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Parameter estimation in physiological models.

Parameter estimation for stochastic differential equations  
from noisy observations. Maximum likelihood and  
filtering techniques.

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# Chapter 1

## I. Hidden Markov models. Exact likelihood.

### 1.1 Introduction

Consider a diffusion process  $(x_t, t \geq 0)$  given as the solution of a stochastic differential equation with unknown parameters in the drift and diffusion coefficients to be estimated. For simplicity, we consider that  $(x_t)$  is one-dimensional but multidimensional processes may be considered too.

At times  $0 \leq t_1 < \dots < t_n < \dots$ , noisy observations  $y_1, y_2, \dots, y_n, \dots$  of  $x_{t_1}, \dots, x_{t_n}, \dots$  are taken: for instance, one observes  $y_i = x_{t_i} + \varepsilon_i$  (additive noise) or  $y_i = x_{t_i} \varepsilon_i$  (multiplicative noise). More generally, the distribution of  $y_i$  given  $x_{t_i}$  may be specified by a kernel  $F(x_{t_i}, dy)$ . We intend to describe the exact likelihood of such observations and focus on models where explicit computations are possible.

In our first lecture, we give the assumptions implying that  $(y_i, i \geq 1)$  is a hidden Markov model. Under these assumptions, we explain how to compute the exact likelihood of  $(y_1, \dots, y_n)$  using the filtering-prediction algorithm. Some theoretical properties of the observed process  $(y_i, i \geq 1)$  and of the exact maximum likelihood estimator are given.

In the second lecture, we study discrete observations of diffusions observed with additive noise. Explicit formulae for the likelihood can be obtained for Gaussian diffusions with Gaussian noise using the Kalman filter approach. Even when the noise is non Gaussian, the previous likelihood can be used as a contrast (quasi-likelihood) to obtain consistent and asymptotically Gaussian estimators.

In the third lecture, we consider discrete observations of diffusions observed with multiplicative noise. The square-root diffusion model (Cox-Ingersoll-Ross diffusion) with a specific multiplicative noise allows to obtain an explicit expression of the likelihood.

In the fourth lecture, we look at other kernels such as Poisson observations or binomial observations with stochastic parameters described by a diffusion.

## 1.2 Hidden Markov models. The framework.

### 1.2.1 The hidden diffusion.

Consider a one-dimensional diffusion process  $(x_t, t \geq 0)$  described by:

$$dx_t = b(\theta, x_t)dt + \sigma(\theta, x_t)dW_t, \quad x_0 = \eta \quad (1.1)$$

where  $(W_t)_{t \geq 0}$  is a Wiener process defined on a probability space  $(\Omega, \mathcal{F}, P)$ ,  $\eta$  is a random variable on  $\Omega$  independent of  $(W_t)$ ,  $b(\theta, \cdot), \sigma(\theta, \cdot)$  are continuous functions on  $\mathbb{R}$ . The parameter  $\theta$  is unknown and belongs to a parameter set  $\Theta \subset \mathbb{R}^p$ . Although  $d$ -dimensional diffusion processes could be considered too, we focus on the one-dimensional case for the sake of simplicity.

Classical assumptions on  $b(\theta, \cdot), \sigma(\theta, \cdot)$  ensure that the stochastic differential equation (1.1) admits a unique strong solution and a unique stationary distribution  $\pi_\theta$  (see *e.g.* Rogers and Williams (1990)). We assume that these assumptions hold and that  $(x_t)$  is in stationary regime, *i.e.* that the initial variable  $\eta$  has distribution  $\pi_\theta$ . In this case,  $(x_t, t \geq 0)$  is strictly stationary, ergodic,  $\beta$ -mixing and therefore  $\alpha$ -mixing. For details, see *e.g.* Genon-Catalot *et al.* (2000). We denote by  $\mathcal{X}$  the state space of  $(x_t)$  which is an interval of the real line.

Let  $P_{\theta,t}(x, dx')$  denote the transition semigroup of  $(x_t)$ . Under standard regularity assumptions on the functions  $b(\theta, \cdot), \sigma(\theta, \cdot)$ , we know that :

$$P_{\theta,t}(x, dx') = p_{\theta,t}(x, x')dx', \quad \pi_\theta(dx) = g_{\theta,t}(x)dx, \quad (1.2)$$

where  $dx$  denotes the Lebesgue measure on the state space  $\mathcal{X}$  and  $p_{\theta,t}(x, x')$  is the transition density of  $P_{\theta,t}(x, dx')$  (see *e.g.* Rogers and Williams (1990)). The transition semigroups of diffusion processes have several nice properties (reversibility, spectral decomposition, eigenvalues, eigenfunctions, ergodicity ...) and this is why it is especially interesting to consider as hidden Markov process a hidden diffusion process.

### 1.2.2 Observations.

At equispaced instants  $t_i = i\Delta$  with  $\Delta > 0$ , one takes measurements  $(x_{t_i})$  but these measurements are not direct. We assume that, at time  $t_i$ , a random variable  $y_i$  is observed and that the following holds:

- (H1) (Conditional independence) Given  $(x_{t_i}, i \geq 0)$ , the random variables  $y_i$  are independent and the conditional distribution of  $y_i$  given  $(x_{t_j}, j \geq 0)$  only depends on  $x_{t_i}$ .

- (H2) (Stationarity) The conditional distribution of  $y_i$  given  $x_{t_i} = x$  does not depend on  $i$ .

Under these assumptions, the process  $(y_i)$  is called a hidden Markov model with hidden chain  $(x_{t_i})$  (see *e.g.* Leroux (1992), Bickel and Ritov (1996), Cappé *et al.* (2005)). The third important assumption concerns the kernel  $F(x, dy)$  specifying the conditional distribution of  $y_i$  given  $x_{t_i} = x$ . We denote by  $\mathcal{Y}$  the state space of  $y_i$ .

- (H3)  $F(x, dy) = f(y|x)\mu(dy)$  for some dominating positive measure  $\mu$  on  $\mathcal{Y}$ .

This means that the observation  $y_i$  is a random function of  $x_{t_i}$ . The existence of the density  $f(y|x)$  is crucial for the computation of the likelihood of  $(y_1, \dots, y_n)$  and of all the conditional distributions involved in filtering and prediction. Note that the transition operator of the hidden chain  $(x_{t_i})$  is  $P_\theta := P_{\theta, \Delta}$  where  $(P_{\theta, t}, t \geq 0)$  is the transition semi-group of  $(x_t)$ . We denote below  $p_\theta(x, x') := p_{\theta, \Delta}(x, x')$  the transition density of the hidden chain.

Let us now give some examples that will be detailed further on.

### 1.2.3 Examples.

- **Example 1: Additive noise.** Let  $(\varepsilon_i)$  be a sequence of real-valued i.i.d. random variables, having density  $f(\cdot)$ , independent of the whole process  $(x_t)$ . Assume that  $y_i = x_{t_i} + \varepsilon_i, i \geq 0$ . Then, assumptions are fulfilled and  $f(y|x) = f(y - x)$ . When  $(x_t)$  is Gaussian and  $f$  is a Gaussian density, we are in the Kalman filter model.
- **Example 2: Multiplicative noise.** With  $(\varepsilon_i)$  as in Example 1, consider  $y_i = x_{t_i}\varepsilon_i$ . Then,  $f(y|x) = \frac{1}{x}f(\frac{y}{x})$ . When  $\varepsilon_i$  has law  $\mathcal{N}(0, 1)$ ,  $(y_i)$  is often called a stochastic volatility models (in discrete time). This model is investigated in Ruiz (1994), with  $(x_t)$  equal to the exponential of an Ornstein-Uhlenbeck process. With another kind of noise and with  $(x_t = \sqrt{r_t})$  and  $(r_t)$  a Cox-Ingersoll-Ross diffusion, the model is treated in Chaleyat-Maurel and Genon-Catalot (2006). With  $(x_t)$  equal to the absolute value of an Ornstein-Uhlenbeck diffusion, the model is studied in Comte *et al.* (2008).
- **Example 3: Other kernels.**
  - (Poisson observations) Suppose that, given  $(x_t)$ ,  $y_i$  has Poisson distribution with parameter  $\lambda(x_{t_i})$  for some continuous function  $\lambda : \mathcal{X} \rightarrow (0, +\infty)$ . Then  $\mathcal{Y} = \mathbb{N}$  and  $F(x, dy) = f(y|x)\mu(dy)$  with

$$f(y|x) = \exp(-\lambda(x)) \frac{\lambda(x)^y}{y!}, \quad \mu(y) = 1, y \in \mathbb{N}.$$

When  $\mathcal{X} = [0, +\infty)$ ,  $\lambda(x) = \lambda x + \mu$  (with  $\lambda > 0, \mu \geq 0$ ) and  $(x_t = r_t)$  with  $(r_t)$  a Cox-Ingersoll-Ross diffusion or the square of an Ornstein-Uhlenbeck process, explicit computations are possible (see Chaleyat-Maurel and Genon-Catalot (2006)). The continuous time version of this model is studied in Boel and Benes (1980).

- (Binomial observations) suppose that  $\mathcal{X} = [0, 1]$  and that, given  $(x_t)$ ,  $y_i$  has binomial distribution with parameters  $N, x_{t_i}$ . In this case,  $\mathcal{Y} = \{0, 1, \dots, N\}$ ,  $\mu(y) = 1$  for  $y = 0, 1, \dots, N$  and  $f(y|x) = \binom{N}{y} x^y (1-x)^{N-y}$ . With  $(x_t)$  a Wright-Fisher diffusion process, the model is studied in Chaleyat-Maurel and Genon-Catalot (2008). With the same unobserved diffusion, geometric observations or negative binomial observations with parameter  $x$  also lead to explicit computations.

### 1.3 Filtering. Prediction. Marginals.

Recall the notations and assumptions. Since  $\Delta$  is fixed, we set  $x_i = x_{i\Delta}$ . The following holds.

- (A0) The chain  $(x_i)$  is strictly stationary, ergodic, with transition operator  $P_\theta(x, dx') = p_\theta(x, x') dx'$  and stationary distribution  $\pi_\theta(dx) = g_\theta(x) dx$  where  $dx$  denotes the Lebesgue measure on the interval  $\mathcal{X}$  (state space of the hidden diffusion process).
- (A1) Given  $(x_i)$ , the random variables  $(y_i)$  are independent and the conditional distribution of  $y_i$  given  $(x_i, j \geq 0)$  when  $x_i = x$  is given by  $F(x, dy) = f(y|x) \mu(dy)$  where  $\mu$  is a positive measure on the state space  $\mathcal{Y}$  of  $y_i$ .

**Proposition 1.3.1.** *The joint process  $(x_i, y_i)$  is Markov with transition kernel*

$$Q(x, y; dx', dy') = p_\theta(x, x') f(y'|x') dx' \mu(dy').$$

Moreover, the process is strictly stationary, ergodic with marginal distribution  $g_\theta(x) f(y|x) dx \mu(dy)$ .

**Proof.** For  $\psi : \mathcal{Y} \rightarrow \mathbb{R}^+$ , set

$$h_\psi(x) = \int_{\mathcal{Y}} \psi(y) f(y|x) \mu(dy).$$

Consider positive functions  $\varphi_i, i = 1, \dots, n$  on  $\mathcal{X}$  and positive functions  $\psi_i, i = 1, \dots, n$  on  $\mathcal{Y}$ . Using successively the conditional independence property and the Markov property of  $(x_i)$ , we get:

$$\mathbb{E} \left( \prod_{i=1}^n \varphi_i(x_i) \psi_i(y_i) \right) = \mathbb{E} \left( \prod_{i=1}^n \varphi_i(x_i) h_{\psi_i}(x_i) \right) = \mathbb{E} \left( \prod_{i=1}^{n-1} \varphi_i(x_i) P_\theta(\varphi_n h_{\psi_n})(x_{n-1}) \right),$$

where

$$P_\theta(\varphi_n h_{\psi_n})(x) = \int_{\mathcal{X} \times \mathcal{Y}} \varphi_n(x') \psi_n(y') p_\theta(x, x') f(y'|x') dx' \mu(dy').$$

So, there appears the transition kernel  $Q(x, y; dx', dy')$ .

The stationarity is immediate. For the ergodicity (which is not immediate), we refer to *e.g.* Leroux (1992) for a direct proof or Genon-Catalot *et al.* (2000) where ergodicity is proved using  $\alpha$ -mixing.  $\square$

Now, we are in position to compute the following distributions:

- (Predictive distributions):  $\nu_{i|i-1:1}(dx) = \mathcal{L}(x_i|y_{i-1}, \dots, y_1), i \geq 1$ , with, by convention,  $\nu_{1|0:1}(dx) = \mathcal{L}(x_1)$ .
- (Filtering distributions):  $\nu_{i|i:1}(dx) = \mathcal{L}(x_i|y_i, \dots, y_1), i \geq 1$
- (Marginal distributions):  $\mu_{i|i-1:1}(dy) = \mathcal{L}(y_i|y_{i-1}, \dots, y_1), i \geq 1$  with, by convention,  $\mu_{1|0:1}(dy) = \mathcal{L}(y_1)$ .

