \, dx\, dt \tag{43}$$
where $Q_{\tau,T} = (\tau,T)\times\Omega$ *and* $S_\tau u^\tau(t,x) = u^\tau(t-\tau,x)$.

Moreover, the function u^τ *satisfies*
$$\int_\Omega u^\tau(t,x)\, dx = \int_\Omega u_0(x)\, dx \qquad a.e.\ t \in (0,T) \tag{44}$$
and for all $t\in(0,T)$
$$\|u^\tau(t,\cdot)\|^2_{\dot H^{\frac12}(\Omega)} + 2\int_{Q_t} f_\varepsilon(u^\tau)(\partial_x I(u^\tau))^2\, dx\, dt \le \|u_0\|^2_{\dot H^{\frac12}(\Omega)} \tag{45}$$
and if $\int_\Omega F_\varepsilon(u_0)\, dx < \infty$*, then for all* $t\in(0,T)$
$$\int_\Omega F_\varepsilon(u^\tau(t,\cdot))\, dx + \int_0^t \|u^\tau\|^2_{\dot H^{\frac32}_N(\Omega)}\, ds \le \int_\Omega F_\varepsilon(u_0)\, dx. \tag{46}$$

We now have to prove that u^τ converges to a solution of (28) as τ goes to zero.

Proof of Proposition 5. The proof is divided in three steps.

Step 1: *a priori* estimates. We summarize the a priori estimates in the following lemma:

Lemma 3 (*A priori* estimates)**.** *The solution* u^τ *constructed in Corollary 2 satisfies*
$$\|u^\tau\|_{L^\infty(0,T,H^{\frac12}(\Omega))} \le \|u_0^\varepsilon\|_{H^{\frac12}(\Omega)}, \tag{47}$$
$$\sqrt{\varepsilon}\|\partial_x I(u^\tau)\|_{L^2(Q)} \le C, \tag{48}$$
$$\left\|\frac{u^\tau - S_\tau u^\tau}{\tau}\right\|_{L^2(\tau,T,W^{-1,r'}(\Omega))} \le C \tag{49}$$
for all $r' \in (2,+\infty)$ *where* C *does not depend on* $\tau>0$ *(but does depend on* r'*).*

Proof. Estimate (47) and (48) are direct consequences of (44) and (45).

Next, we note that
$$\frac{u^\tau - S_\tau u^\tau}{\tau} = \partial_x\Big(-f_\varepsilon(u^\tau)\partial_x I(u^\tau)\Big).$$
The bound (47) and Sobolev embedding theorems imply that $(u^\tau)_\tau$ is bounded in $L^\infty(0,T,L^p(\Omega))$ for all $p<\infty$ and so $f_\varepsilon(u^\tau)$ is bounded in $L^\infty(0,T,L^p(\Omega))$ for all $p<\infty$. Since $\partial_x I(u^\tau)$ is bounded in $L^2(Q)$, we deduce that $f_\varepsilon(u^\tau)\partial_x I(u^\tau)$ is bounded in $L^2(\tau,T,L^r(\Omega))$ for all $r\in[1,2)$. It follows that
$$\partial_x\left(f_\varepsilon(u^\tau)\partial_x I(u^\tau)\right) \text{ is bounded in } L^2(\tau,T,W^{-1,r'}(\Omega))$$
for all $r'\in(2,\infty)$. ☐

Step 2: Compactness result. Thanks to the following imbeddings

$$H^{\frac{1}{2}}(\Omega) \hookrightarrow L^q(\Omega) \to W^{-1,r'}(\Omega)$$

for all $q < \infty$, we can use Aubin's lemma to obtain that $(u^\tau)_\tau$ is relatively compact in $\mathcal{C}^0(0,T,L^q(\Omega))$ for all $q < \infty$.

Remark that $(\partial_x I(u^\tau))_\tau$ is bounded in $L^2(Q)$ and $(u^\tau)_\tau$ is bounded in $L^\infty(0,T;L^1(\Omega))$. It follows that $(u^\tau)_\tau$ is bounded in $L^2(0,T,H^2_N(\Omega))$. Since

$$H^2_N(\Omega) \hookrightarrow H^{\frac{3}{2}}_N(\Omega) \to W^{-1,r'}(\Omega)$$

we deduce that $(u^\tau)_\tau$ is relatively compact in $L^2(0,T;H^{\frac{3}{2}}_N(\Omega))$. Up to a subsequence, we can thus assume that

- $u^\tau \to u^\varepsilon \in L^\infty(0,T,H^{\frac{1}{2}}(\Omega))$ almost everywhere in Q;
- $u^\tau \to u^\varepsilon$ in $L^2(0,T,H^1(\Omega))$ strong;
- $\partial_x I(u^\tau) \rightharpoonup \partial_x I(u^\varepsilon)$ in $L^2(Q)$ weak.

