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OPTION HEDGING BY AN INFLUENT INFORMED INVESTOR

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Abstract

In this paper a model with an influent and informed investor is presented from a hedging point of view. The financial agent is supposed to possess an additional information, and is also supposed to influence the market prices. The problem is modeled by a forward-backward stochastic differential equation (FBSDE), to be solved under an initial enlargement of the Brownian filtration. An existence and uniqueness Theorem is proved under standard assummptions. The financial interpretation is derived, together with an example of such influenced informed model.

Keywords: Enlargement of filtration; FBSDE; insider trading; influent investor; asymmetric information; martingale representation.

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1. Introduction

In this article a hedging problem is studied from an influent informed investor's point of view. The agent is supposed to possess an additional information on the market, and is also supposed to influence market prices. This is a natural extension of the work of Eyraud-Loisel [11], where the informed agent was only supposed to be a small investor, with no influence on asset prices. We now study an influent informed agent, who wants to hedge against an option.

The presence of an asymmetrical information will be modeled by an initial enlargement of the Brownian filtration, as developed by the German school to model insider

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trading, with the works of Schweizer, Föllmer, Imkeller, and also Amendinger and Becherer (for example see [3, 2, 14, 20]). This approach was also studied by Grorud and Pontier at the same period, see [16, 17, 18]. Grorud and Pontier also developed in [19] a model where the informed agent may influence asset prices, one of the first work in which the insider trader is not supposed to be a small investor. This hypothesis (called "influent" or "large" investor in the literature) was introduced by Cuocu and Cvitanic in 1998 [7].

In Eyraud-Loisel [11], in a model without influence, it has been proved that hedging strategies of both agents, informed and uninformed, are identical. The very limits of this modeling is that the informed trader is supposed to be a small agent, with no influence on asset prices. But it is well known that there exists large traders in the market, who may influence the evolution of asset prices, either by their large investment depth, or by their notoriety, when a charter phenomenon appears. Then asset prices may be influenced by certain big agents in the market, and it is quite natural to suppose that such big (or large) agents may have more easily access to additional information on the market. This is the reason it is interesting to develop a model with an influent informed investor. This investor may influence asset prices either by his wealth X_t, or by his portfolio strategy π_t, which may influence the drift b of the volatility σ of prices. The Backward Stochastic Differential Equation (BSDE) driving the wealth process and the investment strategy, modeling the hedging problem, is then fully coupled with the forward equation of prices. This type of equations, called Forward-Backward Stochastic Differential Equations (in short FBSDE) appear when modeling hedging problems for large traders, studied for example by Cuoco and Cvitanic (1998) [7], or Cvitanic et Ma (2000) [8]. As there is an additional information, this FBSDE has to be solved in an enlarged filtration.

In Section 2, we formalize the financial problem in terms of FBSDE. In Section 3, we give, under certain hypotheses, an existence and uniqueness Theorem for such a FBSDE under an initial enlargement of filtration. We derive a proof very similar to the proof in the Brownian filtration case by Pardoux and Tang (1999) [25], whose details have been put in Appendix A. We prove, under similar hypotheses on the driver, coupled with hypothesis ($\mathbf{H_3}$) on the additional information, that the FBSDE has a unique solution in the enlarged space. This result is obtained in 3 cases of influence on the drift and volatility of prices. The first case is a case of "weak influence", where the Lipschitz coefficient of the drift and volatility of prices with respect to the wealth process and the investment strategy are not too large; the second one is satisfied when the payoff is independent of the price process, and the last case concerns models where the investment strategy of the large trader do not influence the volatility of prices. Under such conditions, the influent informed agent has a unique hedging strategy, and we give in Section 4 a financial interpretation of this result in terms of completeness of the informed market, and incompleteness of the market for a non informed trader's point of view.

Finally in Section 5, we present an example of influence satisfying all our hypotheses.

2. Model

2.1. Market model with influence

Let W be a k-dimensional standard Brownian motion, and $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ filtered probability space, where $\Omega = C([0,T];\mathbf{R}^k)$. $(\mathcal{F}_t)_{t\in[0,T]}$ denotes the natural Brownian filtration W, completed with the P-null sets. On the market, there are k risky assets and a riskless asset. The asset prices dynamics is supposed to be influenced by the wealth and portfolio strategy of an agent, called the large investor, or influent investor. X_t denotes the wealth at time t or this agent and π_t her portfolio.
Prices of risky assets are driven by the following diffusion :

$$dP^i_t = b_i(t, P_t, X_t, \pi_t)dt + (\sigma_i(t, P_t, X_t, \pi_t), dW_t), \ \ P^i_0 = p_i > 0, \forall i \in [1, k], \qquad (1)$$

where b and σ are supposed to satisfy :

$$\begin{aligned} b_i(t, P_t, X_t, \pi_t) &= b'_i(t, P_t, X_t, \pi_t) P^i_t, \\ \sigma_i(t, P_t, X_t, \pi_t) &= \sigma'_i(t, P_t, X_t, \pi_t) P^i_t. \end{aligned} \tag{2}$$

The riskless asset evolves according to the following equation :

$$dP^0_t = P^0_t r(t, X_t, \pi_t) dt \ , \ P^0_0 = 1. \tag{3}$$

Functions b'_i, σ'_i and r defined on $\Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k$ are supposed to be $\mathcal{F}$-adapted for fixed p, x, z, respectively $\mathbb{R}^k, \mathbb{R}^{k \times k}, \mathbb{R}$-valued.

The financial agent wants to hedge against a given contingent claim. She tries to determine her positive initial wealth, and her portfolio between 0 and maturity T in order to possess as terminal wealth X_T the payoff ξ of the contingent claim.

In the standard self-financing framework, the wealth of the agent may be written as the solution of the following stochastic differential equation :

$$\begin{aligned} dX_t &= \left((X_t - \sum_{i=0}^{k} \pi^i_t) r(t, X_t, \pi_t) \right) dt + \sum_{i=1}^{k} \pi^i_t b'_i(t, P_t, X_t, \pi_t) dt \\ &\quad + \sum_{i=1}^{k} \pi^i_t < \sigma'_i(t, P_t, X_t, \pi_t), dW_t > . \end{aligned}$$

As prices satisfy $P^i > 0, \ dt \otimes d\mathbb{P}$ almost surely, thanks to the exponential form of SDE (1) and (3), and from the hypothesis on the form of coefficients b_i and σ_i in Equation (2), this is equivalent to

$$X_t = X_T - \int_t^T f(s, P_s, X_s, Z_s) ds - \int_t^T < Z_s, dW_s >, \tag{4}$$

where

$$
\begin{aligned}
f(s,P_s,X_s,Z_s) &= X_s r(s,X_s,Z_s) + < {\sigma'^*}^{-1} Z_s, b'(s,P_s,X_s,Z_s) - r(s,X_s,Z_s).\mathbb{1} >,\\
\text{and } Z_s &= \sigma'^*(s,X_s,P_s,\pi_s)\pi_s
\end{aligned}
$$

The forward equation of prices (1) and the backward equation of wealth (4) are now fully coupled, because of the influence hypothesis :

$$
\begin{cases}
P_t = P_0 + \int_0^t b(s,P_s,X_s,Z_s)ds + \int_0^t < \sigma(s,P_s,X_s,Z_s), dW_s > \\
X_t = X_T - \int_t^T f(s,P_s,X_s,Z_s)ds - \int_t^T < Z_s, dW_s >
\end{cases}. \tag{5}
$$

Hypotheses are still to be specified in order to ensure such that this system admits a unique solution. This is the aim of Section 3.

2.2. Informed agent

The influent agent described in the previous section is also supposed to be an insider trader. She has an additional information L at time $t = 0$. This information is supposed to be $\mathcal{F}_{T'}$-measurable, where $T < T'$: it will be public at time T'. The global information available at time t to the informed agent is not $\mathcal{F}_t$ any more, but it has been augmented with information L. To model it, we introduce the enlarged filtration

$$
\mathcal{Y}_t := \bigcap_{s>t} (\mathcal{F}_s \vee \sigma(L)). \tag{6}
$$

It is an initial enlargement of the Brownian filtration (as developped by Jacod [21]). This kind of information is known as strong initial information. We will work under the following usual hypothesis, introduced by Jeulin [22, 23], and extensively used by Grorud and Pontier [16] and Eyraud-Loisel [11] :

Hypothesis 1. ($\mathbf{H_3}$) *There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that under $\mathbb{Q}$, $\mathcal{F}_t$ and $\sigma(L)$ independent for all $t < T'$.*

This hypothesis ($\mathbf{H_3}$) is known to be equivalent to "the conditional distribution of L given $\mathcal{F}_t$ is equivalent to the distribution of L, for all $t < T'$". Under ($\mathbf{H_3}$), a remarkable property is that W_t is still a $(\mathcal{Y}, \mathbb{Q})$-Brownian motion. Moreover, $\mathbb{Q}$ may be chosen such that, for all $t \leq T < T'$, $\mathbb{Q}|_{\mathcal{F}_t} = \mathbb{P}|_{\mathcal{F}_t}$, and on $\mathcal{Y}_T$, $\mathbb{Q} = \frac{1}{q(T,L)}\mathbb{P}$. For a reference on the existence of such a probability, and on the properties under ($\mathbf{H_3}$), the reader may refer to Jeulin [23], Amendinger [1], Grorud and Pontier [16].

