Spinorial Characterizations of Surfaces into 3-dimensional pseudo-Riemannian Space Forms

Marie-Amélie Lawn* and Julien Roth†

Abstract

We give a spinorial characterization of isometrically immersed surfaces of arbitrary signature into 3-dimensional pseudo-Riemannian space forms. For Lorentzian surfaces, this generalizes a recent work of the first author in $\mathbb{R}^{2,1}$ to other Lorentzian space forms. We also characterize immersions of Riemannian surfaces in these spaces. From this we can deduce analogous results for timelike immersions of Lorentzian surfaces in space forms of corresponding signature, as well as for spacelike and timelike immersions of surfaces of signature (0,2), hence achieving a complete spinorial description for this class of pseudo-Riemannian immersions.

keywords: Dirac Operator, Killing Spinors, Isometric Immersions, Gauss and Codazzi Equations.

subjclass: Differential Geometry, Global Analysis, 53C27, 53C40, 53C80, 58C40.

1 Introduction

A fundamental question in the theory of submanifolds is to know whether a (pseudo-)Riemannian manifold $(M^{p,q}, g)$ can be isometrically immersed into a fixed ambient manifold $(\overline{M}^{r,s}, \overline{g})$. In this paper, we focus on the case of hypersurfaces, that is, codimension 1. When the ambient space is a space form, as the pseudo-Euclidean space $\mathbb{R}^{p,q}$ and the pseudo-spheres $\mathbb{S}^{p,q}$ of positive constant curvature, or the pseudo-hyperbolic spaces $\mathbb{H}^{p,q}$ of negative constant curvature, the answer is given by the well-known fundamental theorem of hypersurfaces:

Theorem. *[9] $(M^{p,q}, g)$ be a pseudo-Riemannian manifold with signature (p, q), $p + q = n$. Let A be a symmetric Codazzi tensor , that is, $d^{\nabla} A = 0$, satisying*

$$R(X,Y)Z = \delta\Big[\langle A(Y), Z\rangle A(X) - \langle A(X), Z\rangle A(Y)\Big] + \kappa\Big[\langle Y, Z\rangle X - \langle X, Z\rangle Y\Big]$$

with $\kappa \in \mathbb{R}$ for all $x \in M$ and $X, Y, Z \in T_xM$.
Then, if $\delta = 1$ (resp. $\delta = -1$), there exists locally a spacelike (resp. timelike) isometric immersion of M in $\mathbb{M}^{p+1,q}(\kappa)$ (resp. $\mathbb{M}^{p,q+1}(\kappa)$).

In the Riemannian case and for small dimensions ($n = 2$ or 3), an other necessary and sufficient condition is now well-known. This condition is expressed

*marie-amelie.lawn@uni.lu
†julien.rothoth@univ-mlv.fr

in spinorial terms, namely, by the existence of a special spinor field. This work initiated by Friedrich [4] in the late 90's for surfaces of $\mathbb{R}^3$ was generalized for surfaces of $\mathbb{S}^3$ and $\mathbb{H}^3$ [8] and other 3-dimensional homogeneous manifolds [10].

The first author [5] uses this approach to give a spinorial characterization of space-like immersions of Lorentzian surfaces in the Minkowski space $\mathbb{R}^{2,1}$. In this paper, we give a generalization of this result to Lorentzian or Riemannian surfaces into one of the three Lorentzian space forms, $\mathbb{R}^{2,1}$, $\mathbb{S}^{2,1}$ or $\widetilde{\mathbb{H}}^{2,1}$. These finally allows us to give a complete spinorial characterization for spacelike as well as for timelike immersions of surfaces of arbitrary signature into pseudo-Riemannian space forms.

We will begin by a section of recalls about extrinsic pseudo-Riemannian spin geometry. For further details, one refers to [1, 2] for basic facts about spin geometry and [1, 3, 7] for the extrinsic aspect.

2 Preliminaries

2.1 Pseudo-Riemannian spin geometry

Let $(M^{p,q}, g)$, $p+q=2$, be an oriented pseudo-Riemannian surface of arbitrary signature isometrically immersed into a three-dimensional pseudo-Riemannian spin manifold $(N^{r,s}, \overline{g})$. We introduce the parameter ε as follows: $\varepsilon = i$ if the immersion is timelike and $\varepsilon = 1$ if the immersion is spacelike. Let ν be a unit vector normal to M. The fact that M is oriented implies that M carries a spin structure induced from the spin structure of N and we have the following identification of the spinor bundles and Clifford multiplications:

$$\begin{cases} \Sigma N_{|M} \equiv \Sigma M. \\ X \cdot \varphi_{|M} = (\varepsilon \nu \bullet X \bullet \varphi)_{|M}, \end{cases}$$

where $\cdot$ and $\bullet$ are the Clifford multiplications, respectively on M and N. Moreover, we have the following well-known spinorial Gauss formula

$$\overline{\nabla}_X \varphi = \nabla_X \varphi - \frac{\varepsilon}{2} A(X) \cdot \varphi, \tag{1}$$

where A is the shape operator of the immersion. Finally, we recall the Ricci identity on M

$$R(e_1, e_2)\varphi = \frac{1}{2} \varepsilon_1 \varepsilon_2 R_{1221} e_1 \cdot e_2 \cdot \varphi, \tag{2}$$

where e_1, e_2 is a local orthonormal frame of M and $\varepsilon_j = g(e_j, e_j)$.
The complex volume element on the surface depends on the signature and is defined by

$$\omega^{\mathbb{C}}_{p,q} = i^{q+1} e_1 \cdot e_2.$$

Obviously ${\omega^{\mathbb{C}}_{p,q}}^2 = 1$ independently of the signature and the action of $\omega^{\mathbb{C}}$ splits ΣM into two eigenspaces $\Sigma^{\pm} M$ of real dimension 2. Therefore, a spinor field φ can be written as $\varphi = \varphi^+ + \varphi^-$ with $\omega^{\mathbb{C}} \cdot \varphi^{\pm} = \pm \varphi^{\pm}$. Finally, we denote $\overline{\varphi} = \omega^{\mathbb{C}} \cdot \varphi = \varphi^+ - \varphi^-$.

2.2 Restricted Killing spinors

Let $(M^{p,q},g)$, $p+q=2$ be a surface of the pseudo-Riemannian space form $\mathbb{M}^{r,s}(\kappa)$, $r+s=3$, $p\leqslant r$, $q\leqslant s$. This space form carries a Killing spinor φ, that is satisfying $\overline{\nabla}_X\varphi=\lambda X\bullet\varphi$, with $\kappa=4\lambda^2$. From the Gauss formula (1), the restriction of φ on M satisfies the equation

$$\nabla_X\varphi=\frac{\varepsilon}{2}A(X)\cdot\varphi+\lambda X\bullet\varphi. \tag{3}$$

But we have

$$X\bullet\varphi=\varepsilon^2\nu\bullet\nu\bullet X\bullet\varphi=-\varepsilon^2\nu\bullet X\bullet\nu\bullet\varphi=-\varepsilon X\cdot(\nu\bullet\varphi).$$

Moreover, the complex volume element $\omega^{\mathbb{C}}_{r,s}=-i^se_1\bullet e_2\bullet\nu$ of $\mathbb{M}^{r,s}(\kappa)$ over M acts as the identity on $\Sigma\mathbb{M}^{r,s}(\kappa)_{|M}\equiv\Sigma M$. Thus, we have

$$\begin{aligned}\nu\bullet\varphi &= \omega^{\mathbb{C}}_{r,s}\bullet\varphi=-i^s\nu\bullet e_1\bullet e_2\bullet\nu\bullet\varphi\\ &= i^s\nu\bullet e_1\bullet\nu\bullet e_2\bullet\varphi\\ &= i^s\varepsilon^2(\varepsilon\nu\bullet e_1)\bullet(\varepsilon\nu e_2\bullet\varphi)\\ &= i^s\varepsilon^2e_1\cdot e_2\cdot\varphi.\end{aligned}$$

Hence a simple case by case computation shows that we have

$$X\bullet\varphi=-i^s\varepsilon^3X\cdot e_1\cdot e_2\cdot\varphi=iX\cdot\omega^{\mathbb{C}}_{p,q}\cdot\varphi=iX\cdot\overline{\varphi}.$$

in the six possible cases (for the three possible signatures (2,0), (1,1), (0,2) of the surface with respectively $\varepsilon=1$ or i).
We will call a spinor solution of equation (3) a real special Killing spinor (RSK)-spinor if $\varepsilon\in\mathbb{R}$, and an imaginary special Killing spinor (ISK)-spinor if $\varepsilon\in i\mathbb{R}$.

