Hermite matrix in Lagrange basis for scaling static output feedback polynomial matrix inequalities

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Abstract

Using Hermite's formulation of polynomial stability conditions, static output feedback (SOF) controller design can be formulated as a polynomial matrix inequality (PMI), a (generally nonconvex) nonlinear semidefinite programming problem that can be solved (locally) with PENNON, an implementation of a penalty method. Typically, Hermite SOF PMI problems are badly scaled and experiments reveal that this has a negative impact on the overall performance of the solver. In this note we recall the algebraic interpretation of Hermite's quadratic form as a particular Bézoutian and we use results on polynomial interpolation to express the Hermite PMI in a Lagrange polynomial basis, as an alternative to the conventional power basis. Numerical experiments on benchmark problem instances show the substantial improvement brought by the approach, in terms of problem scaling, number of iterations and convergence behavior of PENNON.

Keywords: Static output feedback, Hermite stability criterion, Polynomial matrix inequality, Nonlinear semidefinite programming.

1 Introduction

In 1854 the French mathematician Charles Hermite studied quadratic forms for counting the number of roots of a polynomial in the upper half of the complex plane (or, by a simple rotation, in the left half-plane), more than two decades before Routh, who was apparently not aware of Hermite's work, see [8]. Hurwitz himself used some of Hermite's ideas to derive in 1895 his celebrated algebraic criterion for polynomial stability, now called the Routh-Hurwitz criterion and taught to engineering students in tabular form.

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Hermite's criterion can be interpreted as a symmetric formulation of the Routh-Hurwitz criterion. This symmetry can be exploited in a semidefinite programming framework, as shown in [3] and [4] in the context of simultaneous stabilization of linear systems. Along the same vein, in [5] the problem of static output feedback (SOF) design was formulated as a polynomial matrix inequality (PMI) problem. In some cases (e.g. only one input or output available for feedback) this PMI problem simplifies to a bilinear matrix inequality (BMI) that can be solved numerically with PENBMI, a particular instance of PENNON, a general penalty method for nonlinear and semidefinite programming. Only convergence to a local optimum is guaranteed, but experiments reported in [5] show that quite often the approach is viable numerically. In particular, the SOF PMI formulation involves only controller parameters, and does not introduce (a typically large number of) Lyapunov variables.

Our motivation in this paper is to contribute along the lines initiated in [5] and to study the impact of SOF PMI problem formulation on the behavior of PENNON, in particular w.r.t. data scaling and number of iterations. The Hermite matrix depends quadratically on coefficients of the characteristic polynomial, in turn depending polynomially on the controller parameters. As a result, coefficients of a given Hermite matrix typically differ by several orders of magnitude, and experiments reveal that this poor data scaling significantly impacts on the performance of PENNON.

In this paper we use an alternative formulation of the Hermite matrix, using a Lagrange polynomial basis instead of the standard power basis. We build on previous work from the computer algebra and real algebraic geometry communities, recalling the interpretation of Hermite's quadratic form as a particular Bézoutian, the resultant of two polynomials, see [6] and references therein. This interpretation provides a natural choice for the nodes of the Lagrange basis. The construction of the Hermite matrix in this basis is carried out efficiently by interpolation, overcoming difficulties inherent to Vandermonde matrices, as suggested in [12] for general Bézout matrices.

In addition to digesting and tailoring to our needs results from computational algebraic geometry, another contribution of our paper is to extend slightly the characterization of [12] to Hermitian forms with complex and repeated interpolation nodes. In particular, in our SOF design application framework, these nodes are roots of either imaginary or real part of a target characteristic polynomial featuring spectral properties desirable for the closed-loop system. This target polynomial is the main tuning parameter of our approach, and we provide numerical evidence that a suitably choice of target polynomial, compatible with achievable closed-loop dynamics, results in a significant improvement of SOF PMI problem scaling, with positive effects on the overall behavior (convergence, number of outer and inner iterations, linesearch steps) of PENNON. Furthermore, some of the problems that were not solvable in the power basis, see [5], can now be solved in the Lagrange basis. These improvements are illustrated on numerical examples extracted from the publicly available benchmark collection COMPl$_e$ib, see [9].