**Theorem 1.3.1.** *The distributions  $\nu_{i|i-1:1}(dx), \nu_{i|i:1}(dx)\mu_{i|i:1}(dx)$  can be recursively computed using the three following operators:*

1. *Up-dating operator: For  $\nu$  a probability measure on  $\mathcal{X}$ , and  $y \in \mathcal{Y}$ ,  $\varphi_y(\nu)$  is the probability measure on  $\mathcal{X}$  defined by:*

$$\varphi_y(\nu)(dx) = \frac{f(y|x)\nu(dx)}{p_\nu(y)}, \quad \text{with} \quad p_\nu(y) = \int_{\mathcal{X}} \nu(dx)f(y|x). \quad (1.3)$$

2. *Prediction operator: For  $\nu$  a probability measure on  $\mathcal{X}$ ,  $\psi_\theta(\nu) = \nu P_\theta$  is the probability on  $\mathcal{X}$  defined by:*

$$\nu P_\theta(dx') = \left( \int_{\mathcal{X}} \nu(dx)p_\theta(x, x') \right) dx' \quad (1.4)$$

3. *Marginal operator: For  $\nu$  a probability measure on  $\mathcal{X}$ , we define the marginal distribution on  $\mathcal{Y}$ ,  $p_\nu(y)\mu(dy)$  with  $p_\nu(y)$  given above.*

Then, the algorithm is as follows. Set  $\nu_{1|0:1}(dx) = \mathcal{L}(x_1)$ . We have, for all  $i \geq 1$ ,

$$\text{(up-dating)} \quad \nu_{i|i:1} = \varphi_{y_i}(\nu_{i|i-1:1}), \quad \text{(prediction)} \quad \nu_{i+1|i:1} = \nu_{i|i:1} P_\theta,$$

$$\text{(marginal)} \quad \mu_{i|i-1:1}(dy) = p_{\nu_{i|i-1:1}}(y)\mu(dy).$$

**Proof.** To simplify notations, we denote by  $p(z_1, \dots, z_n)$  the density of any  $n$ -tuple  $(z_1, \dots, z_n)$  of random variables and by  $p(z_i|z_1, \dots, z_n)$  the conditional density of a random variable  $z_i$  given  $(z_1, \dots, z_n)$ . We use the symbol  $\propto$  to ignore constants in densities. For the first up-dating, it is immediate that:

$$\nu_{1|1:1}(dx) \propto \nu_{1|0:1}(dx)f(y|x).$$

Hence,  $\nu_{1|1:1} = \varphi_{y_1}(\nu_{1|0:1})$ . Then, we have

$$p(x_i|y_i, \dots, y_1) \propto p(x_i, y_i, \dots, y_1),$$

where

$$\begin{aligned} p(x_i, y_i, \dots, y_1) &= \int_{\mathcal{X}^{i-1}} g_\theta(x_1) f(y_1|x_1) \left( \prod_{j=2}^i p_\theta(x_{j-1}, x_j) f(y_j|x_j) \right) dx_1 \dots dx_{i-1} \\ &= p(x_i, y_{i-1}, \dots, y_1) f(y_i|x_i) \propto \nu_{i|i-1:1}(dx_i) f(y_i|x_i) \propto \varphi_{y_i}(\nu_{i|i-1:1})(dx_i). \end{aligned}$$

For the prediction, we have

$$p(x_{i+1}|y_i, \dots, y_1) \propto p(x_{i+1}, y_i, \dots, y_1),$$

with

$$\begin{aligned} p(x_{i+1}, y_i, \dots, y_1) &= \int_{\mathcal{X}^i} p(x_1, \dots, x_i, x_{i+1}, y_1, \dots, y_i) dx_1 \dots dx_i \\ &= \int_{\mathcal{X}^i} g_\theta(x_1) f(y_1|x_1) \left( \prod_{j=2}^i p_\theta(x_{j-1}, x_j) f(y_j|x_j) \right) p_\theta(x_i, x_{i+1}) dx_1 \dots dx_i \\ &= \int dx_i p(x_i, y_i, \dots, y_1) p_\theta(x_i, x_{i+1}) \propto \nu_{i|i:1} P_\theta(dx_{i+1}). \end{aligned}$$

For marginals, we have

$$\begin{aligned} p(y_i|y_{i-1}, \dots, y_1) &\propto p(y_1, \dots, y_i) = \int_{\mathcal{X}^i} p(x_1, \dots, x_i, y_1, \dots, y_i) dx_1 \dots dx_i \\ &= \int_{\mathcal{X}^i} g_\theta(x_1) f(y_1|x_1) \left( \prod_{j=2}^{i-1} p_\theta(x_{j-1}, x_j) f(y_j|x_j) \right) p_\theta(x_{i-1}, x_i) f(y_i|x_i) dx_1 \dots dx_i \\ &= \int dx_i p(x_i, y_{i-1}, \dots, y_1) f(y_i|x_i) \propto \int \nu_{i|i-1:1}(dx_i) f(y_i|x_i). \end{aligned}$$

Hence, the results.  $\square$

Let us stress that, although the steps of the prediction-filtering algorithm are simple, there are few models where computations can be done explicitly. Let us define the two compound operators:

$$\Phi_y = \psi_\theta \circ \varphi_y, \quad \text{and} \quad \Psi_y = \varphi_y \circ \psi_\theta. \quad (1.5)$$

The iterations of  $\Phi_y$  define the algorithm for predictive distributions. We have

$$\nu_{i|i-1:1} = \Phi_{y_{i-1}} \circ \dots \circ \Phi_{y_1}(\nu_{1|0:1}).$$

The iterations of  $\Psi_y$  define the algorithm for filtering distributions:

$$\nu_{i|i:1} = \Psi_{y_i} \circ \dots \circ \Psi_{y_2}(\nu_{1|1:1}).$$

In filtering theory, authors generally concentrate on the filtering distributions  $\nu_{i|i:1}$ . For statistical inference, the predictive distributions are more important because they give the marginal distributions used to compute the likelihood.

## 1.4 Likelihood.

### 1.4.1 Exact likelihood.

Several formulae are available for the exact likelihood of  $(y_1, \dots, y_n)$ . First, we can integrate the joint density of  $(x_1, \dots, x_n, y_1, \dots, y_n)$  with respect to  $x_1, \dots, x_n$ . This gives:

$$p_n(\theta, y_1, \dots, y_n) = \int_{\mathcal{X}^n} dx_1 \dots dx_n g_\theta(x_1) f(y_1|x_1) \prod_{i=2}^n p_\theta(x_{i-1}, x_i) f(y_i|x_i). \quad (1.6)$$

Setting

$$p_n(\theta, y_1, \dots, y_n|x_1) = f(y_1|x_1) \int_{\mathcal{X}^{n-1}} dx_2 \dots dx_n \prod_{i=2}^n p_\theta(x_{i-1}, x_i) f(y_i|x_i) \quad (1.7)$$

which is the conditional density of  $(y_1, \dots, y_n)$  given  $x_1$ , we obtain:

$$p_n(\theta, y_1, \dots, y_n) = \int_{\mathcal{X}} dx_1 g_\theta(x_1) p_n(\theta, y_1, \dots, y_n|x_1). \quad (1.8)$$

Finally, we have

$$p_n(\theta, y_1, \dots, y_n) = p_1(\theta, y_1) \prod_{i=2}^n p_i(\theta, y_i|y_{i-1}, \dots, y_1), \quad (1.9)$$

where  $p_i(\theta, y_i|y_{i-1}, \dots, y_1) = p_{\nu_{i|i-1:1}}(y_i)$  is the conditional density of  $y_i$  given  $(y_{i-1}, \dots, y_1)$ . The latter formula gives a recursive way of computing the exact likelihood provided that the successive marginal distributions are explicitly computable.

### 1.4.2 The Leroux method.

Formula (1.8) suggests, following Leroux (1992), to consider other functions of  $(y_1, \dots, y_n)$  and  $\theta$  that can be used as contrast functions to build maximum contrast estimators. For  $g$  a probability density on  $\mathcal{X}$ , set

$$p_n^g(\theta, y_1, \dots, y_n) = \int_{\mathcal{X}} dx_1 g(x_1) p_n(\theta, y_1, \dots, y_n|x_1). \quad (1.10)$$

Analogously, we have:

$$p_n^g(\theta, y_1, \dots, y_n) = p_1^g(\theta, y_1) \prod_{i=2}^n p_i^g(\theta, y_i|y_{i-1}, \dots, y_1), \quad (1.11)$$

where the successive terms of the product are computed via the filtering-prediction-marginal algorithm starting with the initial density  $g$ . The interest of this approach is that, for a well-chosen  $g$ ,  $p_n^g$  may be much simpler to compute than the exact likelihood. Moreover, Leroux's

paper, completed by Genon-Catalot and Larédo (2006), proves that the estimators computed by maximizing the  $p_n^g$  have good asymptotic properties like the exact maximum likelihood estimator.

## 1.5 Asymptotic properties of the exact maximum likelihood estimator and related estimators.

Denote by  $\Theta$  the parameter set. Let  $\hat{\theta}_n$  (resp.  $\tilde{\theta}_n^g$ ) be any solution of

$$p_n(\hat{\theta}_n, y_1, \dots, y_n) = \sup_{\theta \in \Theta} p_n(\theta, y_1, \dots, y_n),$$

(resp.

$$p_n^g(\tilde{\theta}_n^g, y_1, \dots, y_n) = \sup_{\theta \in \Theta} p_n^g(\theta, y_1, \dots, y_n).)$$

The study of  $\hat{\theta}_n$  (and  $\tilde{\theta}_n^g$ ) is a difficult problem and several papers have dealt with the subject. We may quote:

- Leroux (1992) proves the strong consistency of  $\hat{\theta}_n$  and  $\tilde{\theta}_n^g$ , under very mild assumptions, when the state space of the hidden chain is finite.
- Bickel *et al.* (1998) prove the asymptotic normality of  $\hat{\theta}_n$  also for  $\mathcal{X}$  finite under additional assumptions.
- Douc and Matias (2001) prove the strong consistency and the asymptotic normality of  $\hat{\theta}_n$  when  $\mathcal{X}$  is compact (see also references given therein).
- Cheng Der Fuh (2006) claims that strong consistency and asymptotic normality of  $\hat{\theta}_n$  for a general state space  $\mathcal{X}$  (including the non compact case) is proved. But, the paper is obscure and contains errors.
- Genon-Catalot and Larédo (2006) study, in the spirit of Leroux's paper, the likelihood (1.8) and the contrast (1.10) for a non compact state space  $\mathcal{X}$  and some related properties. A special attention is given to the Kalman filter model. In particular, an adequate choice of  $g$  in (1.10) simplifies considerably the formula and the study of the associated estimator and gives a clear identifiability assumption. See also Genon-Catalot *et al.* (2003).
- Douc *et al.* (2009) give a proof of consistency of the exact maximum likelihood estimator under general assumptions including the Kalman model and non linear state-space models.

## 1.6 Appendix

Consider a one-dimensional diffusion process  $(x_t, t \geq 0)$  given by:

$$dx_t = b(x_t)dt + \sigma(x_t)dW_t, \quad x_0 = \eta, \quad (1.12)$$

where  $(W_t)$  is a Wiener process,  $\eta$  a real valued random variable independent of  $(W_t)$ . Consider the following assumptions:

- The functions  $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$  are continuous.
- There exists an interval  $(l, r)$  such that  $b, \sigma$  are  $C^1$  on  $(l, r)$  and  $\sigma^2(x) > 0$  for all  $x \in (l, r)$ .
- The scale density given by

$$s(x) = \exp\left(-2 \int_{x_0}^x \frac{b(u)}{\sigma^2(u)} du\right), \quad x_0, x \in (l, r)$$

satisfies  $\int_l^x s(x)dx = +\infty = \int^r s(x)dx$ .

- The speed measure given by

$$m(x) = \frac{1}{\sigma^2(x)s(x)}$$

satisfies  $\int_l^r m(x)dx = M < \infty$ .

- $P(\eta \in (l, r)) = 1$ .

Then, equation (1.12) admits a unique strong solution, which is a positive recurrent diffusion process on the state space  $(l, r)$ . The probability  $\pi(dx) = M^{-1}m(x)1_{(l,r)}(x)dx$  is the unique invariant probability of (1.12). If  $\eta$  has distribution  $\pi$ ,  $(x_t)$  is strictly stationary, ergodic and  $\beta$ -mixing.