2.3 Norm assumptions

In this section, we precise the norm assumptions useful for the statement of the main result. Let $(M^{p,q},g)$ be a pseudo-Riemannian surface and φ spinor field on M. Let $\varepsilon=1$ or i and $\lambda\in\mathbb{R}$ or $i\mathbb{R}$. We say that φ satisfies the norm assumption $\mathcal{N}_\pm(p,q,\lambda,\varepsilon)$ if the following holds:

1. For $p=2,q=0$ or $p=0,q=2$:
   - If $\varepsilon=1$, then $X|\varphi_1|^2=\pm2\Re e\left<i\eta X\cdot\overline{\varphi},\varphi\right>$.
   - If $\varepsilon=i$, then $X\left<\varphi,\overline{\varphi}\right>=\pm2\Re e\left<i\eta X\cdot\varphi_1,\varphi_1\right>$.
2. For $p=1,q=1$: φ is non-isotropic

3 The main result

We now state the main result of the present paper.

Theorem 1. *Let $(M^{p,q},g)$, $p+q=2$ be an oriented pseudo-Riemannian manifold. Let H be a real-valued function. Then, the three following statements are equivalent:*

1. *There exist two nowhere vanishing spinor fields φ_1 and φ_2 satisfying the norm assumptions $\mathcal{N}_+(p,q,\lambda,\varepsilon)$ and $\mathcal{N}_-(p,q,\lambda,\varepsilon)$ respectively and*

$$D\varphi_1 = 2\varepsilon H\varphi_1 + 2i\lambda\overline{\varphi_1} \quad \text{and} \quad D\varphi_2 = -2\varepsilon H\varphi_2 - 2i\lambda\overline{\varphi}_2.$$

2. *There exist two spinor fields φ_1 and φ_2 satisfying*

$$\nabla_X\varphi_1 = \frac{\varepsilon}{2}A(X)\cdot\varphi_1 - i\lambda X\cdot\overline{\varphi}_1, \quad \text{and} \quad \nabla_X\varphi_2 = -\frac{\varepsilon}{2}A(X)\cdot\varphi_1 + i\lambda X\cdot\overline{\varphi}_2,$$

where A is a g-symmetric endomorphism and $H = -\frac{1}{2}\mathrm{tr}\,(A)$.

3. *There exists a local isometric immersion from M into the (pseudo)-Riemannian space form $\mathbb{M}^{p+1,q}(4\lambda^2)$ (resp. $\mathbb{M}^{p,q+1}(4\lambda^2)$) if $\varepsilon = 1$ (resp. $\varepsilon = i$) with mean curvature H and shape operator A.*

Remark 1. *Note that, in this result, two spinor fields are needed to get an isometric immersion. Nevertheless, for the case of Riemannian surfaces in Riemannian space forms (Friedrich [4] and Morel [8]) only one spinor solution of one of the two equations is sufficient. This is also the case for surfaces of signature $(0,2)$ in space forms of signature $(0,3)$.*

In order to prove this theorem, we give two technical lemmas.

Lemma 3.1. *Let $(M^{p,q},g)$ be an oriented (pseudo)-Riemannian surface and η,λ two complex numbers. If M carries a spinor field satisfying*

$$\nabla_X\varphi = \eta A(X)\cdot\varphi + i\lambda X\cdot\overline{\varphi},$$

then, we have

$$\big(\varepsilon_1\varepsilon_2 R_{1212} + 4\eta^2\det(A) - \lambda^2\big)\, e_1\cdot e_2\cdot\varphi = 2\eta d^\nabla A(e_1,e_2)\cdot\varphi.$$

Proof : An easy computation yields

$$\begin{aligned}\nabla_X\nabla_Y\varphi &= \eta\nabla_X A(Y)\cdot\varphi + \eta^2 A(Y)\cdot A(X)\cdot\varphi + i\eta\lambda A(Y)\cdot X\cdot\omega^{\mathbb{C}}\cdot\varphi \\ &\quad + i\lambda\nabla_X Y\cdot\omega^{\mathbb{C}}\cdot\varphi + i\eta\lambda Y\cdot\omega^{\mathbb{C}}\cdot A(X)\cdot\varphi - \lambda^2 Y\cdot\omega^{\mathbb{C}}\cdot X\cdot\omega^{\mathbb{C}}\varphi.\end{aligned}$$

Hence (the other terms vanish by symmetry)

$$\begin{aligned}\mathcal{R}(e_1,e_2)\varphi &= \nabla_{e_1}\nabla_{e_2}\varphi - \nabla_{e_2}\nabla_{e_1}\varphi - \nabla_{[e_1,e_2]}\varphi \\ &= \eta\big(\nabla_{e_1}A(e_2) - \nabla_{e_2}A(e_1) - A([e_1,e_2])\big)\varphi + \eta^2(A(e_2)A(e_1) - A(e_1)A(e_2))\varphi \\ &\quad -\lambda^2(e_2\cdot\omega^{\mathbb{C}}\cdot e_1\omega^{\mathbb{C}} - e_1\cdot\omega^{\mathbb{C}}\cdot e_2\omega^{\mathbb{C}})\varphi.\end{aligned}$$

