

**TIME-SPLITTING SCHEMES AND MEASURE SOURCE TERMS
FOR A QUASILINEAR RELAXING SYSTEM**

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Several singular limits are investigated in the context of a 2×2 system arising for instance in the modeling of chromatographic processes. In particular, we focus on the case where the relaxation term and a L^2 projection operator are concentrated on a discrete lattice by means of Dirac measures. This formulation allows to study more easily some time-splitting numerical schemes.

Keywords: relaxation schemes, nonconservative products, conservation laws.

1. Introduction

Relaxation is a phenomenon which appears in a wide variety of physical situations. In gas dynamics, it occurs when the gas isn't in thermodynamic equilibrium. In elasticity, it is usually referred to as a fading memory mechanism. Hyperbolic conservation laws with relaxation are also served as discrete kinetic models. Relaxation approximations to scalar conservation laws can be considered for purely numerical purposes too; general references are *e.g.* ^{9,11,13,14,15,10,17,16,22,23,26,32}.

In ²³, T.P. Liu studied the following 2×2 simple model which captures the basic features of this physical process:

$$\begin{cases} \partial_t u + \partial_x f(u, v) = 0, \\ \partial_t v + \partial_x g(u, v) = \frac{1}{\varepsilon}(V(u) - v), \end{cases} \quad x \in \mathbb{R}, t > 0 \text{ with } \partial_v f(u, v) < 0. \quad (1.1)$$

The numerical issue in such a system is to handle properly the relaxing process in its infinite strength limit $\varepsilon \rightarrow 0$ on coarse computational grids, see ^{27,12}. This requirement is twofold as it includes some numerical stability properties and a consistency with the expected asymptotic behavior in the context of weak solutions. In order to derive numerical schemes meeting these criteria, attention has been driven onto several kinds of efficient treatments for the relaxation term: see ^{1,8,9,15,25}. Indeed, robust and efficient approximations can be derived based on "time-splitting" algorithms where one alternates at every time-step the resolution of the purely convective part and the handling of the reaction process by any O.D.E. solver.

It can be sometimes shown rigorously that these numerical schemes converge using *e.g.* the BV compactness framework together with appropriate entropy consistency properties. Our main goal here is to complete such a study using slightly different arguments. Roughly speaking, we are about to consider both the projection and the reaction term as being “localized” on a lattice; *formally*, this amounts to going from (1.1) to

$$\begin{cases} \partial_t u + \partial_x f(u, v) = \sum_{n \in \mathbb{N}_*} (\mathcal{P}^h(u) - u) \delta(t - n\Delta t), \\ \partial_t v + \partial_x g(u, v) = \sum_{n \in \mathbb{N}_*} \left(\frac{\Delta t}{\varepsilon} (V(u) - v) + (\mathcal{P}^h(v) - v) \right) \delta(t - n\Delta t), \end{cases} \quad (1.2)$$

where δ stands for the Dirac measure in $t = 0$, $\Delta t > 0$ is a given time-step and \mathcal{P}^h denotes the standard L^2 projection operator on piecewise constant functions related to a mesh-size $h > 0$. Indeed, this new way to handle singular source terms in hyperbolic equations can be found in ^{2,7,8,33} in the context of kinetic equations or semilinear relaxation to the one-dimensional conservation law. Such an approach based on “measure source terms” allows in particular to recover a fix proposed in ¹⁰ to suppress spurious behavior in the stiff regime. Indeed, the idea proposed in ¹⁰ for reactive chemical flows consists in switching the order in which the reaction and projection steps appear inside the Dirac masses, as we shall see in §3.2, §3.3.

In this paper, we plan to work out only the simpler case of a quasilinear 2×2 system with relaxation previously considered in ^{27,32,19}, see (2.3). We shall mainly rely on an extensive use of some *nonconservative products*, ^{5,21,29}, in order to give a *rigorous* definition of the right-hand side, see Propositions 1 and 2. In return, our formulation entails direct stability and convergence proofs exploiting mainly estimates easily proved at the level of the original differential problem; this allows to bypass some lengthy computations.

Hence, this work is organized as follows: in §2, we establish some compactness properties and study the nonconservative singular system obtained as the relaxation and projection terms are concentrated inside Dirac masses, see (2.23). We introduce a notion of *entropy solution* well suited for this kind of problems, see Definition 2. Then, in §3, we present some applications to two types of time-splitting numerical schemes for which we prove convergence towards the “equilibrium equation” as the relaxation parameter diverges, see Theorems 4 and 5.

2. Study of a singular relaxing system

2.1. A quasilinear 2×2 model

We are interested in the Cauchy problem for the following balance laws:

$$\begin{cases} \partial_t(u + v) + \partial_x f(u) = 0, \\ \partial_t v = \mu(A(u) - v), \\ u(., 0) = u_0, v(., 0) = v_0. \end{cases} \quad x \in \mathbb{R}, t > 0 \quad (2.3)$$

This system arises *e.g.* in chromatography, ^{11,26,30}. In this context, $u, v \in [0, 1]$ stand for some species densities, one contained in a fluid flowing through a fixed bed and the other being absorbed by the material on the bed. This adsorption process is modeled by the right-hand side $\mu R(u, v) = \mu(A(u) - v)$, $\mu \gg 1$ where $1/\mu$ is the relaxation time. This model has been extensively studied within the theory of BV functions, ^{6,34}, by the authors of ^{31,32} (see also ^{24,19,26,27}) under the following hypotheses:

$$\begin{cases} f(0) = 0, f'(u) \geq 0; \\ A(0) = 0, A(1) = 1, A'(u) \geq 0. \end{cases} \quad (2.4)$$

We recall here some of their main results.

Definition 1 We say that the pair (u, v) is an **entropy solution** of (2.3) if:

- (i) $u(\cdot, t)$ and $v(\cdot, t)$ lie in $L^1 \cap BV(\mathbb{R})$ for any $t \in \mathbb{R}^+$ and satisfy (2.3) in the sense of distributions;
- (ii) there exists $M \in \mathbb{R}^+$ such that for any $t \geq 0, s \geq 0$:

$$\|u(\cdot, t) - u(\cdot, s)\|_{L^1(\mathbb{R})} + \|v(\cdot, t) - v(\cdot, s)\|_{L^1(\mathbb{R})} \leq M|t - s|;$$

- (iii) there holds for any $k, l \in \mathbb{R}$ and any nonnegative test-function in $\mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$:

$$\begin{aligned} & \partial_t \left[|u - k| + |v - l| \right] + \partial_x \left[f(u) - f(k) \right] \\ & \leq \mu R(u, v) \left[\text{sgn}(v - l) - \text{sgn}(u - k) \right]. \end{aligned} \quad (2.5)$$

The relevance of this notion comes from the forthcoming theorems whose detailed proofs are to be found in ³². We stress however that a more general definition of entropy solution to (2.3) has been proposed within the class of L^∞ functions, ²⁶.

Theorem 1 Let $u_0, v_0 \in L^1 \cap BV(\mathbb{R})$; under the hypotheses (2.4), there exists a unique entropy solution to (2.3) which satisfies furthermore for any $t \in \mathbb{R}^+$:

- (i) $\mu \|R(u, v)(\cdot, t)\|_{L^1(\mathbb{R})} \leq M$ for some $M \in \mathbb{R}^+$;
- (ii) Given another set of initial data \tilde{u}_0, \tilde{v}_0 in $L^1 \cap BV(\mathbb{R})$, there holds:

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^1(\mathbb{R})} + \|v(\cdot, t) - \tilde{v}(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0 - \tilde{u}_0\|_{L^1(\mathbb{R})} + \|v_0 - \tilde{v}_0\|_{L^1(\mathbb{R})}.$$

The theory of Kruřkov, ¹⁸, ensures that there exists a unique entropy solution w to the following ‘‘equilibrium’’ equation since $A' \geq 0$:

$$\begin{cases} \partial_t(w + A(w)) + \partial_x f(w) = 0, & x \in \mathbb{R}, t > 0 \\ w(\cdot, 0) = w_0 \in L^1 \cap BV(\mathbb{R}). \end{cases} \quad (2.6)$$

Theorem 2 Under the assumptions of Theorem 1 and if $w_0 = u_0$, there exists a constant $M' \in \mathbb{R}^+$ such that:

$$\|u(\cdot, t) - w(\cdot, t)\|_{L^1(\mathbb{R})} \leq \frac{M'}{\mu^{\frac{1}{3}}}, \quad t \in \mathbb{R}^+.$$

2.2. Introduction of a “localized” singular system

In all the sequel, Δt and h will stand for fixed positive parameters related to some cartesian computational discretization.