3. Solution of the enlarged FBSDE

3.1. Enlarged FBSDE and space of solutions

We have supposed that the informed investor may have an influence on asset prices. To solve the hedging problem described in Section 2, we have to find a solution of the coupled forward-backward SDE (5). The mathematical problem lays in the space of solutions : this equation has to be solved in the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$. As under hypothesis ($\mathbf{H_3}$), W is still a Brownian motion under $\mathbb{Q}$, the FBSDE is the same in the enlarged space :

$$\begin{cases} P_t = P_0 + \int_0^t b(s, P_s, X_s, Z_s)ds + \int_0^t < \sigma(s, P_s, X_s, Z_s), dW_s > \\ X_t = g(P_T) - \int_t^T f(s, P_s, X_s, Z_s)ds - \int_t^T < Z_s, dW_s > . \end{cases} \tag{7}$$

with W a $(\mathcal{Y}, \mathbb{Q})$-Brownian motion, $P_0 \in \mathcal{Y}_0$ and $\xi = g(P_T) \in L^2(\Omega, \mathcal{Y}, \mathbb{Q})$.
As there exists existence and uniqueness results (see Eyraud-Loisel [11]) for BSDE under initially enlarged filtration, the main difficulty is linked to the coupling between the forward equation of prices and the backward equation of wealth, which can not be solved separately. Pardoux and Tang [25] show the existence of a unique solution in the case of a Brownian filtration (i.e. trivial initial σ-field), which is not the case here. We will have to adapt the proof to the present case.

Remark : Let us remark here the main difference with the model in [11], where all parameters of the price diffusion were adapted to the Brownian filtration, and so

$\mathcal{F}^P = \mathcal{F}^W$, the filtration generated by prices was the Brownian filtration. In the present model, wealth X and portfolio Z may be $\mathcal{Y}$-adapted and depend on L. So the influence hypotheses induces that $\mathcal{F}^P \neq \mathcal{F}^W$. We have now $\mathcal{F}^P \subset \mathcal{Y}$: a part of the additional information of the influent insider is "revealed" by the observation of prices. This is mathematically induced by the coupling of both SDEs.

If $\mathbb{H}$ denotes a general Euclidian space, $M^2(0,T;\mathbb{H})$ denotes the set of all $\mathcal{Y}$-progressively measurable $\mathbb{H}$-valued process $\{u(t),\ 0 \leq t \leq T\}$ such that

$$\| u(.) \|_0 := \left(E_{\mathbb{Q}} \int_0^T |u(s)|^2 \right)^{1/2} < +\infty.$$

For any process in $M^2(0,T;\mathbb{R}^d)$, we denote also by

$$\| u(.) \|_\lambda := \left(E_{\mathbb{Q}} \int_0^T e^{-\lambda s} |u(s)|^2 \right)^{1/2} < +\infty.$$

a family of equivalent norms, indexed by $\mathbb{R}$.

$\mathcal{S}^2_{\mathbb{Q}}$ denotes the set of all $\mathcal{Y}$-progressively measurable process X such that

$$\|X\|^2 := E_{\mathbb{Q}} \left[\sup_{0 \leq t \leq T} |X_t|^2 \right] < +\infty.$$

and $\mathcal{S}^2_{\mathbb{Q},c}$ the subset of continuous processes of $\mathcal{S}^2_{\mathbb{Q}}$.

A strong solution of system (7) is a $\mathcal{Y}$-adapted solution $(P_t, X_t, Z_t)_{0 \leq t \leq T}$ such that :

$$E_{\mathbb{Q}} \int_0^T \| Z_t \|^2 \, dt < +\infty. \tag{8}$$

It financially means that the agent looks for an admissible portfolio hedging the contingent claim.

3.2. Existence and Uniqueness Theorem

b, σ, f and g are supposed to satisfy the following hypotheses (($A1$) to ($A8$)), which are derived from those for which E. Pardoux and S. Tang obtained existence of a unique solution in the case where there is no additional information (trivial initial σ-field):

(A1) σ is invertible $dt \times d\mathbb{P}$-a.s., σ' and $\sigma'^{-1}(b' - r)$ are bounded.

(A2) functions b, f, σ, g are continuous w.r.t. p, x, z in $\mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^k$, for all $(\omega, t) \in \Omega \times [0, T]$.

(A3) $\exists \lambda_1, \lambda_2 \in \mathbb{R}$ tel que $\forall t, p, p_1, p_2, x, x_1, x_2, z$, $\mathbb{P}$-p.s. :

$$\begin{aligned} < b(t,p_1,x,z) - b(t,p_2,x,z), p_1 - p_2 > &\leq \lambda_1 |p_1 - p_2|^2, \\ < f(t,p,x_1,z) - f(t,p,x_2,z), x_1 - x_2 > &\leq \lambda_2 |x_1 - x_2|^2. \end{aligned}$$

(A4) b is globally Lipschitz w.r.t. x and z, and at most linearly increasing w.r.t. p, and f is globally Lipschtiz w.r.t. p and z, and at most linearly increasing w.r.t. x: $\exists k, k_i$ such that $\mathbb{P}$-a.s. $\forall t, p_i, x_i, z_i$

$$\begin{aligned} |b(t,p,x_1,z_1) - b(t,p,x_2,z_2)| &\leq k_1 |x_1 - x_2| + k_2 \parallel z_1 - z_2 \parallel, \\ |b(t,p,x,z)| &\leq |b(t,0,x,z)| + k(1 + |p|), \\ |f(t,p_1,x,z_1) - f(t,p_2,x,z_2)| &\leq k_3 |p_1 - p_2| + k_4 \parallel z_1 - z_2 \parallel, \\ |f(t,p,x,z)| &\leq |f(t,p,0,z)| + k(1 + |x|). \end{aligned}$$

(A5) σ is globally Lipschitz w.r.t. p, x and z : $\exists k_i$ such that $\mathbb{P}$-a.s. $\forall t, p_i, x_i, z_i$

$$\parallel \sigma(t,p_1,x_1,z_1) - \sigma(t,p_2,x_2,z_2) \parallel^2 \leq k_5^2 |p_1 - p_2|^2 + k_6^2 |x_1 - x_2|^2 + k_7^2 \parallel z_1 - z_2 \parallel^2 .$$

(A6) g is globally Lipschitz w.r.t. p : $\exists k_8$ such that $\mathbb{P}$-a.s. $\forall p_i$

$$|g(p_1) - g(p_2)|^2 \leq k_8^2 |p_1 - p_2|^2.$$

(A7) $\forall p,\ x,\ z,\ b(.,p,x,z),\ f(.,p,x,z)$ and $\sigma(.,p,x,z)$ are $\mathcal{F}$-adapted processes and $g(p)$ is $\mathcal{F}_T$-measurable. Moreover :

$$E_{\mathbb{P}}\int_0^T |b(s,0,0,0)|^2 ds + E_{\mathbb{P}}\int_0^T |f(s,0,0,0)|^2 ds \\ +E_{\mathbb{P}}\int_0^T \parallel \sigma(s,0,0,0) \parallel^2 ds + E_{\mathbb{P}}|g(0)|^2 < +\infty. \tag{9}$$

(A8) b is also supposed to be globally Lipschitz w.r.t. p : $\exists k_9$ such that $\mathbb{P}$-a.s. $\forall t, p_i, x, z$

$$|b(t,p_1,x,z) - b(t,p_2,x,z)|^2 \leq k_9^2 |p_1 - p_2|^2.$$

Remarks :

- Hypothesis $(A1)$ guarantees $E_{\mathbb{P}}\left(\exp \frac{1}{2}\int_0^T \sigma_s'^2 ds\right) < +\infty$, which implies that the price process P is a true martingale, as a uniformly integrable local martingale (Doleans exponential of $\sigma'\tilde{W}$ under a risk-neutral probability), and also insures that prices are positive $\mathbb{P}$-a.s. (see D. Lépingle et J. Mémin [24]).
- Moreover, from Hypothesis $(A1)$, $\mathcal{E}(-\sigma'^{-1}(b' - r\mathbb{1}).W)$ is $\mathbb{P}$-integrable. This guarantees the existence of a risk-neutral probability (and so no arbitrage opportunity) when there is no additional information, see F. Delbaen et W. Schachermayer [10].
- We can notice that these two remarks are also true under probability $\mathbb{Q}$ of Hypothesis $(\mathbf{H_3})$.

These hypotheses guarantee existence in $(\Omega, \mathcal{F}, \mathbb{P})$ of a unique solution, and so of a unique portfolio (only in certain cases of coupling, see E. Pardoux et S. Tang [25], and Conditions $(I1)$ to $(I3)$ defined hereafter) in the case where there is no additional information (Brownian filtration). With the additional information L satisfying hypothesis $(\mathbf{H_3})$, on space $(\Omega, \mathcal{Y}, \mathbb{Q})$, is there still a unique solution to the system ?
Hypotheses $(A1)$ to $(A8)$ are still satisfied, except $(A7)$, under the enlarged space $(\Omega, \mathcal{Y}, \mathbb{Q})$, with the new filtration $\mathcal{Y}$ and new probability $\mathbb{Q}$, as $\mathbb{Q}$ and $\mathbb{P}$ are equivalent.