2 PMI formulation of SOF design problem

We briefly recall the polynomial matrix inequality (PMI) formulation of static output feedback (SOF) design problem proposed in [5].

Consider the linear system

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\end{aligned}$$

of order n with m inputs and p outputs, that we want to stabilize by static output feedback (SOF)

$$u = Ky.$$

In other words, given matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, we want to find matrix $K \in \mathbb{R}^{m\times p}$ such that the eigenvalues of closed-loop matrix $A + BKC$ all belong to the left half of the complex plane.

Let $k \in \mathbb{R}^{mp}$ be the vector obtained by stacking the columns of matrix K. Define

$$q(s, k) = \det(sI - A - BKC) = \sum_{i=0}^{n} q_i(k)s^i \tag{1}$$

as the characteristic polynomial of matrix $A+BKC$. Coefficients of increasing powers of indeterminate s in polynomial $q(s, k)$ are multivariate polynomials in k, i.e.

$$q_i(k) = \sum_{\alpha} q_{i\alpha} k^{\alpha} \tag{2}$$

where $\alpha \in \mathbb{N}^{mp}$ describes all monomial powers.

The Routh-Hurwitz criterion for stability of polynomials has a symmetric version called the Hermite criterion. A polynomial is stable if and only if its Hermite matrix, quadratic in the polynomial coefficients, is positive definite. Algebraically, the Hermite matrix can be defined via the Bézoutian, a symmetric form of the resultant.

Let $a(u)$, $b(u)$ be two polynomials of degree n of the indeterminate u. Define the bivariate quadratic form

$$\frac{a(u)b(v) - a(v)b(u)}{u - v} = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij} u^{i-1} v^{j-1}.$$

The n-by-n matrix with entries b_{ij} is the Bézoutian matrix, whose determinant is the resultant of a and b, obtained by eliminating variable u from the system of equations $a(u) = b(u) = 0$.

The Hermite matrix in power basis of $q(s, k)$, denoted by $H^P(k)$, is defined as the Bézoutian matrix of the real and imaginary parts of $q(ju, k)$:

$$\begin{aligned}a(u, k) &= \operatorname{Im} q(jw, k)\\ b(u, k) &= \operatorname{Re} q(jw, k).\end{aligned}$$

The roots of polynomial $q(s, k)$ belongs to the left half-plane if and only if

$$H^P(k) = \sum_{i=0}^{n}\sum_{j=0}^{n} q_i(k)q_j(k)H^P_{ij} \succ 0.$$

The above relation is a matrix inequality depending polynomially on parameters k. Therefore, finding k amounts to solving a polynomial matrix inequality (PMI) problem.

Example 2.1 As an illustrative example, consider problem `NN6` in [9]. The closed-loop characteristic polynomial is (to 8 significant digits):

$$\begin{aligned}
q(s,k) &= s^9 + 23.300000s^8 + (4007.6500 - 14.688300k_2 + 14.685000k_4)s^7 \\
&\quad + (91133.935 - 14.685000k_1 + 14.688300k_3 + 15.132810k_4)s^6 \\
&\quad + (1149834.9 - 57334.489k_2 + 15.132810k_3 + 36171.693k_4)s^5 \\
&\quad + (20216420 - 57334.489k_1 + 36171.693k_3 + 35714.763k_4)s^4 \\
&\quad + (49276365 - 12660338k_2 + 35714.763k_3 + 3174671.8k_4)s^3 \\
&\quad + (-1562.6281 \cdot 10^5 - 12660338k_1 - 3174671.8k_3 + 3133948.9k_4)s^2 \\
&\quad + (-4315.5562 \cdot 10^5 + 95113415k_2 + 3133948.9k_3)s \\
&\quad + 95113415k_1
\end{aligned}$$

