NON LOCAL POINCARÉ INEQUALITIES ON LIE GROUPS WITH POLYNOMIAL VOLUME GROWTH

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Abstract. Let G be a real connected Lie group with polynomial volume growth, endowed with its Haar measure dx. Given a C^2 positive function M on G, we give a sufficient condition for an L^2 Poincaré inequality with respect to the measure $M(x)dx$ to hold on G. We then establish a non-local Poincaré inequality on G with respect to $M(x)dx$.

Contents

1. Introduction

Let G be a unimodular connected Lie group endowed with a measure $M(x)\,dx$ where $M \in L^1(G)$ and dx stands for the Haar measure on G. By "unimodular", we mean that the Haar measure is left and right-invariant. We always assume that $M = e^{-v}$ where v is a C^2 function on G. If we denote by $\mathcal{G}$ the Lie algebra of G, we consider a family

$$\mathbb{X} = \{X_1, ..., X_k\}$$

of left-invariant vector fields on G satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the $X_i's$. A standard metric on G , called the Carnot-Caratheodory metric, is naturally associated with $\mathbb{X}$

and is defined as follows: let $\ell : [0,1] \to G$ be an absolutely continuous path. We say that ℓ is admissible if there exist measurable functions $a_1, ..., a_k : [0,1] \to \mathbb{C}$ such that, for almost every $t \in [0,1]$, one has

$$\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).$$

If ℓ is admissible, its length is defined by

$$|\ell| = \int_0^1 \left(\sum_{i=1}^{k} |a_i(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

For all $x, y \in G$, define $d(x,y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group and x, so that the distance from x to y is equal to $|y^{-1}x|$.

For all $r > 0$, denote by $B(x,r)$ the open ball in G with respect to the Carnot-Caratheodory distance and by $V(r)$ the Haar measure of any ball. There exists $d \in \mathbb{N}^*$ (called the local dimension of $(G, \mathbb{X})$) and $0 < c < C$ such that, for all $r \in (0,1)$,

$$cr^d \leq V(r) \leq Cr^d,$$

see [NSW85]. When $r > 1$, two situations may occur (see [Gui73]):

- Either there exist $c, C, D > 0$ such that, for all $r > 1$,
  $$cr^D \leq V(r) \leq Cr^D$$
  where D is called the dimension at infinity of the group (note that, contrary to d, D does not depend on $\mathbb{X}$). The group is said to have polynomial volume growth.
- Or there exist $c_1, c_2, C_1, C_2 > 0$ such that, for all $r > 1$,
  $$c_1 e^{c_2 r} \leq V(r) \leq C_1 e^{C_2 r}$$
  and the group is said to have exponential volume growth.

When G has polynomial volume growth, it is plain to see that there exists $C > 0$ such that, for all $r > 0$,

$$V(2r) \leq CV(r), \tag{1.1}$$

which implies that there exist $C > 0$ and $\kappa > 0$ such that, for all $r > 0$ and all $\theta > 1$,

$$V(\theta r) \leq C\theta^\kappa V(r). \tag{1.2}$$

Denote by $H^1(G, d\mu_M)$ the Sobolev space of functions $f \in L^2(G, d\mu_M)$ such that $X_i f \in L^2(G, d\mu_M)$ for all $1 \leq i \leq k$. We are interested in L^2 Poincaré inequalities for the measure $d\mu_M$. In order to state sufficient conditions for such an inequality to hold, we introduce the operator

$$L_M f = -M^{-1} \sum_{i=1}^{k} X_i \Big\{ M X_i f \Big\}$$

for all f such that

$$f \in \mathcal{D}(L_M) := \left\{ g \in H^1(G, d\mu_M);\ \frac{1}{\sqrt{M}} X_i \Big\{ M X_i f \Big\} \in L^2(G, dx),\ \forall 1\ \leq i \leq k \right\}.$$

One therefore has, for all $f \in \mathcal{D}(L_M)$ and $g \in H^1(G, d\mu_M)$,

$$\int_G L_M f(x) g(x) d\mu_M(x) = \sum_{i=1}^{k} \int_G X_i f(x) \cdot X_i g(x) d\mu_M(x).$$

In particular, the operator L_M is symmetric on $L^2(G, d\mu_M)$.
Following [BBCG08], say that a C^2 function $W : G \to \mathbb{R}$ is a Lyapunov function if $W(x) \geq 1$ for all $x \in G$ and there exist constants $\theta > 0$, $b \geq 0$ and $R > 0$ such that, for all $x \in G$,

$$-L_M W(x) \leq -\theta W(x) + b \mathbf{1}_{B(e,R)}(x), \tag{1.3}$$

where, for all $A \subset G$, $\mathbf{1}_A$ denotes the characteristic function of A. We first claim:

Theorem 1.1. *Assume that G is unimodular and that there exists a Lyapunov function W on G. Then, $d\mu_M$ satisfies the following L^2 Poincaré inequality: there exists $C > 0$ such that, for all function $f \in H^1(G, d\mu_M)$ with $\int_G f(x) d\mu_M(x) = 0$,*

$$\int_G |f(x)|^2 d\mu_M(x) \leq C \sum_{i=1}^{k} \int_G |X_i f(x)|^2 d\mu_M(x). \tag{1.4}$$

Let us give, as a corollary, a sufficient condition on v for (1.4) to hold:

Corollary 1.2. *Assume that G is unimodular and there exist constants $a \in (0, 1)$, $c > 0$ and $R > 0$ such that, for all $x \in G$ with $|x| > R$,*

$$a \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c. \tag{1.5}$$

Then (1.4) holds.

