QUANTUM TRANSFER OPERATORS AND QUANTUM SCATTERING

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1. Introduction and statement of the result

These notes present a new method, developed in collaboration with Johannes Sjöstrand and Maciej Zworski, the aim of which is a better understanding of quantum scattering systems, in situations where the set of classically trapped trajectories at some energy $E > 0$ is bounded, but can be a complicated fractal set. In particular, we are interested in the situations where this *trapped set* is a "chaotic repeller" hosting a hyperbolic (Axiom A) flow. Such a scattering system belongs to the realm of *quantum chaos*, namely the study of wave or quantum systems, the classical limit of which enjoy chaotic properties. This type of dynamics occurs for instance in the scattering by 3 or more disks in the Euclidean plane [15], but also in scattering by a smooth potential (see fig 1). Chaotic scattering systems are physically relevant: for instance, mesoscopic *quantum dots* are often modelled by open chaotic billiards [17]; the ionization of atoms or molecules in presence of external electric and/or magnetic fields also involves classical chaotic trajectories [2]; Open quantum billiards can also be realized in microwave billiard expermiments [30].

The method we propose is a quantum version of the Poincaré section/Poincaré map construction used to analyze the classical flow (see §1). Namely, around some scattering energy $E > 0$ we will construct a *quantum transfer operator* (or quantum *monodromy* operator), which contains the relevant information of the quantum dynamics at this energy, in a much reduced form: this operator has finite rank (which increases in the semiclassical limit), it allows to characterize the *quantum resonances* of the scattering system in the vicinity of the energy E. The quantum transfer operator is very similar with the *open quantum maps* studied as toy models for chaotic scattering [4, 19, 24].

Our main result (Theorem 1) will be stated in §1.2. In §2 and §3 we sketch the proof of this result. We defer the details of the proofs, as well as some applications of the method, to a forthcoming publication [18].

From flows to maps, and back. Let us recall some facts from classical dynamics. In the theory of dynamical systems, the study of a flow $\Phi^t : Y \to Y$ generated by some vector field (or ODE) on a phase space Y (say, a smooth manifold) can obten be facilitated by considering a *Poincaré section* of that flow, namely a family $\Sigma = \{\Sigma_i,\, i = 1, \ldots, J\}$ of hypersurfaces of Y, which intersect the flow transversely. The successive intersections of the flow with Σ define a first return (or Poincaré) map $\kappa : \Sigma \to \Sigma$ (see fig. 1). This map,

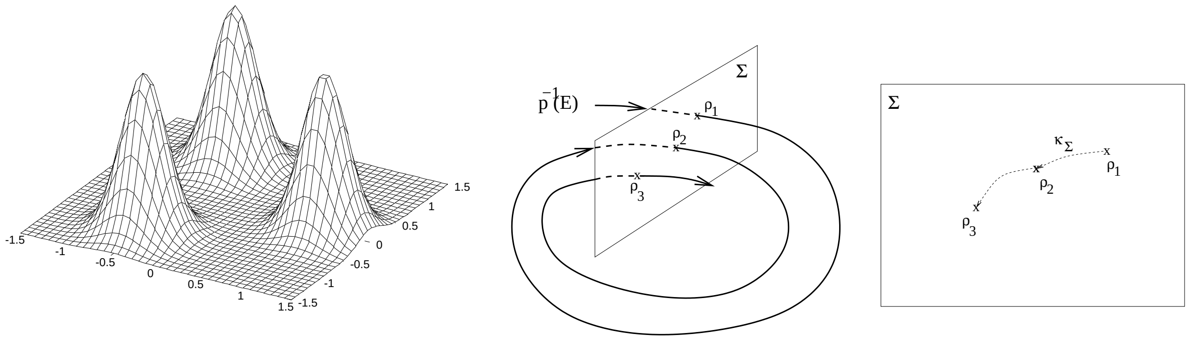

FIGURE 1. Left: a 3-bump potential, which admits a fractal hyperbolic trapped set at intermediate energies [25, Appendix]. Right: schematic representation of a Poincaré section.

defined on a phase space Σ of codimension 1, conveniently represents the flow on Y. Long time properties of κ are often easier to analyze than the corresponding properties of the flow. One can reconstruct the flow Φ^t from the knowledge of κ together with the *return time function* $\tau : \Sigma \to \mathbb{R}_+$, which measures the time spanned between the intersections ρ and $\kappa(\rho)$. Below we will explain how *transfer operators* associated with κ can also help to compute long time properties of the flow.

1.0.1. *Hamiltonian scattering.* The flows we consider are Hamiltonian flows defined on the cotangent space $T^*\mathbb{R}^n$. A Hamiltonian (function) $p \in \mathcal{C}^\infty(T^*\mathbb{R}^n)$ defines a Hamilton vector field H_p on phase space, which generates the flow $\Phi^t = \exp(tH_p)$. For our specific choice (1.1), the flow is complete. It preserves the symplectic structure on $T^*\mathbb{R}^n$, and leaves invariant each energy shell $p^{-1}(E)$, so it makes sense to study the dynamics on each individual shell. A Poincaré section $\Sigma \subset p^{-1}(E)$ naturally inherits a symplectic structure, which is preserved by the Poincaré map κ. Hence, the Poincaré maps we consider are (local) symplectomorphisms on Σ.

We will specifically consider Hamiltonians of the form

$$p(x,\xi) = \frac{|\xi|^2}{2} + V(x), \tag{1.1}$$

with a potential $V \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ (say, supported in a ball $B(0,R_0) \subset \mathbb{R}^n$). Such a Hamiltonian generates a *scattering system*: for any energy $E > 0$, particles can come from infinity, scatter on the the potential, and be sent back towards infinity. Depending on the shape of V and of the energy, some trajectories can also be trapped forever (in the past and/or in the future) inside the ball $B(0,R_0)$. This leads to the definition of the *trapped set* at energy E:

$$K_E \stackrel{\text{def}}{=} \{\rho \in p^{-1}(E) \ :\ \exp(\mathbb{R}H_p)(\rho) \text{ is bounded}\}, \tag{1.2}$$

which is a compact, flow-invariant subset of $p^{-1}(E)$. The interesting long time dynamics takes place in the vicinity of K_E, so the Poincaré section Σ need only represent correctly the flow Φ^t restricted on K_E, or on some neighbourhood of it. The Poincaré map κ will also be defined in some neighbourhood of the *reduced trapped set* $\mathcal{T}_E \stackrel{\rm def}{=} K_E \cap \Sigma$. We will give a more precise description of Σ in §2.1.

1.0.2. *Chaotic dynamics and transfer operators.* Our Theorem 1 will be relevant to the case where the flow on K_E is uniformly hyperbolic (and satisfies Smale's Axiom A). Such a flow is, in a sense "maximally chaotic". Hyperbolicity means that at each point $\rho \in K_E$ the tangent space $T_\rho p^{-1}(E)$ can be split between the flow direction, an unstable and a stable subspaces:

$$T_\rho p^{-1}(E) = \mathbb{R} H_p \oplus E^+(\rho) \oplus E^-(\rho)\,, \tag{1.3}$$

where the (un)stable subspaces are defined by the long time properties of the tangent map: there exist $C, \lambda > 0$ such that, for any $\rho \in K_E$,

$$v \in E^{\mp}(\rho) \Longleftrightarrow \|d\Phi^{\pm t} v\| \leq C\, e^{-\lambda t}, \quad t > 0\,.$$

The Poincaré map κ then inherits the Axiom A property.