- We first introduce a projector onto piecewise constant functions; for any $j \in \mathbb{Z}$, we denote $C_j \stackrel{\text{def}}{=} [(j - \frac{1}{2})h, (j + \frac{1}{2})h[$. Then, we define:

$$\begin{aligned} \mathcal{P}^h : L^1 \cap BV(\mathbb{R}) &\rightarrow L^1 \cap BV(\mathbb{R}) \\ \varphi &\mapsto \left(\frac{1}{h} \int_{C_j} \varphi(x).dx \right)_{j \in \mathbb{Z}} \end{aligned} \quad (2.7)$$

- Let $I^n \stackrel{\text{def}}{=} [n\Delta t, (n+1)\Delta t[$; we define two functions which are Lipschitz continuous with respect to time for $\epsilon > 0$ small enough:

$$\mathbb{R}^+ \ni t \mapsto a^\epsilon(t) = \begin{cases} (n-1)\Delta t, & t \in [(n-1)\Delta t, n\Delta t - \epsilon[\\ n\Delta t + \left(\frac{t - n\Delta t}{\epsilon} \right) \Delta t, & t \in [n\Delta t - \epsilon, n\Delta t[\end{cases} \quad (2.8)$$

and:

$$\mathbb{R}^+ \ni t \mapsto b^\epsilon(t) = \begin{cases} n\Delta t + \left(\frac{t - n\Delta t}{\epsilon} \right) \Delta t, & t \in [n\Delta t, n\Delta t + \epsilon[\\ (n+1)\Delta t, & t \in [n\Delta t + \epsilon, (n+1)\Delta t[\end{cases} \quad (2.9)$$

We can therefore consider the following system for any $\epsilon > 0$, μ, ν in \mathbb{R}^+ within the theory proposed in ³²:

$$\begin{cases} \partial_t u^\epsilon + \partial_x f(u^\epsilon) = \nu(\mathcal{P}^h(u^\epsilon) - u^\epsilon)\partial_t a^\epsilon - \mu R(u^\epsilon, v^\epsilon)\partial_t b^\epsilon, \\ \partial_t v^\epsilon = \nu(\mathcal{P}^h(v^\epsilon) - v^\epsilon)\partial_t a^\epsilon + \mu R(u^\epsilon, v^\epsilon)\partial_t b^\epsilon, \\ u^\epsilon(\cdot, 0) = u_0, v^\epsilon(\cdot, 0) = v_0. \end{cases} \quad x \in \mathbb{R}, t > 0 \quad (2.10)$$

In the limit $\epsilon \rightarrow 0$, both a^ϵ and b^ϵ converge almost everywhere towards the piecewise constant function $t \mapsto \Delta t \sum_{n \in \mathbb{N}_*} Y(t - n\Delta t)$, where Y stands for the classical Heaviside distribution.

We give at once a relaxation estimate for (2.10) which is uniform in ϵ .

Lemma 1 *Under the assumptions of Theorem 1, the entropy solution of (2.10) satisfies the relaxation estimate $\mu \|R(u^\epsilon, v^\epsilon)\partial_t b^\epsilon\|_{\mathcal{M}_{loc}(\mathbb{R} \times \mathbb{R}_+^*)} \leq \bar{C}$ for some $\bar{C} \in \mathbb{R}^+$.*

Proof. We follow the classical ideas from ^{17,26,32} introducing a special entropy/entropy-flux pair for (2.10):

$$\eta(u^\epsilon, v^\epsilon) = \frac{(u^\epsilon)^2}{2} + H(v^\epsilon), \quad H(v^\epsilon) = \int_0^{v^\epsilon} A^{-1}(\xi).d\xi, \quad q(u^\epsilon) = \int^{u^\epsilon} \xi.f'(\xi).d\xi.$$

We observe that under the hypotheses (2.4), the function H is strictly convex. Therefore, for any positive test-function, the following inequality holds:

$$\begin{aligned} \partial_t \eta(u^\epsilon, v^\epsilon) + \partial_x q(u^\epsilon) &\leq -\mu(A(u^\epsilon) - v^\epsilon)(u^\epsilon - A^{-1}(v^\epsilon)) \partial_t b^\epsilon \\ &+ \nu \left((\mathcal{P}^h(u^\epsilon) - u^\epsilon) u^\epsilon + (\mathcal{P}^h(v^\epsilon) - v^\epsilon) A^{-1}(v^\epsilon) \right) \partial_t a^\epsilon. \end{aligned} \quad (2.11)$$

We integrate (2.11) on $\mathbb{R} \times [0, T]$ for any $T \in \mathbb{R}^+$. Using Jensen's inequality, we can get rid of the terms involving the projection operator since $\partial_t a^\epsilon \geq 0$:

$$\int_{\mathbb{R} \times [0, T]} (\mathcal{P}^h(u^\epsilon) - u^\epsilon) u^\epsilon \partial_t a^\epsilon . dx . dt = h \int_{[0, T]} \left(\mathcal{P}^h(u^\epsilon)^2 - \mathcal{P}^h((u^\epsilon)^2) \right) \partial_t a^\epsilon . dt \leq 0.$$

We observe now that since $A \in C^1(0, 1)$ is increasing according to (2.4), the derivative of its inverse mapping satisfies the following bound:

$$\|(A^{-1})'\|_{C^0} \geq \frac{1}{\|A'\|_{C^0}} \stackrel{def}{=} C.$$

And we have for any w, v in $[0, 1]$:

$$(A^{-1}(w) - A^{-1}(v))(w - v) \geq C(w - v)^2.$$

Hence the integral involving $\mathcal{P}^h(v^\epsilon)$ is treated the following way:

$$\begin{aligned} &\int_{\mathbb{R} \times [0, T]} (\mathcal{P}^h(v^\epsilon) - v^\epsilon) A^{-1}(v^\epsilon) \partial_t a^\epsilon . dx . dt = \\ &- \int_{\mathbb{R} \times [0, T]} (\mathcal{P}^h(v^\epsilon) - v^\epsilon) (A^{-1}(\mathcal{P}^h(v^\epsilon)) - A^{-1}(v^\epsilon)) \partial_t a^\epsilon . dx . dt \leq \\ &- C \int_{\mathbb{R} \times [0, T]} (\mathcal{P}^h(v^\epsilon) - v^\epsilon)^2 \partial_t a^\epsilon . dx . dt \leq 0. \end{aligned}$$

Finally, choosing $w = A(u)$, we have for $C > 0$, the lower bound of $(A^{-1})'$:

$$(A(u^\epsilon) - v^\epsilon)(u^\epsilon - A^{-1}(v^\epsilon)) \geq C(A(u^\epsilon) - v^\epsilon)^2.$$

Thus we get the expected relaxation estimate (see also ¹⁷):

$$C\mu \int_{\mathbb{R} \times [0, T]} (A(u^\epsilon) - v^\epsilon)^2 \partial_t b^\epsilon . dx . dt \leq \int_{\mathbb{R}} \left(\eta(u^\epsilon, v^\epsilon)(x, 0) - \eta(u^\epsilon, v^\epsilon) \right) (x, T) . dx.$$

Since $BV(\mathbb{R}) \subset L^\infty(\mathbb{R})$, we are done. \square

Another important feature of (2.10) lies in the following compactness result which will be of constant use in the sequel of the paper.

Lemma 2 *Under the assumptions of Theorem 1, let (u^ϵ, v^ϵ) be a sequence of entropy solutions to (2.10). Then (u^ϵ, v^ϵ) is relatively compact in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ as $\epsilon \rightarrow 0$.*

Proof. It follows the classical BV stability canvas and makes use of the so-called ‘‘quasi-monotonicity’’ property of the relaxation term, ^{1,26,31}.

Therefore, we see that the sequence (u^ϵ, v^ϵ) lies in $BV_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ and it remains to invoke Helly's compactness principle to conclude the proof. \square

We notice that because of the Dirac masses in time arising at the right-hand side of the equations, layers are likely to appear every time $n\Delta t$, $n \in \mathbb{N}_*$ and

$$u^\epsilon, v^\epsilon \notin C^0(\mathbb{R}^+; L^1(\mathbb{R})), \quad \epsilon \rightarrow 0,$$

as in (for instance) ²⁸ in the case of initial data being not "well-prepared"; or it can be evidenced just by taking $u_0 \equiv 0$, $v_0(x) = \sin(\pi x/h)$ inside (2.23).

2.3. A meaning for the ambiguous products

The preceding bounds for u^ϵ imply some compactness for the terms lying in the right-hand side in the weak- \star topology of measures on $\mathbb{R} \times \mathbb{R}_*^+$. More precisely, we plan to shed some light on the products emanating in (2.10) in the limit $\epsilon \rightarrow 0$. First, we state a stabilization result.

Lemma 3 *Under the assumptions of Theorem 1, the sequences of entropy solutions (u^ζ, v^ζ) to the Cauchy problem with initial data $(u_0^\zeta, v_0^\zeta) \in L^1 \cap BV(\mathbb{R})$ for:*

$$\begin{aligned} \partial_t u^\zeta + \zeta \cdot \partial_x f(u^\zeta) &= \nu \Delta t (\mathcal{P}^h(u^\zeta) - u^\zeta), & \text{or} & & \partial_t u^\zeta + \zeta \cdot \partial_x f(u^\zeta) &= -\mu \Delta t R(u^\zeta, v^\zeta), \\ \partial_t v^\zeta &= \nu \Delta t (\mathcal{P}^h(v^\zeta) - v^\zeta), & & & \partial_t v^\zeta &= \mu \Delta t R(u^\zeta, v^\zeta), \end{aligned}$$

are relatively compact in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ and belong to $C^0(\mathbb{R}^+; L^1(\mathbb{R}))$ as $\zeta \rightarrow 0$.