Notice that, if (1.5) holds with $a \in \left(0, \frac{1}{2}\right)$, then the Poincaré inequality (1.4) has the following self-improvement:

Proposition 1.3. *Assume that G is unimodular and that there exist constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that, for all $x \in G$,*

$$\frac{1-\varepsilon}{2} \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c \text{ whenever } |x| > R. \tag{1.6}$$

Then there exists $C > 0$ such that, for all function $f \in H^1(G, d\mu_M)$ such that $\int_G f(x) d\mu_M(x) = 0$:

$$\sum_{i=1}^{k} \int_G |X_i f(x)|^2 \, d\mu_M(x) \geq C \int_G |f(x)|^2 \left(1 + \sum_{i=1}^{k} |X_i v(x)|^2\right) d\mu_M(x) \tag{1.7}$$

We finally obtain a Poincaré inequality for $d\mu_M$ involving a non local term:

Theorem 1.4. *Let G be a unimodular Lie group with polynomial growth. Let $d\mu_M = M dx$ be a measure absolutely continuous with respect to the Haar measure on G where $M = e^{-v} \in L^1(G)$ and $v \in C^2(G)$. Assume that there exist constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that (1.6) holds. Let $\alpha \in (0,2)$. Then there exists $\lambda_\alpha(M) > 0$ such that, for any function $f \in \mathcal{D}(G)$ satisfying $\int_G f(x)\, d\mu_M(x) = 0$,*

$$\begin{aligned} \iint_{G \times G} \frac{|f(x) - f(y)|^2}{V\left(|y^{-1}x|\right) |y^{-1}x|^\alpha} \, dx \, d\mu_M(y) &\geq \lambda_\alpha(M) \\ &\int_{\mathbb{R}^n} |f(x)|^2 \left(1 + \sum_{i=1}^{k} |X_i v(x)|^2\right) d\mu_M(x). \end{aligned} \tag{1.8}$$

Note that (1.8) is an improvement of (1.7) in terms of fractional nonlocal quantities. The proof follows the same line as the paper [MRS09] but we concentrate here on a more geometric context.

In order to prove Theorem 1.4, we need to introduce fractional powers of L_M. This is the object of the following developments. Since the operator L_M is symmetric and non-negative on $L^2(G, d\mu_M)$, we can define the usual power L^β for any $\beta \in (0,1)$ by means of spectral theory.

Section 2 is devoted to the proof of Theorem 1.1 and Corollary 1.2. Then, in Section 3, we check L^2 "off-diagonal" estimates for the resolvent of L_M and use them to establish Theorem 1.4.

2. A proof of the Poincaré inequality for $d\mu_M$

We follow closely the approach of [BBCG08]. Recall first that the following L^2 local Poincaré inequality holds on G for the measure dx: for all $R>0$, there exists $C_R>0$ such that, for all $x\in G$, all $r\in(0,R)$, all ball $B:=B(x,r)$ and all function $f\in C^\infty(B)$,

$$\int_B |f(x)-f_B|^2\,dx \le C_R r^2 \sum_{i=1}^k \int_B |X_i f(x)|^2\,dx, \tag{2.9}$$

where $f_B := \frac{1}{V(r)}\int_B f(x)dx$. In the Euclidean context, Poincaré inequalities for vector-fields satisfying Hörmander conditions were obtained by Jerison in [Jer86]. A proof of (2.9) in the case of unimodular Lie groups can be found in [SC95], but the idea goes back to [Var87]. A nice survey on this topic can be found in [HK00]. Notice that no global growth assumption on the volume of balls is required for (2.9) to hold.

The proof of (1.4) relies on the following inequality:

Lemma 2.1. *For all function $f\in H^1(G,d\mu_M)$ on G,*

$$\int_G \frac{L_M W}{W}(x) f^2(x) d\mu_M(x) \le \sum_{i=1}^k \int_G |X_i f(x)|^2\,d\mu_M(x). \tag{2.10}$$

Proof: Assume first that f is compactly supported on G. Using the definition of L_M, one has

$$\begin{aligned}
\int_G \frac{L_M W}{W}(x) f^2(x) d\mu_M(x) &= \sum_{i=1}^k \int_G X_i\left(\frac{f^2}{W}\right)(x)\cdot X_i W(x) d\mu_M(x)\\
&= 2\sum_{i=1}^k \int_G \frac{f}{W}(x) X_i f(x)\cdot X_i W(x) d\mu_M(x)\\
&\quad -\sum_{i=1}^k \int_G \frac{f^2}{W^2}(x)\,|X_i W(x)|^2\,d\mu_M(x)\\
&= \sum_{i=1}^k \int_G |X_i f(x)|^2\,d\mu_M(x)\\
&\quad -\sum_{i=1}^k \int_G \left|X_i f - \frac{f}{W}X_i W\right|^2 (x) d\mu_M(x)\\
&\le \sum_{i=1}^k \int_G |X_i f(x)|^2\,d\mu_M(x).
\end{aligned}$$

Notice that all the previous integrals are finite because of the support condition on f. Now, if f is as in Lemma 2.1, consider a nondecreasing sequence of smooth compactly supported functions χ_n satisfying

$$\mathbf{1}_{B(e,nR)} \leq \chi_n \leq 1 \text{ and } |X_i\chi_n| \leq 1 \text{ for all } 1 \leq i \leq k.$$