To study the long time properties of such chaotic flow, it has proved convenient to use *transfer operators* associated with κ [1]. Let us give an example of such operators. Given any weight function $f \in C(\Sigma, \mathbb{R})$, one define the transfer operator $\mathcal{L}_f$ by a weighted push-forward on functions $\varphi : \Sigma \to \mathbb{R}$:

$$\mathcal{L}_f\, \varphi(\rho) \stackrel{\rm def}{=} \sum_{\rho' : \kappa(\rho') = \rho} e^{f(\rho')}\, \varphi(\rho')\,.$$

Provided $\mathcal{L}_f$ is applied to some appropriate functional space[1], its *spectrum* can deliver relevant information about the long time dynamics of κ. For instance, the spectral radius $r_{sp}(\mathcal{L}_f)$ determines the *topological pressure* of κ associated with the weight f, which provides statistical information on the long periodic orbits of κ:

$$\log r_{sp}(f) = \mathcal{P}(\kappa, f) \stackrel{\rm def}{=} \lim_{T\to\infty} \frac{1}{T} \log \sum_{|\gamma| \leq T} e^{\int_\gamma f}\,.$$

(here $\int_\gamma f$ is the sum of values of $f(\rho)$ along the periodic orbit γ). The topological pressure of the *flow* Φ^t, associated with a weight $F \in C(X)$, can also be computed through transfer operators. One defines on Σ the function $f(\rho) = \int_0^{\tau(\rho)} F(\Phi^t(\rho))$, that is the accumulated weight from $\rho \in \Sigma$ to its next return $\kappa(\rho)$, and considers the family $\{\mathcal{L}_{f - s\tau},\ s \in \mathbb{R}\}$ of transfer operators. The following relation then relates the pressures of κ and Φ^t:

$$s = \mathcal{P}(\Phi^t, F) \Longleftrightarrow \mathcal{P}(\kappa, f - s\tau) = 0 \Longleftrightarrow r_{sp}(\mathcal{L}_{f - s\tau}) = 1\,.$$

[1] The functional space can be rather complicated, see e.g. [14] for the case of Anosov diffeomorphisms.

The decay of correlations for the Axiom A flow Φ^t is encoded in the *Ruelle-Pollicott resonances*, which are the poles of the Fourier transform of the correlation function [22]. Within some strip $\mathcal{S} \subset \mathbb{C}$, these resonances $\{z_i\}$ can be characterized by using the family $\{\mathcal{L}_{f-z\tau},\ z \in \mathbb{C}\}$ of complex weighted transfer operators: $z_i \in \mathcal{S}$ is a resonance iff $\mathcal{L}_{f-z_i\tau}$ has an eigenvalue equal to 1. This property can be written (abusively, because transfer operators are usually not trace class) as

$$z \in \mathcal{S} \text{ is a Ruelle-Pollicott resonance} \Longleftrightarrow \det(1-\mathcal{L}_{f-z\tau}) = 0\,. \tag{1.4}$$

1.1. **A quantum scattering problem.** Let us now introduce the quantum dynamics we are interested in. The operator

$$P = P(h) = -\frac{h^2\Delta}{2} + V(x)\,,\quad V \in \mathcal{C}_c^\infty(\mathbb{R}^n)\,,$$

generates the Schrödinger dynamics $U(t) = \exp(-itP(h)/h)$ on $L^2(\mathbb{R}^n)$. $P(h)$ is the h-quantization of the classical Hamiltonian (1.1), so the semiclassical behaviour of the quantum dynamics will be strongly influenced by the flow $\exp(tH_p)$. We focus on the dynamics around some positive energy $E > 0$, so the flow we need to understand is $\Phi^t \restriction p^{-1}(E)$. We will assume that

- the flow on $p^{-1}(E)$ has no fixed point: $dp\restriction_{p^{-1}(E)} \neq 0$.
- the trapped set K_E has topological dimension 1. Equivalently, the reduced trapped set $\mathcal{T}_E = K_E \cap \Sigma$ is totally disconnected.

These conditions are satisfied, for example, for a 3-bump potential at intermediate energies (see fig. 1). The second condition was absent in previous studies of such systems [28, 20], it is a technical constraint specific to the approach we develop below (as we explain after Thm 1, the condition required for the method to work is actually weaker).

We are interested in the long time Schrödinger dynamics near energy E, so it is natural to investigate the spectrum of $P(h)$ near E. That operator is self-adjoint on $L^2(\mathbb{R}^n)$, with domain $H^2(\mathbb{R}^n)$; but, due to the bounded support of $V(x)$, the spectrum of $P(h)$ is absolutely continuous on $\mathbb{R}^+$, without any embedded eigenvalue. Nevertheless, the truncated resolvent $\psi(P(h)-z)^{-1}\psi$ (with $\psi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$), well-defined in the quadrant $\{\operatorname{Re} z >, \operatorname{Im} z > 0\}$, can be meromorphically extended across the real axis into $\{\operatorname{Re} z > 0, \operatorname{Im} z < 0\}$. The finite rank poles $\{z_i\}$ in this region are called its *resonances* (they do not depend on the specific cutoff ψ). Resonances are often understood as "generalized eigenvalues": they are associated with *metastable* modes $u_i(x)$ which are not square-integrable, but satisfy the differential equation $P(h)u_i = z_i\, u_i$, so that they decay expontially in time, at a rate given by $|\operatorname{Im} z_i|/h$.

One of our objectives is to better understand the distribution of these resonances in the h-neighbourhood of the energy E, that is in disks $D(E, Ch)$ (see fig. 2). More precisely, we want to investigate:

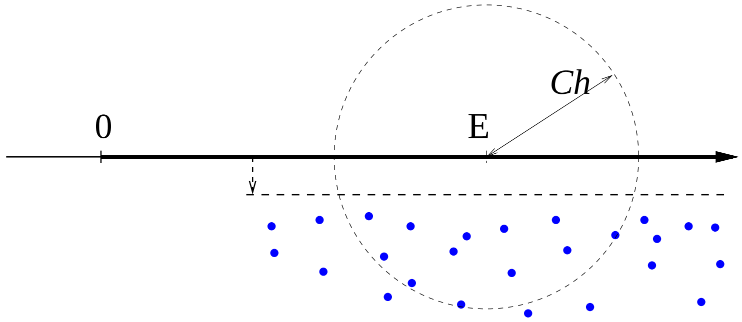

FIGURE 2. Schematic representation of the spectrum of $P(h)$ and its resonances near the energy E.

- the number of resonances in $D(E, Ch)$. So far *fractal upper bounds* have been proven [28]. We wish to investigate whether similar lower bounds can be obtained, at least for a generic system.
- the width of the resonance free strip in $D(E, Ch)$. A lower bound for such a strip has been expressed in terms of some topological pressure [15, 11, 20], but a recent result of Petkov-Stoyanov (for obstacle scattering) shows that this lower bound is in general not sharp [21].

1.2. **Our result.** Our main result is a "quantization" of the Poincaré section method presented above.