Since we have

$$A(e_2)A(e_1) - A(e_1)A(e_2) = -2\det(A)$$

and

$$e_2\cdot\omega^{\mathbb{C}}\cdot e_1\omega^{\mathbb{C}} - e_1\cdot\omega^{\mathbb{C}}\cdot e_2\cdot\omega^{\mathbb{C}} = e_1\cdot e_2\cdot\big(\omega^{\mathbb{C}}\big)^2 - e_2\cdot e_1\cdot\big(\omega^{\mathbb{C}}\big)^2 = 2e_1\cdot e_2,$$

by the Ricci identity (2), we get

$$\frac{1}{2}\varepsilon_1\varepsilon_2 R_{1221}e_1e_2\cdot\varphi = \eta d^\nabla A(e_1,e_2) - 2\eta^2\det(A)e_1\cdot e_2\varphi - 2\lambda^2 e_1\cdot e_2\cdot\varphi,$$

and finally

$$(4)\qquad (-\varepsilon_1\varepsilon_2 R_{1212} + 4\eta^2 \det(A) + 4\lambda^2)e_1 \cdot e_2 \cdot \varphi = 2\eta d^{\nabla} A(e_1, e_2) \cdot \varphi.$$

□

Now, we give this second lemma

Lemma 3.2. *Let $(M^{p,q}, g)$ be an oriented (pseudo)-Riemannian surface and λ a complex number. If M carries a spinor field solution of the equation*

$$(5)\qquad D\varphi = \pm\left(\varepsilon H\varphi + 2i\lambda\overline{\varphi}\right)$$

and satisfying the norm assumption $\mathcal{N}_{\pm}(p, q, \lambda, \varepsilon)$, then this spinor satisfies

$$\nabla_X \varphi = \pm\left(\frac{\varepsilon}{2} A(X) \cdot \varphi - i\lambda X \cdot \overline{\varphi}\right).$$

Proof : We give the proof for the sign $+$. The other case is strictly the same . **Case of signature (1,1)**: We define the endomorphism B_φ by

$$(B_\varphi)^i_j = g(B_\varphi(e_i), e_j) = \beta_\varphi(e_i,\ e_j) := \langle \varepsilon\nabla_{e_i}\varphi,\ e_j \cdot \varphi\rangle.$$

Obviously Using $\frac{e_i\cdot\varphi^{\pm}}{\langle\varphi^+,\varphi^-\rangle}$ as a normalized dual frame of $\Sigma^{\mp}M$ and the same proof as in [5] we can show that

$$\langle \nabla_X\varphi, e_i \cdot \varphi^{\pm}\rangle = \langle \varepsilon\nabla_X\varphi, \varepsilon e_i \cdot \varphi^{\pm}\rangle = -\frac{1}{2\varepsilon\langle\varphi^+,\varphi^-\rangle}\langle B_\varphi(X)\cdot\varphi, e_i\cdot\varphi^{\mp}\rangle.$$

and hence $\nabla_X\varphi = -\frac{1}{2\varepsilon\langle\varphi^+,\varphi^-\rangle} B_\varphi(X)\cdot\varphi$. Moreover

$$\begin{aligned}
\beta_\varphi(e_1,\ e_2) &= \langle \nabla_{e_1}\varphi,\ e_2\cdot\varphi\rangle = -\langle \varepsilon\nabla_{e_1}\varphi,\ e_1^2 \cdot e_2 \cdot \varphi\rangle \\
&= -\langle \varepsilon e_1 \cdot \nabla_{e_1}\varphi,\ e_1\cdot e_2\cdot\varphi\rangle = -\langle \varepsilon D\varphi + \varepsilon e_2 \cdot \nabla_{e_2}\varphi,\ e_1\cdot e_2\cdot\varphi\rangle \\
&= -\varepsilon^2 H\langle\varphi,\ e_1\cdot e_2\cdot\varphi\rangle - \langle 2i\varepsilon\lambda\omega^{\mathbb{C}}\cdot\varphi,\ e_1\cdot e_2\cdot\varphi\rangle + \beta_\varphi(e_2, e_1) \\
&= -\langle 2i\varepsilon\lambda\omega^{\mathbb{C}}\cdot\varphi,\ e_1\cdot e_2\cdot\varphi\rangle + \beta_\varphi(e_2, e_1),
\end{aligned}$$

since for any $\varphi, \psi \in \Gamma(\Sigma M)$

$$\langle\varphi,\ e_1\cdot e_2\cdot\psi\rangle = \langle e_2\cdot e_1\cdot\varphi,\ \psi\rangle = -\langle e_1\cdot e_2\cdot\varphi,\ \psi\rangle = -\langle\varphi,\ e_1\cdot e_2\cdot\psi\rangle = 0.$$