Step 3: Derivation of Equation (28)**.** We want to pass to the limit in (43).

We fix $\varphi \in \mathcal{C}^1_c([0,T),H^1(\Omega))$. Then

$$\iint_{Q_\tau} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi = \iint_Q u^\tau(t,x)\frac{\varphi(t,x)-\varphi(t+\tau,x)}{\tau} - \frac{1}{\tau}\int_0^\tau\int_\Omega u^\tau(t,x)\varphi(t,x)\,dx + \frac{1}{\tau}\int_{T-\tau}^T\int_\Omega u^\tau(t,x)\varphi(t,x)\,dx.$$

We deduce:

$$\iint_{Q_\tau} \frac{u^\tau - S_\tau u^\tau}{\tau}\varphi \to -\iint_Q u^\varepsilon(\partial_t\varphi) - \int_\Omega u^\varepsilon(0,x)\varphi(0,x)\,dx + 0.$$

It remains to pass to the limit in the non-linear term. Let $\eta > 0$. Since $u^\tau \to u^\varepsilon$ almost everywhere in Q, Egorov's theorem yields the existence of a set $A_\eta \subset Q$ such that $|Q \setminus A_\eta| \leq \eta$ and

$$u^\tau \to u^\varepsilon \text{ uniformly in } A_\eta\,.$$

In particular,

$$\sqrt{f_\varepsilon(u^\tau)}\partial_x\varphi \to \sqrt{f_\varepsilon(u^\varepsilon)}\partial_x\varphi \text{ in } L^2(A_\eta)$$

and

$$\sqrt{f_\varepsilon(u^\tau)}\partial_x I(u^\tau) \rightharpoonup \sqrt{f_\varepsilon(u^\varepsilon)}\partial_x I(u^\varepsilon) \text{ in } L^2(A_\eta)\,. \tag{50}$$

Hence

$$\int_{A_\eta} f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi \to \int_{A_\eta} f_\varepsilon(u^\varepsilon)\partial_x I(u^\varepsilon)\partial_x\varphi$$

as τ goes to zero.

Finally, we look at what happens on $Q \setminus A_\eta$. Choose p_1, p_2, p_3 such that $\sum_i p_i^{-1} = 1$ and write

$$\begin{aligned}\int_{Q\setminus A_\eta} &|f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi| \\ &\le \|\partial_x\varphi\|_{L^\infty(0,T,L^{p_1}(\Omega))}\int_0^T \|f_\varepsilon(u^\tau)\partial_x I(u^\tau)\|_{L^{p_2}(\Omega)}\|\mathbf{1}_{Q\setminus A_\eta}\|_{L^{p_3}(\Omega)} \\ &\le \|\partial_x\varphi\|_{L^\infty(0,T,L^{p_1}(\Omega))}\|f_\varepsilon(u^\tau)\partial_x I(u^\tau)\|_{L^2(0,T,L^{p_2}(\Omega))}\|\mathbf{1}_{Q\setminus A_\eta}\|_{L^2(0,T,L^{p_3}(\Omega))}.\end{aligned}$$

We now choose $p_2 \in [1,2)$ (and so $p_1 > 2$ and $p_3 > 2$).

$$\int_{Q\setminus A_\eta} |f_\varepsilon(u^\tau)\partial_x I(u^\tau)\partial_x\varphi| \le C(\varphi)\|\mathbf{1}_{Q\setminus A_\eta}\|_{L^{p_3}(Q)} \le C(\varphi)\eta^{\frac{1}{p_3}}.$$

Since η is arbitrary, the proof is complete.

Step 4: Inequalities. Since $u^\tau \to u^\varepsilon$ in $L^\infty(0,T,L^1(\Omega))$, mass conservation equation (29) follows from (44).