We will assume the following additional integrability condition, in order to have $(A7)$ also satisfied under $\mathbb{Q}$:

$$
\begin{aligned}
(A7') \qquad & E_{\mathbb{Q}} \int_0^T |b(s,0,0,0)|^2 ds + E_{\mathbb{Q}} \int_0^T |f(s,0,0,0)|^2 ds \\
& + E_{\mathbb{Q}} \int_0^T \| \sigma(s,0,0,0) \|^2 \, ds + E_{\mathbb{Q}} |g(0)|^2 < +\infty. \qquad (10)
\end{aligned}
$$

Remark: Let us notice that this condition is always satisfied in our financial framework, as all parameters are null in 0.

Notice also that results are proved in the present section in the general dimension case for processes $(P_t, X_t, Z_t)_{t\in[0,T]}$ taking values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$. In the financial setting $n = k$, $m = 1$ et $d = k$.

Three different influence cases are treated here :

(I1) The influence is weak : the forward and backward equations are weakly coupled: $\exists\, \varepsilon_0 > 0$ depending on $k_3, k_4, k_5, k_8, \lambda_1, \lambda_2$ and T such that $k_1, k_2, k_6, k_7 \in [0, \varepsilon_0)$.

(I2) g, $\mathcal{F}_T$-measurable, is independent from the price process, and λ_1 and λ_2 from hypothesis (A3) satisfy also: $\exists\; C_i > 0, i = 1,2,3,4, C_4 < k_4^{-1}, \theta > 0$ such that

$$
\begin{aligned}
\lambda_1 + \lambda_2 \quad < \quad & -\frac{1}{2}\left[k_3 C_3 \left(\frac{k_2 C_2 + k_7^2}{1 - k_4 C_4} + \frac{k_1 C_1 + k_6^2}{\theta} \right) \right. \\
& \left. + k_1 C_1^{-1} + k_2 C_2^{-1} + k_3 C_3^{-1} + k_4 C_4^{-1} + k_5^2 + \theta \right]. \qquad (11)
\end{aligned}
$$

(I3) σ is independent from z : the portfolio does not influence the volatility of prices and λ_1 and λ_2 from hypothesis (A3) satisfy also: $\exists C_i > 0, i = 1,3,4,\; C_4 < k_4^{-1},\; \theta > 0,\; \alpha > 0$ such that

$$
\begin{aligned}
\lambda_1 + \lambda_2 \quad < \quad & -\frac{1}{2}\left[(1+\alpha) \left[k_1 C_1 + k_6^2 + \frac{k_2^2}{\alpha(1 - k_4 C_4)} \right] \left(k_8^2 + \frac{k_3 C_3}{\theta} \right) \right. \\
& \left. + k_1 C_1^{-1} + k_3 C_3^{-1} + k_4 C_4^{-1} + k_5^2 + \theta \right]. \qquad (12)
\end{aligned}
$$

In these 3 cases, the existence and uniqueness Theorem given by E. Pardoux and S. Tang [25] is still satisfied in the enlarged space.

Theorem 1. *We suppose Hypotheses* $(A1)$ *to* $(A6)$, $(A7')$ *and* $(A8)$ *satisfied, and also one of the cases (I1) to (I3) satisfied. Then the following forward-backward stochastic differential equation:*

$$\begin{cases} P_t = P_0 + \int_0^t b(s, P_s, X_s, Z_s)ds + \int_0^t < \sigma(s, P_s, X_s, Z_s), dW_s > \\ X_t = \xi - \int_t^T f(s, P_s, X_s, Z_s)ds - \int_t^T < Z_s, dW_s > . \end{cases} \tag{13}$$

admits in the space $(\Omega, \mathcal{Y}, Q)$ *a unique solution* (P, X, Z) $\mathcal{Y}$*-adapted such that*

$$E_{\mathbb{Q}} \int_0^T \left(\sup_{0 \le u \le t} |P_t|^2 + \sup_{0 \le u \le t} |X_t|^2 + \parallel Z_t \parallel^2 \right) dt < +\infty. \tag{14}$$

Proof : We derive the four majoration lemma and use the same arguments as in Pardoux and Tang [25]. Theorem 2.1 in Eyraud-Loisel [11] for BSDE with enlarged filtration allows us to construct strict contractions and conclude to the existence and uniqueness of the solution in the enlarged space. Even if the standard proofs in the Brownian case may be adapted to the case of an enlarged filtration, even with a change of filtration and a non trivial initial σ-algebra, there are some difficulties within the adaptation of these proofs. The key point that leads hidden in Theorem 2.1 of [11] is the existence of a martingale representation Theorem under hypothesis $(\mathbf{H_3})$ in the enlarged space. This point is hidden in the problem of solving the FBSDE, but it is nevertheless crucial. See in Appendix A the main steps of this proof. □

4. Financial interpretation

4.1. Bound on the wealth process

We also derived another result, unknown in the literature, up to the knowledge of the author, which may financially be interpreted as a bound on the wealth process.

Proposition 1. *Under Hypotheses* $(A1)$ *and* $(A8)$*, and* $(A7')$*,* $\forall \xi \in L^2(\Omega, \mathcal{Y}, \mathbb{Q})$*, if* k_3*=0 and if* $f(s,0,0,0)$ *is bounded, then* X *solution of the FBSDE (13) satisfies:*

$$|X_t|^2 \leq e^{-\lambda(T-t)} E_{\mathbb{Q}}\left(|\xi|^2 \middle| \mathcal{Y}_t\right) + \frac{1}{\varepsilon} E_{\mathbb{Q}}\left(\int_t^T e^{-\lambda(s-t)} |f(s,0,0,0)|^2 ds \middle| \mathcal{Y}_t \right), \qquad (15)$$

where $\lambda = -(2\lambda_2 + \varepsilon + k_4 C_4^{-1})$*,* $\varepsilon > 0$ *and* $0 < C_4 \leq k_4^{-1}$*.*

Proof: This is obtained by using Itô's formula, taking conditional expectation w.r.t. $\mathcal{Y}_t$ and using standard inequalities, as well as Hypotheses (A1) to (A8).

Apply Itô's lemma to $e^{-\lambda t}|X_t|^2$ between t and T, with $\lambda \in \mathbb{R}$:

$$\begin{aligned} e^{-\lambda t}|X_t|^2 + \int_0^T e^{-\lambda s}|Z_s|^2 ds &= e^{-\lambda T}|\xi|^2 + \lambda \int_t^T e^{-\lambda s}|X_s|^2 ds \\ +2\int_t^T X_s f(s, P_s, X_s, Z_s) e^{-\lambda s} ds &- 2\int_t^T X_s Z_s e^{-\lambda s} dW_s. \end{aligned} \qquad (16)$$

Taking conditional expectation of this expression w.r.t. $\mathcal{Y}_t$ under probability measure $\mathbb{Q}$ eliminates the last term, as the increments of the $(\mathcal{Y}, \mathbb{Q})$-Brownian motion W are independent. Hypotheses $(A3)$ and $(A4)$ give a majoration to the other terms, and as $k_3 = 0$, we obtain, $\forall \varepsilon, C_3, C_4 > 0$:

$$\begin{aligned} |X_t|^2 e^{-\lambda t} &\leq e^{-\lambda T} E_{\mathbb{Q}}\left(|\xi|^2 \middle| \mathcal{Y}_t\right) + \left(\lambda + 2\lambda_2 + \varepsilon + \frac{k_4}{C_4}\right) E_{\mathbb{Q}}\left(\int_t^T e^{-\lambda s}|X_s|^2 ds \middle| \mathcal{Y}_s \right) \\ &+ (k_4 C_4 - 1) E_{\mathbb{Q}}\left(\int_t^T e^{-\lambda s}|Z_s|^2 ds \middle| \mathcal{Y}_s \right) + \frac{1}{\varepsilon} E_{\mathbb{Q}}\left(\int_t^T e^{-\lambda s}|f(s,0,0,0)|^2 ds \middle| \mathcal{Y}_s \right). \end{aligned}$$

Choosing $\lambda = -(2\lambda_2 + \varepsilon + k_4 C_4^{-1})$, and $0 < C_4 \leq k_4^{-1}$, as X solution of FBSDE (13) satisfies $E_Q(\int_0^T |X_t|^2 dt) < +\infty$, we obtain the expected inequality :

$$|X_t|^2 e^{-\lambda t} \leq e^{-\lambda T} E_{\mathbb{Q}}\left(|\xi|^2 \middle| \mathcal{Y}_t\right) + \frac{1}{\varepsilon} E_{\mathbb{Q}}\left(\int_t^T e^{-\lambda s}|f(s,0,0,0)|^2 ds \middle| \mathcal{Y}_s \right).$$

□

Corollary 1. *If moreover ξ is bounded, then process X is bounded.*

Proof: This appears naturally from equation (15) as the right term is bounded. □

Toy example : We can find a simple example to illustrate this property, in the case of prices driven by a geometric Brownian motion (Black-Scholes model), without influence:

$$f(s,p,x,z) = xr + \sigma'^{-1}(b' - r)z.$$

f does not depend on variable p, so $k_3 = 0$, and $f(s,0,0,0) = 0$.
In any cases of ξ to hedge, X satifies :

$$|X_t|^2 \leq e^{-\lambda(T-t)} E_{\mathbb{Q}} \left(|\xi|^2 \middle| \mathcal{F}_t \right). \tag{17}$$

And for a pay-off ξ bounded (for instance a European put option $(K - P_T)_+$, bounded by K), we have :

$$|X_t|^2 \leq e^{-\lambda(T-t)} K^2 \leq e^{-\lambda T} K^2. \tag{18}$$

Wealth is bounded by $K^2 e^{-\lambda T}$.