Theorem 1. *Assume that, for some energy $E > 0$, the trapped set K_E for the flow $\exp(tH_p)$ is topologically one dimensional, and contains no fixed point.*

Then, for $h > 0$ small enough, there exists a family of matrices $\{M(z, h),\ z \in D(0, Ch)\}$ holomorphic w.r.to z, such that the zeros of the function

$$\zeta(z, h) \stackrel{\text{def}}{=} \det(I - M(z, h)) \tag{1.5}$$

give the resonances of $(P(h) - E)$ in $D(0, Ch)$, with correct multiplicities.

The matrices $M(z, h)$ have the following structure. There exists a Poincaré section $\Sigma = \sqcup_{i=1}^{J} \Sigma_i$ and map $\kappa : \Sigma \to \Sigma$, an h-Fourier integral operator $\mathcal{M}(z, h) : L^2(\mathbb{R}^{n-1})^J \circlearrowleft$ quantizing κ, and a projector Π_h of rank $r(h) \asymp h^{-n+1}$, such that

$$M(z, h) = \Pi_h\, \mathcal{M}(z, h)\, \Pi_h + \mathcal{O}(h^N)\,.$$

The remainder estimate holds in the operator norm on $\mathbb{C}^{r(h)}$. The exponent N can be assumed arbitrary large.

Remark 1.1. The 1-dimensional condition we impose on K_E is not strictly necessary. What one needs is the existence of a Poincaré section Σ intersecting $\Phi^t \restriction K_E$, such that $\partial\Sigma \cap K_E = \emptyset$; in particular, we don't need the flow to be hyperbolic on K_E. Still, Axiom

A flows provide the most obvious example for which this condition is satisfied [5]; it holds as well for the broken geodesic flow in the scattering by 3 disks satisfying a no-eclipse condition [15].

This theorem shows that the dynamics generated by the Hamiltonian $P(h)$ near E can be "summarized" in the family of *quantum transfer operators* $\{M(z,h),\ z \in D(0,Ch)\}$. One reason for this terminology is that $M(z,h)$ bears some resemblance with the transfer operators $\mathcal{L}_{f-z\tau}$ briefly described in §1.0.2. The equation (1.5) characterizing quantum resonances is obviously the quantum analogue of the (generally formal) equation (1.4) defining Ruelle-Pollicott resonances. Also, the notation $\zeta(z,h)$ in (1.5) hints at an analogy, or relationship, between this spectral determinant and some form of *semiclassical zeta function* (such functions have been mostly studied in the physics literature, see e.g. [7]).

The operators $M(z,h)$ have the same semiclassical structure as *open quantum maps* studied in the (mostly physical) literature as toy models of quantum scattering systems. For instance, the distribution of resonances and resonant modes has proven to be much easier to study numerically for open quantum maps, than for realistic flows [4, 24, 19, 16]. The novelty here, is that the operators $M(z,h)$ allow to characterize a "physical" resonance spectrum.

1.2.1. *Historical remarks.* Actually, a similar method has been introduced in the theoretical physics literature devoted to "quantum chaos". To the author's knowledge, the first such construction appeared in Bogomolny's work [3] on multidimensional closed quantum systems. In that work, a family of quantum transfer operators $T(E)$ is constructed, which are integral operators defined on a hypersurface in configuration space. The eigenvalues of the bound Hamiltonian are then obtained, in the semiclassical limit, as roots of the equation $\det(1-T(E)) = 0$. This work generated a lot of interest in the quantum chaos community. Smilansky and co-workers derived a similar quantization condition for closed Euclidean 2-dimensional billiards [9], replacing $T(E)$ by a scattering matrix $S(E)$ associated with the dual scattering problem. Bogomolny's method was also extended to study quantum scattering situations [12]. On the other hand, Prosen developed an "exact" (that is, not necessarily semiclassical) quantum surface of section method to study certain closed Hamiltonian systems [23].

In the mathematics literature similar operators appeared in the framework of obstacle scattering [13, 15]: the scattering problem was analyzed through integral operators defined on the obstacle boundaries, which also have the structure of Fourier integral operators associated with the bounce map. More recently, a *monodromy operator* formalism has been introduced in [27] to study the Schrödinger dynamics in the vicinity of a single isolated periodic orbit. This approach has then been used to investigate concentration properties of eigenmodes in the vicinity of such an orbit [6]. The construction we present below heavily borrows from the techniques developed in [27]. It improves them on two aspects: first, our invariant set K_E is more complex than a single periodic orbit. Second, the connection we establish between the operators $(P(h) - E - z)$ and $M(z, h)$ is deeper than previously.

2. Formal construction of the quantum transfer operator

The proof of Thm 1 proceeds in several steps. It uses many tools of h-pseudodifferential calculus (we will use the notations of [8, 10]). We just recall a few of them:

- a state $u = u(h) \in L^2$ is microlocalized in a domain $U \Subset T^*\mathbb{R}^n$ iff, for any function $\chi \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n)$ with $\operatorname{supp}\chi \cap \bar{U} = \emptyset$, one has $\|\chi^w u\|_{L^2} = \mathcal{O}(h^\infty)\|u\|_{L^2}$. (here $\chi^w = \chi^w(x, hD_x)$ denotes the h-Weyl quantization of χ).
- two states u, v are said microlocally equivalent in $U \Subset T^*\mathbb{R}^n$ iff, for any cutoff $\chi \in \mathcal{C}_c^\infty(U)$, one has $\|\chi^w(u - v)\|_{L^2} = \mathcal{O}(h^\infty)$.
- similarly, two operators A, B are said microlocally equivalent in $V \times U$ (with $U, V \Subset T^*\mathbb{R}^n$) iff, for any cutoffs $\chi_1 \in \mathcal{C}_c^\infty(U)$, $\chi_2 \in \mathcal{C}_c^\infty(V)$, one has $\|\chi_2^w(A-B)\chi_1^w\|_{L^2\to L^2} = \mathcal{O}(h^\infty)$.
- an operator A is microlocally defined in $V \times U$ iff it is microlocally equivalent in $V \times U$ to some globally defined operator. "Microlocally defined in U" will mean "microlocally defined in $U \times U$".

The present section constructs the quantum transfer operators microlocally in a neighbourhood of the trapped set K_E, without paying attention to the rest of the phase space. The arguments making the construction globally well-defined will be presented in §3.

The microlocal construction being strongly tied to a Poincaré section, we start by describing the latter in some detail.

2.1. **Description of the Poincaré section.**

According to the assumptions of the theorem, the trapped set K_E is a compact set of topological dimension unity. It is then possible to construct a Poincaré section $\Sigma = \sqcup_{i=1}^J \Sigma_i \subset p^{-1}(E)$ with the following properties:

- each Σ_i is a $(2n-2)$-dimensional topological disk, transverse to the flow.
- the maximal diameter of the Σ_i can be chosen arbitrary small.
- there exists a time $\tau_{\max} > 0$ such that, for any $\rho \in K_E$, the trajectory $\Phi^t(\rho)$ intersects Σ at some time $0 < t \leq \tau_{\max}$.
- the boundary $\partial\Sigma = \sqcup_i\Sigma_i$ does not intersect K_E.

If we restrict ourselves to points in the reduced trapped set $\mathcal{T}_E \stackrel{\text{def}}{=} K_E \cap \Sigma$, the map $\rho \mapsto \rho_+(\rho)$ defines a bicontinuous bijection $\kappa : \mathcal{T} \to \mathcal{T}$.