Proof. It follows from the same arguments than the proof of Lemma 2. \square

Proposition 1 *Under the hypotheses of Lemma 2, $(u^\epsilon, v^\epsilon) \rightarrow (u, v)$ in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ as $\epsilon \rightarrow 0$ up to a subsequence. Moreover, there holds:*

$$\sum_{n \in \mathbb{N}_*} \Delta t \left(\begin{array}{c} \mathcal{P}^h(u^\epsilon) - u^\epsilon \\ \mathcal{P}^h(v^\epsilon) - v^\epsilon \end{array} \right) \partial_t a^\epsilon \stackrel{\text{weak-}\star}{\rightharpoonup} \mathcal{M} \quad (2.15)$$

$$\sum_{n \in \mathbb{N}_*} \Delta t \left(\int_0^1 \left(\begin{array}{c} \mathcal{P}^h(\bar{u}) - \bar{u} \\ \mathcal{P}^h(\bar{v}) - \bar{v} \end{array} \right) (x, \tau) d\tau \right) \delta(t - n\Delta t)$$

where \bar{u}, \bar{v} satisfy the following differential equations for $\tau \in [0, 1]$:

$$\partial_\tau \left(\begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right) = \nu \Delta t \left(\begin{array}{c} \mathcal{P}^h(\bar{u}) - \bar{u} \\ \mathcal{P}^h(\bar{v}) - \bar{v} \end{array} \right), \quad (2.16)$$

together with the initial data, $x \in \mathbb{R}, t = n\Delta t$ for $n \in \mathbb{N}_*$:

$$\bar{u}(x, \tau = 0) = u(x, t = 0), \quad \bar{v}(x, \tau = 0) = v(x, t = 0). \quad (2.17)$$

Proof. We want to compute the value of the following expression for every $\phi \in C_c^0(\mathbb{R} \times \mathbb{R}_*^+)$, the space of continuous functions with compact support in $\mathbb{R} \times \mathbb{R}_*^+$:

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}_*^+} (\mathcal{P}^h(u^\epsilon) - u^\epsilon) \partial_t a^\epsilon \phi(x, t) \cdot dx \cdot dt = \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \int_{n\Delta t - \epsilon}^{n\Delta t} (\mathcal{P}^h(u^\epsilon) - u^\epsilon) \frac{\Delta t}{\epsilon} \phi(x, t) \cdot dx \cdot dt. \end{aligned} \quad (2.18)$$

We pick a $n \in \mathbb{N}_*$ and define $\tau = 1 + \frac{t-n\Delta t}{\epsilon}$. Thus, according to this ‘‘inner variable’’, the system (2.10) rewrites for $\tau \in [0, 1]$:

$$\begin{cases} \partial_\tau \bar{u}^\epsilon + \epsilon \partial_x f(\bar{u}^\epsilon) = \nu \Delta t (\mathcal{P}^h(\bar{u}^\epsilon) - \bar{u}^\epsilon), \\ \partial_\tau \bar{v}^\epsilon = \nu \Delta t (\mathcal{P}^h(\bar{v}^\epsilon) - \bar{v}^\epsilon), \end{cases}$$

together with the initial data:

$$\bar{u}^\epsilon(x, \tau = 0) = u^\epsilon(x, n\Delta t - \epsilon), \quad \bar{v}^\epsilon(x, \tau = 0) = v^\epsilon(x, n\Delta t - \epsilon).$$

We perform the change of variables $t \mapsto \tau$ in (2.18) to get:

$$\begin{aligned} & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \int_{n\Delta t - \epsilon}^{n\Delta t} (\mathcal{P}^h(u^\epsilon) - u^\epsilon) \frac{\Delta t}{\epsilon} \phi(x, t). dx. dt = \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \int_0^1 (\mathcal{P}^h(\bar{u}^\epsilon) - \bar{u}^\epsilon)(x, \tau) \frac{\Delta t}{\epsilon} \phi(x, n\Delta t + \epsilon\tau - \epsilon). dx. \epsilon d\tau. \end{aligned}$$

Lemma 3 implies that \bar{u}^ϵ satisfies (2.15), (2.16) in the limit $\epsilon \rightarrow 0$ and we are done. \square

Remark 1 Thanks to the linear form of the projection term, it is possible to define directly the limit of $(\mathcal{P}^h(u^\epsilon) - u^\epsilon) \partial_t a^\epsilon$ as $\epsilon \rightarrow 0$ as a distribution of order zero (i.e. a bounded measure) provided $\Delta t = O(h)$. Let $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$, we can compute:

$$\sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} (\mathcal{P}^h(u) - u) \phi(x, n\Delta t). dx = \sum_{j, n \in \mathbb{Z} \times \mathbb{N}_*} h \left(\mathcal{P}^h(u) \mathcal{P}^h(\phi) - \mathcal{P}^h(u\phi) \right) (jh, n\Delta t).$$

And we have $\sum_{j \in \mathbb{Z}} |\mathcal{P}^h(u) \mathcal{P}^h(\phi) - \mathcal{P}^h(u\phi)| (jh, t) \leq \|\phi\|_{C^0 TV(u)(\cdot, t)}$.

Concerning the relaxation term, the situation could be different because of its nonlinear structure. Nevertheless, we have the following result.

Proposition 2 Under the hypotheses of Lemma 2, $(u^\epsilon, v^\epsilon) \rightarrow (u, v)$ in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ as $\epsilon \rightarrow 0$ up to a subsequence. Moreover, there holds:

$$\left(A(u^\epsilon) - v^\epsilon \right) \partial_t b^\epsilon \stackrel{weak-\star}{\rightharpoonup} \mathcal{M} \sum_{n \in \mathbb{N}_*} \Delta t \left(\int_0^1 (A(\bar{u}) - \bar{v})(x, \tau) d\tau \right) \delta(t - n\Delta t) \quad (2.19)$$

where \bar{u}, \bar{v} satisfy the following differential equations for $\tau \in [0, 1]$:

$$\partial_\tau \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = \mu \Delta t \begin{pmatrix} -(A(\bar{u}) - \bar{v}) \\ (A(\bar{u}) - \bar{v}) \end{pmatrix}, \quad (2.20)$$

together with the initial data, $x \in \mathbb{R}, t = n\Delta t$ for $n \in \mathbb{N}_*$:

$$\bar{u}(x, \tau = 0) = u(x, t - 0), \quad \bar{v}(x, \tau = 0) = v(x, t - 0). \quad (2.21)$$

Proof. Let $\phi \in C_c^0(\mathbb{R} \times \mathbb{R}_*^+)$, we intend now to pass to the limit $\epsilon \rightarrow 0$ in the following expression:

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}_*^+} (A(u^\epsilon) - v^\epsilon) \partial_t b^\epsilon \phi(x, t). dx. dt = \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \int_{n\Delta t}^{n\Delta t + \epsilon} (A(u^\epsilon) - v^\epsilon) \frac{\Delta t}{\epsilon} \phi(x, t). dx. dt. \end{aligned} \quad (2.22)$$

We pick a $n \in \mathbb{N}_*$ and define $\tau = \frac{t-n\Delta t}{\epsilon}$ as an “inner variable” inside the system (2.10) which reads for $\tau \in [0, 1]$:

$$\begin{cases} \partial_\tau \bar{u}^\epsilon + \epsilon \partial_x f(\bar{u}^\epsilon) = -\mu \Delta t (A(\bar{u}^\epsilon) - \bar{v}^\epsilon), \\ \partial_\tau \bar{v}^\epsilon = \mu \Delta t (A(\bar{u}^\epsilon) - \bar{v}^\epsilon), \end{cases}$$

together with the initial data:

$$\bar{u}^\epsilon(x, \tau = 0) = u^\epsilon(x, n\Delta t), \quad \bar{v}^\epsilon(x, \tau = 0) = v^\epsilon(x, n\Delta t).$$

In order to conclude the proof, we perform the change of variables $t \mapsto \tau$ in (2.22) and invoke Lemma 3 to derive (2.19), (2.20) in the limit $\epsilon \rightarrow 0$. \square

2.4. A contraction result “à la Kružkov”

We can now consider weak solutions of the following singular relaxing system for μ, ν in \mathbb{R}^+ , $x \in \mathbb{R}, t > 0$:

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum_{n \in \mathbb{N}_*} \Delta t \left[\nu (\mathcal{P}^h(u) - u) - \mu R(u, v) \right] \delta(t - n\Delta t), \\ \partial_t v = \sum_{n \in \mathbb{N}_*} \Delta t \left[\nu (\mathcal{P}^h(v) - v) + \mu R(u, v) \right] \delta(t - n\Delta t), \\ u(., 0) = u_0 \in L^1 \cap BV(\mathbb{R}), v(., 0) = v_0 \in L^1 \cap BV(\mathbb{R}), \end{cases} \quad (2.23)$$

where all the measure-valued source terms are **always** to be understood in the sense of distributions according to the results stated in Propositions 1 and 2. We first extend the notion of “entropy solution” to (2.23).

