



## Interpretation and Computation of $\alpha$ -Junctions for Combining Belief Functions

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### Abstract

The  $\alpha$ -junctions are the associative, commutative and linear operators for belief functions with a neutral element. This family of rules includes as particular cases the unnormalized Dempster's rule and the disjunctive rule. Until now, the  $\alpha$ -junctions suffered from two main limitations. First, they did not have an interpretation in the general case. Second, it was difficult to compute a combination by an  $\alpha$ -junction. In this paper, an interpretation for these rules is proposed. It is shown that the  $\alpha$ -junctions correspond to a particular form of knowledge about the truthfulness of the sources providing the belief functions to be combined. Simple means to compute a combination by an  $\alpha$ -junction are also laid bare. These means are based on generalizations of mechanisms that exist to compute the combination by the unnormalized Dempster's rule.

**Keywords.** Transferable Belief Model, Dempster-Shafer Theory, Belief Functions, Information Fusion, Uncertain reasoning.

### 1 Introduction

The Transferable Belief Model (TBM) [16, 12] is a model for quantifying beliefs using belief functions [8]. An essential part of the TBM is the aggregation of belief functions, which is done using so-called combination rules. To accommodate for various information fusion problems, many combination rules have been proposed (see, e.g., [15] for a recent survey) and, in particular, the unnormalized version of Dempster's rule [1], referred to as the conjunctive rule in this paper, the disjunctive rule [3, 9], the exclusive disjunctive rule and its negation [3, 11].

The use of the conjunctive rule is appropriate when one can assume that all sources providing the belief functions to be combined, tell the truth [11]. On the other hand, the disjunctive rule should be used when

it is known that at least one of the sources tells the truth, but it is not known which one [11]. The uses of the exclusive disjunctive rule and its negation are also conditioned by knowledge on the truthfulness of the sources of information: the former fits with the case where exactly one of the sources is known to tell the truth, but it is not known which one, whereas the latter corresponds to a situation where either all or none of the sources are known to tell the truth [11]. Furthermore, all of these four rules assume that the sources are independent, meaning that those sources are assumed to provide distinct pieces of evidence.

In [11], Smets introduced an infinite family of combination rules, which he called  $\alpha$ -junctions. This family basically represents the set of associative, commutative and linear operators for belief functions with a neutral element. It includes as special cases the four rules mentioned above. The behavior of an  $\alpha$ -junction is determined by a parameter  $\alpha$  and by the neutral element. The four special cases are recovered for particular values of  $\alpha$ . For other values of this parameter, the  $\alpha$ -junctions did not have an interpretation.

To our knowledge, this family of rules has never been exploited. This can be explained, at least in part, by the fact that these rules suffered from two main limitations until now. First, those operators did not have an interpretation in the general case. Second, it was difficult to compute a combination by an  $\alpha$ -junction using the methods proposed in [11], as already remarked by Smets [13].

In this paper, this theoretical contribution of Smets is carefully reexamined: some new light on the meaning of the  $\alpha$ -junctions is shed and their mathematics are simplified to make their computation easier. More precisely, it is first shown that these operators correspond to a particular form of knowledge, determined by the parameter  $\alpha$ , on the truthfulness of the sources. The  $\alpha$ -junctions become thus suitable as flexible combination rules that allow us to take into account some particular knowledge about the sources.

Several efficient and simple ways of computing a combination by an  $\alpha$ -junction are then presented, making the practical use of the  $\alpha$ -junctions in applications possible. These new means are based on generalizations of mechanisms that can be used to compute the combination by the conjunctive rule.

The rest of this paper is organized as follows. Necessary concepts of the TBM are first recalled in Section 2. In Section 3, basic notions on  $\alpha$ -junctions are given. An interpretation for the  $\alpha$ -junctions is proposed in Section 4. Several simple means to compute a combination by an  $\alpha$ -junction are then unveiled in Section 5. Section 6 concludes the paper.

Note that due to lack of space, the proofs of the theorems and propositions presented in this paper, are not provided. They can be found in [7].

## 2 Fundamental Concepts of the TBM

### 2.1 Representation of Beliefs

In this paper, the TBM [16, 12] is accepted as a model to quantify uncertainties based on belief functions [8]. Let  $\Omega = \{\omega_1, \dots, \omega_K\}$  denote a finite set of possible values of a variable  $\omega$ ;  $\Omega$  is called the frame of discernment of  $\omega$ . In the TBM, the beliefs held by a rational agent  $Ag$  regarding the actual value  $\omega_0$  taken by  $\omega$  is represented by a basic belief assignment (BBA)  $m$  defined as a mapping from  $2^\Omega$  to  $[0, 1]$  verifying  $\sum_{A \subseteq \Omega} m(A) = 1$ . Subsets  $A$  of  $\Omega$  such that  $m(A) > 0$  are called focal sets of  $m$ . A BBA  $m$  is said to be: *vacuous* if  $\Omega$  is the only focal set, this BBA is denoted by  $m_\Omega$ ; *categorical* if it has only one focal set; *simple* if it has at most two focal sets and, if it has two,  $\Omega$  is one of those. A simple BBA (SBBA)  $m$  such that  $m(A) = 1 - \alpha$  for some  $A \neq \Omega$  and  $m(\Omega) = \alpha$ , can be written  $A^\alpha$ . This notation for SBBA is useful in this paper to shorten some expressions.

A BBA  $m$  can equivalently be represented by its associated belief, plausibility and commonality functions defined, respectively, as:

$$bel(A) = \sum_{\emptyset \neq B \subseteq A} m(B),$$

$$pl(A) = \sum_{B \cap A \neq \emptyset} m(B),$$

and

$$q(A) = \sum_{B \supseteq A} m(B), \quad (1)$$

for all  $A \subseteq \Omega$ . The BBA  $m$  can be recovered from any of these functions. For instance:

$$m(A) = \sum_{B \supseteq A} (-1)^{|B|-|A|} q(B), \quad \forall A \subseteq \Omega,$$

where  $|A|$  denotes the cardinality of  $A$ .

The negation (or complement)  $\bar{m}$  of a BBA  $m$  is defined as the BBA verifying  $\bar{m}(A) = m(\bar{A})$ ,  $\forall A \subseteq \Omega$ , where  $\bar{A}$  denotes the complement of  $A$  [3].  $\bar{m}$  represents the BBA that would be induced if the agent knows that the source providing a BBA  $m$  is not telling the truth, i.e., is lying [11].

Another important concept of the TBM is the least commitment principle (LCP) [9]. This principle postulates that, given a set of BBAs compatible with a set of constraints, the most appropriate BBA is the least informative. The LCP becomes operational through the definition of partial orderings allowing the informational comparison of BBAs. Such orderings were proposed in [17] and [3]. For instance, the  $q$ -ordering is defined as follows: a BBA  $m_1$  is said to be at least as  $q$ -committed, or at least as  $q$ -informed, than a BBA  $m_2$  iff we have  $q_1(A) \leq q_2(A)$ , for all  $A \subseteq \Omega$ .