Let now consider the decomposition $\beta_\varphi(X,Y) = S_\varphi(X,Y) + T_\varphi(X,Y)$ in the symmetric part S_φ and antisymmetric part T_φ. Hence, we see easily that if $\lambda/\varepsilon \in i\mathbb{R}$, then β_φ is symmetric, *i.e.*, $T_\varphi = 0$. and if $\lambda/\varepsilon \in \mathbb{R}$, then $T_\varphi(X) = 2i\lambda/\varepsilon\, \omega^{\mathbb{C}}\cdot X$. In the two cases, we have

$$\nabla_X\varphi = \frac{\varepsilon}{2} A(X)\cdot\varphi - i\lambda X\cdot\overline{\varphi},$$

by setting $A = 2S_\varphi$. We verify easily that $tr(A) = 2tr(S_\varphi) = 2tr(B_\varphi) = -2H$.

Case of signature $(2,0)$ **or** $(0,2)$**:** The proof is fairly standard following the technique used in [4], [8] and [10]. We consider the tensors $Q^{\pm}_\varphi$ defined by

$$Q^{\pm}_\varphi(X,Y) = \Re e\left\langle \varepsilon\nabla_X\varphi^{\pm}, Y\cdot\varphi^{\mp}\right\rangle.$$

Then, we have

$$\mathrm{tr}\,(Q^{\pm}_{\varphi}) = -\Re e\left\langle \varepsilon D\varphi^{\pm}, \varphi^{\mp}\right\rangle = -\Re e\left\langle \varepsilon(\varepsilon H \pm 2i\lambda\varphi^{\mp}, \varphi^{\mp}\right\rangle = -\varepsilon^2\big(H \pm 2\Re e(\lambda)\big)|\varphi^{\mp}|^2.$$

Moreover, we have the following defect of symmetry of $Q^{\pm}_{\varphi}$,

$$\begin{array}{rcl} Q^{\pm}_{\varphi}(e_1,e_2) & = & \Re e\left\langle \varepsilon\nabla_{e_1}\varphi^{\pm}, e_2\cdot\varphi^{\mp}\right\rangle = \Re e\left\langle \varepsilon e_1\cdot\nabla_{e_1}\varphi^{\pm}, e_1\cdot e_2\cdot\varphi^{\mp}\right\rangle \\ & = & \Re e\left\langle \varepsilon D\varphi^{\pm}, e_1\cdot e_2\cdot\varphi^{\mp}\right\rangle - \Re e\left\langle \varepsilon\nabla_{e_2}\varphi^{\pm}, e_1\cdot e_2\cdot\varphi^{\mp}\right\rangle \\ & = & \Re e\left\langle (\varepsilon^2 H \pm 2i\varepsilon\lambda)\varphi^{\mp}, e_1\cdot e_2\cdot\varphi^{\mp}\right\rangle + \Re e\left\langle \varepsilon\nabla_{e_2}\varphi^{\pm}, e_1\cdot\varphi^{\mp}\right\rangle \\ & = & 2\Re e(\varepsilon\lambda)|\varphi^{\mp}|^2 + Q^{\pm}_{\varphi}(e_2,e_1). \end{array}$$

Then, using the fact that $\varepsilon e_1\cdot\dfrac{\varphi^{\pm}}{|\varphi^{\pm}|^2}$ and $\varepsilon e_2\cdot\dfrac{\varphi^{\pm}}{|\varphi^{\pm}|^2}$ form a local orthonormal frame of $\Sigma^{\mp}M$ for the real scalar product $\Re e\left\langle\cdot,\cdot\right\rangle$, we see easily that

$$\nabla_X\varphi^+ = \varepsilon\frac{Q^+_{\varphi}(X)}{|\varphi^-|^2}\cdot\varphi^- \quad \text{and} \quad \nabla_X\varphi^- = \varepsilon\frac{Q^-_{\varphi}(X)}{|\varphi|^+|^2}\cdot\varphi^+.$$