4.2. Market incompleteness derived by the influent informed agent

In these three cases of influence, $(I1)$, $(I2)$ et $(I3)$, the forward-backward stochastic differential equation has a unique adapted solution in the enlarged space. This means that the influent agent has a unique hedging strategy adapted to his information. We have supposed σ invertible, so from Z_t, it is possible to derive the unique portfolio π_t hedging the pay-off ξ : $\pi_t = {\sigma'_t}^{-1} Z_t$. If the solution of the insider trader is adapted to the Brownian filtration, then it is the same as if there was no additional information. Any contingent claim in $L^2(\Omega, \mathcal{Y}, \mathbb{Q})$ satisfying hypothesis $(A6)$ is attainable. In fact, the market is complete for the informed investor, relatively to the enlarged filtration. The hedging problem in the market is reduced to a resolution of a FBSDE, whose coupling depends on the influence of the insider in the market.

We obtain here the same results for the insider's point of view as in [11], for the existence and uniqueness of the solution of a BSDE under an enlarged filtration.
What is different in the model with an influent agent, compared to the model without influence developed in [11], is the behavior of the market from the point of view of a normally informed agent (when the additional information is unknown). The market from a non insider's point of view is incomplete. This incompleteness is due to a lack of information, as in a model developed by H. Föllmer and M. Schweizer (1991) [13]. Our model is an example of complete market which becomes incomplete from a small non informed investor point of view. The study of such a market uses tools of quadratic hedging in incomplete market, or under incomplete information, and will be developed separately in a further work, see Eyraud-Loisel [12].

5. Example

In this last section, we present a model of influence that satisfies all hypotheses of this study. Suppose that the price process is driven by the following dynamics (stochastic volatility model) :

$$dP_t = b'(P_t, X_t, \pi_t)P_t dt + \sigma'_t(\eta)P_t dW_t, \tag{19}$$

where

$$\sigma'_t(\eta) = \sigma^0 \mathcal{I}_{[0,\eta[}(t) + \sigma^1 \mathcal{I}_{[\eta,T]}(t)\ ,\ \sigma^0, \sigma^1 \neq 0. \tag{20}$$

The volatility of this model is piecewise constant, taking two possible values σ^0 and σ^1 fixed by the model, η is a random variable satisfying hypothesis ($\mathbf{H_3}$), taking his values in $[0, T+\varepsilon]$.
The information of the insider trader is $L = \eta\ \ \mathcal{F}_{T+\varepsilon}-$measurable . This is an example of strong initial information.

5.1. Case without influence

If we choose a drift parameter $b'(P_t, X_t, \pi_t) = b_0$ constant, we obtain a model without influence. The coupling between the BSDE and the FSDE vanishes, and it leads to a resolution of BSDE with initial enlargement of filtration. All hypotheses $(A1)$ to $(A8)$ and case $I1$ (no dependence, so a fortiori weak dependence) are satisfied, and even without influence, the market is complete from the insider point of view, and incomplete from the non insider point of view. This gives a toy example where an additional information may "complete" the market.

5.2. Case with influence

The drift parameter is chosen as the following :

$$b'(X_t, P_t, \pi_t) = b_0 + \frac{b_1}{(1+P_t)(1+\pi_t^2)} \ , \quad b_0, b_1 \in \mathbb{R} \text{ fixed.} \tag{21}$$

The interest rate r is supposed to be constant.
Drift b' is bounded, and may vary between two thresholds b_0 and $b_0 + a$. Two cases may appear, depending on the sign of b_1. If $b_1 < 0$, the influence is a positive influence: the bigger is the investment porfolio, the higher is the drift of the prices. This is moderated by the level of prices: the higher are the prices, the lower is the influence. If $b_1 > 0$, it is the converse principle: when the level of the portfolio increases, the drift of the prices decreases, and the influence is stronger when the level of prices is high. Remark that the case $b_1 = 0$ is the case treated in the previous section, without influence.
Depending on the sign of b_1, representing the amplitude of the influence, this influence will have either a leverage effect or a return effect on the drift of the price process around the value b_0. The influence is from the insider's portfolio on the price process, which remains bounded according to the hypotheses.
We can also notice that

$$\sigma_s'^{-1}(b_s' - r_s) = \sigma_s'^{-1}(b_0 - r + \frac{b_1}{(1+P_t)(1+\pi_t^2)})$$

is bounded, as well as σ'. So there exists a risk-neutral probability measure $\tilde{\mathbb{Q}}$ under which $dP_t = \sigma'_t P_t d\tilde{W}_t$ is a positive uniformly integrable martingale (see Lépingle and Mémin [24] Theorems II-2 and III-7).

Remark: We don't have here constraints on the signs of b_0 or b_1, whereas it is often the case in previous influence models developed in the literature, such as in the model introduced by Cuoco and Cvitanic (1998) [7], and treated deeply in Grorud and Pontier (2005) [19] (their influence form is slightly different from the one treated in this work). This may be explained by the fact that we consider a hedging problem, whereas they considered an optimization problem, and therefore we do not need the convexity of the parameters here.

For the present model, considering the hedging of a European call option of maturity T and exercise price K, the parameters are the following :

$$\begin{aligned} f(s, P_s, X_s, \pi_s) &= X_s r + \left(b_0 - r + \frac{b_1}{(1+P_t)(1+\pi_t^2)} \right) \pi_s \ , \\ g(P_T) &= (P_T - K)_+ \end{aligned} \tag{22}$$

which leads to the following FBSDE modeling the hedging problem of the informed agent :

$$\begin{cases} P_t = P_0 + \int_0^t \left(b_0 + \frac{b_1}{(1+P_t)(1+\pi_t^2)} \right) P_s ds + \int_0^t \sigma_s(\eta) P_s dW_s \\ X_t = (P_T - K)_+ - \int_t^T (X_s r + (b_0 + ah(\pi_s)) \pi_s - r\pi_s) ds - \int_t^T \sigma_s(\eta) \pi_s dW_s. \end{cases} \tag{23}$$

We have to check if all hypotheses of Theorem 1 are satisfied.

(A1) σ' is bounded and invertible, since σ^0 and σ^1 are constant and not null. Moreover, $\sigma'^{-1}(b' - r)$ is bounded, since b' and σ'^{-1} are bounded.

(A2) b, σ, f and g are continuous w.r.t. variables p, x, z.

(A3) This hypothesis is satisfied by taking $\lambda_1 = \sup(b_0, b_0 + b_1)$:

$$
\begin{aligned}
&(b(t,p_1,x,z) - b(t,p_2,x,z), p_1 - p_2) \\
&\quad = \left(b_0 p_1 + \frac{b_1}{(1+p_1)(1+\sigma'^{-2}z^2)} - b_0 p_2 - \frac{b_1}{(1+p_2)(1+\sigma'^{-2}z^2)} \right)(p_1 - p_2) \\
&\quad = b_0 (p_1 - p_2)^2 + \frac{b_1}{(1+\sigma'^{-2}z^2)} \times \frac{(p_1-p_2)^2}{(1+p_1)(1+p_2)} \\
&\quad \leq \sup(b_0, b_0 + b_1)(p_1 - p_2)^2, \qquad (24)
\end{aligned}
$$

and $\lambda_2 = r$: $(f(t,p,x_1,z) - f(t,p,x_2,z), x_1 - x_2) = r(x_1 - x_2)^2$.

(A4) $k_1 = 0$ since b does not depend on x.

$b(t,p,x,z)$ is $\mathcal{C}^1$, with first derivative w.r.t. z given by :

$$
\left| \frac{\partial b(t,p,x,z)}{\partial z} \right| = \left| \frac{b_1 p}{1+p} \frac{(-2\sigma'^{-1}z)}{(1+\sigma'^{-2}z^2)^2} \right| \leq |b_1| \left| \frac{1}{1+\sigma'^{-2}z^2} \times \frac{(-2\sigma'^{-1}z)}{(1+\sigma'^{-2}z^2)} \right| \leq |b_1|.
$$

So b is uniformly Lipschitz w.r.t. z, with $k_2 = |b_1|$.