### 2.2 Combination of Beliefs

The beliefs represented by BBAs can be aggregated using appropriate operators, called combination rules. In this section, the definitions of some of these combination rules are provided. Some notions related to these rules, which will be generalized in later parts of this paper, are also given.

The conjunctive rule is denoted by  $\odot$ . It is defined as follows. Let  $m_1$  and  $m_2$  be two BBAs, and let  $m_1 \odot_2$  be the result of their combination by  $\odot$ . We have, for all  $A \subseteq \Omega$ :

$$m_1 \odot_2(A) = \sum_{B \cap C = A} m_1(B) m_2(C). \quad (2)$$

This rule is appropriate when the sources that have induced  $m_1$  and  $m_2$ , are known to tell the truth and to be independent. Furthermore, this rule is commutative, associative and admits a unique neutral element: the vacuous BBA  $m_\Omega$ . Of interest is that this rule has a simple expression in terms of commonality functions. We have:

$$q_1 \odot_2(A) = q_1(A) \cdot q_2(A), \quad \forall A \subseteq \Omega.$$

In the TBM, conditioning by  $B \subseteq \Omega$  is equivalent to conjunctive combination with a categorical BBA  $m_B$  focused on  $B$ , i.e.,  $m_B(B) = 1$ . The result is denoted by  $m[B]$ , with  $m[B] = m \odot m_B$ . The conditional BBA  $m[B]$  quantifies our belief on  $\Omega$ , in a context where  $B$  holds. This operation is called the unnormalized Dempster's rule of conditioning. The combination by the conjunctive rule  $\odot$  admits a simple expression using the unnormalized Dempster's rule of conditioning. Indeed, let  $m_1$  and  $m_2$  be two BBAs. We have, for all

$A \subseteq \Omega$

$$m_{1 \circledast 2}(A) = \sum_{B \subseteq \Omega} m_1[B](A) m_2(B). \quad (3)$$

When it cannot be assumed that all the sources tell the truth, it may be assumed that at least one of them tells the truth, without knowing which one. In such a situation, and provided that the sources are independent, the disjunctive rule [3, 9] is appropriate. The disjunctive rule is denoted by  $\circledast$ . Let  $m_1$  and  $m_2$  be two distinct BBAs, and let  $m_{1 \circledast 2}$  be the result of their combination by  $\circledast$ . We have:

$$m_{1 \circledast 2}(A) = \sum_{B \cup C = A} m_1(B) m_2(C), \quad \forall A \subseteq \Omega.$$

The disjunctive rule is commutative, associative and admits a unique neutral element: the BBA which assigns the total mass of belief to the empty set, i.e.,  $m(\emptyset) = 1$ . This BBA, which we denote by  $m_\emptyset$ , is the negation of the neutral BBA  $m_\Omega$  of the conjunctive rule and is sometimes called the or-vacuous BBA [11]. The dual nature of  $\circledast$  and  $\circledcirc$  becomes apparent when one notices that these operators are linked by De Morgan's laws [3]:

$$\begin{aligned} \overline{m_1 \circledast m_2} &= \overline{m_1} \circledcirc \overline{m_2} \\ \overline{m_1 \circledcirc m_2} &= \overline{m_1} \circledast \overline{m_2}. \end{aligned}$$

Of interest for this paper are two other rules: the exclusive disjunctive rule denoted by  $\circledast$  and its negation denoted by  $\circledcirc$  [11], which are defined as follows. We have, for all  $A \subseteq \Omega$ :

$$m_{1 \circledast 2}(A) = \sum_{A = B \cup C} m_1(B) m_2(C),$$

where  $\cup$  is the exclusive OR (XOR), i.e.,  $B \cup C = (B \cap \overline{C}) \cup (\overline{B} \cap C)$  for all  $B, C \subseteq \Omega$ , and

$$m_{1 \circledcirc 2}(A) = \sum_{A = B \cap C} m_1(B) m_2(C),$$

where  $\cap$  denotes logical equality, i.e.,  $B \cap C = (B \cap C) \cup (\overline{B} \cap \overline{C})$  for all  $B, C \subseteq \Omega$ .

The rules  $\circledast$  and  $\circledcirc$  are commutative, associative and admit a unique neutral element:  $m_\emptyset$  and  $m_\Omega$ , respectively. Furthermore, they are linked by De Morgan's laws. The rule  $\circledcirc$  corresponds to the situation where it is known that exactly one of the sources of information tells the truth, but it is not known which one [11]. The rule  $\circledast$  corresponds to the situation where it is known that either all or none of the sources of information tell the truth [11].

## 2.3 Operations on Product Spaces

In Section 4 of this paper, some operations that allow the manipulation of BBAs defined on product spaces, are needed. They are succinctly presented here. Let  $m^{\Omega \times \Theta}$  denote a BBA defined on the Cartesian product  $\Omega \times \Theta$  of the frames of two variables  $\omega$  and  $\theta$ . The marginal BBA  $m^{\Omega \times \Theta \downarrow \Omega}$  is defined, for all  $A \subseteq \Omega$ , as

$$m^{\Omega \times \Theta \downarrow \Omega}(A) = \sum_{\{B \subseteq \Omega \times \Theta, (B \downarrow \Omega) = A\}} m^{\Omega \times \Theta}(B),$$

where  $(B \downarrow \Omega)$  denotes the projection of  $B$  onto  $\Omega$ , defined as

$$(B \downarrow \Omega) = \{\omega \in \Omega \mid \exists \theta \in \Theta, (\omega, \theta) \in B\}.$$

Conversely, let  $m^\Omega$  be a BBA defined on  $\Omega$ . Its vacuous extension on  $\Omega \times \Theta$  is defined as:

$$m^{\Omega \uparrow \Omega \times \Theta}(B) = \begin{cases} m^\Omega(A) & \text{if } B = A \times \Theta, \\ & \text{for some } A \subseteq \Omega, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Given two BBAs  $m_1^\Omega$  and  $m_2^\Theta$ , their conjunctive combination on  $\Omega \times \Theta$  can be obtained by combining their vacuous extensions on  $\Omega \times \Theta$  using (4). Formally:

$$m_1^\Omega \circledast m_2^\Theta = m_1^{\Omega \uparrow \Omega \times \Theta} \circledast m_2^{\Theta \uparrow \Omega \times \Theta}.$$

Two other operations that have been defined for BBAs on product spaces are the *conditioning* operation, and its inverse operation called the *ballooning extension*. They are defined as follows. Let  $m^{\Omega \times \Theta}$  denote a BBA on  $\Omega \times \Theta$ , and  $m_B^{\Omega \times \Theta}$  the BBA on  $\Omega \times \Theta$  with single focal set  $\Omega \times B$  with  $B \subseteq \Theta$ , i.e.,  $m_B^{\Omega \times \Theta}(\Omega \times B) = 1$ . The conditional BBA on  $\Omega$  given  $\theta \in B$  is defined as:

$$m^\Omega[B] = (m^{\Omega \times \Theta} \circledast m_B^{\Omega \times \Theta}) \downarrow \Omega.$$

Now, let  $m^\Omega[B]$  denote the conditional BBA on  $\Omega$ , given  $\theta \in B \subseteq \Theta$ . The ballooning extension of  $m^\Omega[B]$  on  $\Omega \times \Theta$  is the least committed BBA, whose conditioning on  $B$  yields  $m^\Omega[B]$  [9]. It is obtained as:

$$m^\Omega[B] \uparrow \Omega \times \Theta(C) = m^\Omega[B](A),$$

if  $C = (A \times B) \cup (\Omega \times (\Theta \setminus B))$ , for some  $A \subseteq \Omega$ , and  $m^\Omega[B] \uparrow \Omega \times \Theta(C) = 0$  otherwise. Example 1 illustrates the ballooning extension.