with SOF gain $K = [k_1\, k_2\, k_3\, k_4]$. The 9-by-9 Hermite matrix of this polynomial cannot be displayed entirely for space reasons, so we choose two representative entries:

$$\begin{aligned}
H^P_{3,3}(k) &= 10244466 \cdot 10^8 - 53923375 \cdot 10^7 k_1 + 55487273 \cdot 10^6 k_2 \\
&\quad + 10310826 \cdot 10^7 k_3 - 32624061 \cdot 10^7 k_4 + 16028416 \cdot 10^7 k_1 k_2 \\
&\quad - 27103829 \cdot 10^4 k_1 k_3 - 36752006 \cdot 10^6 k_1 k_4 \\
&\quad - 43632833 \cdot 10^6 k_2 k_3 - 43073807 \cdot 10^6 k_2 k_4 \\
&\quad + 22414163 k_3^2 + 10078541 \cdot 10^6 k_3 k_4 + 99492593 \cdot 10^5 k_4^2
\end{aligned}$$

and

$$H^P_{9,9}(k) = 23.300000.$$

We observe that this Hermite matrix is ill-scaled, in the sense that the coefficients of its entries (multivariate polynomials in k_i) differ by several orders of magnitude. This representation is not suitable for a matrix inequality solver.

3 A simple scaling strategy

A possible remedy to address the poor scaling properties of the Hermite matrix is to scale the frequency variable s, that is, to substitute ρs for s in the characteristic polynomial $q(s,k)$, for a suitable positive scaling ρ. Finding the optimal value of ρ (e.g. in terms of relative scaling of the coefficients of the Hermite matrix) may be formulated as an optimization problem, but numerical experiments indicate that nearly optimal results are achieved when following the basic strategy consisting of choosing ρ such that the constant and highest power polynomial coefficients are both equal to one. For example, this idea was implemented by Huibert Kwakernaak in the `scale` function of the Polynomial Toolbox for Matlab, see [11].

Example 3.1 Consider the simple example `AC4` in [9]. The open-loop characteristic polynomial is

$$\begin{aligned} q(s,0) &= \det(sI - A) \\ &= s^4 + 150.92600s^3 + 130.03210s^2 - 1330.6306s - 66.837750 \end{aligned}$$

with Hermite matrix in power basis

$$H^P = \begin{bmatrix} 88936.354 & 0 & 10087.554 & 0 \\ 0 & -162937.14 & 0 & 1330.631 \\ 10087.554 & 0 & 20955.855 & 0 \\ 0 & 1330.6306 & 0 & 150.92600 \end{bmatrix}.$$

To measure quantitatively the scaling of a matrix X, we may use its condition number. If the matrix is poorly scaled, then its condition number is large. Minimizing the condition number therefore improves the scaling. For the above matrix, its condition number (in the Frobenius norm), defined as $\|H^P\|_F \|(H^P)^{-1}\|_F$, is equal to 1158.2. If we choose $\rho = \sqrt[4]{1.0000/66.840} = 3.5000 \cdot 10^{-1}$, the scaled characteristic polynomial has unit constant and highest coefficient, and the resulting scaled Hermite matrix reads

$$SH^PS = \begin{bmatrix} 163.48864 & 0 & 151.37636 & 0 \\ 0 & -2445.0754 & 0 & 163.00225 \\ 151.37636 & 0 & 2567.0923 & 0 \\ 0 & 163.00225 & 0 & 150.92600 \end{bmatrix}$$

with

$$S = \mathrm{diag}\left(\rho^3, \rho^2, \rho^1, 1\right).$$

The Frobenius condition number of SH^PS is equal to 32.096.

Whereas this simple scaling strategy with one degree of freedom may prove useful for small-degree polynomials and small-size Hermite matrices, a more sophisticated approach is required for larger instances.

4 Hermite matrix in Lagrange basis

In this section we show how the Hermite matrix can be scaled by an appropriate choice of polynomial basis. Moreover, this basis allows for a straightforward entrywise construction of the Hermite matrix.