Moreover, since $b(t,0,x,z) = 0$, we can write

$$
|b(t,p,x,z)| \leq (|b_0| + |b_1|)\,|p|. \qquad (25)
$$

So $k \geq |b_0| + |b_1|$.

f is also $\mathcal{C}^1$, with first derivative w.r.t. p and z given by :

$$
\begin{aligned}
\left| \frac{\partial f(t,p,x,z)}{\partial p} \right| &= \left| b_1 \frac{\pi}{1+\pi^2} \frac{1}{(1+p)^2} \right| \leq \frac{|b_1|}{2}, \\
\left| \frac{\partial f(t,p,x,z)}{\partial z} \right| &= \left| \frac{b_1 \sigma'^{-1}}{(1+p)} \frac{1-\pi^2}{(1+\pi^2)^2} \right| \leq \frac{|b_1|}{\inf(\sigma^0, \sigma^1)}.
\end{aligned}
$$

So f is uniformly Lipschitz w.r.t. p and z, with $k_3 = \frac{|b_1|}{2}$ and $k_4 = \frac{|b_1|}{\inf(\sigma^0,\sigma^1)}$.

Finally f is linear w.r.t. x, so $k \geq r$. $k = \max(r, |b_0| + |b_1|)$ is convenient.

(A5) σ is uniformly Lipschitz w.r.t. P (piecewise linear), and does not depend on x and z, so this hypothesis is satisfied with $k_5 = sup(\sigma^0, \sigma^1)$, $k_6 = 0$, $k_7 = 0$.

(A6) g is uniformly Lipschitz w.r.t. p, with $k_8 = 1$.

(A7) $b(., P, X, \pi)$, $f(., P, X, \pi)$ and $\sigma(., P, X, \pi)$ are $\mathcal{Y}_t$-adapted processes and $g(p)$ is deterministic for fixed p.

Moreover, the integrability equation in 0 is satisfied : all terms are null.

(A8) b is uniformly Lipschitz in p, with coefficient $k_9 = |b_0| + |b_1|$:

$$\begin{aligned} |b(t, p_1, x, z) - b(t, p_2, x, z)| &= \left| b_0(p_1 - p_2) + \frac{b_1(p_1 - p_2)}{(1+\pi^2)(1+p_1)(1+p_2)} \right| \\ &\leq (|b_0| + |b_1|)\, |p_1 - p_2|. \end{aligned}$$

We are in the case of weak influence $(I1)$ of existence and uniqueness Theorem 1. Indeed, whatever is the value of b_1, C_2 may be chosen small enough so that Equation (41) (see proof of the Theorem 1 in Appendix A) is satisfied. In the present example, this equation becomes, as $k_1 = k_6 = k_7 = 0$:

$$|b_1|C_2 < \left[\frac{1 - e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1}{1 - \frac{|b_1|C_4}{\inf(\sigma^0, \sigma^1)}} \right]^{-1} \left((1 \vee e^{-\bar{\lambda}_1 T}) + \frac{|b_1|C_3}{2} \frac{(1 - e^{-\bar{\lambda}_1 T})}{\bar{\lambda}_1} \right)^{-1},$$

which is always true as C_2 may be chosen arbitrarily small.
Let us notice that if b_0 and b_1 are negative enough to satisfy Equation (43), we could also use the case $(I3)$, because there is no influence from the insider on the volatility of prices.
So all hypotheses of Theorem 1 are satisfied, and EDSPR (13) admits a unique solution. This allows us to conclude to the existence of a unique solution of the insider hedging problem.

Remark : Despite several studies developed on the subject, from Chevance (1997) [6], to Delarue (2002) [9], Gobet, Lemor and Warin (2005) [15], or more recently Bouchard and Elie (2008) [4], or Bouchard, Elie and Touzi (2009) [5] with discretization schemes of BSDE and FBSDE, there is still a difficulty that leaves in the use of such equations, which is the difficulty to express explicitely the solutions. But all these new schemes may open a way to simulate and use more extensively such study, because these new works and their future extensions give interesting tracks for efficient schemes for FBSDEs.

Appendix A. Proof of Theorem 1

The first step of the proof of Theorem 1 consist in deriving the first four majoration lemmas on the norms of solutions. These fundamental estimates lemmas are the same as those in Pardoux and Tang [25]. Even if the considered space is not the same (the initial sigma-field is not trivial in our case), these majoration lemmas still hold in the enlarged space. The main reason is that the only tools used to prove these lemmas are Itô Formula, Lipschitz property and linear growth of the different functions and coefficients in the hypotheses $(A1)$ to $(A7)$, which are also supposed to hold under $\mathbb{P}$ and, as seen page 9, remain true under $\mathbb{Q}$. We recall the statement of these four lemmas in the way we use it (slightly different from the lemmas given in [25] called Lemmas 2.1 to 2.4).

Lemma 1. *Suppose* $(A3)-(A5)$, $(A7)$ *and* $(A7')$ *hold. Let* $(X(.),Z(.)) \in M^2(0,T;\mathbb{R}^m) \times M^2(0,T;\mathbb{R}^{m\times d})$, *and* $(P(.)) \in M^2(0,T;\mathbb{R}^n)$ *solutions of forward equation of (13). Then, for all* $\lambda \in \mathbb{R}, \varepsilon, C_1, C_2 > 0$,

$$\begin{aligned} \| P(.) \|_\lambda^2 \quad \leq \quad & \frac{1-e^{-\bar{\lambda_1}T}}{\bar{\lambda_1}} \Big[\big(k_1C_1 + k_6^2(1+\varepsilon)\big) \| X \|_\lambda^2 + \big(k_2C_2 + k_7^2(1+\varepsilon)\big) \| Z \|_\lambda^2 \\ & + E_{\mathbb{Q}}|P_0|^2 + \frac{1}{\varepsilon} \| b(.,0,0,0) \|_\lambda^2 + \left(1+\frac{1}{\varepsilon}\right) \| \sigma(.,0,0,0) \|_\lambda^2 \Big] . \end{aligned} \tag{26}$$

with $\bar{\lambda_1} := \lambda - 2\lambda_1 - k_1C_1^{-1} - k_2C_2^{-1} - k_5^2(1+\varepsilon) - \varepsilon$.
Furthermore, if $\bar{\lambda_1} \geq 0$,

$$\begin{aligned} e^{-\lambda T} E_{\mathbb{Q}}|P_T|^2 \quad \leq \quad & \big(k_1C_1 + k_6^2(1+\varepsilon)\big) \| X \|_\lambda^2 + \big(k_2C_2 + k_7^2(1+\varepsilon)\big) \| Z \|_\lambda^2 \\ & + E_{\mathbb{Q}}|P_0|^2 + \frac{1}{\varepsilon} \| b(.,0,0,0) \|_\lambda^2 + \left(1+\frac{1}{\varepsilon}\right) \| \sigma(.,0,0,0) \|_\lambda^2 . \end{aligned} \tag{27}$$

Lemma 2. *Suppose Hypotheses* $(A3), (A4)$,$(A6), (A7)$ *and* $(A7')$ *hold. Let* $(P(.)) \in M^2(0,T;\mathbb{R}^n)$, *and* $(X(.),Z(.)) \in M^2(0,T;\mathbb{R}^m) \times M^2(0,T;\mathbb{R}^{m\times d})$ *satisfy backward equation of (13). For all*

$\lambda \in \mathbb{R}, \varepsilon, C_3, C_4 > 0$, *such that* $0 < C_4 < k_4^{-1}$, *define* $\bar{\lambda_2} := -\lambda - 2\lambda_2 - k_3 C_3^{-1} - k_4 C_4^{-1} - \varepsilon$.. *We have :*

$$\| X(.) \|_\lambda^2 \leq \frac{1 - e^{-\bar{\lambda_2} T}}{\bar{\lambda_2}} \Bigg[k_8^2 (1+\varepsilon) e^{-\lambda T} E_{\mathbb{Q}} |P_T|^2 + k_3 C_3 \| P(.) \|_\lambda^2 + \left(1 + \frac{1}{\varepsilon}\right) e^{-\lambda T} E_{\mathbb{Q}} |g(0)|^2 + \frac{1}{\varepsilon} \| f(.,0,0,0) \|_\lambda^2 \Bigg]. \tag{28}$$

Furthermore if $\bar{\lambda_2} \geq 0$ *:*

$$\| Z(.) \|_\lambda^2 \leq \frac{1}{1 - k_4 C_4} \Bigg[k_8^2 (1+\varepsilon) e^{-\lambda T} E_{\mathbb{Q}} |P_T|^2 + k_3 C_3 \| P(.) \|_\lambda^2 + \left(1 + \frac{1}{\varepsilon}\right) e^{-\lambda T} E_{\mathbb{Q}} |g(0)|^2 + \frac{1}{\varepsilon} \| f(.,0,0,0) \|_\lambda^2 \Bigg]. \tag{29}$$