**Example 1.** Consider two frames  $\Omega = \{\omega_1, \omega_2\}$  and  $\Theta = \{\theta_1, \theta_2\}$ . Further, let  $m^\Omega[\theta_2]$  be a conditional BBA defined by  $m^\Omega[\theta_2](\{\omega_1\}) = 0.6$  and  $m^\Omega[\theta_2](\Omega) = 0.4$ . The ballooning extension of  $m^\Omega[\theta_2]$  is:

$$\begin{aligned} m^\Omega[\theta_2] \uparrow \Omega \times \Theta(\{(\omega_1, \theta_2), (\omega_1, \theta_1), (\omega_2, \theta_1)\}) &= 0.6, \\ m^\Omega[\theta_2] \uparrow \Omega \times \Theta(\Omega \times \Theta) &= 0.4. \end{aligned}$$

## 2.4 Matrix Notation

The matrix notation can be used to greatly simplify the mathematics of belief function theory. In [13], Smets proposed a review of the application of the matrix calculus to belief functions. This section is devoted to a summary of parts of [13] that are relevant to this paper.

### Belief functions as column vectors

A BBA  $m$  (and its associated functions  $bel$ ,  $pl$  and  $q$ ) defined on  $2^\Omega$  can be seen as a column vector of size  $2^{|\Omega|}$ . The elements of  $m$  can be ordered arbitrarily but the so-called binary order is particularly convenient. The binary order means that the first element of  $m$  is related to the empty set, the next to  $\{a\}$ , the next to  $\{b\}$ , the next to  $\{a, b\}$ , etc. More generally, the  $i$ th element of the vector  $\mathbf{m}$  corresponds to the set with elements indicated by 1 in the binary representation of  $i - 1$ . For instance, let  $\Omega = \{a, b, c, d\}$ . The first element ( $i = 1$ ) of the vector  $\mathbf{m}$  corresponds to the emptyset since the binary representation of  $1 - 1$  is 0000. The twelfth element ( $i = 12$ ) corresponds to  $\{a, b, d\}$  since the binary representation of  $12 - 1$  is 1011.

We use the following conventions. By default, the length of vectors and matrices are  $2^{|\Omega|}$ , and vectors are column vectors. Matrices and vectors are written in bold type, and their elements in normal type, e.g., a matrix is noted  $\mathbf{M}$  and the element on its  $i$ th row and  $j$ th column is noted  $M(i, j)$ . Sometimes a matrix will be defined by its general term, in this case we write  $\mathbf{M} = [M(i, j)]$ . For instance, if  $M(i, j)$  is defined by  $M(i, j) = 0, \forall i, j$ , then  $\mathbf{M}$  is a matrix, whose elements are zeros. Finally,  $\mathbf{I}$  denotes the unit matrix and  $\mathbf{Kron}(\mathbf{A}, \mathbf{B})$  denotes the  $mp \times nq$  matrix resulting from the Kronecker product of a  $m \times n$  matrix  $\mathbf{A}$  with a  $p \times q$  matrix  $\mathbf{B}$ . The matrix  $\mathbf{Kron}(\mathbf{A}, \mathbf{B})$  is defined by:

$$\mathbf{Kron}(\mathbf{A}, \mathbf{B}) = \begin{bmatrix} A(1, 1)\mathbf{B} & \cdots & A(1, n)\mathbf{B} \\ \vdots & \ddots & \vdots \\ A(m, 1)\mathbf{B} & \cdots & A(m, n)\mathbf{B} \end{bmatrix}.$$

The transformation (1) of a BBA  $m$  into its associated commonality function  $q$  can be represented using the matrix notation. We have

$$q(A) = \sum_{B \subseteq \Omega} Q(A, B)m(B),$$

where  $Q(A, B) = 1$  iff  $B \supseteq A$  and 0 otherwise. Letting  $\mathbf{Q} = [Q(A, B)]$ ,  $A, B \subseteq \Omega$ , we have  $\mathbf{q} = \mathbf{Q} \cdot \mathbf{m}$  and  $\mathbf{m} = \mathbf{Q}^{-1} \cdot \mathbf{q}$  [13]. The matrix  $\mathbf{Q}$  may be obtained

in a very simple way using Kronecker multiplication. Indeed, we have:

$$\mathbf{Q}^{i+1} = \mathbf{Kron} \left( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{Q}^i \right), \mathbf{Q}^1 = \mathbf{I},$$

where  $\mathbf{Q}^i$  denotes the matrix  $\mathbf{Q}$  when  $|\Omega| = i$ .

### Transformations of BBA into BBA

In this paragraph, we present how the transformation of a BBA into another BBA, given a piece of evidence, can be expressed using the matrix notation.

**Definition 1.** A stochastic matrix  $\mathbf{M} = [M(i, j)]$  is a square matrix with  $M(i, j) \geq 0$  and  $\sum_i M(i, j) = 1, \forall j$ .

Let  $\mathcal{M}^\Omega$  be the set of BBAs defined on  $\Omega$ . As shown by [13, Theorem 6.1], the set of matrices that map every element of  $\mathcal{M}^\Omega$  into an element of  $\mathcal{M}^\Omega$  is the set of stochastic matrices.

The revision of a BBA  $m_1$  by a piece of evidence  $Ev$  can be represented by a stochastic matrix  $\mathbf{M}_{Ev, m_1}$  that transforms  $m_1$  into  $m_1[Ev]$ :

$$\mathbf{m}_1[Ev] = \mathbf{M}_{Ev, m_1} \cdot \mathbf{m}_1.$$

If the value of the matrix depends only on  $Ev$  and not on  $m_1$  (in which case the pieces of evidence that induced  $m_1$  and  $Ev$  are said 'distinct' [13]), we can write:

$$\mathbf{m}_1[Ev] = \mathbf{M}_{Ev} \cdot \mathbf{m}_1.$$

The combinations by the rules  $\oplus$ ,  $\odot$ ,  $\ominus$  and  $\oslash$  are particular cases of revision. For instance, the conjunctive revision of a BBA  $m_1$  by a distinct piece of evidence inducing a BBA  $m_2$  is achieved by a special kind of matrix, called a Dempsterian specialization matrix [5] and denoted by  $\mathbf{S}_{m_2}$ . This matrix is defined as a function of  $m_2$ : its general term is  $S_{m_2}(A, B) = m_2[B](A)$ ,  $A, B \subseteq \Omega$ . We have  $\mathbf{m}_2 \odot \mathbf{m}_1 = \mathbf{S}_{m_2} \cdot \mathbf{m}_1$ .