We set $W = \frac{Q^+_{\varphi}}{|\varphi^-|^2} - \frac{Q^-_{\varphi}}{|\varphi^+|^2}$. From the above computations, we have immediately that $W + \Re e\,(i\lambda/\varepsilon)\,\mathrm{Id}$ is symmetric and trace-free. Now, we will show that $W + \Re e\,(i\lambda/\varepsilon)\,\mathrm{Id}$ is of rank at most 1. First, we have

$$X|\varphi^+|^2 + \varepsilon^2 X|\varphi^-|^2 = 2\Re e\left\langle \varepsilon W(X)\cdot\varphi^-, \varphi^+\right\rangle.$$

Moreover, from the norm assuption $\mathcal{N}(p,q,\lambda,\varepsilon)$, we have

$$X|\varphi^+|^2 + \varepsilon^2 X|\varphi^-|^2 = 2\Re e\left\langle i\lambda X\cdot\varphi, \varphi\right\rangle = 4\Re e\left\langle i\lambda X\cdot\varphi^-, \varphi^+\right\rangle.$$

We deduce immediately that $W + 2\Re e\,(i\lambda/\varepsilon)\,\mathrm{Id}$ is of rank at most 1 and hence vanishes identically since it is symmetric and trace-free. Thus, we have the following relation

$$|\varphi^+|^2 Q^+_{\varphi} - |\varphi^-|^2 Q^-_{\varphi} = -2\Re e(i\lambda/\varepsilon)|\varphi^+|^2|\varphi^-|^2 g.$$

From now on, we will distinguish two cases.

• *Case 1:* $i\lambda/\varepsilon \in \mathbb{R}$.

Then we are in one of these two possible situations: $\varepsilon = i$ and $\lambda\in\mathbb{R}$ or $\varepsilon = 1$ and $\lambda \in i\mathbb{R}$. The second situation was studied by Morel [8].

So we define the following tensor $F := Q^+_{\varphi} - Q^-_{\varphi} + 2i\varepsilon\lambda(|\varphi^+|^2 - |\varphi^-|^2)g$. We have then

$$\begin{array}{rcl} \nabla_X\varphi & = & \nabla_X\varphi^+ + \nabla_X\varphi^- = \varepsilon\dfrac{Q^+_{\varphi}(X)}{|\varphi^-|^2}\cdot\varphi^+ + \varepsilon\dfrac{Q^-_{\varphi}(X)}{|\varphi^-|^2}\cdot\varphi^- \\ & = & \varepsilon\dfrac{F(X)}{|\varphi|^2}\cdot(\varphi^+ + \varphi^-) - i\lambda X\cdot\varphi^- - i\lambda X\cdot\varphi^+ \\ & = & \dfrac{\varepsilon}{2}A(X)\cdot\varphi - i\lambda X\cdot\overline{\varphi}, \end{array}$$

where we have set $A = \frac{2F}{|\varphi^2|}$. We conclude by noticing that A is a symmetric tensor with $\mathrm{tr}\,(A) = -2H$.

• *Case 2:* $i\lambda/\varepsilon \in i\mathbb{R}$.

Then we are in one of these two possible situations: $\varepsilon = i$ and $\lambda \in i\mathbb{R}$ or $\varepsilon = 1$ and $\lambda \in \mathbb{R}$. The second situation was studied by Morel [8].
In this case, we have from the previous computations that W vanishes identically. So we set

$$F = \frac{Q^+_\varphi}{|\varphi-|^2} = \frac{Q^-_\varphi}{|\varphi^+|^2}$$

and then we have $\nabla_X\varphi = F(X)\cdot\varphi$, where $F(X)$ is defined by $g(F(X),Y) = F(X,Y)$. Nevertheless, F is not symmetric. We define the following symmetric tensor $A(X,Y) = \frac{1}{|\varphi|^2}(F(X,Y)+F(Y,X))$. We compute immediately

$$A(e_1,e_1) = 2F(e_1,e_1)/|\varphi|^2 \quad , \quad A(e_2,e_2) = 2F(e_2,e_2)/|\varphi|^2,$$

$$A(e_1,e_2) = 2F(e_1,e_1)/|\varphi|^2 - 2\lambda/\varepsilon \quad \text{and} \quad A(e_2,e_2) = 2F(e_2,e_2)/|\varphi|^2 + 2\lambda/\varepsilon.$$

Finally, we conclude that

$$\nabla_X\varphi = \frac{\varepsilon}{2}A(X)\cdot\varphi + \lambda X\cdot\omega\cdot\varphi = \frac{\varepsilon}{2}A(X)\cdot\varphi - i\lambda X\cdot\overline{\varphi}.$$

□

Now, we can give the proof of Theorem 1. We have already proven that 3. implies 2. which implies 1. Moreover, Lemma 3.2 shows that 1. implies 2. Now, we will prove that 2. implies 3. For this, we use Lemma 3.1, but we need to distinguish the three cases for the different signatures. Let $\varphi = \varphi^+ + \varphi^-$.