4.1 Distinct interpolation points

Consider n distinct interpolation points $u_i \in \mathbb{C}$, $i = 1, \ldots, n$, and define the j-th Lagrange polynomial

$$l_j(u) = \prod_{i=1, i\neq j}^{n} \frac{u - u_i}{u_j - u_i}$$

which is such that $l_j(u_j) = 1$ and $l_j(u_i) = 0$ if $i \neq j$. In matrix form we can write

$$\begin{bmatrix} 1 \\ u \\ u^2 \\ \vdots \\ u^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_n \\ u_1^2 & u_2^2 & \cdots & u_n^2 \\ \vdots & & & \vdots \\ u_1^{n-1} & u_1^{n-1} & \cdots & u_n^{n-1} \end{bmatrix} \begin{bmatrix} l_1(u) \\ l_2(u) \\ l_3(u) \\ \vdots \\ l_n(u) \end{bmatrix} = V_u l(u) \tag{3}$$

where V_u is a Vandermonde matrix. Given a univariate polynomial $q(s)$ with real coefficients, define

$$\begin{array}{rcl} a(u) & = & \operatorname{Im} q(jw) \\ b(u) & = & \operatorname{Re} q(jw) \end{array} \tag{4}$$

as its imaginary and real parts on the imaginary axis, respectively. In the following, the star denotes transpose conjugation and the prime denotes differentiation, i.e.

$$a'(u) = \frac{da(u)}{du}.$$

Theorem 4.1 *When the interpolation points are distinct (i.e. $u_i \neq u_j$, $i \neq j$, $i,j = 1,\ldots,n$), the Hermite matrix of $q(s)$ in Lagrange basis, denoted by H^L, is given entrywise by*

$$H^L_{i,j} := \begin{cases} \dfrac{a(u_i^*)b(u_j) - a(u_j)b(u_i^*)}{u_i^* - u_j} & \text{if } u_i^* \neq u_j, \\ a'(u_i^*)b(u_j) - a(u_j)b'(u_i^*) & \text{otherwise,} \end{cases}$$

for all $i,j = 1,\ldots,n$.

Proof Let us express the Bézoutian of a and b as a bivariate quadratic form

$$\frac{a(u)b(v) - a(v)b(u)}{u - v} = \begin{bmatrix} 1 \\ v \\ \vdots \\ v^{n-1} \end{bmatrix}^* H^P \begin{bmatrix} 1 \\ u \\ \vdots \\ u^{n-1} \end{bmatrix}$$

where H^P is the Hermite matrix of q in the power basis. Recalling relation (3), the Bézoutian becomes

$$\frac{a(u)b(v) - a(v)b(u)}{u - v} = l(v)^* V_v^* H^P V_u l(u) = l(v)^* H^L l(u)$$

so that the Hermite matrix of q in the Lagrange basis can be expressed as

$$H^L = V_v^* H^P V_u.$$

By evaluation at n distinct interpolation points u_i and v_j, H^L is given entrywise by

$$H^L_{i,j} = \frac{a(u_i^*)b(v_j) - a(v_j)b(u_i^*)}{u_i^* - v_j}. \tag{5}$$

Now let $u_i^* \to v_j$ for all $i,j = 1,\ldots,n$. After adding and subtracting $a(v_j)b(v_j)$ to the numerator of (5), we find

$$H^L_{i,j} = a'(u_i^*)b(u_j) - a(u_j)b'(u_i^*),$$

using a limiting argument. □

4.2 Repeated interpolation points

Let us define the bivariate polynomials

$$c_{i,j}(u,v) := \frac{\partial^{i+j-2}}{\partial u^{i-1}\partial v^{j-1}}\left(\frac{a(u)b(v)-a(v)b(u)}{u-v}\right)$$

for all $i,j = 1,\ldots,n$ and denote by

$$a^{(k)}(u) = \frac{d^k a(u)}{du^k}$$

the k-th derivative of univariate polynomial $a(u)$.