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# Chapter 2

## Gaussian diffusions and additive noise.

### 2.1 Introduction

In this second lecture, we consider a Gaussian diffusion, *i.e.* an Ornstein-Uhlenbeck process, observed in discrete time with an additive perturbation. When the perturbation is Gaussian, this model is the well-known Kalman filter, which is of common use in the field of filtering (on-line estimation of the unobserved data). Likelihood inference in such models is also rather standard. Nevertheless, it is indeed difficult and we intend to focus on some maybe less known features of this model. In particular, we insist on the links with hidden Markov models and the classical Gaussian likelihood theory. We give the expression of the exact likelihood and some related contrasts which yield estimators asymptotically equivalent to the exact maximum likelihood estimator. For simplicity, we consider the one-dimensional Ornstein-Uhlenbeck process first and give indications for the multidimensional case later on. There is a huge number of references on the subject. Therefore, it is difficult to give an exhaustive list. Since we rely mostly on hidden Markov models, we refer to Brockwell and Davies (1991), Cappé *et al.* (2005) for general properties and to specific papers quoted in the text.

### 2.2 Model and observations.

Let  $(x(t), t \geq 0)$  be given by:

$$dx(t) = \alpha x(t)dt + cdW_t, \quad x(0) = \eta \quad (2.1)$$

with  $\eta$  a real random variable independent of the Brownian motion  $W$ . We assume that  $\alpha < 0$  and set  $\theta = (\alpha, c^2)$  for the unknown parameter. This process admits a stationary distribution  $\pi_\theta(dx) = \mathcal{N}(0, \sigma_s^2(\theta))$  with

$$\sigma_s^2(\theta) = \frac{c^2}{2|\alpha|}, \quad (2.2)$$

and we assume that the distribution of  $\eta$  is the stationary distribution so that the process  $(x(t))$  is strictly stationary, Gaussian and ergodic. Solving (2.1) yields, for all  $t, h \geq 0$ :

$$x(t+h) = e^{\alpha h} x(t) + Z_{t,h}, \quad (2.3)$$

where

$$Z_{t,h} = ce^{\alpha(t+h)} \int_t^{t+h} e^{-\alpha s} dW_s$$

is independent of  $\mathcal{F}_t = \sigma(\eta, W_s, s \leq t)$  and has distribution  $\mathcal{N}(0, \beta^2(h))$  with

$$\beta^2(h) = c^2 \frac{e^{2\alpha h} - 1}{2\alpha}. \quad (2.4)$$

Considering a sampling interval  $\Delta$ , the observation times  $t_i = i\Delta$  and setting  $x_i = x(i\Delta)$  for the discretized process, we get:

$$x_i = ax_{i-1} + \beta\eta_i, \quad x_0 = \eta, \quad (2.5)$$

where  $a = e^{\alpha\Delta}$ ,  $\beta = \beta(\Delta)$  and  $(\eta_i \geq 1)$  is a sequence of i.i.d. random variables independent of  $\eta$  with distribution  $\mathcal{N}(0, 1)$ . Note that obviously, for all  $\Delta$ ,

$$\sigma_s^2(\theta) = \frac{\beta^2}{1 - a^2}.$$

At time  $t_i$ , the observation is

$$y_i = x_i + \varepsilon_i, \quad (2.6)$$

where  $(\varepsilon_i)$  is a sequence of i.i.d. random variables with law  $\mathcal{N}(0, \gamma^2)$  independent of  $(x(t))$ . We assume (for simplicity) that  $\gamma^2$  is known. The joint process  $(x_i, y_i)$  is a hidden Markov model (H1-H2 are immediate). The hidden chain  $(x_i)$  has state space  $\mathcal{X} = \mathbb{R}$ , transition kernel  $P_\theta(x, dx') = \mathcal{N}(ax, \beta^2)(dx')$  and transition density

$$p_\theta(x, x') = \frac{1}{\beta\sqrt{2\pi}} \exp\left(-\frac{(x' - ax)^2}{2\beta^2}\right).$$

The observation space is  $\mathcal{Y} = \mathbb{R}$  and  $F(x, dy) = f(y|x)dy$  with

$$f(y|x) = \frac{1}{\gamma\sqrt{2\pi}} \exp\left(-\frac{(y-x)^2}{2\gamma^2}\right).$$

## 2.3 Conditional distributions.

To compute the exact likelihood associated with the observation  $(y_1, \dots, y_n)$ , we need to compute the predictive distributions  $\nu_{i|i-1:1}^\theta(dx) = \mathcal{L}(x_i|y_{i-1}, \dots, y_1)$ ,  $i \geq 1$ , from which we derive the conditional densities

$$p_i(\theta, y_i|y_{i-1}, \dots, y_1) = p_{\nu_{i|i-1:1}^\theta}(y_i) = \int f(y_i|x)\nu_{i|i-1:1}^\theta(dx).$$

Usually, in the case of the Kalman filter model, these distributions are directly computed using the fact that they are all Gaussian. Hence, one can compute directly the conditional means and variances. This approach has a drawback. It cannot be generalized to non Gaussian models. On the contrary, the hidden Markov model approach is general. Moreover, in the case of the Kalman model, it is especially simple. Indeed, we only need to compute the up-dating operator  $\varphi_y$ , the prediction operator  $\psi_\theta$ . Then, the compound operator  $\Phi_y^\theta = \psi_\theta \circ \varphi_y$  gives the algorithm for predictive distributions. We recover the special feature of this model as these operators evolve within the family of Gaussian distributions (with the convention that a Dirac mass  $\delta_x$  is considered as a Gaussian distribution with mean  $x$  and nul variance).

**Proposition 2.3.1.** • (Up-dating operator) If  $\nu = \mathcal{N}(m, \sigma^2)$  and  $y \in \mathbb{R}$ , then,  $\varphi_y(\nu) = \mathcal{N}(\hat{m}(y), \hat{\sigma}^2)$  with

$$\hat{m}(y) = \hat{\sigma}^2 \left( \frac{y}{\gamma^2} + \frac{m}{\sigma^2} \right), \quad \hat{\sigma}^2 = \frac{\sigma^2 \gamma^2}{\sigma^2 + \gamma^2}.$$

• (Prediction operator) If  $\nu = \mathcal{N}(m, \sigma^2)$ , then  $\psi_\theta(\nu) = \nu P_\theta = \mathcal{N}(\bar{m}, \bar{\sigma}^2)$  with

$$\bar{m} = am, \quad \bar{\sigma}^2 = \beta^2 + a^2 \sigma^2.$$

• (Marginal operator) If  $\nu = \mathcal{N}(m, \sigma^2)$ ,  $p_\nu(y) = \mathcal{N}(m, \sigma^2 + \gamma^2)$ .

**Proof.** These results are elementary. For the first and third points, consider  $(X, Y)$  such that  $X \sim \nu = \mathcal{N}(m, \sigma^2)$  and given  $X = x$ ,  $Y \sim \mathcal{N}(x, \gamma^2)$ , then,  $\varphi_y(\nu)$  is exactly the conditional distribution of  $X$  given  $Y = y$  and  $p_\nu(y)dy$  is the marginal distribution of  $Y$ . For the second point, consider  $(X, X')$  with  $X \sim \nu = \mathcal{N}(m, \sigma^2)$  and  $X' = aX + \eta_1$  with  $\eta_1 \sim \mathcal{N}(0, \beta^2)$  independent of  $X$ , then  $\psi_\theta(\nu)$  is the distribution of  $X'$ .  $\square$

**Corollary 2.3.1.** (Operator for predictive distributions) If  $\nu = \mathcal{N}(m, \sigma^2)$ , then,  $\Phi_y^\theta(\nu) = \psi_\theta \circ \varphi_y(\nu) = \mathcal{N}(\tilde{m}, \tilde{\sigma}^2)$  with

$$\tilde{m} := \Phi_y^\theta(m, \sigma^2) = a \left( m\delta(\sigma^2) + y(1 - \delta(\sigma^2)) \right), \quad \text{with} \quad \delta(\sigma^2) = \frac{\gamma^2}{\gamma^2 + \sigma^2},$$

$$\tilde{\sigma}^2 := \Phi^\theta(\sigma^2) = \beta^2 + a^2 \sigma^2 \delta(\sigma^2)$$

The corollary is an obvious consequence of the previous proposition. Let us stress some specific features which appear now. First, since only Gaussian distributions are involved, they are completely specified by their mean and variances. Hence, the operators acting on measures are simplified into operators acting on  $\mathbb{R} \times \mathbb{R}^+$ . This corresponds to the definition of a finite-dimensional filter. Moreover, note that the variance of  $\Phi_y(\nu)$  only depends on  $\sigma^2$  and neither on  $m$  nor on  $y$ ). Now, we state the following useful lemma.

**Lemma 2.3.1.** The function

$$v \rightarrow \Phi^\theta(v) = \beta^2 + a^2 v \delta(v) = \beta^2 + a^2 v \frac{\gamma^2}{\gamma^2 + v}$$

giving the variance of the prediction algorithm is increasing from  $I = [\beta^2, \frac{\beta^2}{1-a^2}]$  onto  $I$  and Lipschitz with constant  $a^2$ . Consequently,  $\Phi^\theta$  admits a unique fixed point  $\sigma_\infty^2(\theta) \in I$  defined as the solution of

$$\Phi^\theta(\sigma_\infty^2) = \sigma_\infty^2(\theta).$$

Moreover, for all  $v \in I$ , the  $n$ -th iterate  $\Phi^\theta \circ \Phi^\theta \circ \dots \circ \Phi^\theta(v)$  tends with exponential rate  $a^{2n}$  to  $\sigma_\infty^2(\theta)$  as  $n$  tends to infinity. In particular, this holds for  $v = \frac{\beta^2}{1-a^2}$ .

The proof is elementary and omitted. The fixed point can be explicitly computed since it solves a simple second degree equation. We do not need its explicit expression.

Now, for further use, we introduce the dependence on the initial distribution (distribution of  $x_1$ ) in the notations. We denote by  $\nu_{i|i-1:1}^\theta(\nu)$  the conditional distribution of  $x_i$  given  $y_{i-1}, \dots, y_1$  when  $\mathcal{L}(x_1) = \nu = \nu_{1|0:1}^\theta(\nu)$ . With  $\nu = \mathcal{N}(m, \sigma^2)$ , we have:

$$\nu_{i|i-1:1}^\theta(\nu) = \mathcal{N}(m_{i|i-1:1}(\theta, (m, \sigma^2)), \sigma_{i|i-1:1}^2(\theta, \sigma^2)) = \Phi_{y_{i-1}}^\theta \circ \dots \circ \Phi_{y_1}^\theta(\nu). \quad (2.7)$$

The predictive conditional mean

$$\mathbb{E}_\nu^\theta(x_{i+1}|y_i, \dots, y_1) = m_{i+1|i:1}(\theta, (m, \sigma^2)) = \Phi_{y_i}^\theta(m_{i|i-1:1}(\theta, (m, \sigma^2))), \quad m_{1|0:1}(\theta, (m, \sigma^2)) = m \quad (2.8)$$

depends on the initial distribution (through  $(m, \sigma^2)$ ) and the previous observations  $(y_i, \dots, y_1)$ . And, the conditional variance of  $x_{i+1}$  given  $y_i, \dots, y_1$

$$\sigma_{i+1|i:1}^2(\theta, \sigma^2) = \Phi^\theta(\sigma_{i|i-1:1}^2(\theta, \sigma^2)), \quad \sigma_{1|0:1}^2(\theta, \sigma^2) = \sigma^2 \quad (2.9)$$

is deterministic. Using Lemma 2.3.1, it converges as  $i$  tends to infinity to the fixed point  $\sigma_\infty^2(\theta)$  of  $\Phi^\theta$ . Finally, the conditional distribution of  $y_i$  given  $y_{i-1}, \dots, y_1$  when the initial distribution is  $\nu = \mathcal{N}(m, \sigma^2)$  is equal to

$$p_i^{(m, \sigma^2)}(\theta, y_i | y_{i-1}, \dots, y_1) = \mathcal{N}(m_{i|i-1:1}(\theta, (m, \sigma^2)), \gamma^2 + \sigma_{i|i-1:1}^2(\theta, \sigma^2)) \quad (2.10)$$

The distribution of  $y_1$  is equal to

$$p_1^{(m, \sigma^2)}(\theta, y_1) = \mathcal{N}(m_{1|0:1}(\theta, (m, \sigma^2)), \gamma^2 + \sigma_{1|0:1}^2(\theta, \sigma^2)) = \mathcal{N}(m, \gamma^2 + \sigma^2).$$

## 2.4 Exact likelihood and related contrasts.

If the initial distribution is  $\nu = \mathcal{N}(m, \sigma^2)$ , then the exact likelihood is

$$p_n^{(m, \sigma^2)}(\theta, y_1, \dots, y_n) = p_1^{(m, \sigma^2)}(\theta, y_1) \prod_{i=2}^n p_i^{(m, \sigma^2)}(\theta, y_i | y_{i-1}, \dots, y_1).$$

Therefore, under our assumptions, the exact likelihood corresponds to  $\nu = \pi_\theta$ , *i.e.*  $(m, \sigma^2) = (0, \sigma_s^2(\theta))$ . For the exact likelihood, we simply omit the superscript in the notation and write

$$p_n^{(0, \sigma_s^2(\theta))}(\theta, y_1, \dots, y_n) = p_n(\theta, y_1, \dots, y_n).$$

The exact maximum likelihood estimator  $\hat{\theta}_n$  is computed as a maximizer of the above function. Nevertheless, the function  $p_n^{(m, \sigma^2)}(\theta, y_1, \dots, y_n)$  can be considered as a contrast function and the associated maximum contrast estimator can be studied.