Definition 2 We say that the pair (u, v) is an **entropy solution** of (2.23) if:

- (i) u and v lie in $L^1 \cap BV(\mathbb{R} \times [0, T])$ for any $T \in \mathbb{R}^+$ and satisfy (2.23) in the sense of distributions;
- (ii) there holds for any $k, l \in \mathbb{R}$ and any nonnegative test-function in $\mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$:

$$\begin{aligned} & \partial_t \left[|u - k| + |v - l| \right] + \partial_x \left| f(u) - f(k) \right| \leq \\ & \sum_{n \in \mathbb{N}_*} \Delta t \left[\nu (\mathcal{P}^h(u) - u) \operatorname{sgn}(u - k) + \nu (\mathcal{P}^h(v) - v) \operatorname{sgn}(v - l) + \right. \\ & \left. \mu R(u, v) (\operatorname{sgn}(v - l) - \operatorname{sgn}(u - k)) \right] \delta(t - n\Delta t). \end{aligned} \quad (2.24)$$

The main result of this section states that system (2.23), (2.24) is $L^1(\mathbb{R})$ -contractive.

Theorem 3 Let $(u_0, v_0) \in L^1 \cap BV(\mathbb{R})$; under the hypotheses (2.4), there exists a unique entropy solution (u, v) to (2.23). Moreover, if $(\tilde{u}_0, \tilde{v}_0)$ stands for another set of initial data in $L^1 \cap BV(\mathbb{R})$ and (\tilde{u}, \tilde{v}) for its associated entropy solution, there holds for any $t \in \mathbb{R}^+$:

$$\|u(., t) - \tilde{u}(., t)\|_{L^1(\mathbb{R})} + \|v(., t) - \tilde{v}(., t)\|_{L^1(\mathbb{R})} \leq \|u_0 - \tilde{u}_0\|_{L^1(\mathbb{R})} + \|v_0 - \tilde{v}_0\|_{L^1(\mathbb{R})}.$$

Despite the fact that the solution of (2.23), (2.24) suffer discontinuities on the lines $t = n\Delta t$, $n \in \mathbb{N}_*$, its BV regularity allows one to define a set $\mathcal{N} \subset [0, T]$ of measure zero containing 0 and T such that each point in the strip $(x, t) \in \mathbb{R} \times [0, T]$, $t \notin \mathcal{N}$ is either an approximate jump or an approximate continuity location,⁶

Proof. We need first to check the regularity criterion (i) in Definition 2. This is indeed a direct consequence of the uniform bounds (2.12), (2.13), (2.14) in the proof of Lemma 2. We start from the entropy inequality (2.5) satisfied by u^ϵ, v^ϵ for $\epsilon > 0$:

$$\begin{aligned} & \partial_t \left[|u^\epsilon - k| + |v^\epsilon - l| \right] + \partial_x \left| f(u^\epsilon) - f(k) \right| \leq \\ & \nu \left[(\mathcal{P}^h(u^\epsilon) - u^\epsilon) \operatorname{sgn}(u^\epsilon - k) + (\mathcal{P}^h(v^\epsilon) - v^\epsilon) \operatorname{sgn}(v^\epsilon - l) \right] \partial_t a^\epsilon + \\ & \mu R(u^\epsilon, v^\epsilon) (\operatorname{sgn}(v^\epsilon - l) - \operatorname{sgn}(u^\epsilon - k)) \partial_t b^\epsilon. \end{aligned} \quad (2.25)$$

Therefore, the inequality (2.24) comes in the limit of (2.25) by passing to the limit $\epsilon \rightarrow 0$ as in the proofs of Propositions 1 and 2.

Concerning the contraction property which implies uniqueness, we follow the ideas of 18,20,32. Let us now introduce a nonnegative test function $\Psi(x, t, y, s) = \psi(x, t) \cdot \zeta(x-y) \cdot \zeta(t-s)$ belonging to $\mathcal{D}((\mathbb{R} \times \mathbb{R}_*^+)^2)$ with ζ the standard approximation of the Dirac mass. We test the preceding entropy inequalities (2.25) for both u^ϵ, v^ϵ and $\tilde{u}^\epsilon, \tilde{v}^\epsilon$ with this particular function, we add and integrate on $(\mathbb{R} \times \mathbb{R}_*^+)^2$. For $\epsilon > 0$, the standard theory of Kruřkov allows to let ζ concentrate to the Dirac measure since a^ϵ, b^ϵ are Lipschitz functions. We obtain:

$$\begin{aligned} & \partial_t \left[|u^\epsilon - \tilde{u}^\epsilon| + |v^\epsilon - \tilde{v}^\epsilon| \right] + \partial_x \left| f(u^\epsilon) - f(\tilde{u}^\epsilon) \right| \leq \\ & \nu \left[((\mathcal{P}^h(u^\epsilon) - u^\epsilon) - (\mathcal{P}^h(\tilde{u}^\epsilon) - \tilde{u}^\epsilon)) \operatorname{sgn}(u^\epsilon - \tilde{u}^\epsilon) + \right. \\ & \left. ((\mathcal{P}^h(v^\epsilon) - v^\epsilon) - (\mathcal{P}^h(\tilde{v}^\epsilon) - \tilde{v}^\epsilon)) \operatorname{sgn}(v^\epsilon - \tilde{v}^\epsilon) \right] \partial_t a^\epsilon + \\ & \mu (R(u^\epsilon, v^\epsilon) - R(\tilde{u}^\epsilon, \tilde{v}^\epsilon)) (\operatorname{sgn}(v^\epsilon - \tilde{v}^\epsilon) - \operatorname{sgn}(u^\epsilon - \tilde{u}^\epsilon)) \partial_t b^\epsilon. \end{aligned} \quad (2.26)$$

We study mainly the zero-order terms. Using the mean-value theorem for the relaxation term leads to

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}_*^+} \left(\operatorname{sgn}(v^\epsilon - \tilde{v}^\epsilon) - \operatorname{sgn}(u^\epsilon - \tilde{u}^\epsilon) \right) \left(A(u^\epsilon) - A(\tilde{u}^\epsilon) - v^\epsilon + \tilde{v}^\epsilon \right) \partial_t b^\epsilon \psi(x, t) . dx . dt \\ & = \int_{\mathbb{R} \times \mathbb{R}_*^+} A'(\xi) \left(\operatorname{sgn}(v^\epsilon - \tilde{v}^\epsilon) (u^\epsilon - \tilde{u}^\epsilon) - |u^\epsilon - \tilde{u}^\epsilon| \right) \partial_t b^\epsilon \psi(x, t) . dx . dt \\ & \quad + \int_{\mathbb{R} \times \mathbb{R}_*^+} \left(\operatorname{sgn}(u^\epsilon - \tilde{u}^\epsilon) (v^\epsilon - \tilde{v}^\epsilon) - |v^\epsilon - \tilde{v}^\epsilon| \right) \partial_t b^\epsilon \psi(x, t) . dx . dt, \end{aligned}$$

for a $\xi \in [\min(u^\epsilon, \tilde{u}^\epsilon), \max(u^\epsilon, \tilde{u}^\epsilon)]$. Since $\partial_t b^\epsilon \geq 0$ and $A' \geq 0$, this term gives a nonpositive contribution on the right-hand side of the entropy inequality (2.26).