Lemma 4.2 *When the interpolation points are all equal (i.e.* $u_i = u_j$ *for all* $i,j = 1,\ldots,n$*), the Hermite matrix of* $q(s)$ *in Lagrange basis is given entrywise by*

$$H^L_{i,j} := \begin{cases} \dfrac{c_{i,j}(u_i^*,u_j)}{(i-1)!(j-1)!} & \text{if } u_i^* \neq u_j, \\ \displaystyle\sum_{k=0}^{i-1} \frac{a^{(j+k)}(u_i^*)b^{(i-k-1)}(u_j) - a^{(i-k-1)}(u_j)b^{(j+k)}(u_i^*)}{(j+k)!(i-k-1)!} & \text{otherwise,} \end{cases}$$

for all $i,j = 1,\ldots,n$.

Proof The proof of this result follows along the same lines as the proof of Theorem 4.1, with additional notational difficulties due to higher-order differentations. □

Example 4.3 Let us choose $n = 3$ equal interpolation points ($u_1 = u_2 = u_3 = x \in \mathbb{R}$). According to Lemma 4.2, H^L has the following entries:

$$\begin{aligned} H^L_{11} &= \tfrac{a'(x)b(x)-a(x)b'(x)}{1!} \\ H^L_{12} &= \tfrac{a^{(2)}(x)b(x)-a(x)b^{(2)}(x)}{2!} \\ H^L_{13} &= \tfrac{a^{(3)}(x)b(x)-a(x)b^{(3)}(x)}{3!} \\ H^L_{22} &= \tfrac{a^{(2)}(x)b'(x)-a'(x)b^{(2)}(x)}{2!} + \tfrac{a^{(3)}(x)b(x)-a(x)b^{(3)}(x)}{3!} \\ H^L_{23} &= \tfrac{a^{(3)}(x)b'(x)-a'(x)b^{(3)}(x)}{3!} + \tfrac{a^{(4)}(x)b(x)-a(x)b^{(4)}(x)}{4!} \\ H^L_{33} &= \tfrac{a^{(3)}(x)b^{(2)}(x)-a^{(2)}(x)b^{(3)}(x)}{3!2!} + \tfrac{a^{(4)}(x)b'(x)-a'(x)b^{(4)}(x)}{4!} + \tfrac{a^{(5)}(x)b(x)-a(x)b^{(5)}(x)}{5!}. \end{aligned}$$

Based on Theorem 4.1 and Lemma 4.2, we leave it to the reader to derive entrywise expressions for the Lagrange basis Hermite matrix in the general case when only some interpolation points are repeated.

In the remainder of the paper we will assume for notational simplicity that the interpolation points are all distinct.

4.3 Scaling

Corollary 4.4 *Let the interpolation points be (distinct) roots of either $a(u)$ or $b(u)$, as defined in (4). Then the Hermite matrix of $q(s)$ in Lagrange basis is block diagonal, with 2×2 blocks corresponding to pairs of complex conjugate points and 1×1 blocks corresponding to real points.*

Proof From Theorem 4.1, all the off-diagonal entries of H^L are given by

$$\frac{a(u_i^*)b(u_j) - a(u_j)b(u_i^*)}{u_i^* - u_j} \tag{6}$$

when interpolation points u_i and u_j are not complex conjugate. Both terms $a(u_i^*)b(u_j)$ and $a(u_j)b(u_i^*)$ are equal to zero in (6) since the interpolation points are the roots of either $a(u)$ or $b(u)$. The diagonal entries are $a'(u_i^*)b(u_j) - a(u_j)b'(u_i^*)$ since it is assumed that interpolation points are distinct. Therefore this part of H^L is 1×1 block-diagonal.