Let us stress the fact that the expression of the exact likelihood is far from simple. Indeed, we have:

$$p_n(\theta, y_1, \dots, y_n) \propto \prod_{i=1}^n (\gamma^2 + \sigma_{i|i-1:1}^2(\theta))^{-1/2} \exp \left[ -\frac{(y_i - m_{i|i-1:1}(\theta))^2}{2(\gamma^2 + \sigma_{i|i-1:1}^2(\theta))} \right], \quad (2.11)$$

where we have set for simplicity:

$$m_{i|i-1:1}(\theta) = m_{i|i-1:1}(\theta, (0, \sigma_s^2(\theta))) = \Phi_{y_{i-1}}^\theta \circ \dots \circ \Phi_{y_1}^\theta((0, \sigma_s^2(\theta))),$$

$$\sigma_{i|i-1:1}^2(\theta) = \sigma_{i|i-1:1}^2(\theta, \sigma_s^2(\theta)) = \Phi^\theta \circ \dots \circ \Phi^\theta(\sigma_s^2(\theta)).$$

The iterations can be solved explicitly but lead to rather complicated sums. Nevertheless, the computation of the exact maximum likelihood estimator is numerically feasible. The score function and the hessian can be recursively computed.

Consistency and asymptotic normality with rate  $\sqrt{n}$  hold for the exact maximum likelihood estimator. This is evidently well known and can be obtained by a classical approach for Gaussian likelihoods (see below).

Following Genon-Catalot and Larédo (2006), instead of looking at the exact likelihood, we can consider the contrasts  $p_n^{(m, \sigma^2)}(\theta, y_1, \dots, y_n)$  for  $(m, \sigma^2) \neq (0, \sigma_s^2(\theta))$ . It can be proved that the associated maximum contrast estimators are asymptotically equivalent to the exact maximum likelihood estimator whatever  $(m, \sigma^2)$  provided that  $\sigma^2 > 0$ . Consequently, we may choose in a convenient way the values  $m, \sigma^2$  and study the associated contrast instead of the exact likelihood. The choice

$$m = 0, \quad \sigma^2 = \sigma_\infty^2(\theta)$$

leads to an noteworthy simplification. We denote by  $\tilde{p}_n(\theta, y_1, \dots, y_n) = p_n^{(0, \sigma_\infty^2(\theta))}(\theta, y_1, \dots, y_n)$  the contrast corresponding to this choice. Let us note that, since the initial variance is equal to the fixed point of  $\Phi^\theta$ , this value remains unchanged along the iterations. As a consequence, the algorithm for the mean is simplified into:

$$H_y^\theta(m) := \Phi_y^\theta(m, \sigma_\infty^2(\theta)) = a \left( \frac{y}{\gamma^2} + \frac{m}{\sigma_\infty^2(\theta)} \right) \frac{\sigma_\infty^2(\theta) \gamma^2}{\sigma_\infty^2(\theta) + \gamma^2}.$$

Let us set

$$\rho = \frac{\sigma_\infty^2(\theta)}{\sigma_\infty^2(\theta) + \gamma^2} \in (0, 1).$$

Then,

$$H_y^\theta(m) = a(\rho y + m(1 - \rho)).$$

Hence,

$$H_{y_{i-1}}^\theta \circ \dots \circ H_{y_1}^\theta(0) = \sum_{j=0}^{i-2} a\rho(a(1 - \rho))^j y_{i-1-j} = x_i(\theta, y_{i-1}, \dots, y_1).$$

Thus

$$\tilde{p}_n(\theta, y_1, \dots, y_n) \propto (\gamma^2 + \sigma_\infty^2(\theta))^{-n/2} \exp\left(-\sum_{i=1}^n \frac{(y_i - x_i(\theta, y_{i-1}, \dots, y_1))^2}{2(\gamma^2 + \sigma_\infty^2(\theta))}\right).$$

This expression is much easier to handle. In particular, it allows to obtain an explicit and simple representation of the limit of the normalized log-likelihood. For details and further results, we refer to Genon-Catalot *et al.* (2003) and Genon-Catalot and Larédo (2006).

## 2.5 ARMA property and consequences.

We establish the links with the standard likelihood theory for stationary Gaussian processes. For this, refer to *e.g.* Dzhaparidze and Yaglom (1983), Dacunha-Castelle and Duflo (1986), Brockwell and Davies (1991).

**Proposition 2.5.1.** *The process  $(y_i)$  is ARMA(1, 1). Its spectral density is equal to*

$$f_\theta(\lambda) = \frac{\beta^2 + \gamma^2(1 + a^2) - 2a\gamma^2 \cos \lambda}{1 + a^2 - 2a \cos \lambda}.$$

**Proof.** Setting  $\xi_i = y_i - ay_{i-1} = \beta\eta_i + \varepsilon_i - a\varepsilon_{i-1}$ , we see that  $\text{Cov}(\xi_i, \xi_{i+k}) = 0$  for  $k \geq 1$ . Thus,  $(\xi_i)$  is MA(1). The ARMA property follows. Since  $\text{Var}\xi_i = \beta^2 + \gamma^2(1 + a^2)$  and  $\text{Cov}(\xi_i, \xi_{i+1}) = -a\gamma^2$ , using the ARMA property, we deduce the spectral density.  $\square$

This simple result has some important consequences. First, it provides another way to deduce the asymptotic behaviour of the exact maximum likelihood estimator since the asymptotic behaviour of exact m.l.e. in stationary ARMA Gaussian processes is well known.

First, after some computations, we have the following identifiability property:

$$\forall \lambda, \quad f_\theta(\lambda) = f_{\theta'}(\lambda) \Rightarrow \theta = \theta'.$$

Then, setting  $\theta = (\theta_1, \theta_2)$  (with  $\theta_1 = a, \theta_2 = \beta^2$ ), the matrix

$$I(\theta) = \left( \int_{-\pi}^{\pi} f_\theta^{-2}(\lambda) \frac{\partial f_\theta}{\partial \theta_i}(\lambda) \frac{\partial f_\theta}{\partial \theta_j}(\lambda) \frac{d\lambda}{2\pi} \right)_{1 \leq i, j \leq 2}$$

is the asymptotic Fisher information matrix. Under some additional standard assumptions,  $\sqrt{n}(\hat{\theta}_n - \theta)$  converges in distribution to  $\mathcal{N}(0, I^{-1}(\theta))$ .

For stationary Gaussian processes, the Whittle approximation of the likelihood defines a contrast which yields minimum contrasts estimators which are asymptotically equivalent to the exact maximum likelihood estimator. Let us define the periodogram:

$$I_n(\lambda) = \frac{1}{n} \left| \sum_{j=1}^n y_j \exp(-ij\lambda) \right|^2.$$

The Whittle contrast is given by:

$$U_n(\theta) = \int_{-\pi}^{\pi} \left( \log f_{\theta}(\lambda) + \frac{I_n(\lambda)}{f_{\theta}(\lambda)} \right) \frac{d\lambda}{2\pi}.$$

The associated minimum contrast estimators are defined as minimisers of  $U_n(\theta)$ .

## 2.6 Non Gaussian noises.

Suppose that the noise  $\varepsilon_i$  is non Gaussian. Then, it is possible to use the Gaussian likelihood as a contrast. This yields consistent and asymptotically Gaussian estimators which are much better than simple moment estimators. This is demonstrated *e.g.* in Ruiz (1994).

## 2.7 Multidimensional extension.

Consider a  $d$ -dimensional Ornstein-Uhlenbeck process  $(X(t))$  satisfying:

$$dX(t) = AX(t)dt + \Sigma dW_t, \quad X(0) = \eta,$$

with  $\eta$  a  $\mathbb{R}^d$ -valued random variable, independent of the Brownian motion  $W$  of  $\mathbb{R}^d$  and  $A$  is a  $(d, d)$  matrix. Assume that the observations are:

$$Y(t_i) = HX(t_i) + \varepsilon_i$$

where  $H$  is a known  $(k, d)$ -matrix  $\varepsilon_i$  is a sequence of i.i.d. variables with law  $\mathcal{N}_k(0, Q)$ , independent of  $(X(t))$ .

We make some simplifying assumptions. Assume that the matrix  $A$  is diagonalisable with negative eigenvalues  $(\lambda_i, i = 1, \dots, d)$ . Denote by  $P$  a matrix of eigenvectors such that  $P^{-1}AP = D := \text{diag}((\lambda_i, i = 1, \dots, d))$ . Then, the process  $Z(t) = P^{-1}X(t)$  satisfies:

$$dZ(t) = DZ(t)dt + \Gamma dW_t, \quad \Gamma = P^{-1}\Sigma.$$

Since  $(X(t))$  is not observed, we can change the model into:

$$Y_i = Y(t_i) = JZ(t_i) + \varepsilon_i, \quad J = HP.$$

It is worth noting that each column of  $P$  (eigenvector of  $P$ ) is defined up to a multiplicative constant. In some cases,  $P$  can be chosen such that  $J = HP$  does not depend on unknown parameters. Let us assume that this holds. Therefore, the unknown parameters are  $\theta = (\lambda_i, i = 1, \dots, d, \gamma_{i,j}, i, j = 1, \dots, d)$  where the  $\gamma_{i,j}$ 's are the elements of  $\Gamma$ .

The discretization of  $(Z(t))$  standardly yields a AR(1) process  $Z_i = Z(i\Delta)$  such that

$$Z_{i+1} = \exp(D\Delta)Z_i + \exp(D(i+1)\Delta) \int_{i\Delta}^{(i+1)\Delta} \exp(-Ds) \Gamma dW_s.$$

Here, the matrix  $\exp(D\Delta) = \text{diag}(\exp(\lambda_i\Delta), i = 1, \dots, d)$  is diagonal. The kernel of the hidden chain  $(Z_i)$  is

$$P_\theta(z, dz') = \mathcal{N}(\exp(D\Delta)z, R)$$

where

$$R = \int_0^\Delta \exp(Du) \Gamma \Gamma' \exp(Du) du = \left( \alpha_{i,j} \frac{e^{(\lambda_i + \lambda_j)\Delta} - 1}{\lambda_i + \lambda_j} \right),$$

where the  $\alpha_{i,j}$ 's are the elements of  $\Gamma \Gamma'$ . The process  $(Z(t))$  admits a unique stationary distribution equal to the law  $\mathcal{N}(0, V)$  with

$$V = \int_0^{+\infty} \exp(Du) \Gamma \Gamma' \exp(Du) du = \left( \frac{-\alpha_{i,j}}{\lambda_i + \lambda_j} \right).$$

The conditional distribution of  $Y_i$  given  $Z_i = z$  is the distribution

$$F(z, dy) = \mathcal{N}(Jz, Q)(dy).$$

Hence, the up-dating and prediction operators can be easily computed. This allows to obtain the exact likelihood. We state the proposition analogous to Proposition 2.3.1:

**Proposition 2.7.1.** • (*Up-dating operator*) If  $\nu = \mathcal{N}_d(m, K)$  and  $y \in \mathbb{R}^k$ , then  $\varphi_y(\nu) = \mathcal{N}_d(\hat{m}(y), \hat{K})$  with

$$\hat{m}(y) = m + K J' (JK J' + Q)^{-1} (y - Jm), \quad \hat{K} = K - K J' (JK J' + Q)^{-1} JK.$$

• (*Prediction*) If  $\nu = \mathcal{N}_d(m, K)$ , then  $\psi_\theta(\nu) = \nu P_\theta = \mathcal{N}_d(\bar{m}, \bar{K})$  with

$$\bar{m} = \exp(D\Delta)m, \quad \bar{K} = \exp(D\Delta)K \exp(D\Delta) + R.$$

• (*marginal operator*) If  $\nu = \mathcal{N}_d(m, K)$ , then  $p_\nu(y) = \mathcal{N}_k(Jm, JK J' + Q)$ .

The algorithm of predictive distributions is ruled by the compound operator  $\Phi_y = \psi_\theta \circ \varphi_y$ . The exact likelihood is obtained recursively. The score function and the Hessian are also obtained recursively.