Concerning the projection operator, we see that

$$\begin{aligned} & \int_{\mathbb{R} \times \mathbb{R}_*^+} \operatorname{sgn}(u^\epsilon - \tilde{u}^\epsilon) \left(\mathcal{P}^h(u^\epsilon - \tilde{u}^\epsilon) - (u^\epsilon - \tilde{u}^\epsilon) \right) \partial_t a^\epsilon \psi(x, t) . dx . dt = \\ & \int_{\mathbb{R} \times \mathbb{R}_*^+} \left(\operatorname{sgn}(u^\epsilon - \tilde{u}^\epsilon) - \operatorname{sgn}(\mathcal{P}^h(u^\epsilon - \tilde{u}^\epsilon)) \right) \left(\mathcal{P}^h(u^\epsilon - \tilde{u}^\epsilon) - (u^\epsilon - \tilde{u}^\epsilon) \right) \partial_t a^\epsilon \psi . dx . dt \\ & \quad + \int_{\mathbb{R} \times \mathbb{R}_*^+} \operatorname{sgn}(\mathcal{P}^h(u^\epsilon - \tilde{u}^\epsilon)) \left(\mathcal{P}^h(u^\epsilon - \tilde{u}^\epsilon) - (u^\epsilon - \tilde{u}^\epsilon) \right) \partial_t a^\epsilon \psi(x, t) . dx . dt. \end{aligned}$$

Since $\mathcal{P}^h(u^\epsilon - \tilde{u}^\epsilon)$ is constant on each cell C_j , the last integral vanishes. Once again, since $\partial_t a^\epsilon \geq 0$, this term gives a nonpositive contribution in (2.26). We end up with:

$$\partial_t \left[|u^\epsilon - \tilde{u}^\epsilon| + |v^\epsilon - \tilde{v}^\epsilon| \right] + \partial_x \left[f(u^\epsilon) - f(\tilde{u}^\epsilon) \right] \leq 0.$$

It remains now to select ψ as in 18,20,3 to complete the proof of Theorem 3. \square

Remark 2 We can notice that this contraction property gives back the $BV(\mathbb{R})$ regularity for $u(\cdot, t)$, $v(\cdot, t)$ as soon as u_0, v_0 belong to $BV(\mathbb{R})$ since the problem (2.23) is translation invariant.

We notice finally that since \mathcal{P}^h is linear, any weak solution of (2.23) also satisfies:

$$\begin{cases} \partial_t(u+v) + \partial_x f(u) = \sum_{n \in \mathbb{N}_*} \nu \Delta t \left[\mathcal{P}^h(u+v) - (u+v) \right] \delta(t - n\Delta t), \\ \partial_t v = \sum_{n \in \mathbb{N}_*} \Delta t \left[\nu (\mathcal{P}^h(v) - v) + \mu R(u, v) \right] \delta(t - n\Delta t). \end{cases}$$

3. Study of the resulting “time-splitting” numerical schemes

3.1. A technical lemma

We point out quickly that the special structure of the “singular projection term” allows us to refine a bit the entropy inequality (2.24).

Lemma 4 In the sense of Proposition 1, the following holds for any $k \in \mathbb{R}$ and any nonnegative continuous compactly supported test-function:

$$\begin{aligned} & \sum_{n \in \mathbb{N}_*} \nu \Delta t \left(\mathcal{P}^h(u) - u \right) \operatorname{sgn}(u - k) \cdot \delta(t - n\Delta t) \leq \\ & \sum_{n \in \mathbb{N}_*} \left(|\mathcal{P}^h(u) - k| - |u - k| \right) (1 - \exp(-\nu \Delta t)) \cdot \delta(t - n\Delta t). \end{aligned}$$

Proof. Let $0 \leq \phi \in C_c^0(\mathbb{R} \times \mathbb{R}_+)$; we perform the following splitting:

$$\begin{cases} \operatorname{sgn}(\bar{u} - k) = \operatorname{sgn}(\mathcal{P}^h(\bar{u}) - k) + (\operatorname{sgn}(\bar{u} - k) - \operatorname{sgn}(\mathcal{P}^h(\bar{u}) - k)), \\ \mathcal{P}^h(\bar{u}) - \bar{u} = (\mathcal{P}^h(\bar{u}) - k) - (\bar{u} - k). \end{cases}$$

Since we have $\operatorname{sgn}(x) \cdot y \leq |y|$ for any x, y in \mathbb{R}^2 , we get out of (2.16) for any $k \in \mathbb{R}$:

$$\partial_\tau |\bar{u} - k| \leq \nu \Delta t \left(|\mathcal{P}^h(\bar{u}) - k| - |\bar{u} - k| \right).$$

We observe that $\partial_\tau |\mathcal{P}^h(\bar{u}) - k| = 0$ along the flow of (2.16) and this implies:

$$|\bar{u} - k|(x, 1) - |\bar{u} - k|(x, 0) \leq (1 - \exp(-\nu \Delta t)) \cdot \left(|\mathcal{P}^h(u) - k| - |u - k| \right)(x, 0).$$

Therefore, we just have to recall the following from (2.15) to conclude the proof:

$$\begin{aligned} & \sum_{n \in \mathbb{N}_*} \nu \Delta t \int_{\mathbb{R}} \left(\int_0^1 \operatorname{sgn}(\bar{u} - k) (\mathcal{P}^h(\bar{u}) - \bar{u})(x, \tau) \cdot d\tau \right) \phi(x, n\Delta t) \cdot dx = \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \left(\int_0^1 \partial_\tau |\bar{u} - k|(x, \tau) \cdot d\tau \right) \phi(x, n\Delta t) \cdot dx. \end{aligned}$$

□

This means in particular that the following inequality holds as a consequence of (2.24) and this last lemma:

$$\begin{aligned} & \partial_t \left[|u - k| + |v - l| \right] + \partial_x \left| f(u) - f(k) \right| \leq \\ & \sum_{n \in \mathbb{N}_*} \left[(|\mathcal{P}^h(u) - k| - |u - k| + |\mathcal{P}^h(v) - l| - |v - l|) (1 - \exp(-\nu \Delta t)) \right. \\ & \left. + \mu \Delta t R(u, v) (\operatorname{sgn}(v - l) - \operatorname{sgn}(u - k)) \right] \delta(t - n \Delta t), \end{aligned} \quad (3.27)$$

in the sense of Proposition 2. One can notice the similarity between this last inequality and the one derived in ³¹ §3; indeed the numerical approximation considered by these authors matches ours in the special case where $\nu \rightarrow +\infty$ and a backward Euler solver is chosen instead of (2.20). This reduces to a time-splitting scheme with some kind of “straight lines” approximation of the quantity (2.19). In contrast, we recall that it has been proved in ³ §5, that one has:

$$\sum_{n \in \mathbb{N}_*} \nu \Delta t \left(\mathcal{P}^h(u) - u \right) \left(\operatorname{sgn}(u - k) - \operatorname{sgn}(\mathcal{P}^h(u) - k) \right) \delta(t - n \Delta t) \leq 0,$$

for any positive continuous test-function in the case where $u \in C^0(\mathbb{R}_*^+; L^1(\mathbb{R}))$.

3.2. The classical time-splitting scheme

We plan to study the behavior of the system (2.23) in the limit $\nu \rightarrow +\infty$. Roughly speaking, this reduces to insert a projection stage every time $n \Delta t$, $n \in \mathbb{N}_*$ before igniting the relaxation mechanism.

Proposition 3 *Under the assumptions of Theorem 3, the sequence of entropy solutions of (2.23) converges strongly in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ as $\nu \rightarrow +\infty$ towards the entropy solution of:*

$$\begin{cases} \partial_t u + \partial_x f(u) = - \sum_{n \in \mathbb{N}_*} \mu \Delta t R(u, v) \delta(t - n \Delta t), \\ \partial_t v = \sum_{n \in \mathbb{N}_*} \mu \Delta t R(u, v) \delta(t - n \Delta t), \end{cases} \quad (3.28)$$

where the right-hand side is still defined by means of (2.19), (2.20), but with the piecewise constant initial data for $t = n \Delta t$, $n \in \mathbb{N}_*$:

$$\bar{u}(x, \tau = 0) = \mathcal{P}^h(u)(x, t - 0), \quad \bar{v}(x, \tau = 0) = \mathcal{P}^h(v)(x, t - 0). \quad (3.29)$$

Moreover, the following entropy inequality holds true for any nonnegative test function in $\mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$:

$$\begin{aligned} & \partial_t \left[|u - k| + |v - l| \right] + \partial_x \left| f(u) - f(k) \right| \leq \\ & \sum_{n \in \mathbb{N}_*} \left\{ (|\mathcal{P}^h(u) - k| - |u - k|) + (|\mathcal{P}^h(v) - l| - |v - l|) + \right. \\ & \left. \mu \Delta t R(u, v) (\operatorname{sgn}(v - l) - \operatorname{sgn}(u - k)) \right\} \delta(t - n \Delta t), \end{aligned} \quad (3.30)$$

with k, l in \mathbb{R} and according to (2.19), (2.20), (2.21).

Proof. We split it into several steps for the sake of clarity.