Lemma 3. *Suppose Hypotheses* $(A3) - (A5)$, $(A7)$ *and* $(A7')$ *hold. Let* $P^i(.)$ *satisfy the forward equation of (13) associated to* $(X(.), Z(.)) = (X^i(.), Z^i(.)) \in M^2(0,T;\mathbb{R}^m) \times M^2(0,T;\mathbb{R}^{m\times d})$, $i = 1, 2$. *Define, for all* $\lambda \in \mathbb{R}, C_1, C_2 > 0$, $\bar{\lambda_1} := \lambda - 2\lambda_1 - k_1 C_1^{-1} - k_2 C_2^{-1} - k_5^2$. *Then*

$$\| \Delta P \|_\lambda^2 \leq \frac{1 - e^{-\bar{\lambda_1} T}}{\bar{\lambda_1}} \left[(k_1 C_1 + k_6^2) \| \Delta X \|_\lambda^2 + (k_2 C_2 + k_7^2) \| \Delta Z \|_\lambda^2 \right], \tag{30}$$

and

$$e^{-\lambda T} E_{\mathbb{Q}} |\Delta P_T|^2 \leq \left[1 \vee e^{-\bar{\lambda_1} T} \right] (k_1 C_1 + k_6^2) \| \Delta X \|_\lambda^2 + (k_2 C_2 + k_7^2) \| \Delta Z \|_\lambda^2 . \tag{31}$$

Moreover if $\lambda_1 \geq 0$, *more simply it leads to :*

$$e^{-\lambda T} E_{\mathbb{Q}} |\Delta P_T|^2 \leq (k_1 C_1 + k_6^2) \| \Delta X \|_\lambda^2 + (k_2 C_2 + k_7^2) \| \Delta Z \|_\lambda^2 . \tag{32}$$

Lemma 4. *Suppose Hypotheses* $(A3), (A4), (A6), (A7)$ *and* $(A7')$ *hold. Let* $(X^i(.), Z^i(.))$ *satisfy the backward equation of (13) associated to* $P(.) = P^i(.) \in M^2(0,T;\mathbb{R}^n)$, $i = 1, 2$. *For all* $\lambda \in \mathbb{R}, C_3, C_4 > 0$ *such that* $0 < C_4 < k_4^{-1}$, *define* $\bar{\lambda_2} := -\lambda - 2\lambda_2 - k_3 C_3^{-1} - k_4 C_4^{-1}$. *Then*

$$\| \Delta X \|_\lambda^2 \leq \frac{1 - e^{-\bar{\lambda_2} T}}{\bar{\lambda_2}} \left[k_8^2 e^{-\lambda T} E_{\mathbb{Q}} |\Delta P_T|^2 + k_3 C_3 \| \Delta P \|_\lambda^2 \right], \tag{33}$$

and

$$\| \Delta \bar{Z} \|_\lambda^2 \leq \frac{k_8^2 e^{-\bar{\lambda_2} T} e^{-\lambda T} E_{\mathbb{Q}} |\Delta P_T|^2 + k_3 C_3 (1 \vee e^{-\bar{\lambda_2} T}) \| \Delta P \|_\lambda^2}{(1 - k_4 C_4)(1 \wedge e^{-\bar{\lambda_2} T})}. \tag{34}$$

Moreover if $\lambda_2 \geq 0$, more simply it leads to :

$$\| \Delta \bar{Z} \|_\lambda^2 \leq \frac{1}{1 - k_4 C_4} \left[k_8^2 e^{-\lambda T} E_{\mathbb{Q}} |\Delta P_T|^2 + k_3 C_3 \| \Delta P \|_\lambda^2 \right]. \tag{35}$$

Proof. These lemmas are derived using several standard inequalities, such as Burkholder-Davis-Gundy inequality, and the proof uses extensively the Lipschitz and linear growth assumptions of hypotheses $(A1)$ to $(A7)$, $(A7')$ and $(A8)$. □

We will suppose from now on that all hypotheses $(A1)$ to $(A7)$, as well as $(A7')$ and $(A8)$ are satisfied. We prove the existence and uniqueness of the solution of the forward equation at fixed solution of the backward equation, and then construct a strict contraction in the space of solutions, which will prove the existence and uniqueness of the entire solution of Equation (13).

Let $P \in S^2_{\mathbb{Q},c}$, $\xi \in L^2(\Omega, \mathcal{Y}, \mathbb{Q})$, $(X, Z) \in M^2(0, T; \mathbb{R}^m) \times M^2(0, T; \mathbb{R}^{m \times d})$. Define $\forall t \in [0, T]$

$$\phi(P)_t := \xi + \int_0^t b(s, P_s, X_s, Z_s) ds + \int_0^t \sigma(s, P_s, X_s, Z_s) dW_s. \tag{36}$$

$\phi(P)$ is well defined and continuous, as $P \in \mathcal{S}^2_{\mathbb{Q},c}$, and thanks to properties of b and σ.

Lemma 5. *$\phi(P)$ belongs to $\mathcal{S}^2_{\mathbb{Q},c}$ as soon as $P \in \mathcal{S}^2_{\mathbb{Q},c}$, which means*

$$\| \phi(P) \|^2 = E_{\mathbb{Q}}[\sup_{0 \leq t \leq T} |\phi(P)_t|^2] < +\infty.$$

Proof of Lemma 5 : Let $P_1, P_2 \in \mathcal{S}^2_{Q,c}$. Using Doob and Hölder inequalities, we have

$$E_{\mathbb{Q}}[\sup\nolimits_{0 \leq t \leq u} |\phi(P_1)_t - \phi(P_2)_t|^2] \leq 2T E_{\mathbb{Q}} \left[\int_0^u |b(s, P_1(s)) - b(s, P_2(s))|^2 ds \right]$$
$$+ 8 E_{\mathbb{Q}} \left[\int_0^u \| \sigma(s, P_1(s)) - \sigma(s, P_2(s)) \|^2 \, ds \right].$$

As b and σ are uniformly Lipschitz, there exists $K > 0$ such that :

$$\begin{aligned} & E_{\mathbb{Q}}[\sup_{0 \leq t \leq u} |\phi(P_1)_t - \phi(P_2)_t|^2] \\ \leq \; & 2K^2 (T+4) T E_{\mathbb{Q}} \left[\int_0^u |P_1(s) - P_2(s)|^2 ds \right] \\ \leq \; & 2K^2 (T+4) T E_{\mathbb{Q}} \left[\int_0^u \sup_{0 \leq t \leq s} |P_1(t) - P_2(t)|^2 ds \right]. \end{aligned} \tag{37}$$

Moreover, using again Doob and Hölder's inequalities,

$$\begin{aligned} &E_{\mathbb{Q}}[\sup_{0\le t\le T}|\phi(0)_t|^2] \\ &\le \quad 3\left(E_{\mathbb{Q}}[\xi^2]+E_Q\left[\sup_{0\le t\le T}|\int_0^t b(s,0)ds|^2\right]+E_{\mathbb{Q}}\left[\sup_{0\le t\le T}|\int_0^t \sigma(s,0)dW_s|^2\right]\right). \end{aligned}$$

Applying $(A4)$ and $(A5)$ between (x,z) and $(0,0)$ as well as $(A7')$ we obtain the following majoration :

$$E_{\mathbb{Q}}[\sup_{0\le t\le T}|\phi(0)_t|^2]\le 3\left(E_{\mathbb{Q}}(\xi^2)+T^2K^2+4TK^2\right).$$

Hence

$$\begin{aligned} &E_Q[\sup_{0\le t\le T}|\phi(P)_t|^2]\le 2E_Q[\sup_{0\le t\le T}|\phi(P)_t-\phi(0)_t|^2+|\phi(0)_t|^2] \\ &\le \quad 4K^2T(T+4)E_Q\left[\sup_{0\le t\le T}|P(t)|^2\right]+3E_Q[\xi^2]+K^2T^2+4K^2T. \end{aligned} \tag{38}$$

And then $\phi(P)_t\in\mathcal{S}^2_{\mathbb{Q},c}$ as soon as $P_t\in\mathcal{S}^2_{\mathbb{Q},c}$. $\square$ Let us prove now that the forward equation of (13) has a unique solution in $M^2(0,T;\mathbb{R}^n)$.

Lemma 6. *Let $(X(.),Z(.))$ fixed in $M^2(0,T,\mathbb{R}^m\times\mathbb{R}^{m\times d})$ and let $\xi\in\mathcal{L}^2(\Omega,\mathcal{Y}_0=\sigma(L),\mathbb{Q})$. Then equation*

$$P(t)=\xi+\int_0^t b(s,P_s,X_s,Z_s)ds+\int_0^t \sigma(s,P_s,X_s,Z_s)dW_s \tag{39}$$

has a unique $\mathcal{Y}$-adapted solution $P(.)$ in $M^2(0,T;\mathbb{R}^n)$.

Proof. The proof is similar to the standard one. Two things change : $P(0)$ is not deterministic any more, and the used filtration is not the Brownian one. We have to solve a standard SDE in $(\Omega,\mathcal{Y},\mathbb{Q})$, as under (H_3), W is a $(\mathcal{F},\mathbb{P})$ and a $(\mathcal{Y},\mathbb{Q})$-Brownian.
Fix X and Z two $\mathcal{Y}$-adapted processes in $M^2(0,T,\mathbb{R}^m\times\mathbb{R}^{m\times d})$.