Note that the following matrix relation holds:

$$(I + K J' Q^{-1} J)^{-1} = I - K J' (JK J' + Q)^{-1} J.$$

(Compare with the one-dimensional relation  $\frac{1}{1+\frac{\alpha}{x}} = 1 - \frac{\alpha}{\alpha+x}$ ).

In Favetto and Samson (2010), a complete study of a partially observed model with biological motivation is treated. The unobserved process is a two-dimensional Ornstein-Uhlenbeck process and the observation is the sum of the components to which a noise is added. A special attention is given to the recursive computation of the score function and the Hessian and to the identifiability of parameters.

Note that Pedersen (1994) gives a description of the computation of the likelihood in the general case.

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## 2.8 Appendix

We detail the iterations of the operator  $\Phi_y^\theta$  for predictive distributions given in Corollary 2.3.1. Starting with  $\nu = \mathcal{N}(m, \sigma^2)$ , let us set:

$$m_{1|0:1} = m, \quad \sigma_{1|0:1}^2 = \sigma^2,$$

$$m_{i|i-1:1}(m, \sigma^2) = \Phi_{y_{i-1}}^\theta \circ \dots \circ \Phi_{y_1}^\theta(m, \sigma^2), \quad \sigma_{i|i-1:1}^2(\sigma^2) = \Phi^\theta \circ \dots \circ \Phi^\theta(\sigma^2)$$

$$\delta_i = \frac{\gamma^2}{\gamma^2 + \sigma_{i|i-1:1}^2(\sigma^2)}.$$

Then,

$$m_{i|i-1:1}(m, \sigma^2) = a^{i-1} \delta_i \delta_{i-1} \dots \delta_1 m + a \sum_{l=1}^{i-1} y_{i-l} (1 - \delta_{i-l}) a^{l-1} \delta_{i-1} \delta_{i-2} \dots \delta_{i-l+1}.$$

The conditional distributions of  $y_i$  given  $y_{i-1}, \dots, y_1$  is the Gaussian distribution with mean  $m_{i|i-1:1}(m, \sigma^2)$  and variance  $\sigma_{i|i-1:1}^2(\sigma^2) + \gamma^2$ .

As seen above, if the iterations start with  $m = 0$  and  $\sigma^2 = \sigma_\infty^2(\theta)$ , the expressions are considerably simpler as  $\delta_i = \delta = 1 - \rho$  remains constant.

# Chapter 3

## Diffusions with multiplicative noise.

### 3.1 Introduction

Consider a one-dimensional diffusion process  $(x(t), t \geq 0)$  and assume that the observations taken at times  $0 \leq t_1 < \dots < t_n < \dots$  are of the form:  $y_i = x(t_i)\varepsilon_i$  where  $(\varepsilon_i)$  is a sequence of i.i.d. random variables independent of the process  $(x(t))$ . The noise  $(\varepsilon_i)$  is multiplicative. As it is natural for scale perturbations, we assume that  $(x(t))$  is non negative. As for the noise, we may assume that the  $\varepsilon_i$ 's are nonnegative or signed and symmetric. We present below models for which the up-dating, prediction and marginal operators can be computed explicitly. Consequently, the exact likelihood is also explicit. As for the Kalman model, explicit computations are obtained for a specific class of diffusion models and for a specific class of noises distributions. For this lecture, we refer to Genon-Catalot (2003), Genon-Catalot and Kessler (2004), Chaleyat-Maurel and Genon-Catalot (2006), Comte *et al.* (2007).

### 3.2 Computable filters.

Computations of the conditional distributions of  $x(t_i)$ ,  $x(t_{i+1})$  or  $y_{i+1}$  given  $y_1, \dots, y_i$  rely on iterations of the up-dating, prediction and marginal operators. These iterations are rapidly intractable unless both the up-dating and the prediction operators evolve within a relatively simple class of distributions on the state space of the hidden process. The ideal situation is when this class is a parametric family, *i.e.* a family of distributions specified by a fixed number of parameters. This is the case of the Kalman filter model: the up-dating and prediction operators both evolve within the family of Gaussian distributions. Hence, it is enough to specify recursively the means and variances. Such an ideal situation is not often encountered. Hence, the idea is to find a larger class built using mixtures of parametric distributions.

Recall the general notations. We have an unobserved Markov chain  $(x_i)$  (which we have

supposed to be a regular discrete sampling of a diffusion process) with state space  $\mathcal{X}$  and transition kernel  $P_\theta(x, dx')$ . Observations  $y_i$  are such that  $\mathcal{L}(y_i|x_i = x) = f(y|x)\mu(dy)$ . The up-dating operator acting on distributions on  $\mathcal{X}$  is the mapping:

$$\nu \rightarrow \varphi_y(\nu) \propto f(y|x)\nu(dx).$$

The proportionality coefficient is the marginal distribution  $p_\nu(y)$ . The prediction operator  $\psi_\theta$  is the mapping:

$$\nu \rightarrow \psi_\theta(\nu) = \nu P_\theta.$$

In Chaleyat-Maurel and Genon-Catalot (2006), sufficient conditions are exhibited in order to obtain a “computable filter” model. We detail these conditions now.

**Proposition 3.2.1.** *Let  $\mathcal{F} = \{\nu_c, c \in C\}$  be a parametric family of distributions on  $\mathcal{X}$  where  $C \subset \mathbb{R}^p$  is a set of parameters (and  $c \neq c'$  implies  $\nu_c \neq \nu_{c'}$ ). Consider the enlarged class*

$$\bar{\mathcal{F}}_f = \left\{ \nu = \sum_{i=0}^L \alpha_i \nu_{c_i}, L \in \mathbb{N}, c_i \in C, \alpha_i \geq 0, i = 0, 1, \dots, L, \sum_{i=0}^L \alpha_i = 1 \right\}$$

composed of finite mixtures of distributions of  $\mathcal{F}$  and the following conditions:

- (C1)  $\mathcal{F}$  is a conjugate class for the family  $y \rightarrow f(y|x), x \in \mathcal{X}$ .
- (C2) If  $\nu \in \mathcal{F}$ , then  $\psi_\theta(\nu) = \nu P_\theta \in \bar{\mathcal{F}}_f$ .

Then, if  $\nu \in \bar{\mathcal{F}}_f$ , then, for all  $y$ ,  $\varphi_y(\nu)$  and  $\psi_\theta(\nu)$  both belong to  $\bar{\mathcal{F}}_f$ .

Condition (C1) means that, if  $c \in C$ , then  $\varphi_y(\nu_c)$  belongs to  $\mathcal{F}$ , i.e.:

$$\varphi_y(\nu_c) = \nu_{\varphi_y(c)} \tag{3.1}$$

where  $\varphi_y(c) \in C$ . Since we can identify a distribution  $\nu_c$  to its parameter  $c$ , the operator  $\varphi_y$  is identified to the mapping:

$$c \in C \rightarrow \varphi_y(c) \in C. \tag{3.2}$$

Condition (C1) is classical in Bayesian statistics for obtaining explicit Bayes estimators. Indeed, interpreting  $\nu$  as a prior on  $x$  for the parametric family of densities ( $y \rightarrow f(y|x), x \in \mathcal{X}$ ),  $\varphi_y(\nu)$  is the corresponding posterior distribution.

Condition (C2) means that, if  $c \in C$ , then  $\psi_\theta(\nu_c)$  is a finite mixture of the form

$$\psi_\theta(\nu_c) = \sum_{i=0}^{L_c} \alpha_i(c) \nu_{\tau_i(c)}, \tag{3.3}$$

where  $L_c, \alpha_i(c), \tau_i(c)$  depend on  $\theta$ . Hence, the image of  $\nu_c$  by the prediction operator  $\psi_\theta$  is not an element of  $\mathcal{F}$ , but an element of  $\bar{\mathcal{F}}_f$ . This is why we are not in the ideal situation of a finite-dimensional filter and we need an extension of this notion, namely, the notion of computable filter.

The proof of Proposition 3.2.1 is simple algebra (using (3.1)-(3.3)) and omitted (see Chaleyat-Maurel and Genon-Catalot (2006)). Nevertheless, the result is important since it induces a recursive algorithm to compute the filtering or predictive distributions. The difficulty lies in finding the adequate class  $\mathcal{F}$ . This is what we give in our examples.

Let us stress some facts. First, a distribution  $\nu = \sum_{i=0}^L \alpha_i \nu_{c_i}$  in the class  $\bar{\mathcal{F}}_f$  is specified by the finite sequence of parameters  $(L, \alpha_0, \dots, \alpha_L, c_0, c_1, \dots, c_L)$ . Hence, it is “computable”. Along the iterations of the prediction and up-dating operators, all these parameters will be up-dated. This includes the length  $L$  of the mixture. Hence, the situation is not the same as in the Kalman filter. The number of parameters to be specified may change along iterations. In this sense, a “computable filter” is not a finite-dimensional filter.

It is also important to note that (C1) only concerns the kernel  $f(y|x)\mu(dy)$  whereas (C2) only concerns the kernel  $P_\theta(x, dx')$ . Therefore, they may be checked separately.

### 3.3 Hidden diffusion model. Prediction operator.

Now, we give a diffusion process  $(x(t))$  and a class of distributions  $\mathcal{F}$  such that condition (C2) is fulfilled.

Consider the one-dimensional diffusion process  $(x_t, t \geq 0)$  described by:

$$dx(t) = (2\theta x(t) + \delta c^2)dt + 2c\sqrt{x(t)}dW_t, \quad x(0) = \eta, \quad (3.4)$$

with  $\eta$  a random variable independent of the Brownian motion  $(W_t)$ . When  $\eta$  is nonnegative,  $\theta \in \mathbb{R}$ ,  $\sigma > 0$ ,  $\delta \geq 0$ ,  $x(t)$  is uniquely defined and  $x(t) \geq 0$  for all  $t \geq 0$ . First, we give a representation of  $(x(t))$  when  $\delta$  is a positive integer.

**Proposition 3.3.1.** *Let  $\delta \geq 1$  an integer, and consider  $(\xi_t^i, i = 1, \dots, \delta)$   $\delta$  i.i.d. Ornstein-Uhlenbeck processes satisfying*

$$d\xi_t^i = \theta \xi_t^i dt + c dW_t^i, \quad \xi_0^i = x^i,$$

where  $(W^i, i = 1, \dots, \delta)$  are independent Wiener processes,  $x^i, i = 1, \dots, \delta$  are real values. Then, setting  $x(t) = \sum_{i=1}^{\delta} (\xi_t^i)^2$ ,

$$dx(t) = (2\theta x(t) + \delta c^2)dt + 2c\sqrt{x(t)}dB_t, \quad x(0) = \sum_{i=1}^{\delta} (x^i)^2,$$

where  $(B_t)$  is a Wiener process.

**Proof.** Applying the Ito formula, we get:

$$dx(t) = (2\theta x(t) + \delta c^2)dt + 2c \sum_{i=1}^{\delta} \xi_t^i dW_t^i.$$

Consider  $\tilde{B}$  a Wiener process independent of  $(W^i, i = 1, \dots, \delta)$  and set

$$B_t = \int_0^t 1_{(x(s)>0)} \frac{\sum_{i=1}^{\delta} \xi_s^i dW_s^i}{x^{1/2}(t)} + \int_0^t 1_{(x(s)=0)} d\tilde{B}_s.$$

We find  $\langle B \rangle_t = t$ , so  $B$  is a Wiener process and

$$\sqrt{x(t)} dB_t = \sum_{i=1}^{\delta} \xi_t^i dW_t^i,$$

which gives the result.  $\square$

For general positive  $\delta$ , the process  $(x(t))$  has been largely popularized by its use for modelling interest rate data (see Cox *et al.* (1985)). Moreover, the CIR process is used to model the volatility in the stochastic volatility model proposed by Heston (1993). In the latter paper, a financial asset price  $S(t)$  is described by:

$$d \log S(t) = \sqrt{x(t)} dB_t$$

with  $(B_t)$  a Wiener process independent of  $(x(t))$ . Therefore,

$$\log^2 \frac{S((i+1)\Delta)}{S(i\Delta)} \simeq x(i\Delta) \varepsilon_i,$$

where  $\varepsilon_i = (B_{(i+1)\Delta} - B_{i\Delta})^2$ . However, with such noises distribution, it is not possible to get explicit computations of the filters. This is why we introduce below other distributions for the noises.