Applying (2.10) to $f\chi_n$ and letting n go to $+\infty$ yields the desired conclusion, by use of the monotone convergence theorem in the left-hand side and the dominated convergence theorem in the right-hand side. □

Let us now establish (1.4). Let g be a smooth function on G and let $f := g - c$ on G where c is a constant to be chosen. By assumption (1.3),

(2.11)

$$\int_G f^2(x)d\mu_M(x) \leq \int_G f^2(x)\frac{L_M W}{\theta W}(x)d\mu_M(x)+\int_{B(e,R)} f^2(x)\frac{b}{\theta W}(x)d\mu_M(x).$$

Lemma 2.1 shows that (2.10) holds. Let us now turn to the second term in the right-hand side of (2.11). Fix c such that $\int_{B(e,R)} f(x)d\mu_M(x) = 0$. By (2.9) applied to f on $B(e,R)$ and the fact that M is bounded from above and below on $B(e,R)$, one has

$$\int_{B(e,R)} f^2(x)d\mu_M(x) \leq CR^2\sum_{i=1}^k \int_{B(e,R)} |X_if(x)|^2\, d\mu_M(x)$$

where the constant C depends on R and M. Therefore, using the fact that $W \geq 1$ on G,

(2.12)

$$\int_{B(e,R)} f^2(x)\frac{b}{\theta W}(x)d\mu_M(x) \leq CR^2\sum_{i=1}^k \int_{B(e,R)} |X_if(x)|^2\, d\mu_M(x)$$

where the constant C depends on R, M, θ and b. Gathering (2.11), (2.10) and (2.12) yields

$$\int_G (g(x)-c)^2 d\mu_M(x) \leq C\sum_{i=1}^k \int_G |X_ig(x)|^2\, d\mu_M(x),$$

which easily implies (1.4) for the function g (and the same dependence for the constant C). □

Proof of Corollary 1.2: according to Theorem 1.1, it is enough to find a Lyapunov function W. Define

$$W(x) := e^{\gamma(v(x)-\inf_G v)}$$

where $\gamma > 0$ will be chosen later. Since

$$-L_M W(x) = \gamma \left(\sum_{i=1}^k X_i^2 v(x) - (1-\gamma) \sum_{i=1}^k \left| X_i v(x) \right|^2 \right) W(x),$$

W is a Lyapunov function for $\gamma := 1 - a$ because of the assumption on v. Indeed, one can take $\theta = c\gamma$ and $b = \max_{B(e,R)} \Big\{ - L_M W + \theta W \Big\}$ (recall that M is a C^2 function). □

Let us now prove Proposition 1.3. Observe first that, since v is C^2 on G and (1.6) holds, there exists $\alpha \in \mathbb{R}$ such that, for all $x \in G$,

$$\frac{1-\varepsilon}{2} \sum_{i=1}^k \left| X_i v(x) \right|^2 - \sum_{i=1}^k X_i^2 v(x) \geq \alpha. \tag{2.13}$$

Let f be as in the statement of Proposition 1.3 and let $g := f M^{\frac{1}{2}}$. Since, for all $1 \leq i \leq k$,

$$X_i f = M^{-\frac{1}{2}} X_i g - \frac{1}{2} g M^{-\frac{3}{2}} X_i M.$$

Assumption (2.13) yields two positive constants β, γ such that

$$\begin{aligned} \sum_{i=1}^k \int_G \left| X_i f(x) \right|^2 (x)\, d\mu_M(x) = \\ \sum_{i=1}^k \int_G \left(\left| X_i g(x) \right|^2 + \frac{1}{4} g^2(x) \left| X_i v(x) \right|^2 + g(x) X_i g(x) X_i v(x) \right) dx \\ = \sum_{i=1}^k \int_G \left(\left| X_i g(x) \right|^2 + \frac{1}{4} g^2(x) \left| X_i v(x) \right|^2 + \frac{1}{2} X_i \left(g^2 \right)(x) X_i v(x) \right) dx \\ \geq \sum_{i=1}^k \int_G g^2(x) \left(\frac{1}{4} \left| X_i v(x) \right|^2 - \frac{1}{2} X_i^2 v(x) \right) dx \\ \geq \sum_{i=1}^k \int_G f^2(x) \left(\beta \left| X_i v(x) \right|^2 - \gamma \right) d\mu_M(x). \end{aligned} \tag{2.14}$$

The conjunction of (1.4), which holds because of (1.6), and (2.14) yields the desired conclusion. □

3. Proof of Theorem 1.4

We divide the proof into several steps.

3.1. **Rewriting the improved Poincaré inequality.** By the definition of L_M, the conclusion of Proposition 1.3 means, in terms of operators in $L^2(G, d\mu_M)$, that, for some $\lambda > 0$,

$$L_M \geq \lambda \mu, \tag{3.15}$$

where μ is the multiplication operator by $1 + \sum_{i=1}^k |X_i v|^2$. Using a functional calculus argument (see [Dav80], p. 110), one deduces from (3.15) that, for any $\alpha \in (0, 2)$,

$$L_M^{\alpha/2} \geq \lambda^{\alpha/2} \mu^{\alpha/2}$$

which implies, thanks to the fact $L_M^{\alpha/2} = (L_M^{\alpha/4})^2$ and the symmetry of $L_M^{\alpha/4}$ on $L^2(G, d\mu_M)$, that

$$\int_G |f(x)|^2 \left(1 + \sum_{i=1}^k |X_i v(x)|^2\right)^{\alpha/2} d\mu_M(x) \leq$$

$$C \int_G \left| L_M^{\alpha/4} f(x) \right|^2 d\mu_M(x) = C \left\| L_M^{\alpha/4} f \right\|_{L^2(G, d\mu_M)}^2 .$$

The conclusion of Theorem 1.4 will follow by estimating the quantity $\left\| L^{\alpha/4} f \right\|_{L^2(G, d\mu_M)}^2$.