(i) Construction of the solution of (2.16) by a fixed point argument; for $\tau \in [0, 1]$, we consider any function $w \in C^0(0, 1; L^1(\mathbb{R}))$ together with:

$$\partial_\tau \bar{u} = \nu \Delta t (\mathcal{P}^h(w) - \bar{u}), \quad \bar{u}(\cdot, 0) \in L^1(\mathbb{R})$$

whose solution is obvious:

$$\bar{u}(x, \tau) = \exp(-\nu \Delta t \tau) \bar{u}(x, 0) + \nu \Delta t \exp(-\nu \Delta t \tau) \int_0^\tau \exp(\nu \Delta t s) \mathcal{P}^h(w)(x, s) . ds.$$

This defines a linear mapping $w \mapsto \bar{u}$ in $C^0(0, 1; L^1(\mathbb{R}))$ for any choice of the initial datum. We want to establish a contraction property: for all w_1, w_2 and any $\tau \in [0, 1]$, we observe that

$$\begin{aligned} \int_{\mathbb{R}} |\bar{u}_1 - \bar{u}_2|(\tau) . dx &\leq \nu \Delta t \int_0^\tau \exp(\nu \Delta t (s - \tau)) \int_{\mathbb{R}} |\mathcal{P}^h(w_1) - \mathcal{P}^h(w_2)|(x, s) . dx . ds \\ &\leq (1 - \exp(-\nu \Delta t)) \|w_1 - w_2\|_{C^0(0, 1; L^1(\mathbb{R}))} \end{aligned}$$

By Picard's fixed point theorem, we obtain a unique solution in $C^0(0, 1; L^1(\mathbb{R}))$ for any $\nu \in \mathbb{R}^+$ to:

$$\partial_\tau \bar{u} = \nu \Delta t (\mathcal{P}^h(\bar{u}) - \bar{u}), \quad \bar{u}(\cdot, 0) \in L^1 \cap BV(\mathbb{R})$$

Now, since $\partial_\tau \mathcal{P}^h(\bar{u}) = 0$ along its flow, this differential equation therefore admits an explicit solution:

$$\bar{u}(\cdot, \tau) = \exp(-\nu \Delta t \tau) \bar{u}(\cdot, 0) + (1 - \exp(-\nu \Delta t \tau)) \mathcal{P}^h(\bar{u})(\cdot, 0).$$

In particular, we get:

$$\left(\bar{u} - \mathcal{P}^h(\bar{u}) \right)(\cdot, \tau) = \exp(-\nu \Delta t \tau) \left(\bar{u} - \mathcal{P}^h(\bar{u}) \right)(\cdot, 0). \quad (3.31)$$

(ii) Limit as $\nu \rightarrow +\infty$ of the measure (2.15); for any $\phi \in C_c^0(\mathbb{R} \times \mathbb{R}_*^+)$, we have:

$$\begin{aligned} &\sum_{n \in \mathbb{N}_*} \nu \Delta t \int_{\mathbb{R}} \left(\int_0^1 (\mathcal{P}^h(\bar{u}) - \bar{u})(x, \tau) . d\tau \right) \phi(x, n \Delta t) . dx = \\ &\sum_{n \in \mathbb{N}_*} \nu \Delta t \int_{\mathbb{R}} (\mathcal{P}^h(u) - u)(x, n \Delta t - 0) \left(\int_0^1 \exp(-\nu \tau) . d\tau \right) \phi(x, n \Delta t) . dx \\ &\quad \xrightarrow{\nu \rightarrow +\infty} \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} (\mathcal{P}^h(u) - u)(x, n \Delta t - 0) \phi(x, n \Delta t) . dx, \end{aligned}$$

which is a finite sum provided Δt remains strictly positive.

(iii) Strong compactness as $\nu \rightarrow +\infty$: the bound (2.14) seems to blow up despite the fact that (2.12), (2.13) still hold. Making use of (3.31) in the last step of the proof of Lemma 2, we notice that for any $T \in \mathbb{R}^+$ and $\xi > 0$:

$$\begin{aligned} \int_{\mathbb{R} \times [t, t + \xi]} \sum_{n=1}^{1 + [T/\Delta t]} \nu \Delta t \left\{ \int_0^1 \left| \bar{u} - \mathcal{P}^h(\bar{u}) \right| + \left| \bar{v} - \mathcal{P}^h(\bar{v}) \right| (x, \tau) d\tau \right\} . dx . ds \leq \\ \xi \frac{h}{\Delta t} (T + \Delta t) (TV(u_0) + TV(v_0)) (1 - \exp(-\nu \Delta t)). \end{aligned} \quad (3.32)$$

Therefore, exploiting (2.15) and Lemma 1, we can deduce a fine estimate for the weak solutions of (2.23), (2.15), (2.16) which is uniform in both ν and μ :

$$\sup_{\xi \neq 0} \int_{\mathbb{R} \times [0, T]} \frac{|u(x, t + \xi) - u(x, t)|}{\xi} + \frac{|v(x, t + \xi) - v(x, t)|}{\xi} .dx .dt \leq \quad (3.33)$$

$$(T + \Delta t) \left\{ (h/\Delta t + Lip(f)) (TV(u_0) + TV(v_0)) + O(1) \right\}$$

Together with (2.12), (2.13), this provides compactness in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ by means of Helly's theorem. And in the limit, we get the following system: (recall from (1.2))

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum_{n \in \mathbb{N}_*} \left[(\mathcal{P}^h(u) - u)(\cdot, n\Delta t - 0) - \mu \Delta t R(u, v) \right] \delta(t - n\Delta t), \\ \partial_t v = \sum_{n \in \mathbb{N}_*} \left[(\mathcal{P}^h(v) - v)(\cdot, n\Delta t - 0) + \mu \Delta t R(u, v) \right] \delta(t - n\Delta t). \end{cases}$$

Since we know that $a^\epsilon(t) = b^\epsilon(t - \epsilon)$, we must handle the projection step first and the initial data for (2.20) becomes (3.29). The preceding system then rewrites like (3.28).

(iv) Entropy inequality: Using the same arguments together with Lemma 4, we derive for any $k \in \mathbb{R}$:

$$\begin{aligned} & \sum_{n \in \mathbb{N}_*} \nu \Delta t \int_{\mathbb{R}} \left(\int_0^1 \text{sgn}(\bar{u} - k) (\mathcal{P}^h(\bar{u}) - \bar{u})(x, \tau) .d\tau \right) \phi(x, n\Delta t) .dx = \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \left(|\bar{u} - k|(x, 1) - |\bar{u} - k|(x, 0) \right) \phi(x, n\Delta t) .dx \leq \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \left(|\mathcal{P}^h(u) - k| - |u - k| \right) (x, n\Delta t - 0) (1 - \exp(-\nu \Delta t)) \phi(x, n\Delta t) .dx \\ & \xrightarrow{\nu \rightarrow +\infty} \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \left(|\mathcal{P}^h(u) - k| - |u - k| \right) (x, n\Delta t - 0) \phi(x, n\Delta t) .dx. \end{aligned}$$

This way, we get (3.30) out of (2.24) and the contraction property of Theorem 3 is also preserved. \square

One can observe that as soon as the initial data of (3.28), (2.19), (2.20), (3.29) is \mathcal{P}^h -invariant and the CFL condition $\Delta t \|f'(u)\|_{L^\infty} \leq h$ holds, its entropy solution matches the piecewise constant approximation generated by a classical time-splitting Godunov scheme, see *e.g.* 1.31 and 9.

Theorem 4 *Under the assumptions (2.4), the CFL restriction $\Delta t \|f'(u)\|_{L^\infty} = h$ and for $v_0 = A(u_0)$, $u_0 \in L^1 \cap BV(\mathbb{R})$, the entropy solutions of (3.28), (3.30), (2.19), (2.20), (3.29) converge strongly in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ towards u , the entropy solution in the sense of Kruřkov to:*

$$\partial_t (u + A(u)) + \partial_x f(u) = 0, \quad u(x, 0) = u_0.$$

as $\mu \Delta t \rightarrow +\infty, h \rightarrow 0$.

Proof. As we did before concerning Proposition 3, we are about to split the forthcoming proof for the sake of clarity.

(i) Study of the solution of (2.20); for $\tau \in [0, 1]$, we rewrite its second equation as:

$$\partial_\tau \bar{v} = \mu \Delta t (\tilde{A}(\bar{v}) - \bar{v}), \quad \tilde{A}(\bar{v}) = A(\bar{u}(x, 0) + \bar{v}(x, 0) - \bar{v}(x, \tau)). \quad (3.34)$$

We follow the classical approach, see *e.g.* ⁴, and we introduce the following Banach space for some $k \in \mathbb{R}^+$ to be fixed later:

$$X = \left\{ \varphi \in C^0(0, 1; L^1(\mathbb{R})) \text{ such that } \sup_{t \in [0, 1]} \left(\exp(-k.t) \|\varphi(\cdot, t)\|_{L^1(\mathbb{R})} \right) < +\infty \right\}.$$

For any $w \in X$, we consider the mapping $w \mapsto \bar{v} \in X$,

$$\bar{v}(x, \tau) = \bar{v}(x, 0) \exp(-\mu \Delta t \tau) + \int_0^\tau \mu \Delta t \exp(\mu \Delta t (s - \tau)) \tilde{A}(w)(x, s). ds,$$

which turns out to be a contraction for $k > \mu \Delta t (\text{Lip}(A) - 1)$. By Picard's fixed point theorem, we get a unique solution to (3.34) $\bar{v} \in X$ for any $\mu \in \mathbb{R}^+$. Moreover, it satisfies

$$\partial_\tau (\tilde{A}(\bar{v}) - \bar{v})(x, \tau) = -\mu \left(1 + A'(\bar{u}(x, 0) + \bar{v}(x, 0) - \bar{v}(x, \tau)) \right) (\tilde{A}(\bar{v}) - \bar{v})(x, \tau),$$

which leads to:

$$|A(\bar{u}) - \bar{v}|(x, \tau) \leq \exp(-\mu \Delta t \tau) |A(\bar{u}) - \bar{v}|(x, 0).$$