Let us introduce

$$a(t) = e^{\theta t}, \quad \beta(t) = c \left( \frac{e^{\theta t} - 1}{2\theta} \right)^{1/2}. \quad (3.5)$$

The transition density of  $(x(t))$  can be written as a mixture of distributions: for  $x \geq 0$ ,

$$p_t(x, x') = \sum_{i \geq 0} w_i \left( \frac{a^2(t)x}{\beta^2(t)} \right) g_{i, \beta(t)}(x'), \quad (3.6)$$

where, for  $i \geq 0$ , the weights  $w_i$  are given by

$$u \in \mathbb{R}, \quad w_i(u) = \exp(-u/2) (u/2)^i / i! \quad (3.7)$$

and the function

$$g_{i, \sigma}(x) = 1_{(x>0)} \frac{1}{\sigma \sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma^2}\right) \frac{x^{i-1+\frac{\delta}{2}}}{C_{2i+\delta-1} \sigma^{2i+\delta-1}}, \quad (3.8)$$

is the density of a Gamma distribution  $G(i + \delta/2, 1/2\sigma^2)$  with scale parameter  $1/2\sigma^2$  and location parameter  $(i + \delta/2)$ . The normalising constant  $C_a$ , is simply the absolute moment of order  $a$  of a Gaussian standard variable, i.e. , for  $a \geq 0$  or  $b \geq 1/2$ ,

$$C_a = \mathbb{E}(|X|^a) = \frac{2^{a/2}}{\sqrt{\pi}} \Gamma\left(\frac{a+1}{2}\right), \quad \Gamma(b) = \frac{\sqrt{2\pi}}{2^b} C_{2b-1}$$

for  $X$  a standard Gaussian variable ( $\Gamma$  is the usual Gamma function). The transition density  $p_t(x, x')$  is exactly a  $\beta^2(t)\chi'^2(\frac{a^2(t)x}{\beta^2(t)}, \delta)$ .

For  $\delta = 1$ ,  $x(t) = \xi_t^2$  is the square of an Ornstein-Uhlenbeck process. When  $\theta < 0$ , this process admits a unique stationary distribution  $\pi_\vartheta$  equal to the distribution of  $\xi^2$  with  $\xi \sim \mathcal{N}(0, \sigma_s^2(\vartheta))$  and

$$\sigma_s^2(\vartheta) = \frac{c^2}{2|\theta|}. \quad (3.9)$$

The invariant distribution is therefore

$$\pi_\vartheta(dx) = 1_{(x>0)} \frac{1}{\sigma_s(\theta)\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma_s^2(\theta)}\right) x^{-\frac{1}{2}} dx = G(1/2, \frac{1}{2\sigma_s^2(\theta)}).$$

When  $\delta \geq 2$  and  $\theta < 0$ ,  $x(t)$  is positive recurrent on  $(0, +\infty)$ . In this case, its unique stationary distribution obtained by normalizing its speed measure is  $\pi_\vartheta(dx) = G(\delta/2, \frac{1}{2\sigma_s^2(\theta)})$ .

Let us note that formula (3.6) still holds in the case  $x = 0$  and we have:

$$p_t(0, x') = g_{0,\beta(t)}(x').$$

The special structure of the transition density of  $(x(t))$  suggests to introduce the class

$$\mathcal{F}^\delta = \{\nu_{i,\sigma} = g_{i,\sigma}(x)dx, i \in \mathbb{N}, \sigma \geq 0\}, \quad (3.10)$$

where, by convention  $\nu_{i,0} = \delta_0$  for all  $i$  (note that, as  $\sigma \rightarrow 0$ ,  $\nu_{i,0} \Rightarrow \delta_0$ ). The stationary distribution belongs to  $\mathcal{F}^\delta$  and corresponds to  $i = 0, \sigma = \sigma_s(\vartheta)$ . We define the extended class  $\bar{\mathcal{F}}^\delta$  as the class of mixtures of distributions  $\nu_{i,\sigma}$  having the same scale parameter:

$$\bar{\mathcal{F}}^\delta = \{\nu = \nu_{\alpha,\sigma} = \sum_{i \geq 0} \alpha_i \nu_{i,\sigma}, \alpha = (\alpha_i), \alpha_i \geq 0, \sum_{i \geq 0} \alpha_i = 1, \sigma \geq 0\}. \quad (3.11)$$

The sub-class  $\bar{\mathcal{F}}_f^\delta$  is composed of all distributions  $\nu_{\alpha,\sigma}$  having finite-length mixture parameter, *i.e.* such that

$$L(\alpha) = \sup\{i; \alpha_i > 0\} < \infty.$$

Note that the transition density belongs to the class  $\bar{\mathcal{F}}^\delta$  of infinite mixtures.

Now, we state the result concerning the prediction operator.

**Proposition 3.3.2.** (*Prediction operator*) Let  $\vartheta = (\theta, c^2, \delta)$  denote the unknown parameters. For  $\Delta > 0$ , let  $P_\Delta^\vartheta$  denote the transition operator of  $(x(t))$  with step  $\Delta$ . The prediction operator  $\psi_\vartheta$  satisfies the following property.

- If  $\nu_{i,\sigma}$  belongs to  $\mathcal{F}^\delta$  with  $\sigma > 0$ , then

$$\psi_\vartheta(\nu_{i,\sigma}) = \nu_{i,\sigma} P_\Delta^\vartheta = \sum_{l=0}^i \alpha_l^{i,\sigma} \nu_{l,\tau(\sigma)},$$

with  $\tau^2(\sigma) = \beta^2(\Delta) + a^2(\Delta)\sigma^2$  and for  $l = 0, \dots, i$ ,

$$\alpha_i^{i,\sigma} = \binom{i}{l} \left(1 - \frac{\beta^2(\Delta)}{\tau^2(\sigma)}\right)^l \left(\frac{\beta^2(\Delta)}{\tau^2(\sigma)}\right)^{i-l}.$$

If  $\sigma = 0$ , then,  $\psi_\vartheta(\delta_0) = \nu_{0,\beta(\Delta)}$ .

- If  $\nu = \nu_{\alpha,\sigma}$  belongs to  $\bar{\mathcal{F}}_f^\delta$  with  $\sigma > 0$ ,

$$\psi_\vartheta(\nu_{\alpha,\sigma}) = \nu_{\alpha,\sigma} P_\Delta^\vartheta = \sum_{l \geq 0} \bar{\alpha}_l \nu_{l,\tau(\sigma)}$$

with, for  $l \geq 0$ ,

$$\bar{\alpha}_l = \bar{\alpha}_l(\alpha, \sigma) = \sum_{i \geq l} \alpha_i \alpha_i^{i,\sigma}.$$

Moreover,  $L(\bar{\alpha}) = L(\alpha)$ . If  $\sigma = 0$ , then,  $\psi_\vartheta(\delta_0) = \nu_{0,\beta(\Delta)}$  and  $L(\bar{\alpha}) = 0$ .

Hence, condition (C2) holds for the class  $\mathcal{F}^\delta$  and the extended class  $\bar{\mathcal{F}}_f^\delta$ .

Note that the moments of a distribution  $\nu_{\alpha,\sigma}$  are obtained easily: for  $r \geq 0$ ,

$$\int x^r \nu_{\alpha,\sigma}(dx) = \sigma^{2r} \sum_{i \geq 0} \alpha_i \frac{C_{2(i+r)+\delta-1}}{C_{2i+\delta-1}}. \quad (3.12)$$

It can be simplified using the classical relation  $C_{a+1} = aC_{a-1}$ . In particular, for  $r = 1$ ,

$$\int x \nu_{\alpha,\sigma}(dx) = \sigma^2 \sum_{i \geq 0} \alpha_i (2i + \delta) = \sigma^2 (\delta + 2 \sum_{i \geq 0} i \alpha_i).$$

For all  $r \geq 1$ ,

$$\int x^r \nu_{\alpha,\sigma}(dx) = \sigma^{2r} \sum_{i \geq 0} \alpha_i ([2i + 2(r-1) + \delta][2i + 2(r-2) + \delta] \dots [2i + \delta]). \quad (3.13)$$

The variance is given by:

$$\text{Var} \nu_{\alpha,\sigma} = \sigma^4 [2\delta + 4 \sum_{i \geq 0} i(i+1) \alpha_i - 4(\sum_{i \geq 0} i \alpha_i)^2].$$

### 3.4 Noise distribution. Up-dating operator.

Recall that we observe  $y_i = x_i \varepsilon_i$  with  $x_i = x(i\Delta)$ . In order to fulfill condition (C1), we choose the noise distribution as follows. The random variables  $\varepsilon_i$  are chosen to have Inverse Gamma

distribution  $\text{Inv}G(k, \lambda)$  with scale parameter  $\lambda$  and location parameter  $k$  which must be a positive integer, *i.e.* the distribution of  $y_i$  given  $x_i = x$  has density (w.r.t.  $dy$ )

$$f(y|x) = 1_{y>0} e^{-\frac{\lambda x}{y}} \frac{x^k \lambda^k}{y^{k+1} \Gamma(k)}. \quad (3.14)$$

This is the density of  $\lambda x/G(k, 1)$ .

It is worth noting that  $\mathbb{E}(\varepsilon^r) < \infty$  holds if and only if  $r < k$ . In this case,

$$\mathbb{E}(\varepsilon^r) = \frac{\lambda^r \Gamma(k-r)}{\Gamma(k)}.$$

**Proposition 3.4.1.** (*up-dating and marginal operators*)

- If  $\nu_{i,\sigma}$  belongs to  $\mathcal{F}^\delta$  and  $\sigma > 0$ , then, for all positive  $y$ ,

$$\varphi_y(\nu_{i,\sigma}) = \nu_{i+k, \varphi_y(\sigma)}$$

with

$$\frac{1}{\varphi_y^2(\sigma)} = \frac{1}{\sigma^2} + \frac{2\lambda}{y}.$$

If  $y = 0$  or  $\sigma = 0$ ,  $\varphi_y(\nu) = \delta_0$ . Hence, the following holds, for all non negative  $y$ :

$$\varphi_y^2(\sigma) = \frac{\sigma^2 y}{y + 2\lambda\sigma^2}.$$

The corresponding marginal distribution has density (when  $\sigma > 0$ ):

$$p_{\nu_{i,\sigma}}(y) = \frac{\lambda^k}{\Gamma(k)} \frac{C_{2(k+i)+\delta-1}}{C_{2i+\delta-1}} \frac{\sigma^{2k}}{y} \frac{y^{i+\frac{\delta}{2}}}{(y + 2\lambda\sigma^2)^{i+k+\frac{\delta}{2}}}$$

If  $y$  or  $\sigma$  equal 0, then, the marginal distribution is  $\delta_0$ .

- If  $\nu = \nu_{\alpha,\sigma}$  belongs to  $\bar{\mathcal{F}}_f^\delta$ ,

$$\varphi_y(\nu) = \sum_{i \geq 0} \hat{\alpha}_i \nu_{i, \varphi_y(\sigma)}$$

where

$$\hat{\alpha}_i = \hat{\alpha}_i(\alpha, \sigma) \propto 1_{i \geq k} \alpha_{i-k} \frac{C_{2i+\delta-1}}{C_{2(i-k)+\delta-1}} \left( \frac{\varphi_y^2(\sigma)}{\sigma^2} \right)^{i-k}$$

Moreover,  $L(\hat{\alpha}) = L(\alpha) + k$  except if  $y = 0$  in which case  $\varphi_y(\nu) = \delta_0$  (with  $\hat{\alpha}(k) = 1$ ).

For the marginal distribution, we have, when  $\sigma > 0$ ,

$$p_\nu(y) = \sum_{i \geq 0} \alpha_i p_{\nu_{i,\sigma}}(y).$$

Condition (C1) holds for the class  $\mathcal{F}^\delta$  and the extended class  $\bar{\mathcal{F}}_f^\delta$ .

### 3.5 Algorithm for predictive distributions.

Now, we have to compute the iterations of  $\Phi_y^\vartheta = \psi_\vartheta \circ \varphi_y$ . By the previous paragraph, when the distribution of  $x_1$  belongs to the class  $\bar{\mathcal{F}}_f^\delta$ , for all  $i$ , the distributions  $\nu_{i|i-1:1}(dx), \nu_{i|i:1}(dx)$  also belong to the class  $\bar{\mathcal{F}}_f^\delta$ . It is therefore enough to specify these distributions by their parameters  $\alpha_{i|i-1:1}, \alpha_{i|i:1}$  (mixture parameters),  $L_{i|i-1:1} = L(\alpha_{i|i-1:1}), L_{i|i:1} = L(\alpha_{i|i:1})$  (lengths of the mixture parameters),  $\sigma_{i|i-1:1}, \sigma_{i|i:1}$  (scale parameters)

**Proposition 3.5.1.** *Assume that  $\nu_{1|0:1}$  belongs to the class  $\bar{\mathcal{F}}_f^\delta$ . The scale and mixture and lengths parameters of  $\nu_{i|i-1:1}(dx), \nu_{i|i:1}(dx)$  can be recursively computed as follows.*

- (up-dating) For  $i \geq 1$ ,

$$\sigma_{i|i:1}^2 = \frac{\sigma_{i|i-1:1}^2 y_i}{y_i + 2\lambda \sigma_{i|i-1:1}^2}.$$

For  $j \geq 0$ ,

$$\alpha_{i|i:1}(j) \propto 1_{j \geq k} \alpha_{i|i-1:1}(j-k) \frac{C_{2j+\delta-1}}{C_{2(j-k)+\delta-1}} \left( \frac{\sigma_{i|i:1}^2}{\sigma_{i|i-1:1}^2} \right)^{j-k}.$$

If  $y_i \neq 0$ ,

$$L_{i|i:1} = L_{i|i-1:1} + k.$$

If  $y_i = 0$ ,  $\alpha_{i|i:1}(k) = 1$  and  $L_{i|i:1} = k$ .