## 3 $\alpha$ -Junctions: Basic Notions

In [11], Smets studies the set of possible associative, commutative and linear combination rules with a neutral element. Smets calls this set the  $\alpha$ -junctions because they cover the conjunction, the disjunction and the exclusive disjunction. We report in this section the summary of [11] given in [13].

Let  $\mathbf{m}_1$  and  $\mathbf{m}_2$  be two BBAs on  $\Omega$ . Suppose we want to build a BBA  $\mathbf{m}_{12}$  such that  $\mathbf{m}_{12} = f(\mathbf{m}_1, \mathbf{m}_2)$ , i.e.,  $\mathbf{m}_{12}$  depends only on  $\mathbf{m}_1$  and  $\mathbf{m}_2$ . Smets [11] determines the operators that map  $\mathcal{M}^\Omega \times \mathcal{M}^\Omega$  to  $\mathcal{M}^\Omega$  and that satisfy the following requirements (the origins of those requirements are summarized in [13, p.25]).

- Linearity<sup>1</sup>:  $f(\mathbf{m}, p\mathbf{m}_1 + q\mathbf{m}_2) = pf(\mathbf{m}, \mathbf{m}_1) + qf(\mathbf{m}, \mathbf{m}_2)$ ,  $p \in [0, 1]$ ,  $q = 1 - p$ .
- Commutativity:  $f(\mathbf{m}_1, \mathbf{m}_2) = f(\mathbf{m}_2, \mathbf{m}_1)$ .
- Associativity:  
 $f(f(\mathbf{m}_1, \mathbf{m}_2), \mathbf{m}_3) = f(\mathbf{m}_1, f(\mathbf{m}_2, \mathbf{m}_3))$ .
- Neutral element: existence of a belief function  $\mathbf{m}_{vac}$  such that  $f(\mathbf{m}, \mathbf{m}_{vac}) = \mathbf{m}$  for any  $\mathbf{m}$ .
- Anonymity: relabeling the elements of  $\Omega$  does not affect the results.
- Context preservation: if  $pl_1(X) = 0$  and  $pl_2(X) = 0$  for some  $X \subseteq \Omega$ , then  $pl_{12}(X) = 0$ .

It is shown in [11] that the solutions are stochastic matrices. We have:

$$\mathbf{m}_{12} = \mathbf{K}_{m_1} \cdot \mathbf{m}_2, \quad (5)$$

where

$$\mathbf{K}_{m_1} = \sum_{X \subseteq \Omega} m_1(X) \cdot \mathbf{K}_X. \quad (6)$$

Smets [11] proves that the  $2^{|\Omega|} \times 2^{|\Omega|}$  matrices  $\mathbf{K}_X$  depend only on  $\mathbf{m}_{vac}$  and one parameter  $\alpha \in [0, 1]$ . Furthermore, he shows that there are only two solutions for  $\mathbf{m}_{vac}$ : either  $\mathbf{m}_{vac} = \mathbf{m}_\Omega$  or  $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$ . Hence, there are only two sets of solutions, which are presented now.

### 3.1 Case $\mathbf{m}_{vac} = \mathbf{m}_\Omega$

The definition of the matrices  $\mathbf{K}_X$  that satisfy the above requirements when  $\mathbf{m}_{vac} = \mathbf{m}_\Omega$  is the following.

$$\begin{aligned} \mathbf{K}_\Omega &= \mathbf{I}, \\ \mathbf{K}_X &= \prod_{x \notin X} \mathbf{K}_{\{\bar{x}\}}, \quad \forall X \subset \Omega, \end{aligned}$$

where

$$\mathbf{K}_{\{\bar{x}\}} = [k_{\bar{x}}(A, B)], \quad \forall x \in \Omega,$$

with

$$k_{\bar{x}}(A, B) = \begin{cases} 1 & \text{if } x \notin A, \quad B = A \cup \{x\}, \\ \alpha & \text{if } x \notin B, \quad B = A, \\ 1 - \alpha & \text{if } x \notin B, \quad A = B \cup \{x\}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha \in [0, 1]$  and is constant for all  $\mathbf{K}_X$ . Example 2 illustrates the various matrices  $\mathbf{K}_X$  when  $\Omega = \{a, b\}$  and  $\mathbf{m}_{vac} = \mathbf{m}_\Omega$ .

<sup>1</sup>Smets referred to this property as ‘‘linearity’’. However, it is not real linearity, as it is only valid for convex combinations. We have kept the same terminology for lack of a more appropriate term.

**Example 2.** From (5) and (6), we have (in the matrices below, dots replace zeros and  $\bar{\alpha} = 1 - \alpha$ )

$$\begin{aligned} \mathbf{m}_{12} &= (m_1(\emptyset) \cdot \mathbf{K}_\emptyset + m_1(a) \cdot \mathbf{K}_a \\ &\quad + m_1(b) \cdot \mathbf{K}_b + m_1(\Omega) \cdot \mathbf{K}_\Omega) \cdot \mathbf{m}_2 \\ &= (m_1(\emptyset) \cdot \mathbf{K}_{\{\bar{a}\}} \cdot \mathbf{K}_{\{\bar{b}\}} + m_1(a) \cdot \mathbf{K}_{\{\bar{b}\}} \\ &\quad + m_1(b) \cdot \mathbf{K}_{\{\bar{a}\}} + m_1(\Omega) \cdot \mathbf{I}) \cdot \mathbf{m}_2 \\ &= (m_1(\emptyset) \cdot \begin{bmatrix} \alpha & 1 & \cdot & \cdot \\ \bar{\alpha} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & 1 \\ \cdot & \cdot & \bar{\alpha} & \cdot \end{bmatrix} \cdot \begin{bmatrix} \alpha & \cdot & 1 & \cdot \\ \cdot & \alpha & \cdot & 1 \\ \bar{\alpha} & \cdot & \cdot & \cdot \\ \cdot & \bar{\alpha} & \cdot & \cdot \end{bmatrix} \\ &\quad + m_1(a) \cdot \begin{bmatrix} \alpha & \cdot & 1 & \cdot \\ \cdot & \alpha & \cdot & 1 \\ \bar{\alpha} & \cdot & \cdot & \cdot \\ \cdot & \bar{\alpha} & \cdot & \cdot \end{bmatrix} \\ &\quad + m_1(b) \cdot \begin{bmatrix} \alpha & 1 & \cdot & \cdot \\ \bar{\alpha} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \alpha & 1 \\ \cdot & \cdot & \bar{\alpha} & \cdot \end{bmatrix} \\ &\quad + m_1(\Omega) \cdot \mathbf{I}) \cdot \mathbf{m}_2. \end{aligned}$$

When  $\mathbf{m}_{vac} = \mathbf{m}_\Omega$  and  $\alpha = 1$ , the matrix  $\mathbf{K}_{m_1}$  computed using (6) becomes the Dempsterian specialization matrix and we have  $\mathbf{K}_{m_1} \cdot \mathbf{m}_2 = \mathbf{m}_1 \odot_2 \mathbf{m}_2$  [13]. The case  $\alpha = 0$  corresponds to the rule  $\odot$ . When  $\mathbf{m}_{vac} = \mathbf{m}_\Omega$ , an  $\alpha$ -junction is referred to as an  $\alpha$ -conjunction by Smets since  $\mathbf{m}_\Omega$  is the neutral element of the conjunction [11]. The result of the  $\alpha$ -conjunction of two BBAs  $m_1$  and  $m_2$  is written  $m_1 \odot^\alpha m_2$ . Let us remark that despite what the appellation ‘‘ $\alpha$ -conjunction’’ might lead one to think, an  $\alpha$ -conjunction do not necessarily exhibit a conjunctive behavior. For instance, consider a frame  $\Omega = \{\omega_1, \omega_2\}$  and two precise BBAs  $m_1$  and  $m_2$  such that  $m_1(\{\omega_1\}) = m_2(\{\omega_1\}) = 1$ . We have  $m_1 \odot_2(\Omega) = 1$ , which is the most imprecise BBA.