Each component of the reduced trapped set, $\mathcal{T}_i \stackrel{\text{def}}{=} K_E \cap \Sigma_i$, splits in two different ways:

$$\mathcal{T}_i = \sqcup_j D_{ji}, \quad \text{where} \quad D_{ji} = \{\rho \in \mathcal{T}_i,\ \kappa(\rho) \in \mathcal{T}_j\} \tag{2.1}$$

$$\mathcal{T}_i = \sqcup_j A_{ij}, \quad \text{where} \quad A_{ij} = \{\rho \in \mathcal{T}_i,\ \kappa^{-1}(\rho) \in \mathcal{T}_j\} \tag{2.2}$$

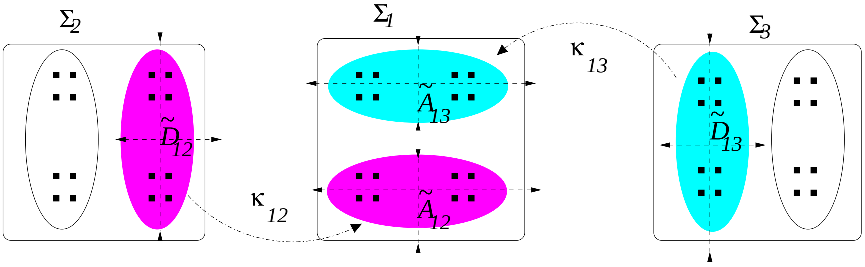

FIGURE 3. Schematic representation of a hyperbolic Poincaré map. The light blue (resp. pink) regions represent $\widetilde{D}_{31}$ and $\widetilde{A}_{31}$ (resp. $\widetilde{D}_{21}$ and $\widetilde{A}_{21}$). The (un)stable directions are represented by the dashed horizontal and vertical lines. The black squares show a coarse-graining of $\mathcal{T}_E$.

We will denote by $J_+(i)$ (resp. $J_-(i)$) the set of indices in the "outflow" (resp. "inflow") of $\mathcal{T}_i$, that is such that D_{ji} and A_{ji} (resp. D_{ij} and A_{ij}) are not empty. The map κ is the union of components κ_{ij}, which relate bijectively D_{ij} with A_{ij}. Since $\mathcal{T}_i$ lies in the interior of Σ_i, the components D_{ji} (resp. A_{ij}) are disconnected from one another. Hence, each κ_{ij} can be extended to be a bijection $\kappa_{ij} : \widetilde{D}_{ij} \to \widetilde{A}_{ij}$, where $\widetilde{D}_{ij}$, $\widetilde{A}_{ij}$ are open neighborhoods of D_{ij} and A_{ij}, respectively in Σ_j and Σ_i. The extended map $\kappa_{ij} : \widetilde{D}_{ij} \to \widetilde{A}_{ij}$ is a symplectomorphism (see fig. 3 for a sketch).

2.2. **Microlocal solutions.** In this section we show that, for $z \in D(0, Ch)$, any solution to the equation $(P(h) - E - z)u = 0$, microlocally near some part of K_E, can be "encoded" by a transversal function $w \in L^2(\mathbb{R}^{n-1})$, which "lives" on one component Σ_i of the Poincaré section.

Take such a component Σ_i. From the assumption $dp\restriction_{p^{-1}(E)} \neq 0$, there exists an open neighbourhood V_i of Σ_i, and a set of symplectic coordinates $(y_1, \ldots, y_n; , \eta_1, \ldots, \eta_n)$ on V_i, such that

- the Hamiltonian $p(\rho) = E + \eta_1$ for any $\rho \in V_i$
- the section Σ_i is locally defined by $\{y_1 = \eta_1 = 0\}$, and the origin $y = \eta = 0$ corresponds to a point in $\mathcal{T}_i$.

Equivalently, there exists a neighbourhood $(0, 0) \in \tilde{V}_i \in T^*\mathbb{R}^n$ and a symplectomorphism $\tilde{\kappa}_i : \tilde{V}_i \to V_i$, such that $p \circ \tilde{\kappa}_i(y, \eta) = E + \eta_1$, etc.

The change of coordinates $\tilde{\kappa}_i$ can be h-quantized into an h-Fourier integral operator $\mathcal{U}_i : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$, microlocally defined and unitary near $V_i \times \tilde{V}_i$, such that

$$\mathcal{U}_i^* \left(P(h) - E\right) \mathcal{U}_i \quad \text{is microlocally equivalent with } hD_{y_1} \text{ in } \tilde{V}_i \times \tilde{V}_i. \tag{2.3}$$

This "quantum change of coordinates" allows one to easily characterize, for $z \in D(0, Ch)$, the microlocal solutions to the equation

$$(P(h) - E - z)u = 0 \qquad \text{microlocally in } V_i\,. \tag{2.4}$$

Indeed, the equation

$$(hD_{y_1} - z)v = 0 \tag{2.5}$$

is obviously solved by

$$v(y_1, y') = e^{izy_1/h}\, w(y'), \quad w \in L^2(\mathbb{R}^{n-1})\,, \tag{2.6}$$

that is by extending some "transversal data" w. We denote this extension by $v = \mathcal{K}(z)w$. Conversely, the solution v can easily be "projected" onto the data w: consider some monotone $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $\chi(y) = 0$ for $y_1 < \epsilon$, $\chi(y) = 1$ for $y_1 > \epsilon$. Then, w can be recovered from v through

$$w(y') = \int_{\mathbb{R}} e^{-izy_1/h}\, \partial_{y_1}\chi(y)\, v(y)\, dy_1\,.$$

In a more compact form, we write $w = \mathcal{K}(\bar{z})^*\chi'\, v$, with $\chi' = \frac{i}{h}[hD_{y_1}, \chi]$.

The solutions of (2.4) are then given by selecting some $w \in L^2(\mathbb{R}^{n-1})$ (microlocalized near the origin), and take

$$u = \mathcal{U}_i\mathcal{K}(z)w \stackrel{\text{def}}{=} \mathcal{K}_i(z)\, w\,. \tag{2.7}$$

That is, the operator $\mathcal{K}_i(z)$ builds a microlocal solution of (2.4) near Σ_i, starting from "transversal data" $w \in L^2(\mathbb{R}^{n-1})$. The latter can be interpreted as a quantum state living in the reduced phase space Σ_i. The converse "projection" is given by

$$w = \mathcal{K}(\bar{z})^*\chi'\, \mathcal{U}_i^*\, u = \mathcal{K}_i(\bar{z})^*\chi_i'\, u \stackrel{\text{def}}{=} R_{+i}(z)\, u\,. \tag{2.8}$$

Here χ_i is the cutoff corresponding to χ near the section Σ_i. To get a consistent definition for χ_i, we must assume that it jumps back down to 0 a little further along the flow, but the precise position will be irrelevant. Indeed, the commutator $\frac{i}{h}[P(h), \chi_i^w(x, hD_x)]$ is equal (microlocally near K_E) to the sum of two pseudodifferential operators with disjoint wavefront sets. The first one is microlocalized near Σ_i (in the region where χ_i jumps from 0 to 1), we will denote it by $\chi_i' = \frac{i}{h}[P(h), \chi_i^w(x, hD_x)]_i$; the second component "lives" in the region where χ_i decreases from 1 to 0, and will not play any role.