3.2. **Off-diagonal L^2 estimates for the resolvent of L_M.** The crucial estimates to derive the desired inequality are some L^2 "off-diagonal" estimates for the resolvent of L_M, in the spirit of [Gaf59] . This is the object of the following lemma.

Lemma 3.1. *There exists C with the following property: for all closed disjoint subsets $E, F \subset G$ with $d(E, F) =: d > 0$, all function $f \in L^2(G, d\mu_M)$ supported in E and all $t > 0$,*

$$\left\| (I + t\, L_M)^{-1} f \right\|_{L^2(F, d\mu_M)} + \left\| t\, L_M (I + t\, L_M)^{-1} f \right\|_{L^2(F, d\mu_M)} \leq$$

$$8\, e^{-C \frac{d}{\sqrt{t}}} \left\| f \right\|_{L^2(E, d\mu_M)} .$$

Proof. We argue as in [AHL$^+$02], Lemma 1.1. From the fact that L_M is self-adjoint on $L^2(G, d\mu_M)$ we have

$$\|(L_M - \mu)^{-1}\|_{L^2(G, d\mu_M)} \leq \frac{1}{\operatorname{dist}(\mu, \Sigma(L_M))}$$

where $\Sigma(L_M)$ denotes the spectrum of L_M, and $\mu \notin \Sigma(L_M)$. Then we deduce that $(\mathrm{I} + t\, L_M)^{-1}$ is bounded with norm less than 1 for all $t > 0$, and it is clearly enough to argue when $0 < t < d$.

In the following computations, we will make explicit the dependence of the measure $d\mu_M$ in terms of M for sake of clarity. Define $u_t = (\mathrm{I} + t\,L_M)^{-1} f$, so that, for all function $v \in H^1(G, d\mu_M)$,

$$\begin{aligned} \int_G u_t(x)\, v(x)\, M(x)\, dx + \\ t \sum_{i=1}^k \int_G X_i u_t(x) \cdot X_i v(x)\, M(x)\, dx = \\ \int_G f(x)\, v(x)\, M(x)\, dx. \end{aligned} \tag{3.16}$$

Fix now a nonnegative function $\eta \in \mathcal{D}(G)$ vanishing on E. Since f is supported in E, applying (3.16) with $v = \eta^2\, u_t$ (remember that $u_t \in H^1(G, d\mu_M)$) yields

$$\int_G \eta^2(x)\, |u_t(x)|^2\, M(x)\, dx + t \sum_{i=1}^k \int_G X_i u_t(x) \cdot X_i(\eta^2 u_t)\, M(x)\, dx = 0,$$

which implies

$$\begin{aligned} \int_G \eta^2(x)\, |u_t(x)|^2\, M(x)\, dx + t \int_G \eta^2(x) \sum_{i=1}^k |X_i u_t(x)|^2\, M(x)\, dx \\ = -2\,t \sum_{i=1}^k \int_G \eta(x)\, u_t(x)\, X_i\eta(x) \cdot X_i u_t(x)\, M(x)\, dx \\ \leq t \int_G |u_t(x)|^2 \sum_{i=1}^k |X_i\eta(x)|^2\, M(x)\, dx + \\ t \int_G \eta^2(x) \sum_{i=1}^k |X_i u_t(x)|^2\, M(x)\, dx, \end{aligned}$$

hence

$$\int_G \eta^2(x)\, |u_t(x)|^2\, M(x)\, dx \leq t \int_G |u_t(x)|^2 \sum_{i=1}^k |X_i\eta(x)|^2\, M(x)\, dx. \tag{3.17}$$

Let ζ be a nonnegative smooth function on G such that $\zeta = 0$ on E, so that $\eta := e^{\alpha\,\zeta} - 1 \geq 0$ and η vanishes on E for some $\alpha > 0$ to be chosen. Choosing this particular η in (3.17) with $\alpha > 0$ gives

$$\int_G \left| e^{\alpha\,\zeta(x)} - 1 \right|^2 |u_t(x)|^2\, M(x)\, dx \leq$$

$$\alpha^2\, t \int_G |u_t(x)|^2 \sum_{i=1}^k |X_i\zeta(x)|^2\; e^{2\,\alpha\,\zeta(x)}\, M(x)\, dx.$$

Taking $\alpha = 1/(2\sqrt{t}\ \max_i \|X_i\zeta\|_\infty)$, one obtains

$$\int_G \left|e^{\alpha\,\zeta(x)} - 1\right|^2 |u_t(x)|^2\; M(x)\, dx \leq \frac{1}{4}\int_G |u_t(x)|^2\, e^{2\,\alpha\,\zeta(x)}\, M(x)\, dx.$$

