A weak product on partially ordered sets to define a compromise between multiobjective optimization problems

Benoît Guédas[1], Xavier Gandibleux[2], and Philippe Dépincé[1]

[1] IRCCyN, 1, rue de la Noë - BP 92 101 - F 44321 Nantes Cedex 03, France.
{Benoit.Guedas,Philippe.Depince}@irccyn.ec-nantes.fr
[2] LINA, 2, rue de la Houssinière - BP 92208 - F 44322 Nantes Cedex 03, France.
Xavier.Gandibleux@univ-nantes.fr

Context

Multidisciplinary design optimization (MDO) deals with complex engineering problems which are decomposed into several subproblems called disciplines. According to Tosserams *et al.* [8], "Examples of advanced engineering systems can be found in aerospace and automotive industry, and an emerging field is microelectromechanical systems[.] For instance, an aircraft may be partitionned with respect to various physics involved (mechanics, aerodynamics, control, etc.), or with respect to its structural components (fuselage, wing tail, panels, spars, ribs, etc.)." The disciplines are often hierarchically organized. Moreover, each discipline may have many conflicting objectives to achieve at the same time. Thus, at each level of the hierarchy, trade-offs have to be found between multiobjective optimization problems.

Multiobjective multidisciplinary methods designed to solve hierarchical multiobjective optimization problems such as EM-MOGA [7], MORDACE [5] or COSMOS [3] are searching for the whole Pareto set which corresponds to the multiobjective optimization problem that bring all the objectives of the problem together. But the solutions are just in a subset of the whole Pareto set. So, Engau and Wiecek [4] proposed an interactive method aiming to find a compromise that can be on the whole Pareto set based on hierarchical decomposition. Still, research in MDO focuses more on decomposition of large systems into simpler ones [1,8] and do not take into account the hierarchical organization: the wanted compromise is a trade-off between objectives and not between disciplines. In this paper, we propose a definition of compromise between disciplines which are multiobjective optimization problems.

Current work

In multiobjective optimization, the compromise between p objective $f_i : \mathbb{R}^n \to \mathbb{R}$ is defined as the Pareto set which can be formalized with ordered sets theory. Let $\mathcal{X} \subseteq \mathbb{R}^n$ the feasible set of the decision space, $\mathcal{Y} := \{f(x) : x \in \mathcal{X}\}$ the objective space with $f := (f_1, \ldots, f_p)^T$ the objective function, and respectively $\mathcal{X}_E$ and $\mathcal{Y}_N$ the efficient solutions and the non dominated points sets. We will note $\mathcal{Y}^i \subseteq \mathbb{R}$ the objective spaces of each objective. The Pareto optimal solutions are the minima of the set $\mathcal{Y}$ which is the cartesian product of the $\mathcal{Y}^i$ ordered with the relation $\leq_{\mathbb{R}^p}$ defined as follows:

$$a \leq_{\mathbb{R}^p} b \Leftrightarrow \forall i \in \{1, \ldots, p\}, a_i \leq_{\mathbb{R}} b_i \tag{1}$$

The Pareto dominance relation $\lneq_{\mathbb{R}^p}$ is thus the weak order associated to $\leq_{\mathbb{R}^p}$. The ordered set $(\mathcal{Y}, \leq_{\mathbb{R}^p})$ is then the result of the product of the p orders $(\mathcal{Y}^i, \leq_{\mathbb{R}})$: $(\mathcal{Y}, \leq_{\mathbb{R}^p}) = (\mathcal{Y}^i, \leq_{\mathbb{R}})^p$, and the solutions are $\mathcal{X}_E = \{x \in \mathcal{X} : f(x) \in \min(\mathcal{Y}, \leq_{\mathbb{R}^n})\}$ which can be written as follows:

$$\mathcal{X}_E = \{x^* \in \mathcal{X} : \forall f(x) \in \mathcal{Y}, f(x) \leq_{\mathbb{R}} f(x^*) \Rightarrow f(x^*) = f(x)\} \tag{2}$$