The same construction can be performed independently near each Σ_j, $j = 1, \ldots, J$. We will call w_j the transversal data associated with the section Σ_j, and $\mathcal{K}_j(z)$, $R_{+j}(z)$ the corresponding operators.

2.3. **From one transversal parametrization the next.** The the solution (2.7) is microlocalized in V_i, since $\mathcal{U}_i$ is only defined microlocally in $V_i \times \tilde{V}_i$. However, this solution can be extended in a forward cylinder $\cup_{0\leq t\leq T}\Phi^t\Sigma_i$ by using the propagator $e^{-it(P-E-z)/h}$: if u is a solution near $\rho \in \Sigma_i$, then $e^{-it(P-E-z)/h}u$ is the extension of this solution near $\Phi^t(\rho)$.

This way, we can extend u up to the vicinity of the sections Σ_j in the outflow of Σ_i. This extended solution will still be denoted by $u = \mathcal{K}_i(z)w_i$. Near Σ_j, this solution can also be parametrized by the "transversal" function $w_j = R_{+j}\, u \in L^2(\mathbb{R}^{n-1})$. The map $w_i \mapsto w_j$, which amounts to changing the transversal parametrization for a single solution u, defines our quantum Poincaré map:

$$\mathcal{M}_{ji}(z,h) \stackrel{\text{def}}{=} R_{+j}(z)\mathcal{K}_i(z)\,. \tag{2.9}$$

This operator is a Fourier integral operator quantizing the Poincaré map κ_{ji}; it is microlocally defined, and microlocally unitary, on $\widetilde{D}_{ji} \times \widetilde{A}_{ji}$.

Let us see how the operators $\mathcal{M}_{ji}(z)$ can be used. Assume $E + z$ is a resonance of $P(h)$, with $z \in D(0, Ch)$. Then, there exists a metastable state $u \in L^2_{loc}$, global solution to the equation $(P - E - z)u = 0$. The above procedure associates to this solution J parametrizations $w_i = R_{+i}(z)u$, microlocally defined near $\mathcal{T}_j$. For any i and $j \in J_+(i)$, these parametrizations satisfy $w_j = \mathcal{M}_{ji}(z)w_i$: can be written in the compact form

$$w = \mathcal{M}(z)\, w\,, \tag{2.10}$$

where $w = (w_i)_i$ is the column vector of all J local parametrizations, and $\mathcal{M}(z)$ is the operator valued matrix $\big(\mathcal{M}_{ij}(z)\big)$.

In the next subsections we prove that the converse statement holds as well: the existence of a solution of $(\mathcal{M}(z) - Id)w = 0$ microlocally near $\mathcal{T}_E$ implies the existence of a solution of $(P - E - z)u = 0$ microlocally near K_E. To prove this we will set up a formal Grushin problem, in which the operator $(\mathcal{M}(z) - I)$ will appear as the "effective Hamiltonian" for the original operator $(P - E - z)$.

2.4. **Grushin problems.** A *Grushin problem* for the family of operators[2] $\{(P - E - z) : H^2_h(\mathbb{R}^n) \to L^2(\mathbb{R}^n),\ z \in D(0, Ch)\}$ consists in the insertion of that operator inside an operator valued matrix

$$\mathcal{P}(z) = \begin{pmatrix} \frac{i}{h}(P - E - z) & R_-(z) \\ R_+(z) & 0 \end{pmatrix} : H^2_h(\mathbb{R}^) \times \mathcal{H} \to L^2(\mathbb{R}^) \times \mathcal{H}\,, \tag{2.11}$$

in a way such that $\mathcal{P}(z)$ is invertible (see e.g. [29] or [10, Appendix] for a general presentation of this method). Ideally, the auxiliary space $\mathcal{H}$ is "much smaller" than L^2 or H^2_h (in

[2] $H^2_h(\mathbb{R}^n)$ is the semiclassical Sobolev space of norm $\|u\|_{H^2_h} = \int |\tilde{u}(\xi)|^2(1 + |h\xi|^2)^2\, d\xi$, with $\tilde{u}$ the Fourier transform of u.

our final version, $\mathcal{H}$ will be finite dimensional). The inverse of $\mathcal{P}(z)$ is traditionally written in the form

$$\mathcal{P}(z)^{-1} = \begin{pmatrix} E(z) & E_+(z) \\ E_-(z) & E_{-+}(z) \end{pmatrix} .$$

The invertibility of $(P-E-z)$ is then equivalent with that of the operator $E_{-+}(z)$: Schur's complement formula shows that

(2.12)
$$\frac{h}{i}(P-E-z)^{-1} = E(z)-E_+(z)E_{-+}(z)^{-1}E_-(z)\,, \qquad E_{-+}(z)^{-1} = -\frac{h}{i}R_+(z)(P-E-z)^{-1}R_-(z)\,,$$

so that $\dim\ker(P-E-z) = \dim\ker E_{-+}(z)$. For this reason, $E_{-+}(z)$ is called an *effective Hamiltonian* associated with $(P(h)-E-z)$. It has a smaller rank than $P(h)$, but its dependence in the spectral parameter z is nonlinear.

2.5. **Our formal Grushin problem.** We will first build our Grushin problem microlocally near K_E (so we can identify H_h^2 with L^2). Our auxiliary space $\mathcal{H}$ will contain local "transversal data" $w_i \in L^2(\mathbb{R}^{n-1})$, one for each section Σ_i, so we have formally $\mathcal{H} = L^2(\mathbb{R}^{n-1})^J$. The auxiliary operators are then vectors of operators: $R_+(z) = (R_{+1}, \ldots, R_{+J})$, $R_-(z) = {}^t(R_{-1}, \ldots, R_{-J})$, which will for now be defined only microlocally:

- $R_{+i}(z)$ is the "projector" (2.8) of $L^2(\mathbb{R}^n)$ onto the parametrization w_i living on Σ_i. We will rebaptize $\chi_i \stackrel{\text{def}}{=} \chi_i^f$ (for *f*orward) the cutoff used in the definition of R_{+i}.
- on the opposite, $R_{-i}(z)$ takes the data $w_i \in L^2(\mathbb{R}^{n-1})$ to produce a microlocal solution, and cuts off this solution by applying the derivative of another cutoff χ_i^b:

$$R_{-i}(z) = \chi_i^{b\prime}\mathcal{K}_i(z)\,. \tag{2.13}$$

  The cutoff χ_i^b (for *b*ackward) is similar with χ_i^f, and $\chi_i^{b\prime}$ is, as before, the component of $[\frac{i}{h}P(h), (\chi_i^b)^w]$ microlocalized near Σ_i. We require that the jump of χ_i^b occurs *before* that of χ_i^f, and that the whole family $\{\chi_i^b,\ i = 1, \ldots, J\}$ satisfies a local resolution of identity near K_E:

$$\sum_i \chi_i^b = 1\,, \quad \text{in some neighbourhood of } K_E. \tag{2.14}$$