When interpolation points u_i and u_j are complex conjugate, there is only one non-zero entry (i, j) which is equal to $a'(u_i^*)b(u_j) - a(u_j)b'(u_i^*)$ and located in the off-diagonal entry, according to pairness. The diagonal entries of this case are equal to zero by virtue of equation (6). Therefore this part of H^L is 2×2 block-diagonal.$\square$

From Corollary 4.4 it follows that we can easily find a block-diagonal scaling matrix S such that the scaled Lagrange Hermite matrix

$$H^S = SH^LS$$

has smaller condition number. Nonzero entries of S are given by

$$S_{i,j} := \left| \sqrt{H^L_{i,j}} \right|^{-1}$$

whenever $H^L_{i,j}$ is a nonzero entry (i, j) of H^L.

Example 4.5 As an illustrative example, consider problem `NN5` in [9]. The open-loop characteristic polynomial is

$$\begin{aligned} q(s) &= s^7 + 10.171000s^6 + 96.515330s^5 + 458.42510s^4 \\ &\quad +2249.4849s^3 + 1.2196400s^2 - 448.72180s + 6.3000000. \end{aligned}$$

The Hermite matrix in power basis has the following entries:

$$\begin{array}{llllll} H^P_{1,1} & = & -2826.9473 & H^P_{1,3} & = & -14171.755 \\ H^P_{1,5} & = & 608.04658 & H^P_{1,7} & = & -6.3000000 \\ H^P_{2,2} & = & -14719.034 & H^P_{2,4} & = & 206313.38 \\ H^P_{2,6} & = & -4570.2494 & H^P_{3,3} & = & 209056.94 \\ H^P_{3,5} & = & -4687.9634 & H^P_{3,7} & = & 1.2196400 \\ H^P_{4,4} & = & 1026532.4 & H^P_{4,6} & = & -22878.291 \\ H^P_{5,5} & = & 21366.759 & H^P_{5,7} & = & -458.42510 \\ H^P_{6,6} & = & 523.23232 & H^P_{7,7} & = & 10.171000, \end{array}$$

remaining nonzero entries being deduced by symmetry. Apparently, this matrix is ill-scaled. Choosing interpolation points u_i as roots of $a(u)$, the imaginary part of $q(s)$ along the imaginary axis, we use Theorem 4.1 to build the Hermite matrix in Lagrange basis:

$$\begin{aligned} H^L &= \operatorname{diag}(-2826.9473,\ 41032866 \cdot 10^3,\ 44286011 \cdot 10^2,\ 41032866 \cdot 10^3, \\ &\quad 44286011 \cdot 10^2, \begin{bmatrix} 0 & 22222.878 \\ 22222.878 & 0 \end{bmatrix}). \end{aligned}$$

This matrix is still ill-scaled (with Frobenius condition number equal to $2.0983 \cdot 10^7$), but it is almost diagonal. Using an elementary diagonal scaling matrix S, we obtain

$$H^S = SH^LS = \operatorname{diag}(-1,\ 1,\ 1,\ 1,\ 1, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix})$$

which is a well-scaled representation of the Hermite matrix, with Frobenius condition number equal to 7.

4.4 Target polynomial

In our control application, let us introduce our main tuning tool which we call target polynomial, denoted by $q(s)$. The target polynomial provides the interpolation points required to build well-scaled Hermite matrix in the SOF problem. These points are defined as in Corollary 4.4 as the roots of either the real or imaginary part of $q(s)$ when evaluated along the imaginary axis.

In the context of SOF design, the target polynomial may be either choosen as

- a valid closed-loop characteristic polynomial (1) for a specific value of k, or
- a polynomial with desired pole distribution for the closed-loop system.

Furthermore, we invoke a continuity argument to observe that the condition and/or scaling of the Hermite matrix does not change abruptly in a neighborhood of a given target polynomial.