1. Proof of existence

By induction we construct a sequence od processus in $\mathcal{S}^2_{\mathbb{Q},c}$:

$$P^0=0,\ P^{n+1}=\phi(P^n),\ \forall n\ge 0.$$

Then using majoration (37) and Cauchy formula we deduce :

$$E_{\mathbb{Q}}\left[\sup_{0\le t\le T}|P_t^{n+1}-P_t^n|^2\right]\le 2K^2(T+4)T\int_0^T E_{\mathbb{Q}}\left[\sup_{0\le t\le T}|P_t^{n}-P_t^{n-1}|^2\right]$$
$$\le C^n\frac{1}{(n-1)!}\int_0^T(T-s)^{n-1}E_{\mathbb{Q}}\left[\sup_{0\le t\le s}|P_t^1|^2\right]ds\le\frac{C^nT^n}{n!}E_{\mathbb{Q}}\left[\sup_{0\le T\le s}|P_t^1|^2\right].$$

Denote by D the majorant in Equation (38) associated to $P=P^0=0$,

$$E_{\mathbb{Q}}\left[\sup_{0\le t\le T}|P_t^{n+1}-P_t^n|^2\right]\le D\frac{C^nT^n}{n!}.$$

Then it follows

$$\begin{aligned}\sum_{n\ge 0}\|\sup_{0\le t\le T}|P_t^{n+1}-P_t^n|\,\|_{L^1} &\le \sum_{n\ge 0}\|\sup_{0\le t\le T}|P_t^{n+1}-P_t^n|\,\|_{L^2}\\ &\le \sqrt{D}\sum_{n\ge 0}\frac{(CT)^{n/2}}{\sqrt{n!}}<+\infty.\end{aligned}$$

The sequence converges $\mathbb{Q}$-a.s., so P^n converges uniformly $\mathbb{Q}$-a.s. on $[0,T]$ to $\tilde{P}$, which is continuous. Moreover $\tilde{P}\in\mathcal{S}^2_{\mathbb{Q},c}$ since the convergence is in $\mathcal{S}^2_{\mathbb{Q}}$. Finally $\tilde{P}$ satisfies SDE (39) est solution de l'EDS par passage à la limite dans $P^{n+1}=\phi(P^n)$.

2. Proof of Uniqueness

Let P^1 and P^2 be two solutions of SDE (39) in $\mathcal{S}^2_{\mathbb{Q},c}$. We have

$$E_Q\left[\sup_{0\le t\le u}|P_t^1-P_t^2|^2\right]\le 2K^2(T+4)\int_0^u E_Q\left[\sup_{0\le t\le r}|P_t^1-P_t^2|^2\right]dr,$$

so applying Gronwall's Lemma, there are indistinguishable:

$$E_Q\left[\sup_{0\le t\le u}|P_t^1-P_t^2|^2\right]=0.$$

This proves the uniqueness of the solution of the forward equation of (13) at fixed (X,Z) $\in M^2(0,T,\mathbb{R}^m\times\mathbb{R}^{m\times d})$. □

The forward equation induces an application

$$M_1:(X,Z)\in M^2(0,T,\mathbb{R}^m\times\mathbb{R}^{m\times d})\mapsto M_1(X(.),Z(.))=P(.),$$

unique solution of the forward equation at fixed $(X(.), Z(.))$. This is justified by Lemma 6. In the same way, we define

$$M_2 : P \in M^2(0,T,\mathbb{R}^n) \mapsto M_2(P(.)) = (X(.), Z(.)) \in M^2(0,T,\mathbb{R}^m \times \mathbb{R}^{m\times d}),$$

the unique solution of the backward equation at fixed $P(.)$. This is possible thanks to the existence and uniqueness Theorem 2.1 given in Eyraud-Loisel [11].
Finally this allows us to define the two following operators

$$\Gamma_1 := M_2 \circ M_1 \text{ et } \Gamma_2 := M_1 \circ M_2.$$

From the previous existence and uniqueness results we deduce in particular that Γ_1 maps $M^2(0,T;\mathbb{R}^{m\times d})$ into itself, and Γ_2 maps $M^2(0,T;\mathbb{R}^n)$ into itself.
Let $(X^i, Z^i) \in M^2(0,T;\mathbb{R}^m \times \mathbb{R}^{m\times d})$, $i = 1, 2$. Define

$$P^i := M_1(X^i, Z^i) \text{ and } (\bar{X}^i, \bar{Z}^i) = \Gamma_1((X^i, Z^i)),\ i = 1, 2.$$

The existence and uniqueness of the solution of the enlarged FBSDE will be obtained using the majoration Lemmas in order to prove that Γ_1 or Γ_2 is a strict contraction, so have a unique fixed point, which is, by construction, the unique solution of the enlarged FBSDE. We will have to distinguish three cases. In each case, we will need to put an upper bound on $\lambda_1 + \lambda_2$ depending on the k_i.

(I1)/ Weak coupling between forward equation and backward equation

We prove here that Γ is a strict contraction for norm $\| \,.\, \|_\lambda$, for $\lambda \in \mathbb{R}$.
From majorations (30), (31), (33) and (34), we can write :

$$\begin{aligned} &\| \Delta\bar{X} \|_\lambda^2 + \| \Delta\bar{Z} \|_\lambda^2 \\ &\leq \left[\frac{1 - e^{-\bar{\lambda_2} T}}{\bar{\lambda_2}} + \frac{1}{1 - k_4 C_4} \right] \left(k_8^2 (1 \vee e^{-\bar{\lambda_1} T}) + k_3 C_3 \frac{1 - e^{-\bar{\lambda_1} T}}{\bar{\lambda_1}} \right) \\ &\qquad \times \left((k_1 C_1 + k_6^2) \| \Delta X \|_\lambda^2 + (k_2 C_2 + k_7^2) \| \Delta Z \|_\lambda^2 \right). \end{aligned} \tag{40}$$

Then, as soon as

$$\begin{aligned} k_1C_1 + k_6^2 &< \left[\frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1}{1-k_4C_4}\right]^{-1} \left(k_8^2(1\vee e^{-\bar{\lambda}_1 T}) + k_3C_3\frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}\right)^{-1}, \\ k_2C_2 + k_7^2 &< \left[\frac{1-e^{-\bar{\lambda}_2 T}}{\bar{\lambda}_2} + \frac{1}{1-k_4C_4}\right]^{-1} \left(k_8^2(1\vee e^{-\bar{\lambda}_1 T}) + k_3C_3\frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1}\right)^{-1}, \end{aligned}$$

Γ_1 is a strict contraction. So the existence of ε_0 which depend on $k_3, k_4, k_8, \lambda_1, \lambda_2, T$ such that $k_1, k_2, k_6, k_7 \in [0, \varepsilon_0)$ is enough to insure that Γ_1 is a strict contraction, which ends the proof of Theorem 1 by proving that there exists a unique fixed point which is the unique solution of enlarged FBSDE (13).

(I2)/ The contingent claim is independent of prices.

We suppose here that g is a $\mathcal{F}_T$-measurable random variable independent of prices, which means $k_8 = 0$. Suppose moreover that $(A3)$ is such that λ_1 and λ_2 satisfy
$\exists\ C_i > 0, i = 1,2,3,4, C_4 < k_4^{-1}, \theta > 0$ such that

$$\lambda_1 + \lambda_2 < -\frac{1}{2}\left[k_3C_3\left(\frac{k_2C_2 + k_7^2}{1-k_4C_4} + \frac{k_1C_1 + k_6^2}{\theta}\right) + \frac{k_1}{C_1} + \frac{k_2}{C_2} + \frac{k_3}{C_3} + \frac{k_4}{C_4} + k_5^2 + \theta\right]. \quad (41)$$

Choose first $\lambda = -(2\lambda_2 + k_3C_3^{-1} + k_4C_4^{-1} + \theta)$.
Then by definition of $\bar{\lambda}_2$ in Lemma 2 and of $\bar{\lambda}_1$ in Lemma 1,

$$\begin{aligned} \bar{\lambda}_2 &= \theta > 0, \\ \bar{\lambda}_1 &= -(2\lambda_1 + 2\lambda_2 + k_1C_1^{-1} + k_2C_2^{-1} + k_3C_3^{-1} + k_4C_4^{-1} + k_5^2 + \theta). \end{aligned}$$

So from (41),

$$\bar{\lambda}_1 > k_3C_3(\frac{k_2C_2 + k_7^2}{1-k_4C_4} + \frac{k_1C_1 + k_6^2}{\theta}) > 0.$$

From Equations (30) and (33) in Lemmas 3 and 4, we get :

$$\parallel \Delta\bar{P} \parallel_\lambda^2 \leq \frac{1-e^{-\bar{\lambda}_1 T}}{\bar{\lambda}_1} k_3C_3\left(\frac{k_2C_2 + k_7^2}{1-k_4C_4} + \frac{k_1C_1 + k_6^2}{\theta}\right) \parallel \Delta P \parallel_\lambda^2 . \quad (42)$$

As soon as (41) is satisfied, the coefficient of $\parallel \Delta P \parallel_\lambda^2$ is nonnegative and strictly bounded by 1. Hence Γ_2 is a strict contraction, and admits a unique fixed point, which is the only solution of FBSDE (13). This ends the proof of Theorem 1 in case $(I2)$.