2.5.1. *Homogeneous problem.* Let us now try to invert the matrix $\mathcal{P}(z)$ we have just defined, at least microlocally near $K_E \times \prod_i \mathcal{T}_i$. First we consider arbitrary transversal data $w = (w_i)$, and try to solve (in $u \in L^2(\mathbb{R}^n)$, $u_- \in L^2(\mathbb{R}^{n-1})^J$) the system

$$\frac{i}{h}(P-E-z)u + \sum_{i=1}^J R_{-i}(z)u_{-i} = 0 \tag{2.15}$$

$$R_{+i}(z)u = w_i, \quad i = 1, \ldots, J\,. \tag{2.16}$$

Eq. (2.16) suggests that u could be a microlocal solution parametrized by w_i, at least in the region where χ_j^f jumps from 0 to 1. Since $\chi_i^b \equiv 1$ in this region, we take the Ansatz

$$u = \sum_{i=1}^{J} (\chi_i^b)^w \mathcal{K}_i(z)\, w_i \stackrel{\text{def}}{=} \sum_i E_{+i}(z)\, w_i\,. \tag{2.17}$$

Injecting this Ansatz in (2.15), we obtain

$$\sum_{i=1}^{J} \frac{i}{h}[P,(\chi_i^b)^w]\mathcal{K}_i(z)\, w_i + \sum_{i=1}^{J} R_{-i}(z)u_{-i} = 0\,, \tag{2.18}$$

which we want to solve in (u_{-i}). Each commutator $\frac{i}{h}[P,(\chi_i^b)^w]$ is the sum of a component $\chi_i^{b\prime} = \frac{i}{h}[P,(\chi_i^b)^w]_i$ microlocalized near Σ_i, and of components $\frac{i}{h}[P,(\chi_i^b)^w]_j$ microlocalized near $\widetilde{A}_{ji} \subset \Sigma_j$, for each index $j \in J_+(i)$. The resolution of identity (2.14) shows that near we have $H_p\chi_i^{b\prime} + H_p\chi_j^{b\prime} = 0$, the quantum version of which reads $\frac{i}{h}[P,(\chi_i^b)^w]_j + \frac{i}{h}[P,(\chi_j^b)^w]_j = 0$ microlocally near $\widetilde{A}_{ji}$. As a result (2.18) can be rewritten as

$$\sum_{i=1}^{J} \chi_i^{b\prime}\mathcal{K}_i(z)\, w_i - \sum_{i=1}^{J}\sum_{j\in J_+(i)} \chi_j^{b\prime}\mathcal{K}_i(z)\, w_i + \sum_{i=1}^{J} R_{-i}(z)u_{-i} = 0\,.$$

Near each Σ_j, $j \in J_+(i)$ we have $\mathcal{K}_i(z)w_i = \mathcal{K}_j(z)\mathcal{M}_{ji}(z)\, w_i$. For each i we can group together the terms localized near Σ_i, and get:

$$R_{-i}(z)\, w_i - \sum_{i\in J_+(j)} R_{-i}(z)\mathcal{M}_{ij}(z)\, w_j + R_{-i}(z)u_{-i} = 0\,.$$

This leads to the microlocal solution

$$u_{-i} = -w_i + \sum_{i\in J_+(j)} \mathcal{M}_{ij}(z)\, w_j \stackrel{\text{def}}{=} \sum_j E_{-+ij}(z)w_j\,. \tag{2.19}$$

We have thus solved the system (2.15,2.16) microlocally near $K_E \times \prod_i \mathcal{T}_i$, and provided explicit expressions for the operators $E_+(z)$ and $E_{-+}(z) = \mathcal{M}(z) - Id$, microlocally near the trapped set.

2.5.2. *Nonhomogeneous problem.* To complete the microlocal inversion of $\mathcal{P}(z)$, we now take $v \in L^2(\mathbb{R}^n)$ microlocalized near K_E, and try to solve (in u, u_-, microlocally near K_E)

$$\frac{i}{h}(P - E - z)u + \sum_{i=1}^{J} R_{-i}(z)u_{-i} = v\,. \tag{2.20}$$

Let us first assume that v is microlocalized inside the region $\{\chi_i^b(\rho) = 1\}$ for some index i. We then take the truncated parametrix $\widetilde{E}(z) = \int_0^T e^{-it(P-E-z)/h}\, dt$, with T large enough so that $e^{-iTP/h}v$ is microlocalized beyond $\operatorname{supp}\chi_i^b$, and define the Ansatz $u = (\chi_i^b)^w\,\widetilde{E}(z)\,v$. The latter satisfies

$$\frac{i}{h}(P-E-z)u = v + \frac{i}{h}[P,(\chi_i^b)^w]\widetilde{E}(z)\,v\,dt \tag{2.21}$$

$$= v + \chi_i^{b\prime}\widetilde{E}(z)v - \sum_{j\in J_+(i)} \chi_j^{b\prime}\widetilde{E}(z)v\,. \tag{2.22}$$

(we have used the splitting of the commutator explained above). Now, provided T is not too large, the state $\widetilde{E}(z)v$ is microlocalized away from Σ_i so that $\chi_i^{b\prime}\widetilde{E}(z)v = \mathcal{O}(h^\infty)$ On the opposite, for each $j \in J_+(i)$ that state is a microlocal solution of $(P-E-z)u-0$ near $\widetilde{A}_{ji}$, which can then be written as $\mathcal{K}_j(z)u_{-j}$ with $u_{-j} = R_{+j}\widetilde{E}(z)v$. The above equality becomes

$$\frac{i}{h}(P-E-z)\,u = v - \sum_{j\in J_+(i)} R_{-j}u_{-j}\,,$$

and solves (2.20) microlocally.

If v is microlocalized near Σ_i, we cutoff $\widetilde{E}(z)v$ by $(\sum_{j\in J_-(i)}\chi_j^b + \chi_i^b)^w$, which is equivalent to identity near $\mathcal{T}_i$, and take as above $u_{-j} = R_{+j}\widetilde{E}(z)v$, $j \in J_+(i)$.

We have now fully inverted $\mathcal{P}(z)$ microlocally near $K_E \times \prod_i \mathcal{T}_i$, and the norm of the inverse can be shown to be of order unity. The effective Hamiltonian reads $E_{-+}(z) = \mathcal{M}(z) - Id$. Hence, as anticipated above, the existence of a nontrivial state w satisfying $w = \mathcal{M}(z)\,w$ is *equivalent* with that of a microlocal solution to $(P-E-z)u = 0$ near K_E.

To prove Thm 1, we must define our Grushin problem *globally*, that is properly define the auxiliary space $\mathcal{H}$ and the operators $R_\pm$, in such a way that $\mathcal{P}(z)$ is invertible. One then says that the Grushin problem is *well-posed*.

3. From the formal to the well-posed Grushin problem

In order to make our Grushin problem well-posed, we will first "deform" the original Schrödinger operator $P(h)$ in order to transform its resonances z_i into *bona fide* L^2 eigenfunctions of the deformed operator $P_\theta(h)$. This deformation is performed through a "complex scaling" of $P(h)$ far away from the scattering region. The operator $(P_\theta(h) - E)$ will now be elliptic outside a large ball $B(0,R)$. In order to enlarge this zone of ellipticity to the complement of a smaller neighbourhood of K_E, we will then modify the Hilbert structure of our auxiliary states, using an appropriate *escape function* $G(x,\xi)$. After these two modifications, we will be able to complete the construction of a well-posed Grushin problem, with finite dimensional auxiliary spaces.