### 3.2 Case $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$

The definition of the matrices  $\mathbf{K}_X$  that satisfy the above requirements when  $\mathbf{m}_{vac} = \mathbf{m}_\emptyset$  is the following.

$$\begin{aligned} \mathbf{K}_\emptyset &= \mathbf{I}, \\ \mathbf{K}_X &= \prod_{x \in X} \mathbf{K}_{\{x\}}, \quad \forall X \in 2^\Omega \setminus \{\emptyset\}, \end{aligned}$$

where

$$\mathbf{K}_{\{x\}} = [k_x(A, B)], \quad \forall x \in \Omega,$$

with

$$k_x(A, B) = \begin{cases} 1 & \text{if } x \notin B, \quad A = B \cup \{x\}, \\ \alpha & \text{if } x \in B, \quad B = A, \\ 1 - \alpha & \text{if } x \notin A, \quad B = A \cup \{x\}, \\ 0 & \text{otherwise,} \end{cases}$$

where  $\alpha \in [0, 1]$  and is constant for all  $\mathbf{K}_X$ .

When  $\mathbf{m}_{vac} = \mathbf{m}_0$ , an  $\alpha$ -junction is referred to as an  $\alpha$ -disjunction since  $\mathbf{m}_0$  is the neutral element of the disjunction [11]; we denote an  $\alpha$ -disjunctive rule by  $\odot^\alpha$ . Furthermore, when  $\mathbf{m}_{vac} = \mathbf{m}_0$  and  $\alpha = 1$ , we have  $\mathbf{K}_{m_1} \cdot \mathbf{m}_2 = \mathbf{m}_1 \odot_2$ . The case  $\alpha = 0$  corresponds to the rule  $\odot$ .

Finally, we have, for any  $\alpha \in [0, 1]$  [13, Theorem 12.2]:

$$\begin{aligned} \overline{m_1 \odot^\alpha m_2} &= \overline{m_1} \odot^\alpha \overline{m_2}, \\ \overline{m_1 \odot m_2} &= \overline{m_1} \odot m_2, \end{aligned} \quad (7)$$

i.e.,  $\alpha$ -conjunctive rules and  $\alpha$ -disjunctive rules are linked by De Morgan laws. In particular, the De Morgan duality between the conjunctive and disjunctive rules is recovered by setting  $\alpha = 1$  in (7).

## 4 Interpretation

In this section, an interpretation for the  $\alpha$ -junctions is proposed. This interpretation relies on the concept of the truthfulness of the sources of information.

### 4.1 Truthfulness of the Sources

Let  $\omega$  be a variable, which takes its values in a frame  $\Omega$ . Suppose an agent who does not know anything about the actual value  $\omega_0$  taken by  $\omega$ . Suppose a source  $S_1$  that tells the agent that the actual value  $\omega_0$  is in  $A \subseteq \Omega$ , i.e.,  $\omega_0 \in A$ . If the source tells the truth or, equivalently, is truthful, then the agent believes  $\omega_0 \in A$ . If the source does not tell the truth, then the agent believes  $\omega_0 \in \overline{A}$ .

Let  $\tau$  be a variable taking its values in a frame  $T = \{t, f\}$ . We use  $\tau$  to denote the truthfulness of the source. The information  $\omega_0 \in A$  provided by  $S_1$  can be modeled by a BBA  $m_1^\Omega$  such that  $m_1^\Omega(A) = 1$ . The information *when the source tells the truth*,  $\omega_0$  must be in  $A$ , and *when the source does not tell the truth*,  $\omega_0$  must be in  $\overline{A}$ , may be modeled by a BBA noted  $m_{1'}^{\Omega \times T}$  and defined on the product space  $\Omega \times T$  by

$$m_{1'}^{\Omega \times T}(A \times \{t\} \cup \overline{A} \times \{f\}) = 1. \quad (8)$$

Note that we use the index  $1'$  in  $m_{1'}^{\Omega \times T}$ , i.e., the source number followed by the prime symbol, to highlight that the BBA  $m_{1'}^{\Omega \times T}$  is obtained from the source  $S_1$ , as is the case of the BBA  $m_1^\Omega$ , but that it conveys a different information from the BBA  $m_1^\Omega$ .

One verifies that the BBA  $m_{1'}^{\Omega \times T}$  is appropriate to model the information available in this scenario since

- combining  $m_{1'}^{\Omega \times T}$  with a BBA  $m_t^T$  defined on  $T$  by  $m_t^T(t) = 1$ , and then marginalizing on  $\Omega$ , i.e.,

performing

$$(m_{1'}^{\Omega \times T} \odot m_t^T) \downarrow^\Omega, \quad (9)$$

yields a BBA  $m_{Ag}^\Omega$  such that  $m_{Ag}^\Omega = m_1^\Omega$ , which means that the agent's beliefs are equated to the source's beliefs if the agent believes that the source tells the truth;

- combining  $m_{1'}^{\Omega \times T}$  with a BBA  $m_f^T$  defined on  $T$  by  $m_f^T(f) = 1$ , and then marginalizing on  $\Omega$ , i.e., performing

$$(m_{1'}^{\Omega \times T} \odot m_f^T) \downarrow^\Omega, \quad (10)$$

yields a BBA  $m_{Ag}^\Omega$  such that  $m_{Ag}^\Omega = \overline{m_1^\Omega}$ , which is sound since  $\overline{m_1^\Omega}$  represents the BBA that would be induced if the agent knows that a source providing a BBA  $m$  is not telling the truth [11], as mentioned in Section 2.1.

This reasoning may be generalized when the source produces an information in the form of a BBA rather than a set, in which case the BBA  $m_{1'}^{\Omega \times T}$  is such that

$$m_{1'}^{\Omega \times T}(A \times \{t\} \cup \overline{A} \times \{f\}) = m_1^\Omega(A), \quad \forall A \subseteq \Omega. \quad (11)$$

Here again, if we perform (9) and (10), we find  $m_{Ag}^\Omega = m_1^\Omega$  and  $m_{Ag}^\Omega = \overline{m_1^\Omega}$ , respectively, which means that, as expected, the agent's beliefs are equated to what the source says if the source tells the truth, and the agent's beliefs are equal to the negation of what the source says if the source does not tell the truth.