Example 4.6 Consider again Example 2.1 and let the target polynomial be an achievable closed-loop characteristic polynomial $q(s,k) = \det(sI - A - BKC)$, where

$$K = [-4.3264 \cdot 10^{-1},\ -1.6656,\ 1.2537 \cdot 10^{-1},\ 2.8772 \cdot 10^{-1}]$$

is a random feedback gain. The roots of the imaginary part of $q(s,k)$ are chosen as interpolation points

$$u = (0,\ \pm 60.847,\ \pm 16.007,\ \pm 9.2218, \pm 2.7034i)\,.$$

Here are two representative entries of the resulting Lagrange basis Hermite matrix:

$$\begin{aligned} H^S_{3,3}(k) \; &= \; 9.4439251 \cdot 10^{-1} + 1.9763715 \cdot 10^{-4} k_1 - 8.9049916 \cdot 10^{-4} k_2 \\ & - 8.6909277 \cdot 10^{-3} k_3 + 1.9212126 \cdot 10^{-1} k_4 \\ & + 3.8300306 \cdot 10^{-9} k_1 k_2 - 1.0276186 \cdot 10^{-8} k_1 k_3 \\ & + 3.3905595 \cdot 10^{-5} k_1 k_4 - 3.4222179 \cdot 10^{-5} k_2 k_3 \\ & - 3.8046300 \cdot 10^{-5} k_2 k_4 + 2.7420115 \cdot 10^{-9} k_3^2 \\ & + 6.5442491 \cdot 10^{-6} k_3 k_4 + 1.015195648 \cdot 10^{-5} k_4^2 \end{aligned}$$

and

$$H^S_{1,1}(k) = -1.6918611 k_1 + 3.7288052 \cdot 10^{-1} k_1 k_2 + 1.2286264 \cdot 10^{-2} k_1 k_3.$$

Comparing with the entries of the power basis Hermite matrix $H^P(k)$ given in Example 2.1, we observe a significant improvement in terms of coefficient scaling.

5 Numerical examples

In this section, we present the benefits of Lagrange basis against power basis when solving SOF PMI problems found in the database COMPleib, see [9]. Even though Michal Kočvara and Michael Stingl informed us that an AMPL interface to PENNON is now available to solve PMI problems, in this paper for simplicity we consider only BMIs (i.e. quadratic PMIs) and the PENBMI solver (a particular instance of PENNON focusing on BMIs) under the YALMIP modeling interface, see [10]. The numerical examples are processed with YALMIP R20070523 and PENBMI 2.1 under Matlab R2007a running on a Pentium D 3.4GHz system with 1GB ram. We set the PENBMI penalty parameter `P0` by default to 0.001 (note that this is not the default YALMIP setting).

As in [5], the optimization problem to be solved is

$$\begin{array}{ll} \min_{k,\lambda} & \mu\|k\| - \lambda \\ \text{s.t.} & H(k) \succeq \lambda I \end{array}$$

where $H(k)$ is the Hermite matrix in power or Lagrange basis, $\mu > 0$ is a parameter and $\|.\|$ is the Euclidean norm. Parameter μ allows to trade off between feasibility of the BMI and a moderate norm of the feedback gain, which is generally desirable in practice, to avoid large feedback signals. This adjustment is necessary in many examples. Indeed, the smallest values of $\|k\|$ are typically located at the boundary of the feasibility set, so the resulting closed-loop system is fragile and a small perturbation on system parameters may be destabilizing.

PENBMI is a local optimization solver. Therefore, the choice of initial guess k_0, λ_0 is critical. In most of the examples we choose the origin as the initial point. However this is not always an appropriate choice, as illustrated below. In addition to this, PENBMI does not directly handle complex numbers (unless the real and imaginary parts are split off, resulting in a real coefficient problem of double size), so we restrict the interpolation points to be real numbers.

As a result of the root interlacing property, the roots of real and imaginary parts of a stable polynomial are real (and interlacing). Owing to this fact, if we choose a stable target polynomial $q(s)$ the resulting interpolation points are necessarily real.