$(I3)/$ *The volatility σ does not depend on the portfolio.*

We suppose here that $k_7 = 0$ and also that λ_1 and λ_2 in (A_3) satisfy $\exists C_i > 0, i = 1, 3, 4,\ C_4 < k_4^{-1},\ \theta > 0,\ \alpha > 0$ such that

$$\lambda_1+\lambda_2 < -\frac{1}{2}\left[(1+\alpha)\left(k_1C_1 + k_6^2 + \frac{k_2^2}{\alpha(1-k_4C_4)}\right)\left(k_8^2 + \frac{k_3C_3}{\theta}\right) + \frac{k_1}{C_1} + \frac{k_3}{C_3} + \frac{k_4}{C_4} + k_5^2 + \theta\right] \tag{43}$$

Choose first $\lambda = 2\lambda_1 + k_1C_1^{-1} + k_2C_2^{-1} + k_5^2 + \theta$. Then

$$\begin{aligned} \bar{\lambda_1} &= \theta > 0 \\ \bar{\lambda_2} &= -(2\lambda_1 + 2\lambda_2 + k_1C_1^{-1} + k_2C_2^{-1} + k_3C_3^{-1} + k_4C_4^{-1} + k_5^2 + \theta). \end{aligned}$$

From (43), $\bar{\lambda_2} + k_2C_2^{-1} > (1+\alpha)\left[k_1C_1 + k_6^2 + \frac{k_2^2}{\alpha(1-k_4C_4)}\right]\left(k_8^2 + \frac{k_3C_3}{\theta}\right) > 0.$

As this is true for any $C_2 > 0$, we deduce that $\bar{\lambda_2} > 0$. Moreover, as $k_7 = 0$ and $\bar{\lambda}_1,\bar{\lambda_2} > 0$, we deduce from majorations (30), (32), (33) and (35) from Lemmas 3 and 4 :

$$\begin{aligned} \| \Delta\bar{X} \|_\lambda^2 &\leq \frac{1}{\bar{\lambda_2}}\left(k_8^2 + k_3C_3\frac{1-e^{-\bar{\lambda_1}T}}{\bar{\lambda_1}}\right)\left[(k_1C_1 + k_6^2) \| \Delta X \|_\lambda^2 + (k_2C_2) \| \Delta Z \|_\lambda^2\right], \\ \| \Delta\bar{Z} \|_\lambda^2 &\leq \frac{1}{1-k_4C_4}\left(k_8^2 + k_3C_3\frac{1-e^{-\bar{\lambda_1}T}}{\bar{\lambda_1}}\right)\left[(k_1C_1 + k_6^2) \| \Delta X \|_\lambda^2 + (k_2C_2) \| \Delta Z \|_\lambda^2\right]. \end{aligned}$$

Let $\gamma = \alpha(1-k_4C_4)(\bar{\lambda_2})^{-1}$. We obtain :

$$\begin{aligned} \| \Delta\bar{X} \|_\lambda^2 + \gamma \| \Delta\bar{Z} \|_\lambda^2 \leq & \frac{1+\alpha}{\bar{\lambda_2}}\left(k_8^2 + k_3C_3\frac{1-e^{-\bar{\lambda_1}T}}{\bar{\lambda_1}}\right) \\ & \times(k_1C_1 + k_6^2)\left[\| \Delta X \|_\lambda^2 + \frac{k_2C_2}{\alpha(1-k_4C_4)(k_1C_1+k_6^2)}\bar{\lambda_2}\gamma \| \Delta Z \|_\lambda^2\right]. \end{aligned}$$

Choosing $C_2^{-1} = \frac{k_2\bar{\lambda_2}}{\alpha(1-k_4C_4)(k_1C_1+k_6^2)}$, and combining it with previous inequalities leads to

$$\| \Delta\bar{X} \|_\lambda^2 + \gamma \| \Delta\bar{Z} \|_\lambda^2 \leq \frac{1+\alpha}{\bar{\lambda_2}}\left(k_8^2 + \frac{k_3C_3}{\theta}\right)(k_1C_1 + k_6^2)\left[\| \Delta X \|_\lambda^2 + \gamma \| \bar{Z} \|_\lambda^2\right].$$

and

$$\bar{\lambda_2} > (1+\alpha)\left(k_8^2 + \frac{k_3C_3}{\theta}\right)\left(k_1C_1 + k_6^2\right),$$

such that Γ_1 is a strict contraction, as soon as Equation (43) is satisfied. It admits a unique fixed point, which is the unique solution of FBSDE (13). This ends the proof of Theorem 1.

References

[1] J. Amendinger. Initial enlargement of filtrations and additionnal information of financial markets. *Doctoral Thesis*, 1999.

[2] J. Amendinger, D. Becherer, and M. Schweizer. A monetary value for initial information in portfolio optimization. *Finance Stoch.*, 7(1):29–46, 2003.

[3] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stochastic Process. Appl.*, 75(2):263–286, 1998.

[4] B. Bouchard and R. Elie. Discrete time approximation of decoupled forward-backward sde with jumps. *Stoch. Proc. and App.*, 118(1):53–75, 2008.

[5] B. Bouchard, R. Elie, and N. Touzi. Discrete-time approximation of bsdes and probabilistic schemes for fully nonlinear pdes.

[6] D. Chevance. *Résolution numériques des équations différentielles stochastiques rétrogrades*. PhD thesis, Université de Provence – Aix–Marseille I, Marseille, 1997.

[7] D. Cuoco and J. Cvitanić. Optimal consumption choices for a “large” investor. *J. Econom. Dynam. Control*, 22(3):401–436, 1998.

[8] J. Cvitanić and J. Ma. Hedging options for a large investor and forward-backward SDE's. *Ann. Appl. Probab.*, 6(2):370–398, 1996.

[9] F. Delarue. On the existence and uniqueness of solutions to fbsdes in a non-degenerate case. *Stochastic Process. Appl.*, 99(2):209–286, 2002.

[10] F. Delbaen and W. Schachermayer. Non-arbitrage and the fundamental theorem of asset pricing: summary of main results. In *Introduction to mathematical finance (San Diego, CA, 1997)*, volume 57 of *Proc. Sympos. Appl. Math.*, pages 49–58. Amer. Math. Soc., Providence, RI, 1999.

[11] A. Eyraud-Loisel. Backward stochastic differential equations with enlarged filtration. option hedging of an insider trader in a financial market with jumps. *Stochastic Processes and their Applications*, 115(11):1745–1763, 2005.

[12] A. Eyraud-Loisel. Quadratic hedging in an incomplete market derived by an influent informed investor. *Working Paper*, 2009.

[13] H. Föllmer and M. Schweizer. Hedging of contingent claims under incomplete information. In *Applied stochastic analysis (London, 1989)*, volume 5 of *Stochastics Monogr.*, pages 389–414. Gordon and Breach, New York, 1991.

[14] H. Föllmer, C.-T. Wu, and M. Yor. Canonical decomposition of linear transformations of two independent Brownian motions motivated by models of insider trading. *Stochastic Process. Appl.*, 84(1):137–164, 1999.

[15] E. Gobet, J-P. Lemor, and X. Warin. A regression-based monte-carlo method to solve backward stochastic differential equations. *To appear in Annals of Applied Probability*, 2005.

[16] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *Int. Journal of Theor. and App. Fin.*, 1(3):331–347, 1998.

[17] A. Grorud and M. Pontier. Probabilités neutres au risque et asymétrie d'information. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(1):1009–1014, 1999.

[18] A. Grorud and M. Pontier. Asymetrical information and incomplete markets. *Int. Journal of Theor. and App. Fin.*, 4(2):285–302, 2001.

[19] A. Grorud and M. Pontier. Fiancial market model with influential informed investors. *submitted*, 2005.

[20] P. Imkeller, M. Pontier, and F. Weisz. Free lunch and arbitrage possibilities in a financial market model with an insider. *Stochastic Process. Appl.*, 92(1):103–130, 2001.

[21] J. Jacod. *Grossissement Initial, Hypothèse H' et Théorème de Girsanov, Séminaire de calcul stochastique* $1982-83$*, Paris*, volume 1118 of *Lecture Notes in Math.* Springer-Verlag, Berlin, 1985.

[22] T. Jeulin. Grossissement d'une filtration et applications. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Math.*, pages 574–609. Springer, Berlin, 1979.

[23] T. Jeulin. Comportement des semi-martingales dans un grossissement de filtration. *Z. Wahrsch. Verw. Gebiete*, 52(2):149–182, 1980.

[24] D. Lépingle and J. Mémin. Sur l'intégrabilité uniforme des martingales exponentielles. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 42(3):175–203, 1978.

[25] E. Pardoux and S. Tang. Forward-backward stochastic differential equations and quasilinear parabolic PDEs. *Probab. Theory Related Fields*, 114(2):123–150, 1999.