Using the BBA  $m_{1'}^{\Omega \times T}$ , as defined by (11), to represent the agent's beliefs when it receives a BBA  $m_1^\Omega$  from a source  $S_1$ , we may now derive an interpretation for the  $\alpha$ -junctions.

### 4.2 Interpretation of the $\alpha$ -Conjunctions

Suppose two distinct sources  $S_1$  and  $S_2$  that induce two BBAs  $m_1^\Omega$  and  $m_2^\Omega$  on  $\Omega$ . Let  $T_1 = \{t_1, f_1\}$  and  $T_2 = \{t_2, f_2\}$ ; these two frames will be used to model beliefs on the truthfulness of  $S_1$  and  $S_2$ , respectively. Suppose we want to quantify the agent's beliefs on  $\Omega$  given  $m_1^\Omega$ ,  $m_2^\Omega$  and the following distinct pieces of evidence.

- A piece of evidence stating that both or none of the sources tell the truth. This piece of evidence may be modeled by a BBA  $m_{xand}^{T_1 \times T_2}$  defined by

$$m_{xand}^{T_1 \times T_2}(\{(t_1, t_2), (f_1, f_2)\}) = 1.$$

- Distinct items of evidence for all  $x \in \Omega$  of the form

$$pl^{T_1 \times T_2}[x](\{(f_1, f_2)\}) = 1 - \alpha, \quad (12)$$

indicating that if  $\omega_0 = x$ , then it is plausible with strength  $1 - \alpha$  that none of the sources tell the truth.

To compute the agent's beliefs on  $\Omega$  given these distinct pieces of evidence, the items of evidence of the form given by (12), must be transformed into BBAs. In the TBM, this may be done using the LCP. The least committed BBA  $m^{T_1 \times T_2}[x]$  corresponding to (12) is the SBBA  $m^{T_1 \times T_2}[x] = \{(t_1, t_2), (f_1, t_2), (t_1, f_2)\}^{1-\alpha}$ . Using all these distinct items of evidence, the agent's belief  $m_{Ag}^\Omega$  on  $\Omega$  is then equal to

$$m_{Ag}^\Omega = (m_{1'}^{\Omega \times T_1} \odot m_{2'}^{\Omega \times T_2} \odot m_{xand}^{T_1 \times T_2} \odot (\odot_{x \in \Omega} m^{T_1 \times T_2}[x]^{\uparrow \Omega \times T_1 \times T_2}))^{\downarrow \Omega}, \quad (13)$$

with, for  $i = 1$  and  $i = 2$  and all  $A \subseteq \Omega$

$$m_{i'}^{\Omega \times T_i}(A \times \{t_i\} \cup \bar{A} \times \{f_i\}) = m_i^\Omega(A), \quad (14)$$

and, for all  $x \in \Omega$

$$m^{T_1 \times T_2}[x] = \{(t_1, t_2), (f_1, t_2), (t_1, f_2)\}^{1-\alpha},$$

and

$$m_{xand}^{T_1 \times T_2}(\{(t_1, t_2), (f_1, f_2)\}) = 1.$$

**Theorem 1.** *Let  $m_1^\Omega$  and  $m_2^\Omega$  be two BBAs. The BBA  $m_{Ag}^\Omega$  defined by (13) verifies*

$$m_{Ag}^\Omega = m_1^\Omega \odot^\alpha m_2^\Omega.$$

This theorem may be illustrated with a simple valuation network [6] (see Figure 1), which is a graphical display of a set of BBAs, where variables are represented by circular nodes and BBAs are represented by square nodes.

As shown by Theorem 1, an  $\alpha$ -conjunction is equivalent to the pooling by the conjunctive rule of some simple pieces of evidence, which can all be interpreted and that are, moreover, all related to the truthfulness of the sources. In particular, the parameter  $\alpha$  involved in the  $\alpha$ -conjunctions can be interpreted in terms of the plausibility, given  $\omega_0 = x$ , that the sources lie, since this plausibility is equal to  $1 - \alpha$ . Note that since the BBA  $m_{xand}$  excludes the fact that one and only one source tells the truth, we clearly see, from the interpretation given to  $\alpha$ , that we pass from the conjunctive rule to the rule  $\odot$  as  $\alpha$  varies from 1 to 0. Finally, we may remark that, since (12) is logically equivalent to

$$bel^{T_1 \times T_2}[x](\{(t_1, t_2), (f_1, t_2), (t_1, f_2)\}) = \alpha,$$

then the parameter  $\alpha$  involved in the  $\alpha$ -conjunctions is equal to the belief, given  $\omega_0 = x$ , that at least one of the sources tells the truth.

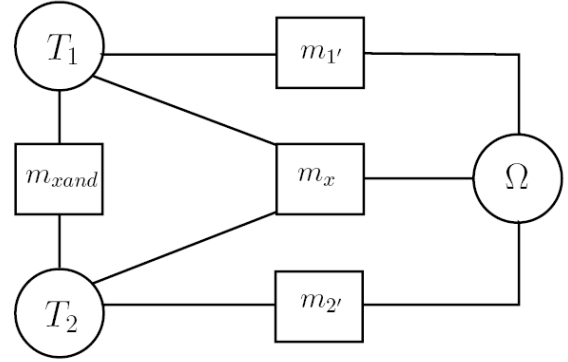


Figure 1: Valuation network for the  $\alpha$ -conjunction of two BBAs  $m_1$  and  $m_2$ . In the network, the term  $(\odot_{x \in \Omega} m^{T_1 \times T_2}[x]^{\uparrow \Omega \times T_1 \times T_2})$  appearing in (13), is replaced by a BBA  $m_x$  defined on  $\Omega \times T_1 \times T_2$ .

Let us eventually remark that Theorem 1 does not extend to more than two sources. Indeed, let  $m_1$ ,  $m_2$  and  $m_3$  be three BBAs. The combination  $m_1^\Omega \odot^\alpha m_2^\Omega \odot^\alpha m_3^\Omega$  is in general not equal to

$$(m_{1'}^{\Omega \times T_1} \odot m_{2'}^{\Omega \times T_2} \odot m_{3'}^{\Omega \times T_3} \odot m_{xand}^{T_1 \times T_2 \times T_3} \odot (\odot_{x \in \Omega} m^{T_1 \times T_2 \times T_3}[x]^{\uparrow \Omega \times T_1 \times T_2 \times T_3}))^{\downarrow \Omega},$$

with  $m_{i'}^{\Omega \times T_i}$ ,  $i = 1, 2, 3$ , defined by (14), and where  $m^{T_1 \times T_2 \times T_3}[x]$  is the least committed BBA corresponding to  $pl^{T_1 \times T_2 \times T_3}[x](\{(f_1, f_2, f_3)\}) = 1 - \alpha$ , and with  $m_{xand}^{T_1 \times T_2 \times T_3}(\{(t_1, t_2, t_3), (f_1, f_2, f_3)\}) = 1$ .