Estimate (30) follows from (45). Indeed, since $u^\tau \to u^\varepsilon$ almost everywhere, Proposition 4 and Fatou's lemma imply that for almost $t \in (0,T)$

$$\|u^\varepsilon(t)\|^2_{\dot H^{\frac12}(\Omega)} \le \liminf_{\tau\to 0}\|u^\tau(t)\|^2_{\dot H^{\frac12}(\Omega)}.$$

Thanks to (50), we also have

$$\begin{aligned}\int_0^T\int_\Omega f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2 \le \liminf_{\tau\to 0}&\int_0^T\int_\Omega f_\varepsilon(u^\tau)(\partial_x I(u^\tau))^2\,dx\,dt \\ &+\int_0^T\int_{\Omega\setminus A_\eta} f_\varepsilon(u^\varepsilon)|\partial_x I(u^\varepsilon)|^2.\end{aligned}$$

Letting $\eta \to 0$ permits to conclude.

To derive (31) we note that $F_\varepsilon(u^\tau) \to F_\varepsilon(u^\varepsilon)$ almost everywhere. So Fatou's Lemma implies for almost every $t \in (0,T)$

$$\int_\Omega F_\varepsilon(u^\varepsilon(t,x))\,dx \le \liminf_{\tau\to 0}\int_\Omega F_\varepsilon(u^\tau(t,x))\,dx \le \int_\Omega F_\varepsilon(u_0)\,dx.$$

Finally, since $(u^\tau)_\tau$ is relatively compact in $L^2(0,T;H^{\frac32}_N(\Omega))$, we have

$$\int_0^t ||u^\varepsilon(s)||^2_{\dot H^{\frac32}}\,ds = \lim_{\tau\to 0}\int_0^t ||u^\tau(s)||^2_{\dot H^{\frac32}}\,ds$$

and so (31) follows from (46). □

5 Proof of Theorem 1

Proof of Theorem 1. Proposition 5 provides the existence of a solution $u^\varepsilon \in L^\infty(0,T;H^{\frac{1}{2}}(\Omega)) \cap L^2(0,T;H^2_N(\Omega))$ of (28). Our goal is now to pass to the limit $\varepsilon \to 0$. We point out that at this point, the solution u^ε may change sign and that it is only at the limit $\varepsilon \to 0$ that we are able to recover a non-negative solution.

Step 1: Compactness result. Arguing as in Lemma 3, we can show that $(u^\varepsilon)_\varepsilon$ is bounded in $L^\infty(0,T,H^{\frac{1}{2}}(\Omega))$ and $\partial_t u^\varepsilon$ is bounded in $L^2(0,T,W^{-1,r'}(\Omega))$ for all $r' \in (1,2)$. Then $(u^\varepsilon)_\varepsilon$ is relatively compact in $\mathcal{C}^0(0,T,L^r(\Omega))$ for all $r \in (2,+\infty)$. Hence, we can extract a subsequence that converges to u in $\mathcal{C}^0(0,T,L^r(\Omega))$ for all $r \geq 1$ and almost everywhere in Q.

Note in particular that u satisfies (8) (by passing to the limit in (29)).

Step 2: Derivation of Equation (5)**.** We now have to pass to the limit in (28). We fix $\varphi \in \mathcal{D}(Q)$. Since $u^\varepsilon \to u$ in $\mathcal{C}^0(0,T,L^1(\Omega))$, we have

$$\iint_Q u^\varepsilon \partial_t \varphi \, dx \to \iint_Q u \, \partial_t \varphi \, dx.$$

Next, we remark that, since $H^{\frac{1}{2}}(\Omega) \to L^2(\Omega)$, (30) implies

$$\varepsilon \iint_Q (\partial_x I(u^\varepsilon))^2 + \varepsilon \iint_Q (\partial_x u^\varepsilon)^2 \leq C_0$$

for some constant C_0 only depending u_0. Cauchy-Schwarz inequality thus implies

$$\iint_Q \varepsilon \partial_x I(u^\varepsilon) \partial_x \varphi \quad \to \quad 0\,.$$

Finally, since $(u^\varepsilon)_+^{\frac{3}{2}} \partial_x I(u^\varepsilon)$ is bounded in $L^2(0,T,L^2(\Omega))$ and u^ε is bounded in $L^\infty(0,T;L^p(\Omega))$ for all $p < \infty$ we have that $(u^\varepsilon)_+^3 \partial_x I(u^\varepsilon)$ is bounded in $L^2(0,T;L^r(\Omega))$ for all $r \in [1,2)$ and so

$$h^\varepsilon := (u^\varepsilon)_+^3 \partial_x I(u^\varepsilon) \rightharpoonup h \quad \text{ in } L^2(0,T;L^r(\Omega))\text{-weak.}$$