Using the fact that the norm of $(I+tL_M)^{-1}$ is bounded by 1 uniformly in $t>0$, this gives

$$\begin{aligned}\left\|e^{\alpha\zeta}\, u_t\right\|_{L^2(G,d\mu_M)} &\leq \left\|\left(e^{\alpha\zeta} - 1\right)\, u_t\right\|_{L^2(G,d\mu_M)} + \|u_t\|_{L^2(G,d\mu_M)}\\ &\leq \frac{1}{2}\left\|e^{\alpha\zeta}\, u_t\right\|_{L^2(G,d\mu_M)} + \|f\|_{L^2(G,d\mu_M)},\end{aligned}$$

therefore

$$\int_G \left|e^{\alpha\,\zeta(x)}\right|^2 |u_t(x)|^2\; M(x)\, dx \;\leq\; 4\int_G |f(x)|^2\; M(x)\, dx.$$

We choose now ζ such that $\zeta = 0$ on E as before and additionnally that $\zeta = 1$ on F. It can furthermore be chosen with $\max_{i=1,\ldots k}\|X_i\zeta\|_\infty \leq C/d$, which yields the desired conclusion for the L^2 norm of $(I + tL_M)^{-1}f$ with a factor 4 in the right-hand side. Since $t\,L_M(\mathrm{I}+t\,L_M)^{-1}f = f - (\mathrm{I} + t\,L_M)^{-1}f$, the desired inequality with a factor 8 readily follows. $\square$

3.3. **Control of $\left\|L_M^{\alpha/4}f\right\|_{L^2(G,d\mu_M)}$ and conclusion of the proof of Theorem 1.4.** This is now the heart of the proof to reach the conclusion of Theorem 1.4. The following first lemma is a standard quadratic estimate on powers of subelliptic operators. It is based on spectral theory.

Lemma 3.2. *Let $\alpha \in (0,2)$. There exists $C>0$ such that, for all $f \in \mathcal{D}(L_M)$,*

(3.18)

$$\left\|L_M^{\alpha/4}f\right\|^2_{L^2(G,d\mu_M)} \leq C_3 \int_0^{+\infty} t^{-1-\alpha/2}\left\|t\,L_M\,(I+t\,L_M)^{-1}f\right\|^2_{L^2(G,d\mu_M)}\, dt.$$

We now come to the desired estimate.

Lemma 3.3. *Let $\alpha \in (0,2)$. There exists $C>0$ such that, for all $f \in \mathcal{D}(G)$,*

$$\int_0^\infty t^{-1-\alpha/2}\left\|t\,L_M\,(I+t\,L_M)^{-1}f\right\|^2_{L^2(G,d\mu_M)}\, dt \leq$$

$$C\iint_{G\times G} \frac{|f(x)-f(y)|^2}{V\left(|y^{-1}x|\right)|y^{-1}x|^\alpha}\, M(x)\; dx\, dy.$$

Proof. Fix $t \in (0, +\infty)$. Following Lemma 3.2, we give an upper bound of

$$\left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} f \right\|^2_{L^2(G,d\mu_M)}$$

involving first order differences for f. Using (1.1), one can pick up a countable family x_j^t, $j \in \mathbb{N}$, such that the balls $B\left(x_j^t, \sqrt{t}\right)$ are pairwise disjoint and

$$G = \bigcup_{j \in \mathbb{N}} B\left(x_j^t, 2\sqrt{t}\right). \tag{3.19}$$

By Lemma 4.1 in Appendix A, there exists a constant $\widetilde{C} > 0$ such that for all $\theta > 1$ and all $x \in G$, there are at most $\widetilde{C}\,\theta^{2\kappa}$ indexes j such that $|x^{-1}x_j^t| \leq \theta\sqrt{t}$ where κ is given by (1.2).

For fixed j, one has

$$t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} f = t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} g^{j,t}$$

where, for all $x \in G$,

$$g^{j,t}(x) := f(x) - m^{j,t}$$

and $m^{j,t}$ is defined by

$$m^{j,t} := \frac{1}{V\left(2\sqrt{t}\right)} \int_{B\left(x_j^t, 2\sqrt{t}\right)} f(y)dy$$

Note that, here, the mean value of f is computed with respect to the Haar measure on G. Since (3.19) holds, one clearly has

$$\begin{aligned}\left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} f \right\|^2_{L^2(G,d\mu_M)} &\leq \sum_{j \in \mathbb{N}} \left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} f \right\|^2_{L^2\left(B(x_j^t, 2\sqrt{t}), d\mu_M\right)} \\ &= \sum_{j \in \mathbb{N}} \left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} g^{j,t} \right\|^2_{L^2\left(B\left(x_j^t, 2\sqrt{t}\right), d\mu_M\right)},\end{aligned}$$

and we are left with the task of estimating

$$\left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} g^{j,t} \right\|^2_{L^2\left(B\left(x_j^t, 2\sqrt{t}\right), d\mu_M\right)}.$$