3.1. **Complex scaling.** Here we use the fact that outside a ball $B(0,R_0) \ni \operatorname{supp} V$, one has $P(h) = -\frac{h^2}{2}\sum_{k=1}^n \frac{\partial^2}{\partial x_i^2}$. In that region that operator can be holomorphically exended into $\widetilde{P} = -\frac{h^2}{2}\sum_{k=1}^{n}\frac{\partial^2}{\partial z_i^2}$, acting on functions on $\mathbb{C}^n$. For $\theta > 0$ small, we deform $\mathbb{R}^n \subset \mathbb{C}^n$ into a smooth contour $\Gamma_\theta \subset \mathbb{C}^n$:

$$\Gamma_\theta \cap B_{\mathbb{C}^n}(0, R_0) = B_{\mathbb{R}^n}(0, R_0)\,,$$
$$\Gamma_\theta \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 2R_0) = e^{i\theta}\mathbb{R}^n \cap \mathbb{C}^n \setminus B_{\mathbb{C}^n}(0, 2R_0)\,.$$

We then define the operator $P_\theta(h)$ acting on $u \in \mathcal{C}_c^\infty(\Gamma_\theta)$, by $P_\theta u = \widetilde{P}(\tilde{u})\restriction_{\Gamma_\theta}$, where $\tilde{u}$ is an almost analytic extension of u. Through the identification $\Gamma_\theta \ni x \longleftrightarrow (\sin\theta)^{-1}\operatorname{Re} x \in \mathbb{R}^n$, the operator P_θ can considered as acting on functions in $\mathcal{C}_c^\infty(\mathbb{R}^n)$, with the action $-e^{-2i\theta}\frac{h^2\Delta}{2}$ outside $B(0, 2R_0)$. One can then show [26] that the resolvent $(P_\theta - z)^{-1} : L^2 \to H_h^2$ is meromorphic in the region $\{\arg(z) > -2\theta\}$. The L^2 spectrum of $P_\theta(h)$ in that region is discrete, independent of θ and R, and consists of the resonances of the initial operator $P(h)$.

Since $P_\theta(h) = P(h)$ inside the ball $B(0, R_0) \supset \pi K_E$, our formal Grushin problem remains unchanged if we replace $P(h)$ by $P_\theta(h)$.

Below we will take values of θ of the form $\theta \sim C\,h\log(1/h)$, $C > 0$ fixed.

3.2. **Finite dimensional auxiliary spaces.** We have built in §2.5 a Grushin problem which is invertible microlocally near the trapped set. To make that Grushin problem well-posed, we need to make definite choices for the auxiliary spaces, that is for each $i = 1, \ldots, J$ define a subspace of $\mathcal{H}_i \subset L^2(\mathbb{R}^{n-1})$ containing the transversal data. This subspace should contain states microlocalized in some neighbourhood S_i of $\mathcal{T}_i$, small enough to lie in the domain $\cup_{j\in J_+(i)}\widetilde{A}_{ji}$ where κ is defined. To construct this subspace explicitly, we may define the neighbourhood S_i as $S_i = \{q_i(\rho) < 0\}$, for a well-chosen $q_i \in \mathcal{C}^\infty(\mathbb{R}^{n-1})$ satisfying $\liminf_{\rho\to\infty} q_i(\rho) > 0$. The subspace $\mathcal{H}_i$ can then be defined as the range of the spectral projector

$$\Pi_i \mathbb{1}_{\mathbb{R}_-}(q_i^w(y, hD_y))\,. \tag{3.1}$$

According to Weyl's law, for h small enough the space $\mathcal{H}_i$ has a finite dimension $\sim \operatorname{vol}(S_i)\,h^{-n+1}$.

One can then consider the Grushin problem (2.11), with P replaced by P_θ, the auxiliary space $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ and the operators

$$R_{+i}(z) \stackrel{\text{def}}{=} \Pi_i\mathcal{K}_i^*(\bar{z})\chi_i^{b\prime}\,, \qquad R_{-i} \stackrel{\text{def}}{=} \chi_i^{b\prime}\mathcal{K}_i(z)\Pi_i\,. \tag{3.2}$$

Unfortunately, when trying to solve this new Grushin problem, that is invert $\mathcal{P}(z)$, one encouters difficulties. Some of them are due to the fact that κ does not leave the neighbourhoods S_i invariant (see fig. 3). As a result, an initial datum $w_i \in \mathcal{H}_i$ is propagated through $\mathcal{M}_{ji}(z)$ into a state $\mathcal{M}_{ji}(z)w_i$ which, in general, is not microlocalized in S_j, and thus cannot belong to $\mathcal{H}_j$. Brutally applying the projector Π_j to $\mathcal{M}_{ji}(z)w_i$ produces an extra term $(1 - \Pi_j)\mathcal{M}_{ji}w_i$, which is difficult to solve away. Another difficulty arises when trying to solve the unhomogeneous problem (2.20) for data v microlocalized at some distance from K_E.

3.3. **Escape functions and modified norms.** These difficulties can be tackled by modifying the Hilbert norms on $H_h^2(\mathbb{R}^n)$ and the auxiliary space $L^2(\mathbb{R}^{n-1})^J$. The new norms will be defined in terms of well-chosen *escape functions* $G \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n)$, $G^i \in \mathcal{C}_c^\infty(T^*\mathbb{R}^{n-1})$. Using these new norms, the problems mentioned above will disappear, because the states microlocalized away from K_E will become easily solvable.

Let us describe the escape function. For some small $\delta > 0$, we consider the thickened energy shell $\widehat{p^{-1}(E)} = \bigcup_{|s|\leq\delta} p^{-1}(E+s)$ and trapped set $\widehat{K}_E = \bigcup_{|s|\leq\delta} K_{E+s}$. It is shown in [28, §§4.1,4.2,7.3] and [20, §6.1] that, for any small $\delta_0 > 0$ and large $R > 0$, and any neighbourhoods $U \subset \overline{U} \subset V$ of $\widehat{K}_E$, one can construct a function $G_0 \in \mathcal{C}_c^\infty(T^*\mathbb{R}^n)$ such that

$$G_0 = 0 \quad \text{on } U, \qquad H_p G_0 \geq 0 \quad \text{on } T^*_{B(0,3R)}\mathbb{R}^n, \tag{3.3}$$

$$H_p G_0 \geq 1 \quad \text{on } T^*_{B(0,3R)}\mathbb{R}^n \cap (\widehat{p^{-1}(E)} \setminus V), \qquad H_p G_0 \geq -\delta_0 \quad \text{on } T^*\mathbb{R}^n\,. \tag{3.4}$$

It is convenient[3] to slightly modify this function in the neighbourhood of the sets $S_i \subset \Sigma_i$. Namely, we consider open neighbourhoods $\tilde{W}_i \Subset W_i$ of S_i in $T^*\mathbb{R}^n$, and modify G_0 into a function G_1, such that $H_p G_1 = 0$ in $\tilde{W}_i$ while $H_p G_1 \geq 1$ on $T^*_{B(0,3R)}\mathbb{R}^n \cap (\widehat{p^{-1}(E)} \setminus (V \cup \bigcup_i W_i)$.