- (prediction)

$$\sigma_{i+1|i:1}^2 = \beta^2(\Delta) + a^2(\Delta) \sigma_{i|i:1}^2.$$

For  $j \geq 0$ ,

$$\alpha_{i+1|i:1}(j) = \sum_{l \geq j} \alpha_{i|i:1}(l) \kappa_j^{(l)}(\Delta)$$

with

$$\kappa_j^{(l)}(\Delta) = \binom{l}{j} \left( 1 - \frac{\beta^2(\Delta)}{\sigma_{i+1|i:1}^2} \right)^j \left( \frac{\beta^2(\Delta)}{\sigma_{i+1|i:1}^2} \right)^{l-j}.$$

If  $\sigma_{i|i:1}^2 > 0$ ,  $L_{i+1|i:1} = L_{i|i:1}$ . If  $\sigma_{i|i:1}^2 = 0$ ,  $L_{i+1|i:1} = 0$ .

- (Marginal distributions):

$$p_{\nu_{i|i-1:1}}(y_i) = p_i(y_i | y_{i-1}, \dots, y_1) = \sum_{j=0}^{L_{i|i-1:1}} \alpha_{i|i-1:1}(j) p_{\nu_{j, \sigma_{i|i-1:1}}}(y_i).$$

Since it always hold that  $\sigma_{i|i-1:1}^2 \geq \beta^2(\Delta) > 0$ , this distribution always has density.

A special attention must be given to the length of the mixture parameters. Starting with a length  $L_{1|0:1}$ , if  $y_1 \neq 0$ ,  $L_{1|1:1} = L_{1|0:1} + k$  and  $L_{2|1:1} = L_{1|1:1} = L_{1|0:1} + k$ . If  $y_1 = 0$ ,  $\alpha_{1|1:1}(k) = 1$  and  $\alpha_{2|1:1}(0) = 1$  which implies  $L_{2|1:1} = 0$ . More generally,

$$L_{i|i-1:1} \leq L_{1|0:1} + (i-1)k.$$

Each time a new observation is equal to 0, the filtering distribution is  $\delta_0$ . Then, the length of the mixture parameter in the predictive distribution is reset to 0 and the scale parameter is reset to  $\beta^2(\Delta)$  (this means that the predictive distribution is not a mixture and is simply a Gamma  $G(\delta/2, 1/2\beta^2(\Delta))$ ). Thus, the length of the mixture parameter may be much smaller than the above upper bound. This can be seen on simulated data where only two or three mixture parameters are significantly non nul.

The scale parameter of the predictive distributions is bounded from below and positive. The scale parameter of the filtering distributions is not bounded from below and may be nul.

It must be stressed that the scale parameter of predictive distributions evolve in a relatively simple way. Indeed, we have

$$\sigma_{i+1|i:1}^2 = F_{y_i}(\sigma_{i|i-1:1}^2) \quad \text{with} \quad F_y(v) = \beta^2(\Delta) + a^2(\Delta) \frac{vy}{y + 2\lambda v}.$$

When  $\theta < 0$  ( $a^2(\Delta) < 1$ ), the function  $F_y$  is increasing from  $I = [\beta^2(\Delta), \frac{\beta^2(\Delta)}{1-a^2(\Delta)}]$  onto  $I$  and Lipschitz with constant  $a^2(\Delta)$ . This allows to obtain stability properties for the scale parameter  $\sigma_{i|i-1:1}^2$  (see Genon-Catalot and Kessler (2004)).

Using the formula for moments (3.12), we obtain:

$$\begin{aligned} \mathbb{E}(x_i | y_{i-1}, \dots, y_1) &= \sigma_{i|i-1:1}^2 \left( \delta + 2 \sum_{j=0}^{L_{i|i-1:1}} j \alpha_{i|i-1:1}(j) \right), \\ \mathbb{E}(x_i | y_i, \dots, y_1) &= \sigma_{i|i:1}^2 \left( \delta + 2 \sum_{j=0}^{L_{i|i:1}} j \alpha_{i|i:1}(j) \right). \end{aligned}$$

The other conditional moments are obtained analogously.

### 3.6 Exact likelihood.

Let us now assume that  $\theta < 0$  and either  $\delta = 1$  ( $x(t)$  is the square of an Ornstein-Uhlenbeck process) or  $\delta \geq 2$  ( $x(t)$  is positive recurrent on  $(0, +\infty)$ ). Moreover, let us assume that the initial variable  $\eta$  has the stationary distribution  $\pi_\vartheta = G(\delta/2, \frac{1}{2\sigma_s^2(\vartheta)})$ . Hence, the computation of the predictive distributions starts with

$$\nu_{1|0:1} = \pi_\vartheta = \nu_{0, \sigma_s(\vartheta)}$$

which belongs to  $\bar{\mathcal{F}}_f^\delta$  and has parameters  $L_{1|0:1} = 0$  and  $\sigma = \sigma_s(\vartheta)$ . The process  $(x_i, y_i)$  is strictly stationary. The exact likelihood is computed by:

$$p_n(\vartheta, y_1, \dots, y_n) = p_1(\vartheta, y_1) \prod_{i=2}^n p_i(\vartheta, y_i | y_{i-1}, \dots, y_1)$$

where  $p_1(\vartheta, y_1)$  is the density of  $y_1$  (and of all  $y_i$ 's) and is equal to

$$p_1(\vartheta, y_1) = p_{\nu_{1|0:1}}(y_1) = p_{\nu_{0, \sigma_s(\vartheta)}}(y_1) = \frac{\lambda^k}{\Gamma(k)} \frac{C_{2k+\delta-1}}{C_{\delta-1}} \frac{\sigma_s^{2k}(\vartheta) y^{\frac{\delta}{2}-1}}{(y + 2\lambda\sigma_s^2(\vartheta))^{k+\frac{\delta}{2}}}$$

and

$$p_i(\vartheta, y_i | y_{i-1}, \dots, y_1) = p_{\nu_{i|i-1:1}}(y_i) = \sum_{0 \leq j \leq L_{i|i-1:1}} \alpha_{i|i-1:1}(j) p_{\nu_{j, \sigma_{i|i-1:1}}} (y_i)$$

where  $L_{i|i-1:1} \leq (i-1)k$  and

$$p_{\nu_{j, \sigma_{i|i-1:1}}}(y_i) = \frac{\lambda^k}{\Gamma(k)} \frac{C_{2(k+j)+\delta-1}}{C_{2j+\delta-1}} \frac{\sigma_{i|i-1:1}^{2k}}{y_i} \frac{y_i^{j+\frac{\delta}{2}}}{(y_i + 2\lambda\sigma_{i|i-1:1}^2)^{j+k+\frac{\delta}{2}}}$$

One can compute the maximum likelihood estimators of  $(\theta, \sigma, \delta)$ . The parameters  $k, \lambda$  come from the noise distribution. They are supposed to be known. The other parameters and the observations  $y_{i-1}, \dots, y_1$  are included in the formulae of  $\sigma_{i|i-1:1}^2, \alpha_{i|i-1:1}$ .

Note that the density of  $y_i$  is equal to the density of  $G/G'$  with  $G, G'$  independent,  $G \sim G(\delta/2, 1/2\sigma_s^2(\vartheta))$  and  $G' \sim G(k, \lambda)$ . This is the distribution of  $2\lambda\sigma_s^2(\vartheta)F$  where  $F$  has Fisher distribution  $F(\delta, 2k)$ .

### 3.7 Related models.

Analogous results can be obtained for the following models:

$$y_i = \sqrt{x_i} \varepsilon_i$$

where  $\varepsilon_i$  has distribution  $1/\sqrt{G(k, \lambda)}$ . Or,

$$y_i = \sqrt{x_i} \varepsilon_i$$

where  $\varepsilon_i$  has the symmetric distribution  $\varepsilon/\sqrt{G(k, \lambda)}$  with  $\varepsilon$  independent of  $G(k, \lambda)$  and with symmetric Bernoulli distribution ( $P(\varepsilon = 1) = P(\varepsilon = -1) = 1/2$ ).

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# Chapter 4

## Other kernels.

### 4.1 Introduction

In this lecture, we present two families of models:

- The hidden process is the same as in the previous lecture:  $x(t)$  is either the square of an Ornstein-Uhlenbeck process or the CIR diffusion process. Given the state  $x(t_i)$ , the observation  $y_i$  has Poisson distribution with parameter  $\lambda(x(t_i)) = \lambda x(t_i)$ .
- The hidden process ( $x(t)$ ) is a Wright-Fisher diffusion process. Given the state  $x(t_i)$ , the observation  $y_i$  has Bernoulli distribution with parameter  $x(t_i)$  (or binomial distribution).

We refer to Chaleyat-Maurel and Genon-Catalot (2006, 2009) for more details.

### 4.2 Conditional Poisson observations.

Let us assume that observations  $y_i$  are such that the conditional distribution of  $y_i$  given  $x(t_i) = x$  is  $F(x, dy) = f(y|x)\mu(dy)$  where  $\mu(dy)$  is the counting measure on  $\mathbb{N}$  simply given by  $\mu(y) = 1$  for all  $y \in \mathbb{N}$  and

$$f(y|x) = \exp(-\lambda x) \frac{(\lambda x)^y}{y!} \quad y \in \mathbb{N}.$$

The hidden diffusion process ( $x(t)$ ) is given by

$$dx(t) = (2\theta x(t) + \delta c^2)dt + 2c\sqrt{x(t)}dW_t, \quad x(0) = \eta, \quad (4.1)$$

with  $\eta$  a random variable independent of the Brownian motion ( $W_t$ ). We assume that  $\delta = 1$  or  $\delta \geq 2$ .

We need to check the conditions (C1)-(C2) for computable filters presented in the previous lecture. Since the hidden process is the same, condition (C2) holds for the class

$$\mathcal{F}^\delta = \{\nu_{i,\sigma} = g_{i,\sigma}(x)dx, i \in \mathbb{N}, \sigma \geq 0\}, \quad (4.2)$$

where the density  $g_{i,\sigma}(x)$  is given by

$$g_{i,\sigma}(x) = 1_{(x>0)} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x}{2\sigma^2}\right) \frac{x^{i-1+\frac{\delta}{2}}}{C_{2i+\delta-1}\sigma^{2i+\delta-1}}. \quad (4.3)$$

It is the density of a Gamma distribution  $G(i + \delta/2, 1/2\sigma^2)$  with scale parameter  $1/2\sigma^2$  and location parameter  $(i + \delta/2)$ . The normalising constant  $C_a$ , is simply the absolute moment of order  $a$  of a Gaussian standard variable.

The extended class  $\bar{\mathcal{F}}^\delta$  is defined as the class of mixtures of distributions  $\nu_{i,\sigma}$  having the same scale parameter:

$$\bar{\mathcal{F}}^\delta = \{\nu = \nu_{\alpha,\sigma} = \sum_{i \geq 0} \alpha_i \nu_{i,\sigma}, \alpha = (\alpha_i), \alpha_i \geq 0, \sum_{i \geq 0} \alpha_i = 1, \sigma \geq 0\}. \quad (4.4)$$

The sub-class  $\bar{\mathcal{F}}_f^\delta$  is composed of all distributions  $\nu_{\alpha,\sigma}$  having finite-length mixture parameter, *i.e.* such that

$$L(\alpha) = \sup\{i; \alpha_i > 0\} < \infty.$$

If  $\sigma = 0$ ,  $\nu\{\alpha, 0\} = \delta_0$  whatever  $\alpha$ .

We only need to check condition (C1) for the same class of distributions.