To that purpose, set

$$C_0^{j,t} = B\left(x_j^t, 4\sqrt{t}\right) \quad \text{and} \quad C_k^{j,t} = B\left(x_j^t, 2^{k+2}\sqrt{t}\right) \backslash B\left(x_j^t, 2^{k+1}\sqrt{t}\right), \; \forall\, k \geq 1,$$

and $g_k^{j,t} := g^{j,t}\, \mathbf{1}_{C_k^{j,t}}$, $k \geq 0$, where, for any subset $A \subset G$, $\mathbf{1}_A$ is the usual characteristic function of A. Since $g^{j,t} = \sum_{k \geq 0} g_k^{j,t}$ one has

$$\begin{aligned}&\left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} g^{j,t} \right\|_{L^2\left(B\left(x_j^t, 2\sqrt{t}\right), d\mu_M\right)} \leq \\ &\sum_{k \geq 0} \left\| t\, L_M \left(\mathrm{I} + t\, L_M\right)^{-1} g_k^{j,t} \right\|_{L^2\left(B\left(x_j^t, 2\sqrt{t}\right), d\mu_M\right)}\end{aligned} \tag{3.20}$$

and, using Lemma 3.1, one obtains (for some constants $C, c > 0$)

$$\begin{gathered} \left\| t\, L_M\, (\mathrm{I} + t\, L_M)^{-1} g^{j,t} \right\|_{L^2\left(B\left(x_j^t, 2\sqrt{t}\right), d\mu_M\right)} \leq \\ C \left(\left\| g_0^{j,t} \right\|_{L^2(C_0^{j,t}, d\mu_M)} + \sum_{k\geq 1} e^{-c\, 2^k} \left\| g_k^{j,t} \right\|_{L^2(C_k^{j,t}, d\mu_M)} \right). \end{gathered} \tag{3.21}$$

By Cauchy-Schwarz's inequality, we deduce (for another constant $C' > 0$)

$$\begin{gathered} \left\| t\, L_M\, (\mathrm{I} + t\, L_M)^{-1} g^{j,t} \right\|^2_{L^2\left(B\left(x_j^t, 2\sqrt{t}\right), d\mu_M\right)} \leq \\ C' \left(\left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t}, d\mu_M)} + \sum_{k\geq 1} e^{-c\, 2^k} \left\| g_k^{j,t} \right\|^2_{L^2(C_k^{j,t}, d\mu_M)} \right). \end{gathered} \tag{3.22}$$

As a consequence, we have

$$\begin{aligned} &\int_0^\infty t^{-1-\alpha/2} \left\| t\, L_M\, (\mathrm{I} + t\, L_M)^{-1} f \right\|^2_{L^2(G, d\mu_M)}\, dt \leq \\ &C' \int_0^\infty t^{-1-\alpha/2} \sum_{j\geq 0} \left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t}, d\mu_M)}\, dt + \\ &C' \int_0^\infty t^{-1-\alpha/2} \sum_{k\geq 1} e^{-c\, 2^k} \sum_{j\geq 0} \left\| g_k^{j,t} \right\|^2_{L^2(C_k^{j,t}, d\mu_M)}\, dt. \end{aligned} \tag{3.23}$$

We claim that, and we pospone the proof into Appendix B:

Lemma 3.4. *There exists $\bar{C} > 0$ such that, for all $t > 0$ and all $j \in \mathbb{N}$:*

A. *For the first term:*

$$\left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t}, M)} \leq \frac{\bar{C}}{V(\sqrt{t})} \int_{B\left(x_j^t, 4\sqrt{t}\right)} \int_{B\left(x_j^t, 4\sqrt{t}\right)} |f(x) - f(y)|^2\, d\mu_M(x)\, dy.$$

B. *For all $k \geq 1$,*

$$\left\| g_k^{j,t} \right\|^2_{L^2(C_k^{j,t}, d\mu_M)} \leq$$

$$\frac{\bar{C}}{V(2^k\sqrt{t})} \int_{x\in B(x_j^t, 2^{k+2}\sqrt{t})} \int_{y\in B(x_j^t, 2^{k+2}\sqrt{t})} |f(x) - f(y)|^2\, d\mu_M(x)\, dy.$$

We finish the proof of the theorem. Using Assertion **A** in Lemma 3.4, summing up on $j \geq 0$ and integrating over $(0,\infty)$, we get

$$\int_0^\infty t^{-1-\alpha/2} \sum_{j\geq 0} \left\| g_0^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, d\mu_M\right)} dt = \sum_{j\geq 0} \int_0^\infty t^{-1-\alpha/2} \left\| g_0^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, d\mu_M\right)} dt$$

$$\leq \bar{C} \sum_{j\geq 0} \int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} \left(\int_{B\left(x_j^t, 4\sqrt{t}\right)} \int_{B\left(x_j^t, 4\sqrt{t}\right)} |f(x)-f(y)|^2 \, d\mu_M(x)\, dy \right) dt$$

$$\leq \bar{C} \sum_{j\geq 0} \iint_{(x,y)\in G\times G} |f(x)-f(y)|^2 M(x) \times$$

$$\left(\int_{t\geq \max\left\{ \frac{\left|x^{-1}x_j^t\right|^2}{16}\, ;\, \frac{\left|y^{-1}x_j^t\right|^2}{16} \right\}} \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} dt \right) dx\, dy.$$

The Fubini theorem now shows

$$\sum_{j\geq 0} \int_{t\geq \max\left\{ \frac{\left|x^{-1}x_j^t\right|^2}{16}\, ;\, \frac{\left|y^{-1}x_j^t\right|^2}{16} \right\}} \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} dt =$$

$$\int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})} \sum_{j\geq 0} \mathbf{1}_{\left(\max\left\{ \frac{\left|x^{-1}x_j^t\right|^2}{16}\, ;\, \frac{\left|y^{-1}x_j^t\right|^2}{16} \right\}, +\infty \right)}(t)\, dt.$$