(ii) Behavior of (3.30) as $\mu \rightarrow +\infty$; the bound (3.33) can be simplified by means of the CFL condition:

$$\sup_{\xi \neq 0} \int_{\mathbb{R} \times [0, T]} \frac{|u(x, t + \xi) - u(x, t)|}{\xi} + \frac{|v(x, t + \xi) - v(x, t)|}{\xi} . dx . dt \leq (T + \Delta t) \left\{ 2 \text{Lip}(f) (TV(u_0) + TV(v_0)) + O(1) \right\}. \quad (3.35)$$

This provides strong L^1_{loc} compactness and for any positive continuous test function $\phi \in C_c^0(\mathbb{R} \times \mathbb{R}_+^*)$, we have:

$$\begin{aligned} & \sum_{n \in \mathbb{N}_*} \Delta t \int_{\mathbb{R}} \mu \left(\int_0^1 (A(\bar{u}) - \bar{v}) (\text{sgn}(\bar{v} - l) - \text{sgn}(\bar{u} - k))(x, \tau) . d\tau \right) \phi(x, n\Delta t) . dx \\ & \leq 2 \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} |A(\mathcal{P}^h(u)) - \mathcal{P}^h(v)| (x, n\Delta t) \left(\int_0^1 \mu \Delta t . \exp(-\mu \Delta t \tau) . d\tau \right) \phi(x, n\Delta t) . dx \\ & \leq O \left(\frac{1 - \exp(-\mu \Delta t)}{\mu \Delta t} \right) \|\phi\|_{C^0}. \end{aligned}$$

We used the preceding estimates in conjunction with the bound in Lemma 1. Therefore, this term vanishes in the limit $\mu \Delta t \rightarrow +\infty, h \rightarrow 0$. Moreover, by Jensen's inequality for convex functions, we also have thanks to (2.13) and the CFL condition,

$$\begin{aligned} & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} \left(|\mathcal{P}^h(u) - k| - |u - k| \right) (x, n\Delta t - 0) \phi(x, n\Delta t) . dx \leq \\ & \sum_{n \in \mathbb{N}_*} \int_{\mathbb{R}} |\mathcal{P}^h(u) - u| (x, n\Delta t - 0) |\mathcal{P}^h(\phi) - \phi| (x, n\Delta t) . dx \leq \\ & h \sum_{n \in \mathbb{N}_*} \Delta t \text{Lip}(f) (TV(u_0) + TV(v_0)) \text{Lip}(\phi) \xrightarrow{h \rightarrow 0} 0. \end{aligned}$$

(iii) It remains to pass to the limit in (3.30); we perform the following splitting for $x \in \mathbb{R}$ and $t \in [n\Delta t, (n+1)\Delta t[$

$$|v - A(u)|(x, t) \leq |v(x, t) - v(x, n\Delta t)| + |v - \mathcal{P}^h(v)|(x, n\Delta t) + \\ |A(\mathcal{P}^h(u)) - \mathcal{P}^h(v)|(x, n\Delta t) + \\ |A(\mathcal{P}^h(u)) - A(u)|(x, n\Delta t) + |A(u)(x, t) - A(u)(x, n\Delta t)|,$$

each term converging to zero in L^1 as $\mu\Delta t \rightarrow +\infty$, $h \rightarrow 0$ under the prescribed CFL condition. Therefore, we fix $l = A(k) \in \mathbb{R}$ and we observe that for any positive $\psi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$

$$\int_{\mathbb{R} \times \mathbb{R}_*^+} |v - l| \partial_t \psi(x, t). dx. dt \rightarrow \int_{\mathbb{R} \times \mathbb{R}_*^+} |A(u) - A(k)| \partial_t \psi(x, t). dx. dt,$$

which leads to the integral form of Kruřkov's entropy inequality:

$$\int_{\mathbb{R} \times \mathbb{R}_*^+} \left(|u - k| + |A(u) - A(k)| \right) \partial_t \psi(x, t) + \left| f(u) - f(k) \right| \partial_x \psi(x, t). dx. dt \geq 0.$$

□

3.3. A modified time-splitting scheme

It is clear that the estimates of Lemmas 2 and 1 still hold for the slightly modified system (compare with (2.10)):

$$\begin{cases} \partial_t u^\epsilon + \partial_x f(u^\epsilon) = -\mu R(u^\epsilon, v^\epsilon) \partial_t a^\epsilon + \nu(\mathcal{P}^h(u^\epsilon) - u^\epsilon) \partial_t b^\epsilon, \\ \partial_t v^\epsilon = \mu R(u^\epsilon, v^\epsilon) \partial_t a^\epsilon + \nu(\mathcal{P}^h(v^\epsilon) - v^\epsilon) \partial_t b^\epsilon, \\ u^\epsilon(\cdot, 0) = u_0, v^\epsilon(\cdot, 0) = v_0. \end{cases} \quad x \in \mathbb{R}, t > 0 \quad (3.36)$$

The Cauchy problem for (3.36) still can be studied within the framework of ³² for any $\epsilon > 0$. According to the same ideas, we send $\epsilon \rightarrow 0$ to get the following result.

Proposition 4 *Let $(u_0, v_0) \in L^1 \cap BV(\mathbb{R})$; under the assumptions (2.4), the sequence (u^ϵ, v^ϵ) of entropy solutions to (3.36) converges in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ towards the entropy solution in the sense of Definition 2 to*

$$\begin{cases} \partial_t u + \partial_x f(u) = \sum_{n \in \mathbb{N}_*} \Delta t \left[-\mu R(u, v) + \nu(\mathcal{P}^h(u) - u) \right] \delta(t - n\Delta t), \\ \partial_t v = \sum_{n \in \mathbb{N}_*} \Delta t \left[\mu R(u, v) + \nu(\mathcal{P}^h(v) - v) \right] \delta(t - n\Delta t), \end{cases} \quad (3.37)$$

which satisfies for any $k, l \in \mathbb{R}$ and any nonnegative test function in $\mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$:

$$\begin{aligned} & \partial_t \left[|u - k| + |v - l| \right] + \partial_x \left| f(u) - f(k) \right| \leq \\ & \sum_{n \in \mathbb{N}_*} \Delta t \left[\mu R(u, v) (\text{sgn}(v - l) - \text{sgn}(u - k)) \right. \\ & \left. + \nu(\mathcal{P}^h(u) - u) \text{sgn}(u - k) + \nu(\mathcal{P}^h(v) - v) \text{sgn}(v - l) \right] \delta(t - n\Delta t), \end{aligned} \quad (3.38)$$

in the sense of Propositions 1 and 2. Given any other set of initial data $(\tilde{u}_0, \tilde{v}_0)$ in $L^1 \cap BV(\mathbb{R})$, this last inequality implies the contraction property for any $t \in \mathbb{R}^+$:

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^1(\mathbb{R})} + \|v(\cdot, t) - \tilde{v}(\cdot, t)\|_{L^1(\mathbb{R})} \leq \|u_0 - \tilde{u}_0\|_{L^1(\mathbb{R})} + \|v_0 - \tilde{v}_0\|_{L^1(\mathbb{R})}.$$

Proof. Similar estimates than the ones shown in the proof of Lemma 2 ensure the strong L^1_{loc} compactness. The inequality (3.38) comes from (2.25) together with Propositions 1 and 2. The contraction property in $L^1(\mathbb{R})$ is proved the same way than for Theorem 3. \square

Remark 3 *It is important to notice that the entropy solutions of (3.37) do not coincide with the ones of (2.23) since the relaxation and the projection steps are not applied in the same order. This means in particular that one has to be careful when dealing with the right-hand sides as different processes are concentrated inside the Dirac masses. Hence there is no uniqueness problem for both systems (2.23), (3.37) relying on Propositions 1 and 2 together with the inequalities (2.24) and (3.38).*

We plan now to carry out the same program as we did for the classical time-splitting scheme in the preceding section. By letting $\nu \rightarrow +\infty$, we derive an equation whose entropy solution coincide with the piecewise constant approximation generated by a numerical scheme designed in the context of reactive Euler equations for which the reaction process operates *before* the projection stage, ¹⁰.