Case of signature (2,0): Here, $\omega^{\mathbb{C}} = ie_1e_2$, hence $e_1\cdot e_2\cdot\varphi = -i\omega^{\mathbb{C}}\cdot\varphi = -i\bar{\varphi}$. Hence formula (4) becomes

$$-i\underbrace{\left(-R_{1212} + \varepsilon^2\det(A) + 4\lambda^2\right)}_{G_{2,0}}\bar{\varphi} = \varepsilon\underbrace{d^\nabla A(e_1,e_2)}_{C_{2,0}}\cdot\varphi.$$

or equivalently $\varepsilon C_{2,0}\cdot\varphi^\pm = \pm iG_{2,0}\varphi^\mp$. Applying two times this relation we have finally

$$\varepsilon^2||C_{2,0}||^2\varphi^\pm = -G^2_{2,0}\varphi^\pm.$$

Again we have two cases.
• *Spacelike immersion:* $\varepsilon = 1$, $M^{2,0} \hookrightarrow \mathbb{M}^{3,0}$.
We refer to [4] for the immersion in $\mathbb{R}^{3,0}$ and to [8] for $\mathbb{S}^3$ and $\mathbb{H}^3$. Only one (RSK)-spinor is needed.
• *Timelike immersion*: $\varepsilon = i$, $M^{2,0} \hookrightarrow \mathbb{M}^{2,1}$.
Two (ISK)-spinors are needed. We deduce from the above relations between $\varphi_1^\pm$ and $\varphi_2^\pm$ that $\langle C_{2,0}\cdot\varphi_1,\varphi_2\rangle = 0$. Moreover, in this case we have $\langle\varphi_1,\varphi_2\rangle = 0$. Thus, since the spinor bundle ΣM is of complex rank 2, we have $C_{2,0}\cdot\varphi_1 = f\varphi_1$ where f is a complex-valued function over M. By taking the inner product by φ_1, we see immediately that f only takes imaginary values, that is $f = ih$ with h real-valued. Thus, we have $\pm G_{2,0}\varphi_1^\pm = ih\varphi_1^\pm$. Since $\varphi_1^\pm$ do not vanish simultaneously, we deduce that h and $G_{2,0}$ vanish identically. Thus C vanishes too and the Gauss and Codazzi equation are satisfied. Then, we get the conclusion by the fundamental theorem of hypersurfaces given above.

Case of signature (1,1): $\omega^{\mathbb{C}} = -e_1e_2$, hence $e_1 \cdot e_2 \cdot \varphi = -\omega^{\mathbb{C}} \cdot \varphi = -\bar{\varphi}$. Hence formula (4) becomes

$$-\underbrace{\left(R_{1212} + \varepsilon^2 \det(A) + 4\lambda^2\right)}_{G_{1,1}} \bar{\varphi} = \varepsilon \underbrace{d^\nabla A(e_1, e_2)}_{C_{1,1}} \cdot \varphi.$$

or equivalently $\varepsilon C_{1,1} \cdot \varphi^\pm = G_{1,1}\varphi^\mp$. Applying two times this relation we have finally

$$\varepsilon^2 ||C_{1,1}||^2 \varphi^\pm = G_{1,1}^2 \varphi^\pm.$$

• *Spacelike immersion:* $\varepsilon = 1$, $M^{1,1} \hookrightarrow \mathbb{M}^{2,1}$.
We refer to [5] for the immersion in $\mathbb{R}^{2,1}$. Let us consider the other space forms. Here again, we need two (RSK)-spinors. Since $\varphi_1^\pm$ do not vanish at the same point, we have clearly that $||C_{1,1}|| = G_{1,1}^2 \geqslant 0$. Moreover, we have

$$\begin{array}{rcl} -||C_{1,1}||^2 \left\langle \varphi_1, \varphi_2 \right\rangle & = & \left\langle C_{1,1} \cdot \varphi_1, C_{1,1} \cdot \varphi_2 \right\rangle \\ & = & -G_{1,1}^2 \left\langle e_1 \cdot e_2 \varphi_1, e_1 \cdot e_2 \cdot \varphi_2 \right\rangle \\ & = & G_{1,1}^2 \left\langle \varphi_1, \varphi_2 \right\rangle. \end{array}$$

Since $\langle \varphi_1, \varphi_2 \rangle$ never vanishes, we deduce that $||C_{1,1}|| = -G_{1,1}^2 \leqslant 0$. Consequently, $||C_{1,1}|| = G_{1,1} = 0$. Moreover, $C_{1,1}$ is not isotropic. Indeed, since $G_{1,1} = 0$, we have $C_{1,1} \cdot \varphi_1 = 0$ and thus $C_{1,1}$ automatically vanishes as proved in [5].
• *Timelike immersion*: $\varepsilon = i$, $M^{1,1} \hookrightarrow \mathbb{M}^{1,2}$. It is easy to see that computations similar to the one for the previous case give the result.
Two (ISK)-spinors are needed.