Passing to the limit in (28), we deduce:

$$\iint_Q u \, \partial_t \varphi \, dx\, dt + \iint_Q h \, \partial_x \varphi \, dx\, dt = -\iint_Q u_0 \varphi(0,\cdot) \, dx\, dt$$

for all $\varphi \in \mathcal{D}(\overline{Q})$. In order to get (5), it only remains to show that

$$h = u_+^3 \partial_x I(u)$$

in the following sense:

$$\iint_Q h\,\phi\,dx\,dt = -\iint_Q 3u_+^2\partial_x u\; I(u)\,\phi\,dx\,dt - \iint_Q u_+^3 I(u)\;\partial_x\phi\,dx\,dt \qquad (51)$$

for all test function ϕ such that $\phi|_{\partial\Omega} = 0$; that is

$$h = \partial_x(u_+^3 I(u)) - 3u_+^2(\partial_x u)I(u) \quad \text{ in } \mathcal{D}'(\Omega).$$

For that we note that since

$$\int_\Omega F_\varepsilon(u_0)\,dx \le C,$$

Inequality (31) implies that $(u^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T;H^{\frac{3}{2}}(\Omega))$. Recall that $(\partial_t u^\varepsilon)_\varepsilon$ is bounded in $L^2(0,T,W^{-1,r'}(\Omega))$ for all $r' \in [1,2)$. Aubin's lemma then implies that u^ε is relatively compact in $L^2(0,T;H^s(\Omega))$ for $s < 3/2$. In particular, we can assume that

$$I(u^\varepsilon) \to I(u) \quad \text{ in } L^2(0,T;L^2(\Omega))$$

and

$$\partial_x u^\varepsilon \to \partial_x u \quad \text{ in } L^2(0,T;L^p(\Omega)), \text{ for all } p<\infty.$$

Writing

$$\begin{aligned}\iint_Q h^\varepsilon\phi &= \iint_Q (u^\varepsilon)_+^3\partial_x I(u^\varepsilon)\;\phi\,dx\,dt\\ &= -\iint_Q 3(u^\varepsilon)_+^2\partial_x u^\varepsilon\; I(u^\varepsilon)\;\phi\,dx\,dt - \iint_Q (u^\varepsilon)_+^3\; I(u^\varepsilon)\;\partial_x\phi\,dx\,dt,\end{aligned}$$

we see that those estimates, together with the fact that u^ε converges to u in $L^\infty(0,T;L^p(\Omega))$ for all $p<\infty$, are enough to pass to the limit and get (51).

Step 4: Properties of u. It is readily seen that u satisfies (8) and the lower semicontinuity of the norm implies (10).

Next, we denote $g^\varepsilon = (u_+^\varepsilon)^{\frac{3}{2}}\partial_x I(u^\varepsilon)$. Inequality (30) implies that g^ε converges weakly in $L^2((0,T)\times\Omega)$ to a function g, and the lower semicontinuity of the norm implies (9). Proceeding as above we now easily show that

$$g = \partial_x(u^{\frac{3}{2}}I(u)) - \frac{3}{2}u^{\frac{1}{2}}\partial_x u\; I(u) \quad \text{ in } \mathcal{D}'(\Omega).$$

Step 5: non-negative solutions. It remains to prove that u is non-negative. This will be a consequence of (31). Indeed, we have

$$\int_\Omega F_\varepsilon(u^\varepsilon(T,\cdot)) \le \int_\Omega F_\varepsilon(u_0)$$

where F_ε is such that $F_\varepsilon'' = \frac{1}{r_+^3+\varepsilon}$. We thus take

$$F_\varepsilon(r) = \int_r^\infty \int_s^{+\infty} F_\varepsilon''(t)dtds.$$

In particular, $F_\varepsilon(r)$ is a decreasing convex function of $r \in \mathbb{R}$ for all ε, and F_ε is decreasing with respect to ε (so $F_\varepsilon(r) \leq F_0(r)$ for all $\varepsilon > 0$). Note that

$$F_0(r) = \begin{cases} \frac{1}{r} & \text{if } r > 0\,, \\ +\infty & \text{if } r \leq 0\,. \end{cases}$$