### 4.3 Interpretation of the $\alpha$ -Disjunctions

The  $\alpha$ -disjunctions can be interpreted in a similar way. Suppose two distinct sources  $S_1$  and  $S_2$  that induce two BBAs  $m_1^\Omega$  and  $m_2^\Omega$  on  $\Omega$ . Suppose we want to compute the agent's beliefs on  $\Omega$  given  $m_1^\Omega$ ,  $m_2^\Omega$  and the following distinct pieces of evidence.

- A piece of evidence stating that the sources do not lie simultaneously. This piece of evidence may be modeled by a BBA  $m_{or}^{T_1 \times T_2}$  defined by

$$m_{or}^{T_1 \times T_2}(\{(t_1, t_2), (t_1, f_2), (f_1, t_2)\}) = 1.$$

- Distinct items of evidence for all  $x \in \Omega$  of the form

$$pl^{T_1 \times T_2}[x](\{(t_1, t_2)\}) = \alpha, \quad (15)$$

indicating that if  $\omega_0 = x$ , then it is plausible with strength  $\alpha$  that both sources tell the truth.

The least committed BBA  $m^{T_1 \times T_2}[x]$  corresponding to (15) is the SBBA  $m^{T_1 \times T_2}[x] =$

$\{(f_1, t_2), (t_1, f_2), (f_1, f_2)\}^\alpha$ . Using all these distinct items of evidence, the agent's belief  $m_{Ag}^\Omega$  on  $\Omega$  is then equal to

$$m_{Ag}^\Omega = (m_{1'}^{\Omega \times T_1} \odot m_{2'}^{\Omega \times T_2} \odot m_{or}^{T_1 \times T_2} \odot (\odot_{x \in \Omega} m^{T_1 \times T_2}[x]^{\uparrow \Omega \times T_1 \times T_2}))^{\downarrow \Omega}, \quad (16)$$

with  $m_{i'}^{\Omega \times T_i}$ ,  $i = 1, 2$ , defined by (14), and where  $m^{T_1 \times T_2}[x] = \{(f_1, t_2), (t_1, f_2), (f_1, f_2)\}^\alpha$  for all  $x \in \Omega$ , and with  $m_{or}^{T_1 \times T_2}(\{(t_1, t_2), (t_1, f_2), (f_1, t_2)\}) = 1$ .

**Theorem 2.** *Let  $m_1^\Omega$  and  $m_2^\Omega$  be two BBAs. The BBA  $m_{Ag}^\Omega$  defined by (16) verifies*

$$m_{Ag}^\Omega = m_1^\Omega \odot^\alpha m_2^\Omega.$$

As shown by Theorem 2, an  $\alpha$ -disjunction is equivalent to the pooling by the conjunctive rule of some simple pieces of evidence. In particular, the parameter  $\alpha$  involved in the  $\alpha$ -disjunctions is equal to the plausibility that the sources tell the truth given  $\omega_0 = x$ . Note that since the BBA  $m_{or}$  excludes the fact that both sources lie, we clearly see, from the interpretation given to  $\alpha$ , that we pass from the disjunctive rule to the exclusive disjunctive rule as  $\alpha$  varies from 1 to 0.

To complete this section on the interpretation of the  $\alpha$ -junctions, we may note that the idea of recovering the disjunctive rule and the exclusive disjunctive rule through the use of the conjunctive rule and BBAs defined on product spaces was investigated by Haenni in [4]. The difference between Haenni's approach and ours is that Haenni used the notion of the reliability of the sources, rather than their truthfulness. The main difference between a reliable source and a truthful source is the following. Suppose a source tells  $\omega_0 \in A$ . If the source is lying, then the agent believes  $\omega_0 \in \bar{A}$ , whereas when the source is unreliable, the agent believes  $\omega_0 \in \Omega$ . As stated in [11] and as may easily be shown using the degenerate case  $\alpha = 0$  in Theorem 2, the exclusive disjunctive rule corresponds to the situation where exactly one of the sources tells the truth, without knowing which one. However, as shown in [7], this rule does not correspond to the situation where exactly one of the sources is reliable, without knowing which one, as wrongly claimed without proof by Theorem 3.3 of [4]. As a matter of fact, it can even be shown that it is actually the disjunctive rule that corresponds to that particular situation [7].

## 5 Computation

In addition to lacking an interpretation, the  $\alpha$ -junctions suffered in [11] from another limitation: they were hard to compute. Indeed, the definitions

of the matrices underlying the  $\alpha$ -junctions are "quite laborious" [13] and thus using an  $\alpha$ -junctive rule looks like a complicated task. It seems thus interesting to have simpler mechanisms to perform a combination by an  $\alpha$ -junctive rule. As shown by Theorem 1, it is possible to compute the combination by an  $\alpha$ -conjunctive rule using the conjunctive rule and BBAs defined on product spaces. In this section, several other new and simple means are provided to compute the combination by an  $\alpha$ -conjunction. These new methods are based on generalizations of mechanisms that can be used to compute a combination by the conjunctive rule. Note that, although not provided in this paper, similar new means exist for the computation of the combination by an  $\alpha$ -disjunction.

### 5.1 $\alpha$ -Conditioning Operation

Definition 2 below introduces a new notion, called  $\alpha$ -conditioning, which will be useful to uncover a simple expression for the  $\alpha$ -conjunctions.

**Definition 2.** *The  $\alpha$ -conditioning of a BBA by a subset  $B \subseteq \Omega$  is equal to the  $\alpha$ -conjunction of this BBA with a categorical BBA focused on  $B$ .*

The result of the  $\alpha$ -conditioning operation on a BBA  $m$  given a subset  $B \subseteq \Omega$ , i.e., the result of  $m \odot^\alpha m_B$  with  $m_B$  the categorical BBA focused on  $B$ , is denoted by  $m[B]^\alpha$ . We use the term " $\alpha$ -conditioning" because  $m[B]^\alpha = m[B]$  when  $\alpha = 1$ .

The following proposition provides an expression for the  $\alpha$ -conditioning operation.

**Proposition 1.** *Let  $B \subseteq \Omega$ . We have, for all  $X \subseteq \Omega$ ,*

$$m[B]^\alpha(X) = \sum_{(A \cap B) \cup (\bar{A} \cap \bar{B} \cap C) = X} m(A) m_\alpha(C),$$

where  $m_\alpha$  is a BBA defined by, for all  $A \subseteq \Omega$ ,  $m_\alpha(A) = \alpha^{|\bar{A}|} (1 - \alpha)^{|A|}$ .

The following proposition introduces a new way to compute a combination by an  $\alpha$ -conjunction, through the use of the  $\alpha$ -conditioning operation.

**Proposition 2.** *Let  $m_1$  and  $m_2$  be two BBAs. We have, for all  $A \subseteq \Omega$*

$$m_1 \odot^\alpha m_2(A) = \sum_{B \subseteq \Omega} m_1[B]^\alpha(A) m_2(B). \quad (17)$$

Note that, when  $\alpha = 1$ , Equation (17) becomes equivalent to (3). Hence, Equation (17) may be seen as a generalization of (3).

## 5.2 “Classical” Expression

Using Propositions 1 and 2, it may be shown that the following proposition holds.