The same reasoning can be applied to the compromise between n multiobjective problems which have p_i objectives each. Then, the compromise solutions of the optimization problem

$\mathcal{X}_E$ would be defined as:

$$\mathcal{X}_E := \{x \in \mathcal{X} : f(x) \in \min \prod_{i=1}^{n} \prod_{j=1}^{p_i} (\mathcal{Y}^{i,j}, \leq_{\mathbb{R}})\} \tag{3}$$

with $\mathcal{Y}^{i,j} := \{f_{i,j}(x) : x \in \mathcal{X}\}$ the objective space of the j^{th} objective function of the i^{th} problem, $\mathcal{X}$ the feasible set of the decision space, and $f := (f_{1,1}, \ldots, f_{1,p_1}, \ldots, f_{n,1}, \ldots, f_{n,p_n})^T$ the objective function. We notice that this compromise is in fact the product of all the objectives put all together (Eq. 2) because the ordered set product is associative.

In multiobjective optimization, the efficient set is defined with products of *totally* ordered sets, but in multidisciplinary design optimization, it should be defined with product of ordered sets which are themselves products of ordered sets and then *partially* ordered sets. In this paper, we propose another product that we call weak product of ordered sets which takes into account the incomparabilities that can arise in partially ordered sets. It says that if an element a is lower than another element b in at least one component of the product and is incomparable of lower in all the other components, then a is lower than b. This can be translated as:

$$a \lneq_{\mathbb{R}^n} b \Longleftrightarrow \begin{cases} \exists i \in \{1, \ldots, n\} & a_i \leq_{\mathbb{R}^{p_i}} b_i \\ \nexists j \in \{1, \ldots, n\} & b_j \lneq_{\mathbb{R}^{p_i}} a_j \end{cases} \tag{4}$$

We notice that if each component of the product are totally ordered, it is equivalent to the Pareto-dominance relation.

Properties and numerical experiments

This dominance relation is compared to other compromise definitions, and in particular the direct product and its extensions presented in [6]. Some properties of these definitions are given and examples on some problems of the literature are presented. As multidisciplinary multi-objective solvers are often based on evolutionnary multiobjective optimization algorithms [2], we discuss the issues that can arise when such algorithms are used with these definitions of compromise as fitness function.

Keywords: multidisciplinary design optimization; multiobjective optimization; compromise; ordered sets

References

1. J. Agte, O. de Weck, J. Sobieszczanski-Sobieski, P. Arendsen, A. Morris, and M. Spieck. MDO: assessment and direction for advancement - an opinion of one international group. *Structural and Multidisciplinary Optimization*, 38(online first), April 2009.
2. K. Deb. *Multi-objective optimization using evolutionary algorithms.* Wiley-Intersciences series in systems and optimization. Wiley, June 2001.
3. P. Dépincé, S. Rabeau, and F. Bennis. Collaborative optimization strategy for multi-objective design. In *Proceedings of DETC'2005*, pages 24–28, September 2005.
4. A. Engau and M. M. Wiecek. 2D decision-making for multicriteria design optimization. *Structural and Multidisciplinary Optimization*, 34(4):301–315, October 2007.
5. A. Giassi. *Optimisation et conception collaborative dans le cadre de l'ingénierie simultanée.* PhD thesis, École Centrale de Nantes, September 2004. (in french)
6. B. Guédas and P. Dépincé. A compromise method for multidisciplinary multi-objective design problems. In *8th World Congress on Structural and Multidisciplinary Optimization*, June 2009.
7. S. Gunawan, A. Farhang-Mehr, and S. Azarm. *Multi-level Multi-objective Genetic Algorithm Using Entropy to Preserve Diversity*, volume 2632 of *Lecture Notes in Computer Science*, page 10. Springer Berlin / Heidelberg, January 2003.
8. S. Tosserams, L. F. P. Etman, and J. E. Rooda. A classification of methods for distributed system optimization based on formulation structure. *Structural and Multidisciplinary Optimization*, 38(online first), January 2009.