Observe that, by Lemma 4.1, there is a constant $N \in \mathbb{N}$ such that, for all $t>0$, there are at most N indexes j such that $\left|x^{-1}x_j^t\right|^2 < 16\,t$ and $\left|y^{-1}x_j^t\right|^2 < 16\,t$, and for these indexes j, one has $|x^{-1}y| < 8\sqrt{t}$. It therefore follows that

$$\sum_{j\geq 0} \mathbf{1}_{\left(\max\left\{ \frac{\left|x^{-1}x_j^t\right|^2}{16}\, ;\, \frac{\left|y^{-1}x_j^t\right|^2}{16} \right\}, +\infty \right)}(t) \leq N\, \mathbf{1}_{\left(|x^{-1}y|^2/64, +\infty\right)}(t),$$

so that, by (1.1),

$$\int_0^\infty t^{-1-\alpha/2} \sum_j \left\| g_0^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, d\mu_M\right)} dt \tag{3.24}$$

$$\leq \bar{C}\, N \iint_{G\times G} |f(x)-f(y)|^2 M(x) \left(\int_{|x^{-1}y|^2/64}^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(\sqrt{t})}\, dt \right) dx\, dy$$

$$\leq \bar{C}\, N \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V\left(|x^{-1}y|\right) |x^{-1}y|^\alpha} \, d\mu_M(x)\, dy.$$

Using now Assertion **B** in Lemma 3.4, we obtain, for all $j \geq 0$ and all $k \geq 1$,

$$
\begin{aligned}
&\int_0^\infty t^{-1-\alpha/2} \sum_{j\geq 0} \left\| g_k^{j,t} \right\|_2^2 dt \\
&\leq \bar{C} \sum_{j\geq 0} \int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V\left(2^k\sqrt{t}\right)} \left(\iint_{B(x_j^t, 2^{k+2}\sqrt{t}) \times B(x_j^t, 2^{k+2}\sqrt{t})} \left| f(x) - f(y) \right|^2 \, M(x)\, dx\, dy \right) dt \\
&\leq \bar{C} \sum_{j\geq 0} \iint_{x,y\in G} \left| f(x) - f(y) \right|^2 \, M(x) \times \\
&\left(\int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(2^k\sqrt{t})} \mathbf{1}_{\left(\max\left\{ \frac{\left|x^{-1}x_j^t\right|^2}{4^{k+2}}, \frac{\left|y^{-1}x_j^t\right|^2}{4^{k+2}} \right\}, +\infty\right)} (t)\, dt \right) dx\, dy.
\end{aligned}
$$

But, given $t > 0$, $x, y \in G$, by Lemma 4.1 again, there exist at most $\widetilde{C}\, 2^{2k\kappa}$ indexes j such that

$$
\left| x^{-1} x_j^t \right| \leq 2^{k+2}\sqrt{t} \ \text{ and } \ \left| y^{-1} x_j^t \right| \leq 2^{k+2}\sqrt{t},
$$

and for these indexes j, $|x^{-1}y| \leq 2^{k+3}\sqrt{t}$. As a consequence,

$$
\begin{aligned}
&\int_0^\infty \frac{t^{-1-\frac{\alpha}{2}}}{V(2^k\sqrt{t})} \sum_{j\geq 0} \mathbf{1}_{\left(\max\left\{ \frac{\left|x^{-1}x_j^t\right|^2}{4^{k+2}}, \frac{\left|x^{-1}x_j^t\right|^2}{4^{k+2}} \right\}, +\infty\right)} (t)\, dt \leq \\
&\widetilde{C}\, 2^{2k\kappa} \int_{t \geq \frac{|x^{-1}y|^2}{4^{k+3}}} \frac{t^{-1-\frac{\alpha}{2}}}{V(2^k\sqrt{t})} dt \leq \\
&\widetilde{C}' \frac{2^{k(2\kappa+\alpha)}}{V\left(|x^{-1}y|\right) |x^{-1}y|^\alpha},
\end{aligned}
\tag{3.25}
$$

for some other constant $\widetilde{C}' > 0$, and therefore

$$
\int_0^\infty \frac{t^{-1-\alpha/2}}{V\left(2^k\sqrt{t}\right)} \sum_j \left\| g_k^{j,t} \right\|^2_{L^2\left(C_0^{j,t}, d\mu_M\right)} dt \leq
$$

$$
\bar{C}\, \widetilde{C}'\, 2^{k(2\kappa+\alpha)} \iint_{G\times G} \frac{\left| f(x) - f(y) \right|^2}{V\left(|x^{-1}y|\right) |x^{-1}y|^\alpha} M(x)\, dx\, dy.
$$

We can now conclude the proof of Lemma 3.3, using Lemma 3.2, (3.21), (3.24) and (3.25). We have proved, by reconsidering (3.23):

$$\int_0^\infty t^{-1-\alpha/2} \left\| t\, L_M\, (\mathrm{I} + t\, L_M)^{-1} f \right\|^2_{L^2(G, d\mu_M)}\, dt \leq \tag{3.26}$$

$$C'\, \bar{C}\, N \iint_{G\times G} \frac{|f(x) - f(y)|^2}{V\left(|x^{-1}y|\right) |x-y|^\alpha}\, M(x)\, dx\, dy$$

$$+ \sum_{k\geq 1} C'\, \bar{C}\, \widetilde{C}'\, 2^{k(2\kappa+\alpha)}\, e^{-c\, 2^k} \iint_{G\times G} \frac{|f(x) - f(y)|^2}{V\left(|x^{-1}y|\right) |x^{-1}y|^\alpha}\, M(x)\, dx\, dy$$