Hence

$$\int_\Omega F_\varepsilon(u_0)\,dx \leq \int_\Omega F_0(u_0)\,dx = \int_\Omega \frac{1}{u_0}\,dx < +\infty\,.$$

By the monotone convergence theorem, we conclude that

$$\limsup_{\varepsilon\to 0} \int_\Omega F_\varepsilon(u^\varepsilon(T,\cdot)) < +\infty\,. \tag{52}$$

Next, we recall that $u^\varepsilon(T,\cdot)$ converges strongly in $L^p(\Omega)$ to $u(T,\cdot)$. We can thus assume that it also converges almost everywhere. Egorov's theorem then implies the existence of a set $A_\eta \subset \Omega$ such that $u^\varepsilon(T,\cdot) \to u$ uniformly in A_η and $|\Omega \setminus A_\eta| < \eta$. For some $\delta > 0$, we now consider

$$C_{\eta,\delta} = A_\eta \cap \{u(T,\cdot) \leq -2\delta\}.$$

For every η, $\delta > 0$ there exists $\varepsilon_0(\eta,\delta)$ such that if $\varepsilon \leq \varepsilon_0(\eta,\delta)$, then $u^\varepsilon(T,\cdot) \leq -\delta$ in $C_{\eta,\delta}$.

But this implies that $C_{\eta,\delta}$ has measure zero. Indeed, if not, then for $\varepsilon \leq \varepsilon_0(\eta,\delta)$ we have

$$F_\varepsilon(u^\varepsilon(T,x)) \geq F_\varepsilon(-\delta) \longrightarrow F_0(-\delta) = +\infty \text{ for all } x \in C_{\eta,\delta}$$

and by the monotone convergence theorem, we deduce

$$\lim_{\varepsilon\to 0} \int_{C_{\eta,\delta}} F_\varepsilon(u^\varepsilon(T,x))\,dx = +\infty$$

which contradicts (52).

We deduce that for all $\delta > 0$ and all $\eta > 0$ we have

$$|\{u(T,\cdot) \leq -2\delta\}| \leq |C_{\eta,\delta}| + |\Omega \setminus A_\eta| \leq \eta$$

and so $|\{u(T,\cdot) \leq -2\delta\}| = 0$ for all $\delta > 0$. We can conclude that

$$\{u(T,\cdot) < 0\} = \bigcap_{n\geq 1} \left\{u(T,\cdot) < -\frac{1}{n}\right\}$$

has measure zero. □

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References

[1] J. I. Adachi and E. Detournay, *Plane-strain propagation of a fluid-driven fracture: finite toughness self-similar solution*, Proc. Roy. Soc. London Series A, (1994).

[2] J. I. Adachi and A. P. Peirce, *Asymptotic analysis of an elasticity equation for a finger-like hydraulic fracture*, J. Elasticity, 90 (2008), pp. 43–69.

[3] R. A. Adams and J. J. Fournier, *Sobolev spaces*, Academic Press, 2003. Pure and Applied Mathematics, Vol. 140.

[4] M. S. Agranovich and B. A. Amosov, *On Fourier series in eigenfunctions of elliptic boundary value problems*, Georgian Math. J., 10 (2003), pp. 401–410. Dedicated to the 100th birthday anniversary of Professor Victor Kupradze.

[5] E. Beretta, M. Bertsch, and R. Dal Passo, *Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation*, Arch. Rational Mech. Anal., 129 (1995), pp. 175–200.

[6] F. Bernis, *Finite speed of propagation and asymptotic rates for some nonlinear higher order parabolic equations with absorption*, Proc. Roy. Soc. Edinburgh Sect. A, 104 (1986), pp. 1–19.

[7] ———, *Viscous flows, fourth order nonlinear degenerate parabolic equations and singular elliptic problems*, in Free boundary problems: theory and applications (Toledo, 1993), vol. 323 of Pitman Res. Notes Math. Ser., Longman Sci. Tech., Harlow, 1995, pp. 40–56.

[8] F. Bernis, *Finite speed of propagation and continuity of the interface for thin viscous flows*, Adv. Differential Equations, 1 (1996), pp. 337–368.

[9] ———, *Finite speed of propagation for thin viscous flows when* $2 \leq n < 3$, C. R. Acad. Sci. Paris Sér. I Math., 322 (1996), pp. 1169–1174.