Proposition 5 *Assume (2.4) and $(u_0, v_0) \in L^1 \cap BV(\mathbb{R})$, the sequence of entropy solutions to (3.37) converges strongly in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ as $\nu \rightarrow +\infty$ towards the entropy solution of:*

$$\begin{aligned} \partial_t u + \partial_x f(u) &= \sum_{n \in \mathbb{N}_*} \left[-\mu \Delta t R(u, v) + (\mathcal{P}^h(u) - u)(\cdot, n\Delta t + 0) \right] \delta(t - n\Delta t), \\ \partial_t v &= \sum_{n \in \mathbb{N}_*} \left[\mu \Delta t R(u, v) + (\mathcal{P}^h(v) - v)(\cdot, n\Delta t + 0) \right] \delta(t - n\Delta t), \end{aligned} \quad (3.39)$$

where the right-hand sides are defined by means of (2.19), (2.20), (2.21). Moreover, the following entropy inequality holds true for any nonnegative test function in $\mathcal{D}(\mathbb{R} \times \mathbb{R}_*^+)$ with k, l in \mathbb{R} :

$$\begin{aligned} &\partial_t \left[|u - k| + |v - l| \right] + \partial_x \left| f(u) - f(k) \right| \leq \\ &\sum_{n \in \mathbb{N}_*} \left\{ \mu \Delta t R(u, v) (\text{sgn}(v - l) - \text{sgn}(u - k)) + \right. \\ &\left. (|\mathcal{P}^h(u) - k| - |u - k|) + (|\mathcal{P}^h(v) - l| - |v - l|) \right\} \delta(t - n\Delta t), \end{aligned} \quad (3.40)$$

The final statement is concerned with the relaxation limit in (3.39).

Theorem 5 *Under the assumptions (2.4), the CFL restriction $\Delta t \|f'(u)\|_{L^\infty} = h$ and for $v_0 = A(u_0)$, $u_0 \in L^1 \cap BV(\mathbb{R})$, the entropy solutions of (3.39), (3.40), (2.19), (2.20), (2.21) converge strongly in $L^1_{loc}(\mathbb{R} \times \mathbb{R}_*^+)$ towards u , the entropy solution in the sense of Kruřkov to:*

$$\partial_t (u + A(u)) + \partial_x f(u) = 0, \quad u(x, 0) = u_0.$$

as $\mu \Delta t \rightarrow +\infty, h \rightarrow 0$.

The proofs of these last two statements are completely similar to the ones of Proposition 3 and Theorem 4 hence we skip them.

4. Conclusion

In this paper, we considered some “measure source terms” for a quasilinear relaxation system whose weak solutions coincide with piecewise constant approximations generated by commonly-used time-splitting numerical schemes. This allows to establish convergence results directly relying on properties of the underlying original system and bypassing some heavy computations required to show the stability of the numerical processes (see *e.g.* §2.5 in ⁷).

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References

1. D. AREGBA-DRIOLLET AND R. NATALINI, *Convergence of relaxation schemes for conservation laws*, Appl. Anal., **61** (1996), pp. 163–196.
2. F. BOUCHUT, *Construction of BGK models with a family of kinetic entropies for a given system of conservation laws*, J. Stat. Phys. **95** (1999), pp. 113–170.
3. F. BOUCHUT AND B. PERTHAME, *Kružkov’s inequalities for scalar conservation laws revisited*, Trans. Amer. Math. Soc. **350** (1998), pp. 2847–2870.
4. H. BREZIS, *Analyse fonctionnelle: théorie et applications*, Masson, Paris (1983).
5. J.F. COLOMBEAU AND A. HEIBIG, *Nonconservative products in bounded variation functions*, SIAM J. Math. Anal., **23** (1992) pp. 941 – 949.
6. C.M. DAFERMOS, *Hyperbolic Conservation Laws in Continuum Physics*, Grundlehren der Mathematischen Wissenschaften, **325** (1999) Springer Verlag
7. L. GOSSE, *Localization effects and measure source terms in numerical schemes for balance laws*, Math. Comp. **71** (2002) pp. 553–582.
8. L. GOSSE AND G. TOSCANI, *An Asymptotic-Preserving Well-Balanced scheme for the hyperbolic heat equations*, C.R. Acad. Sc. Paris Série I, **334** (2002) pp. 337 – 342.
9. L. GOSSE AND A.E. TZAVARAS, *Convergence of relaxation approximations to the equations of elastodynamics*, Math. Comp. **70** (2001) pp. 555 – 577.
10. C. HELZEL, R.J. LEVEQUE AND G. WARNECKE, *A modified fractional step method for the accurate approximation of detonation waves*, SIAM J. Sci. Comp. **22** (2000) pp. 1489 – 1510.
11. F. JAMES, *Convergence results for some conservation laws with a reflux boundary condition and a relaxation term arising in chemical engineering*, SIAM J. Math. Anal. **29** (1998) pp. 1200–1223.
12. S. JIN, *Runge-Kutta Methods for Hyperbolic Conservation Laws with Stiff Relaxation Terms*, J. Comp. Phys., **122** (1995) pp. 51 – 67.
13. S. JIN, L. PARESCHI, G. TOSCANI, *Diffusive relaxation schemes for multiscale discrete-velocity kinetic equations*, SIAM J. Numer. Anal. **35** (1998) pp. 2405–2439
14. S. JIN, L. PARESCHI, G. TOSCANI, *Uniformly accurate diffusive relaxation schemes for multiscale transport equations*, SIAM J. Numer. Anal. **38** (2000) pp. 913–936,.
15. S. JIN AND Z. XIN, *The relaxing schemes for systems of conservation laws in arbitrary space dimensions*, Comm. Pure Appl. Math., **48** (1995) pp. 235 – 277.
16. M.A. KATSOULAKIS, G.T. KOSSIORIS AND CH. MAKRIDAKIS, *Convergence and error estimates of relaxation schemes for multidimensional conservation laws*, Comm. Part. Diff. Eqns., **24** (1999) pp 395 – 424.

17. M.A. KATSOULAKIS AND A.E. TZAVARAS, *Contractive relaxation systems and the scalar multidimensional conservation law*, Comm. Part. Diff. Eqns., **22** (1997), pp. 195 – 233.
18. S.N. KRUŽKOV, *First order quasilinear equations in several independent space variables*, Math. USSR Sbornik, **10** (1970) pp. 217 – 243.
19. A. KURGANOV AND E. TADMOR, *Stiff systems of hyperbolic conservation laws: convergence and error estimates*, SIAM J. Math. Anal. **28** (1997), pp. 1446–1456.
20. N.N. KUZNEČOV, *The accuracy of certain approximate methods for the computation of weak solutions of a first order quasilinear equation* (Russian) Ž. Vychisl. Mat. i Mat. Fiz. **16** (1976), pp. 1489–1502, 1627.
21. P.G. LEFLOCH AND A.E. TZAVARAS, *Representation of weak limits and definition of non-conservative products*, SIAM J. Math. Anal. **30** (1999), pp. 1309–1342.
22. P.L. LIONS AND G. TOSCANI, *Diffusive limits for finite velocity Boltzmann kinetic models*, Rev. Mat. Iberoamericana **13** (1997), pp. 1–34
23. T.P. LIU, *Hyperbolic conservation laws with relaxation*, Comm. Math. Phys. **108** (1987), pp. 153–175.
24. C. KLINGENBERG AND Y.G. LU, *Cauchy problem for hyperbolic conservation laws with a relaxation term*, Proc. Roy. Soc. Edinburgh Sect. A **126** (1996) pp. 821–828.
25. G. NALDI, L. PARESCHI, *Numerical Schemes for Hyperbolic Systems of Conservation Laws with Stiff Diffusive Relaxation*, SIAM J. Numer. Anal. **37** (2000) pp. 1246 – 1270.
26. R. NATALINI, *Recent results on hyperbolic relaxation problems*, in **Analysis of systems of conservation laws** (Aachen, 1997) pp. 128–198, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, 1999.
27. R. PEMBER, *Numerical methods for hyperbolic conservation laws with stiff relaxation I. Spurious solutions*, SIAM J. Appl. Math. **53** (1993) pp. 1293 – 1330.
28. B. PERTHAME, E. TADMOR, *A kinetic equation with kinetic entropy functions for scalar conservation laws*, Comm. Math. Phys. **136** (1991) 501 – 517.
29. J.P. RAYMOND, *A new definition of nonconservative products and weak stability results*, Boll. Un. Mat. Ital. B (7) **10**, (1996), pp. 681–699.
30. K. RHEE, R. ARIS AND N.R. AMUNDSEN, *First-order partial differential equations – I & II*, Prentice Hall International Series 1986, 1989.
31. H.J. SCHROLL, A. TVEITO AND R. WINTHER, *An L^1 -error bound for a semi-implicit difference scheme applied to a stiff system of conservation laws*, SIAM J. Numer. Anal. **34** (1997) pp. 1152–1166.
32. A. TVEITO AND R. WINTHER, *On the rate of convergence to equilibrium for a system of conservation laws with a relaxation term*, SIAM J. Math. Anal. **28** (1997), pp. 136–161.
33. A. VASSEUR, *Kinetic semi-discretization of scalar conservation laws and convergence by using averaging lemmas*, SIAM J. Numer. Anal. **36** (1999) pp. 465–474
34. A.I. VOL'PERT, *The space BV and quasilinear equations*, Math. USSR Sbornik, **2** (1967) pp. 225 – 267.