Case of signature (0,2) $\omega^{\mathbb{C}} = -ie_1e_2$, hence $e_1 \cdot e_2 \cdot \varphi = i\omega^{\mathbb{C}} \cdot \varphi = i\bar{\varphi}$. Hence formula (4) becomes

$$\underbrace{i\left(-R_{1212} + \varepsilon^2 \det(A) + 4\lambda^2\right)}_{G_{0,2}} \bar{\varphi} = \varepsilon \underbrace{d^\nabla A(e_1, e_2)}_{C_{0,2}} \cdot \varphi.$$

or equivalently $\varepsilon C_{0,2} \cdot \varphi^\pm = \pm i G_{0,2}\varphi^\mp$. Applying two times this relation we have finally

$$\varepsilon^2 ||C_{0,2}||^2 \varphi^\pm = -G_{0,2}^2 \varphi^\pm.$$

• *Spacelike immersion:* $\varepsilon = 1$, $M^{0,2} \hookrightarrow \mathbb{M}^{1,2}$.
Similar computations to the case $M^{2,0} \hookrightarrow \mathbb{M}^{2,1}$ give the result. Two (ISK)-spinors are needed.
• *Timelike immersion*: $\varepsilon = i$, $M^{0,2} \hookrightarrow \mathbb{M}^{0,3}$.
We get $||C_{0,2}||^2 \varphi^\pm = G_{0,2}^2 \varphi^\pm$, hence $C_{0,2} = G_{0,2}^2 = 0$ as the norm of $C_{0,2}$ is negative definite. This is a similar computation to the case $M^{2,0} \hookrightarrow \mathbb{M}^{3,0}$. Only one (ISK)-spinor is needed. □

Let us summarize these results. In the tabular below we give the number of (RSK)-(resp. (ISK)-)spinors on the surface $M^{p,q}$ solutions of the special Killing equation (3), or equivalently of the Dirac equation (5), which is sufficient for the surface to be immersed, depending on the signature (p, q) and on the type ε of the immersion.

ε (p,q)	1	i
(0,2)	2 RSK-spinor	1 ISK-spinor
(1,1)	2 RSK-spinors	2 ISK-spinors
(2,0)	1 RSK-spinor	2 ISK-spinors

Table 1: *Number of spinors needed*

References

[1] C. Bär, P. Gauduchon, and A. Moroianu, Generalized cylinders in semi-Riemannian and spin geometry, Math. Z. 249 (2005), no. 3, 545580.

[2] H. Baum, Spin-Strukturen und Dirac Operatoren über pseudo-Riemannschen Mannigfaltgkeiten, Teubner-Texte zur Mathematik, Bd. 41 Teubner-Verlag, Leipzig, 1981.

[3] H. Baum and O. Müller, Codazzi spinors and global ly hyperbolic manifolds with special holonomy, Math. Z. 258 (2008), no. 1, 185211.

[4] T. Friedrich, On the spinor representation of surfaces in Euclidean 3-space, J. Geom. Phys. 28 (1998), 143157.

[5] M.A. Lawn, A spinorial representation for lorentzian surfaces in $\mathbb{R}^{2,1}$, J. Geom. Phys. 58 (2008) no. 6, 683-700.

[6] M.A. Lawn, J. Roth, Isometric immersions of Hypersurfaces into 4-dimensional manifolds via spinors, to appear in Diff. Geom. Appl.

[7] B. Lawson and M.-L. Michelson, Spin Geometry, Princeton University Press, 1989.

[8] B. Morel, Surfaces in $\mathbb{S}^3$ and $\mathbb{H}^3$ via spinors, Actes du séminaire de théorie spectrale et géométrie, Institut Fourier, Grenoble 23 (2005), 922.

[9] B. ONeill, Semi-riemannian geometry with applications to relativity, Academic Press, New York, 1983.

[10] J. Roth, Spinorial characterizations of surfaces into 3-homogeneous manifolds, to appear in J. Geom. Phys.