We then set $G \stackrel{\text{def}}{=} Nh\log(1/h) G_1$, with $N > 0$ fixed but arbitrary large. The exponential $\exp(G^w(x,hD)/h)$ is a pseudodifferential operator in some mildly exotic class, bounded and of bounded inverse on L^2, with norms $\mathcal{O}(h^{-CN})$. We call H_G the vector space $H_h^2(\mathbb{R}^n)$ equipped with the Hilbert norm

$$\|u\|_{H_G} \stackrel{\text{def}}{=} \|\exp(-G^w(x,hD_x)/h)u\|_{H_h^2}\,. \tag{3.5}$$

Similarly, we consider functions $G^i \in \mathcal{C}_c^\infty(T^*\mathbb{R}^{n-1})$ such that (using the coordinate change $\tilde{\kappa}_i$ near Σ_i) $G^i(y',\eta') = G \circ \tilde{\kappa}_i(0,y';0,\eta')$ in some neighbourhood of S_i, and modify the Hilbert norms on the space $L^2(\mathbb{R}^{n-1})$ attached to the section Σ_i by

$$\|w_i\|_{H_{G^i}} \stackrel{\text{def}}{=} \|e^{-(G^i)^w(y',hD_{y'})/h} w_i\|_{L^2(\mathbb{R}^{n-1})}\,. \tag{3.6}$$

3.4. **How these norms resolve our problems.** Let us explain how this change of norm helps us. The action of $P_\theta(h)$ on the Hilbert space H_G is equivalent to the action of $P_{\theta,G}(h) \stackrel{\text{def}}{=} e^{-G^w/h}\, P_\theta(h)\, e^{G^w/h}$ on $H_h^2(\mathbb{R}^n)$, which is a pseudodifferential operator with symbol

$$p_{\theta,G}(\rho) = p(\rho) - iNh\log(1/h) H_p G_1(\rho) + \mathcal{O}(h^2\log^2(1/h))\,, \quad \rho \in T^*_{B(0,R)}\mathbb{R}^n\,.$$

[3] The role of this modification is to ultimately keep the norms $\|R_{+i}(z)\|_{H_G\to\mathcal{H}_i}$, $\|R_{-i}(z)\|_{\mathcal{H}_j\to H_G}$, $\|M_{ji}(z)\|_{\mathcal{H}_i\to\mathcal{H}_j}$ uniformly bounded

Provided we have chosen a dilation angle $\theta \ll \delta_0 N h \log(1/h)$, the properties of G_1 show that

$$\forall \rho \notin (V \cup \bigcup_i W_i), \qquad |\operatorname{Re} p_{\theta,G}(\rho) - E| \leq \delta/2 \Longrightarrow \operatorname{Im} p_{\theta,G} \leq -\theta/C\,, \tag{3.7}$$

This shows that, for any $z \in D(0, Ch)$, the symbol $(p_{\theta,G} - E - z)$ is invertible outside $V \cup \bigcup_i W_i$, with inverse of order $(h \log h^{-1})^{-1}$. Hence, for any $v \in L^2$ microlocalized outside $V \cup \bigcup_i W_i$, the equation $(P_{\theta,G} - E - z)u = v$ can be solved up to $\mathcal{O}(h^\infty)$, with a solution u microlocalized outside $V \cup \bigcup_i W_i$. This remark basically tackles the second problem mentioned at the end of §3.2.

The first problem (the fact that $\kappa_{ji}(S_i)$ is not contained in S_j) is also resolved through this change of norms. Indeed, the escape function G_1 can be chosen such that it *uniformly increases* (say, by some $2C > 0$) along all trajectories of the form $\rho \in S_i \mapsto \kappa_{ji}(\rho) \in \Sigma_j \setminus \in S_j$, so that $\frac{e^{-G(\kappa_{ji}(\rho))/h}}{e^{-G(\rho)/h}} \leq h^{2NC}$. This implies that, for any state w_i microlocalized near such a point ρ, one gets (taking the definition (2.8) for R_{+j})

$$\|R_{+j}(z)\mathcal{K}_i(z) w_i\|_{H_{G^j}} = \mathcal{O}(h^{NC})\, \|w_i\|_{H_{G^i}}\,. \tag{3.8}$$

We then need to modify the finite rank projectors (3.1) defining our auxiliary states, such as to make them orthogonal w.r.to the new norms (otherwise $\|\Pi_i\|_{H_{G^i} \to H_{G^i}}$ could be very large). This modification only amounts to adding a subprincipal (complex valued) term to the function q_i, such that q_i^w becomes selfadjoint on H_{G^i}, and the spectral projector (3.1) orthogonal. The space $\mathcal{H}_i \stackrel{\text{def}}{=} \Pi_i H_{G^i}$ is still made of states microlocalized in S_i, and has dimension $\sim \operatorname{vol}(S_i) h^{-n+1}$. Our operators $R_{\pm i}$ will be defined by (3.2).

Let us reconsider the homogeneous problem §2.5.1 with data $w_i \in \mathcal{H}_i$ in our new Grushin problem. For $j \in J_+(i)$, the state $\mathcal{M}_{ji}(z)\, w_i$ does not a priori belong to $\mathcal{H}_j$. However,, the estimate (3.8) shows that the component of $\mathcal{M}_{ji}(z)\, w_i$ microlocalized outside S_j has an H_{G^j}-norm of order $\mathcal{O}(h^{NC})$. As a result, defining the finite rank operators

$$M_{ji}(z) \stackrel{\text{def}}{=} \Pi_j\, \mathcal{M}_{ji}(z) : \mathcal{H}_i \to \mathcal{H}_j\,, \tag{3.9}$$

we find that

$$u_{-i} \stackrel{\text{def}}{=} -w_i + \sum_{j \in J_+(i)} M_{ij}(z) w_j \in \mathcal{H}_i$$

provides a solution to the homogeneous problem, up to an error $\mathcal{O}(h^{NC})(\sum_i \|w_i\|_{\mathcal{H}_i})$.

The nonhomogeneous problem (2.20) can be solved as well, up to a comparable error (see [18] for details).

To summarize, our globally defined Grushin problem has an approximate inverse $\mathcal{E}(z)$:

$$\mathcal{P}(z)\mathcal{E}(z) = I + \mathcal{R}(z), \qquad \|\mathcal{R}(z)\|_{L^2 \times \mathcal{H} \to L^2 \times \mathcal{H}} = \mathcal{O}(h^{NC})\,,$$

where we insist on the fact that N can be chosen arbitrary large (it comes from the factor in front of the escape function G). Hence, for h small enough this operator has the exact inverse $\tilde{\mathcal{E}}(z) = \mathcal{E}(z)(I+\mathcal{R}(z))^{-1} = \mathcal{E}(z)+\mathcal{O}_{L^2\times\mathcal{H}\to H_h^2\times\mathcal{H}}(h^{NC})$. In particular, the lower-right entry of $\tilde{\mathcal{E}}(z)$ (that is, the exact effective Hamiltonian) reads

$$\widetilde{E}_{-+}(z) = I - M(z) + \mathcal{O}_{\mathcal{H}\to\mathcal{H}}(h^{NC}),$$

where $M(z)$ is the matrix composed of the finite dimensional operators (3.9).

As explained in §2.4, this exact inversion implies that the eigenvalues $\{z_i\}$ of $(P_\theta - E)$ in $D(0, Ch)$ coincide (with multiplicities) with the zeros of $\det(E_{-+}(z))$. $\square$

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