[10] F. Bernis and A. Friedman, *Higher order nonlinear degenerate parabolic equations*, J. Differential Equations, 83 (1990), pp. 179–206.

[11] F. Bernis, J. Hulshof, and F. Quirós, *The "linear" limit of thin film flows as an obstacle-type free boundary problem*, SIAM J. Appl. Math., 61 (2000), pp. 1062–1079 (electronic).

[12] A. L. Bertozzi and M. Pugh, *The lubrication approximation for thin viscous films: the moving contact line with a "porous media" cut-off of van der Waals interactions*, Nonlinearity, 7 (1994), pp. 1535–1564.

[13] ———, *The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions*, Comm. Pure Appl. Math., 49 (1996), pp. 85–123.

[14] B. Buffoni, E. N. Dancer, and J. F. Toland, *The regularity and local bifurcation of steady periodic water waves*, Arch. Ration. Mech. Anal., 152 (2000), pp. 207–240.

[15] X. Cabré and J. Tan, *Positive solutions of nonlinear problems involving the square root of the laplacian*, Preprint, (2009).

[16] ——, *Positive solutions of nonlinear problems involving the square root of the laplacian.* Preprint, 2009.

[17] L. A. Caffarelli and A. Friedman, *Regularity of the free boundary of a gas flow in an n-dimensional porous medium*, Indiana Univ. Math. J., 29 (1980), pp. 361–391.

[18] J. Desroches, E. Detournay, B. Lenoach, P. Papanastasiou, J. R. A. Pearson, M. Thiercelin, and A. Cheng, *The crack tip region in hydraulic fracturing*, Proc. R. Soc. Lond. A, 447 (1994), pp. 39–48.

[19] J. Dolbeault, I. Gentil, and A. Jüngel, *A logarithmic fourth-order parabolic equation and related logarithmic Sobolev inequalities*, Commun. Math. Sci., 4 (2006), pp. 275–290.

[20] A. Friedman, *Variational principles and free-boundary problems*, Robert E. Krieger Publishing Co. Inc., Malabar, FL, second ed., 1988.

[21] J. Geertsma and F. de Klerk, *A rapid method of predicting width and extent of hydraulically induced fractures*, Journal of Petroleum Technology, 21 (1969), pp. 1571–1581.

[22] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[23] V. G. Maz'ja, *Sobolev spaces*, Springer-Verlag, 1985.

[24] S. L. Mitchell, R. Kuske, and A. P. Peirce, *An asymptotic framework for finite hydraulic fractures including leak-off*, SIAM J. Appl. Math., 67 (2006/07), pp. 364–386 (electronic).

[25] O. A. Oleinik, A. S. Kalashnikov, and C. Yiu-Lin, *The Cauchy problem and boundary problems for equations of the type of unsteady filtration*, Izv. Akad. Nauk. SSSR Ser. Mat., 22 (1958), pp. 667–704.

[26] A. Peirce and E. Detournay, *An Eulerian moving front algorithm with weak-form tip asymptotics for modeling hydraulically driven fractures*, Comm. Numer. Methods Engrg., 25 (2009), pp. 185–200.

[27] A. P. Peirce and E. Siebrits, *A dual mesh multigrid preconditioner for the efficient solution of hydraulically driven fracture problems*, Internat. J. Numer. Methods Engrg., 63 (2005), pp. 1797–1823.

[28] A. P. Peirce and E. Siebrits, *An Eulerian finite volume method for hydraulic fracture problems*, in Finite volumes for complex applications IV, ISTE, London, 2005, pp. 655–664.

[29] J. Simon, *Compact sets in the space* $L^p(0,T;B)$, Ann. Math. Pura Appl., 146 (1987), pp. 65–96.

[30] D. A. Spence and P. Sharp, *Self-similar solutions for elastohydrodynamic cavity flow*, Proc. Roy. Soc. London Ser. A, 400 (1985), pp. 289–313.

[31] J. L. Vázquez, *The porous medium equation*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, Oxford, 2007. Mathematical theory.

[32] Y. P. Zheltov and S. A. Khristianovich, *On hydraulic fracturing of an oil-bearing stratum*, Izv. Akad. Nauk SSSR. Otdel Tekhn. Nuk, 5 (1955), pp. 3–41.

[33] A. Zygmund, *Trigonometric series. Vol. I, II*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, third ed., 2002. With a foreword by Robert A. Fefferman.