**Proposition 3.** *Let  $m_1$  and  $m_2$  be two BBAs. Let  $m_{1\odot^\alpha 2}$  denote  $m_1\odot^\alpha m_2$ . We have, for all  $X \subseteq \Omega$ ,*

$$m_{1\odot^\alpha 2}(X) = \sum_{(A \cap B) \cup (\bar{A} \cap \bar{B} \cap C) = X} m_1(A) m_2(B) m_\alpha(C), \quad (18)$$

where  $m_\alpha(A) = \alpha^{|\bar{A}|} (1 - \alpha)^{|A|}$ , for all  $A \subseteq \Omega$ .

This proposition gives us yet another new expression for the  $\alpha$ -conjunctions. We call (18) the “classical” expression for the  $\alpha$ -conjunction since (18) is a generalization of the classical, or most often encountered, definition of the conjunctive rule given by Equation (2). Indeed, if  $\alpha = 1$ , then the BBA  $m_\alpha$  of Proposition 3 is such that  $m_\alpha(\emptyset) = 1$  and thus the term on the right side of (18) reduces to

$$\begin{aligned} & \sum_{(A \cap B) \cup (\bar{A} \cap \bar{B} \cap \emptyset) = X} m_1(A) m_2(B) m_\alpha(\emptyset) \\ &= \sum_{(A \cap B) = X} m_1(A) m_2(B) \\ &= m_{1\odot} m_2(X), \end{aligned}$$

as expected. Similarly, if  $\alpha = 0$ , then  $m_\alpha(\Omega) = 1$ , and thus the term on the right side of (18) reduces to  $m_{1\odot} m_2(X)$ , as expected.

## 5.3 $\alpha$ -Commonality Function

Using the eigendecomposition of  $\mathbf{K}_m$  when  $\mathbf{m}_{vac} = \mathbf{m}_\Omega$ , Smets [11] showed that we have

$$g_{1\odot^\alpha 2} = g_1 \cdot g_2 \quad (19)$$

with

$$\mathbf{g}_{1\odot^\alpha 2} = \mathbf{G} \cdot \mathbf{m}_{1\odot^\alpha 2}, \quad (20)$$

and  $\mathbf{g}_1 = \mathbf{G} \cdot \mathbf{m}_1$  and  $\mathbf{g}_2 = \mathbf{G} \cdot \mathbf{m}_2$ , where  $\mathbf{G}$  is a matrix of eigenvectors of  $\mathbf{K}_m$  (due to lack of space, we refer the reader to [13, p. 26] for the definition of  $\mathbf{G}$ ). From (19) and (20), we obtain

$$\mathbf{m}_{1\odot^\alpha 2} = \mathbf{G}^{-1} \cdot \mathbf{Diag}(\mathbf{g}_1) \cdot \mathbf{g}_2, \quad (21)$$

where  $\mathbf{Diag}(\mathbf{g}_1)$  denotes the diagonal matrix, whose diagonal elements are the elements of the vector  $\mathbf{g}_1$ . As shown by (21), the combination of two BBAs  $m_1$  and  $m_2$  by an  $\alpha$ -conjunctive rule can be simply expressed as the pointwise product of the functions  $g_1$  and  $g_2$  associated, respectively, to  $m_1$  and  $m_2$ . This is a first step in the simplification of the computation

by an  $\alpha$ -conjunction. However, the definition of the matrix  $\mathbf{G}$  is as tedious as the definition of the matrix  $\mathbf{K}_m$ . Fortunately, Theorem 3 shows that it is possible to obtain the matrix  $\mathbf{G}$  in a simple manner.

**Theorem 3.** *The matrix  $\mathbf{G}$  may be obtained using Kronecker multiplication. We have:*

$$\mathbf{G}^{i+1} = \mathbf{Kron} \left( \begin{bmatrix} 1 & 1 \\ \alpha - 1 & 1 \end{bmatrix}, \mathbf{G}^i \right), \mathbf{G}^1 = \mathbf{1},$$

where  $\mathbf{G}^i$  denotes the matrix  $\mathbf{G}$  when  $|\Omega| = i$ .

We now have a very simple way to compute an  $\alpha$ -conjunction, i.e., pointwise product of functions  $g$  which may themselves be obtained by a simple Kronecker product. Furthermore, it may now easily be seen that the  $\mathbf{G}$  matrix generalizes the  $\mathbf{Q}$  matrix in that we have  $\mathbf{G} = \mathbf{Q}$  when  $\alpha = 1$  and thus  $g = q$  in this case. The fact that the function  $g$  generalizes the commonality function can be used to call  $g$  the  $\alpha$ -commonality function associated to a BBA  $m$ .

## 5.4 Comparison of the Computation Methods

In this section, the various new means proposed for the computation of the combination by an  $\alpha$ -conjunctive rule, are briefly compared.

We have laid bare four new ways of performing such a combination: (1) using the  $\alpha$ -conditioning operation (see Proposition 2), (2) using a “classical” expression (see Proposition 3), (3) using the conjunctive rule and BBAs defined on product spaces (see Theorem 1) and (4) using the  $\alpha$ -commonality function obtained from a Kronecker product (see (21) and Theorem 3).

Each of these techniques has some advantages and some drawbacks. Method 4 is arguably the simplest one to implement. However, it may rapidly become impossible to use if the frame of discernment  $\Omega$  is too big, since this method requires computing matrices  $\mathbf{G}$  of size  $2^{|\Omega|} \times 2^{|\Omega|}$ , which are, in addition, not sparse, and it requires performing the pointwise product of vectors  $g$  of size  $2^{|\Omega|}$ . Method 3 is also rather simple to implement, since we merely need to perform combinations by the conjunctive rule. However, it requires working in the space  $\Omega \times T_1 \times T_2$ . Method 1 and 2 share the same characteristics: they are more efficient than method 4 when the frame is big, since they do not require to work with vectors of size  $2^{|\Omega|}$  as  $m_1$  and  $m_2$  may have only a few focal sets, but they are harder to implement.

## 6 Conclusion

The  $\alpha$ -junctions represent the set of associative, commutative and linear combination operators for belief

functions with a neutral element. They include as particular cases familiar combination rules such as the conjunctive and disjunctive rules. They have never been used in the literature due, most certainly, to two limiting factors: in the original article of Smets [11], they lacked (1) an interpretation and (2) simple means to compute them. This paper has proposed solutions to these two issues.

It was first shown that the  $\alpha$ -junctions correspond to some particular form of knowledge about the truthfulness of the sources, making the  $\alpha$ -junctions interesting for applications where such kind of knowledge may be available. This might for instance be the case when dealing with automatic deceiving agents [14]. Then, it was shown that various notions that can be used to perform the computation by the conjunctive rule can be generalized to the  $\alpha$ -junctions. This allowed us to uncover simple methods to perform a combination by an  $\alpha$ -junctive rule. The  $\alpha$ -junctions become thus more usable in practice and potentially useful, irrespective of their meaning, for, e.g., classification applications, as demonstrated in [7].

To conclude, let us mention that, as suggested in [13] and shown in [7], it is possible to obtain  $\alpha$ -junctive canonical decompositions of a belief function, generalizing the conjunctive and disjunctive canonical decompositions [10, 2].

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