and we deduce that

$$\int_0^\infty t^{-1-\alpha/2} \left\| t\, L_M\, (\mathrm{I} + t\, L_M)^{-1} f \right\|^2_{L^2(G, d\mu_M)}\, dt \leq$$

$$C \iint_{G\times G} \frac{|f(x) - f(y)|^2}{V\left(|x^{-1}y|\right) |x^{-1}y|^\alpha}\, d\mu_M(x)\, dy$$

for some constant C as claimed in the statement. □

Remark 3.5. *In the Euclidean context, Strichartz proved in (*[Str67]*) that, when* $0 < \alpha < 2$, *for all* $p \in (1, +\infty)$,

$$\left\| (-\Delta)^{\alpha/4} f \right\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,p} \left\| S_\alpha f \right\|_{L^p(\mathbb{R}^n)} \tag{3.27}$$

where

$$S_\alpha f(x) = \left(\int_0^{+\infty} \left(\int_B |f(x+ry) - f(x)|\, dy \right)^2 \frac{dr}{r^{1+\alpha}} \right)^{\frac{1}{2}},$$

and also ([Ste61]*)*

$$\left\| (-\Delta)^{\alpha/4} f \right\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha,p} \left\| D_\alpha f \right\|_{L^p(\mathbb{R}^n)} \tag{3.28}$$

where

$$D_\alpha f(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x+y) - f(x)|^2}{|y|^{n+\alpha}} dy \right)^{\frac{1}{2}}.$$

In [CRTN01]*, these inequalities were extended to the setting of a unimodular Lie group endowed with a sub-laplacian* Δ*, relying on semigroups techniques and Littlewood-Paley-Stein functionals. In particular, in* [CRTN01]*, the authors use pointwise estimates of the kernel of the semigroup generated by* Δ*. In the present paper, we deal with the operator* L_M *for which these pointwise estimates are not available, but it turns out that* L^2 *off-diagonal estimates are enough for our purpose. Note that we do not obtain* L^p *inequalities here.*

4. Appendix A: Technical lemma

We prove the following lemma.

Lemma 4.1. *Let G and the x_j^t be as in the proof of Lemma 3.3 . Then there exists a constant $\widetilde{C} > 0$ with the following property: for all $\theta > 1$ and all $x \in G$, there are at most $\widetilde{C}\,\theta^{2\kappa}$ indexes j such that $\left|x^{-1}x_j^t\right| \leq \theta\sqrt{t}$.*

Proof of Lemma 4.1. The argument is very simple (see [Kan85]) and we give it for the sake of completeness. Let $x \in G$ and denote

$$I(x) := \left\{ j \in \mathbb{N}\,;\ \left|x^{-1}x_j^t\right| \leq \theta\sqrt{t} \right\}.$$

Since, for all $j \in I(x)$

$$B\left(x_j^t, \sqrt{t}\right) \subset B\left(x, (1+\theta)\sqrt{t}\right),$$

and

$$B\left(x, \sqrt{t}\right) \subset B\left(x_j^t, (1+\theta)\sqrt{t}\right),$$

one has by (1.2) and the fact that the balls $B\left(x_j^t, \sqrt{t}\right)$ are pairwise disjoint,

$$\begin{aligned} |I(x)|\, V\left(x, \sqrt{t}\right) &\leq \sum_{j \in I(x)} V\left(x_j^t, (1+\theta)\sqrt{t}\right) \\ &\leq C(1+\theta)^\kappa \sum_{j \in I(x)} V\left(x_j^t, \sqrt{t}\right) \\ &\leq C(1+\theta)^\kappa V\left(x, (1+\theta)\sqrt{t}\right) \\ &\leq C(1+\theta)^{2\kappa} V\left(x, \sqrt{t}\right) \end{aligned}$$

and we get the desired conclusion. □

5. Appendix B: Estimates for g_j^t

We prove Lemma 3.4. For all $x \in G$,

$$\begin{aligned} g_0^{j,t}(x) &= f(x) - \frac{1}{V(2\sqrt{t})} \int_{B\left(x_j^t, 2\sqrt{t}\right)} f(y)\, dy \\ &= \frac{1}{V(2\sqrt{t})} \int_{B\left(x_j^t, 2\sqrt{t}\right)} (f(x) - f(y))\, dy. \end{aligned}$$

By Cauchy-Schwarz inequality and (1.1), it follows that

$$\left|g_0^{j,t}(x)\right|^2 \leq \frac{C}{V(\sqrt{t})} \int_{B\left(x_j^t, 4\sqrt{t}\right)} |f(x) - f(y)|^2\, dy.$$

Therefore,

$$\left\| g_0^{j,t} \right\|^2_{L^2(C_0^{j,t},M)} \leq \frac{C}{V(\sqrt{t})} \int_{B\left(x_j^t, 4\sqrt{t}\right)} \int_{B\left(x_j^t, 4\sqrt{t}\right)} \left| f(x) - f(y) \right|^2 \, d\mu_M(x) \, dy,$$

which shows Assertion **A**. We argue similarly for Assertion **B** and obtain

$$\left\| g_k^{j,t} \right\|^2_{L^2(C_k^{j,t},M)} \leq \frac{C}{V(2^k\sqrt{t})} \int_{x \in B(x_j^t, 2^{k+2}\sqrt{t})} \int_{y \in B(x_j^t, 2^{k+2}\sqrt{t})} \left| f(x) - f(y) \right|^2 \, d\mu_M(x) \, dy,$$

which ends the proof.

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