**Proposition 4.2.1.** (*up-dating and marginal operators*)

- If  $\nu_{i,\sigma}$  belongs to  $\mathcal{F}^\delta$ , then, for all integer  $y$

$$\varphi_y(\nu_{i,\sigma}) = \nu_{t_y(i), T_y(\sigma)}$$

with

$$t_y(i) = y + i \quad \text{and} \quad \frac{1}{T_y^2(\sigma)} = \frac{1}{T^2(\sigma)} = \frac{1}{\sigma^2} + 2\lambda$$

( $T^2(\sigma) = \frac{\sigma^2}{1+2\lambda\sigma^2}$ ). The up-dating operator is therefore:

$$(i, \sigma) \rightarrow (t_y(i) = y + i, T(\sigma) = \frac{\sigma}{(1 + 2\lambda\sigma^2)^{1/2}}).$$

The corresponding marginal distribution has density, w.r.t. the counting measure on  $\mathbb{N}$ ,

$$p_{\nu_{i,\sigma}}(y) = \frac{(\lambda T^2(\sigma))^y}{y!} \frac{C_{2(y+i)+\delta-1}}{C_{2i+\delta-1}} \left(\frac{T(\sigma)}{\sigma}\right)^{2i+\delta}$$

- If  $\nu = \nu_{\alpha, \sigma}$  belongs to  $\bar{\mathcal{F}}_f^\delta$ ,

$$\varphi_y(\nu) = \sum_{i \geq 0} \hat{\alpha}_i \nu_{i, T(\sigma)}$$

where

$$\hat{\alpha}_i = \hat{\alpha}_i(\alpha, \sigma) \propto 1_{i \geq y} \alpha_{i-y} p_{\nu_{i-y, \sigma}}(y).$$

We have  $L(\hat{\alpha}) = L(\alpha) + y$ . Condition (C1) holds for the class  $\mathcal{F}^\delta$  and the extended class  $\bar{\mathcal{F}}_f^\delta$ .

**Proof.** Noting that:

$$f(y|x)g_{i, \sigma}(x) \propto x^{y+i-1+\delta/2} \exp\left(-\left(\lambda + \frac{1}{2\sigma^2}\right)x\right),$$

we get the results.  $\square$

Note that, here, the scale parameter does not depend on the new observation  $y$ .

#### 4.2.1 Algorithm for predictive distributions.

We denote the parameters of a distribution  $\nu_{i|i-1:1}(dx)$  by  $\alpha_{i|i-1:1}$ ,  $\sigma_{i|i-1:1}$  and  $L_{i|i-1:1}$  for the length of the mixture parameter. We use analogous notations for  $\nu_{i|i:1}(dx)$ . Let us stress here the interest of conditions (C1)-(C2). Since (C2) only concerns the hidden Markov process, there are no changes at all for the prediction step.

**Proposition 4.2.2.** *Assume that  $\nu_{1|0:1}$  belongs to the class  $\bar{\mathcal{F}}_f^\delta$ . The scale, mixture and length parameters of  $\nu_{i|i-1:1}(dx)$ ,  $\nu_{i|i:1}(dx)$  can be recursively computed as follows.*

- (up-dating) For  $i \geq 1$ ,

$$\sigma_{i|i:1}^2 = \frac{\sigma_{i|i-1:1}^2}{1 + 2\lambda\sigma_{i|i-1:1}^2}.$$

For  $j \geq 0$ ,

$$\alpha_{i|i:1}(j) \propto 1_{j \geq y_i} \alpha_{i|i-1:1}(j - y_i) \frac{C_{2j+\delta-1}}{C_{2(j-y_i)+\delta-1}} \left( \frac{\sigma_{i|i:1}^2}{\sigma_{i|i-1:1}^2} \right)^{j-y_i}.$$

The lengths of the mixture parameters satisfy  $L_{i|i:1} = L_{i|i-1:1} + y_i$ .

- (prediction)

$$\sigma_{i+1|i:1}^2 = \beta^2(\Delta) + a^2(\Delta)\sigma_{i|i:1}^2.$$

For  $j \geq 0$ ,

$$\alpha_{i+1|i:1}(j) = \sum_{k \geq j} \alpha_{i|i:1}(k) \kappa_j^{(k)}(\Delta)$$

with

$$\kappa_j^{(k)}(\Delta) = \binom{k}{j} \left(1 - \frac{\beta^2(\Delta)}{\sigma_{i+1|i:1}^2}\right)^j \left(\frac{\beta^2(\Delta)}{\sigma_{i+1|i:1}^2}\right)^{k-j}.$$

The lengths of the mixture parameters satisfy  $L_{i+1|i:1} = L_{i|i:1} = L_{1|0:1} + y_1 + \dots + y_i$ .

- (Marginal distributions):

$$p_{\nu_{i|i-1:1}}(y_i) = p_i(y_i|y_{i-1}, \dots, y_1) = \sum_{j=0}^{L_{i|i-1:1}} \alpha_{i|i-1:1}(j) p_{\nu_{j, \sigma_{i|i-1:1}}}.$$

## 4.2.2 Exact likelihood.

We assume now that  $\theta < 0$  and that the process  $(x(t))$  is in stationary regime with marginal distribution

$$\nu_{1|0:1} = \pi_\vartheta = G(\delta/2, \frac{1}{2\sigma_s^2(\vartheta)}) = \nu_{0, \sigma_s(\vartheta)}.$$

The joint process  $(x_i, y_i)$  is therefore strictly stationary and the distribution of  $y_i$  is given by

$$p_1(\vartheta, y) = p_{\nu_{1|0:1}}(y) = \frac{(\lambda T^2(\sigma_s(\vartheta)))^y C_{2y+\delta-1}}{y! C_{\delta-1}} \left(\frac{T(\sigma_s(\vartheta))}{\sigma_s(\vartheta)}\right)^\delta$$

The exact likelihood is computed by:

$$p_n(\vartheta, y_1, \dots, y_n) = p_1(\vartheta, y_1) \prod_{i=2}^n p_i(\vartheta, y_i|y_{i-1}, \dots, y_1)$$

where

$$p_i(\vartheta, y_i|y_{i-1}, \dots, y_1) = p_{\nu_{i|i-1:1}}(y_i) = \sum_{0 \leq j \leq L_{i|i-1:1}} \alpha_{i|i-1:1}(j) p_{\nu_{j, \sigma_{i|i-1:1}}}(y_i)$$

where  $L_{i|i-1:1} = y_1 + \dots + y_{i-1}$  and

$$p_{\nu_{j, \sigma_{i|i-1:1}}}(y_i) = \frac{\lambda^{y_i} \sigma_{i|i-1:1}^{2y_i} C_{2(y_i+j)+\delta-1}}{y_i! C_{2j+\delta-1}} \left(\frac{\sigma_{i|i-1:1}}{\sigma_{i|i-1:1}}\right)^{2j+\delta}$$

## 4.2.3 Extension.

It is possible to consider a more general observation kernel. We may assume that the observations  $y_i$  are such that the conditional distribution of  $y_i$  given  $x_i = x$  is Poisson with parameter  $\lambda x_i + \lambda_0$  for  $\lambda_0 > 0$  a constant. Then,  $F(x, dy) = f(y|x)\mu(dy)$  with

$$\begin{aligned} f(y|x) &= \exp(-\lambda x + \lambda_0) \frac{(\lambda x + \lambda_0)^y}{y!} \\ &= \sum_{i=0}^y \exp(-\lambda_0) \frac{\lambda_0^{y-i} \lambda^i}{(y-i)! i!} x^i \exp(-\lambda x). \end{aligned}$$

The previous study can be generalized.

Note that this is a discrete time version of the continuous time model proposed in Boel and Benes (1980).

### 4.3 Wright-Fisher diffusion and conditional Bernoulli observations.

Consider the following stochastic differential equation

$$dx(t) = [-\delta x(t) + \delta'(1 - x(t))]dt + 2[x(t)(1 - x(t))]^{1/2}dW_t, \quad x(0) = \eta, \quad (4.5)$$

where  $(W_t)$  is a standard one-dimensional Brownian motion and  $\eta$  is a random variable with values in  $(0, 1)$  independent of  $(W_t)$ . This process is known as the Wright-Fisher gene frequency diffusion model with mutation effects. It has values in the interval  $(0, 1)$ . It appears as the diffusion approximation of the discrete time and space Wright-Fisher Markov chain and is used to model the frequency of an allele A in a population of genes composed of two distinct alleles A and a (see *e.g.* Karlin and Taylor (1981, p. 176-179 and 221-222) or Wai-Yuan (2002, Chap. 6)). Suppose now that the current state  $x(t)$  cannot be directly observed. Instead, at times  $t_1, t_2, \dots, t_n$  with  $0 \leq t_1 < t_2 \dots < t_n$ , we have observations  $y(t_i)$  such that, given the whole process  $(x(t))$ , the random variables  $y(t_i)$  are independent and the conditional distribution of  $y(t_i)$  only depends on the corresponding state variable  $x(t_i)$ . More precisely, we consider the following discrete conditional distribution: a binomial distribution, *i.e.*, for  $N \geq 1$  an integer,

$$P(y(t_i) = y | x(t_i) = x) = \binom{N}{y} x^y (1 - x)^{N-y}, \quad y = 0, 1, \dots, N, \quad (4.6)$$

For simplicity, we focus on the Bernoulli case where  $N = 1$ . Let us set

$$P(y(t_i) = y | x(t_i) = x) = f_x(y) = x^y (1 - x)^{1-y}, \quad y = 0, 1, x \in (0, 1). \quad (4.7)$$

We consider, on the finite set  $\{0, 1\}$ , the dominating measure  $\mu(y) = 1, y = 0, 1$ .

The scale density model (4.5) is given by:

$$s(x) = \exp\left(-\frac{1}{2} \int^x \frac{-\delta u + \delta'(1 - u)}{u(1 - u)} du\right) = x^{-\delta'/2} (1 - x)^{-\delta/2}, \quad x \in (0, 1).$$

It satisfies  $\int_0^1 s(x) dx = \infty = \int^1 s(x) dx$  if and only if  $\delta \geq 2$  and  $\delta' \geq 2$ , conditions that we assume from now on. The speed density is equal to  $m(x) = x^{\delta'/2-1} (1 - x)^{\delta/2-1}$ ,  $x \in (0, 1)$ . Therefore, the unique stationary distribution of (4.5) is the Beta distribution with parameters  $\delta'/2, \delta/2$  which has density

$$\pi(x) = \frac{x^{\frac{\delta'}{2}-1} (1 - x)^{\frac{\delta}{2}-1}}{B(\frac{\delta'}{2}, \frac{\delta}{2})} 1_{(0,1)}(x). \quad (4.8)$$

The instants of observations are equally spaced with sampling interval  $\Delta$ , *i.e.*  $t_n = n\Delta$ ,  $n \geq 1$ . Hence, the process  $(x(t_n))$  is a time-homogeneous Markov chain. We denote by  $p_\Delta(x, x')$  its transition density and by  $P_\Delta$  its transition operator. The transition density is not explicitly known. However, it has a precise spectral expansion (see *e.g.* Karlin and Taylor, 1981, p.335-336: Note that  $2x(t) - 1$  is a Jacobi diffusion process).

The proper parametric class of distributions on  $(0, 1)$  for checking (C1)-(C2) is the following one. Let us define the class of distributions indexed by  $\Theta = \mathbb{N} \times \mathbb{N}$ :

$$\mathcal{F} = \{\nu_{i,j}(dx) \propto h_{i,j}(x)\pi(x)dx, (i, j) \in \mathbb{N} \times \mathbb{N}\}, \quad (4.9)$$

where

$$h_{i,j}(x) = x^i(1-x)^j. \quad (4.10)$$

Hence, each distribution in  $\mathcal{F}$  is a Beta distribution with parameters  $(i + \frac{\delta'}{2}, j + \frac{\delta}{2})$  and (see (4.8))

$$\nu_{i,j}(dx) = \frac{x^{i+\frac{\delta'}{2}-1}(1-x)^{j+\frac{\delta}{2}-1}}{B(i+\frac{\delta'}{2}, j+\frac{\delta}{2})} \mathbf{1}_{(0,1)}(x)dx. \quad (4.11)$$

The extended class is:

$$\bar{\mathcal{F}}_f = \left\{ \sum_{(i,j) \in \Lambda} \alpha_{i,j} \nu_{i,j}, \Lambda \subset \mathbb{N} \times \mathbb{N}, |\Lambda| < +\infty, \alpha = (\alpha_{i,j}, (i, j) \in \Lambda) \in S_f \right\}, \quad (4.12)$$

where

$$S_f = \left\{ \alpha = (\alpha_{i,j}, (i, j) \in \Lambda), \Lambda \subset \mathbb{N} \times \mathbb{N}, |\Lambda| < +\infty, \forall (i, j), \alpha_{i,j} \geq 0, \sum_{(i,j) \in \Lambda} \alpha_{i,j} = 1 \right\} \quad (4.13)$$

is the set of finite mixture parameters. It is worth noting that the stationary distribution  $\pi(x)dx = \nu_{0,0}(dx)$  belongs to  $\mathcal{F}$ . Thus, in the important case where the initial distribution, *i.e.* the distribution of  $\eta$  (see (4.5)), is the stationary distribution, the exact and optimal filters have an explicit formula. The exact formulae for the filtering-prediction algorithm and the likelihood can be obtained from Chaleyat-Maurel and Genon-Catalot (2009). Although explicit, they are not simple and not given here.

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