Example 5.1 Consider again problem AC4, with characteristic polynomial

$$\begin{aligned} q(s,k) &= s^4 + 150.92600s^3 + (130.03210 - 18.135000k_1 - 19612.500k_2)s^2 \\ &\quad - (1330.6306 + 19613.407k_1 + 18322.789k_2)s - (66.837750 + 980.62500k_1 + 867.10818k_2) \end{aligned}$$

and power basis Hermite matrix with entries

$$\begin{aligned} H^P_{1,1} &= 88936.354 + 2615765.6k_1 + 2378454.6k_2 \\ &\quad + 19233397k_1^2 + 34974730k_1k_2 + 15887840k_2^2 \\ H^P_{1,3} &= 10087.554 + 148001.81k_1 + 130869.17k_2 \\ H^P_{2,2} &= -162937.14 - 2378239.7k_1 + 23845311k_2 \\ &\quad + 355689.13k_1^2 + 38500022 \cdot 10^1 k_1k_2 + 35935569 \cdot 10^1 k_2^2 \\ H^P_{2,4} &= 1330.6306 + 19613.407k_1 + 18322.789k_2 \\ H^P_{3,3} &= 20955.855 + 16876.364k_1 - 2941713.4k_2 \\ H^P_{4,4} &= 150.92600. \end{aligned}$$

Open-loop poles of the system are $(2.5792\,, -5.0000 \cdot 10^{-2}\,, -3.4552,\ -150.00)$. If we define our target polynomial roots as $(-5.0000 \cdot 10^{-2}\,, -5.0000 \cdot 10^{-2}\,, -3.4552\,, -150.00)$, keeping the stable open-loop poles and shifting the unstable open-loop pole to the left of the imaginary axis, our 4 interpolation points (roots of the real part of the target polynomial) are $u = (\pm 23.100\,, \pm 4.9276 \cdot 10^{-2})$ and the resulting Lagrange basis Hermite matrix has entries

$$\begin{aligned} H^S_{1,1} &= 6.3432594 \cdot 10^{-1} + 3.1878941 \cdot 10^{-1}k_1 - 17.462079k_2 \\ &\quad + 4.4822907k_1^2 + 4.4060140k_1k_2 + 4.1121739k_2^2 \\ H^S_{1,2} &= -3.7795293 \cdot 10^{-1} - 1.0581354k_1 - 18.455273k_2 \\ &\quad - 3.6574685 \cdot 10^{-3}k_1^2 - 4.4045142k_1k_2 - 4.1114926k_2^2 \\ H^S_{1,3} &= 2.4459288 + 36.285639k_1 + 42.619220k_2 \\ &\quad + 7.8459729k_1^2 + 189.06139k_1k_2 + 169.77324k_2^2 \\ H^S_{1,4} &= 1.9481147 + 28.929191k_1 + 12.037605k_2 \\ &\quad + 7.5224565k_1^2 - 161.11487k_1k_2 - 157.07808k_2^2 \\ H^S_{2,2} &= 6.3432594 \cdot 10^{-1} + 3.1878941 \cdot 10^{-1}k_1 - 17.462079k_2 \\ &\quad + 4.4822907 \cdot 10^{-3}k_1^2 + 4.4060140k_1k_2 + 4.1121739k_2^2 \end{aligned}$$

$$
\begin{aligned}
H^S_{2,3} &= 1.9481074 + 28.929083k_1 + 12.037568k_2 \\
&\quad + 7.5224134k_1^2 - 161.11415k_1k_2 - 157.07737k_2^2 \\
H^S_{2,4} &= 2.4459288 + 36.285639k_1 + 42.619220k_2 \\
&\quad + 7.8459729k_1^2 + 189.06139k_1k_2 + 169.77324k_2^2 \\
H^S_{3,3} &= 659.47243 + 19434.408k_1 + 18140.083k_2 \\
&\quad + 143181.92k_1^2 + 267314.59k_1k_2 + 124766.18k_2^2 \\
H^S_{3,4} &= 665.36241 + 19520.378k_1 + 17279.067k_2 \\
&\quad + 143169.06k_1^2 + 253396.76k_1k_2 + 111775.41k_2^2 \\
H^S_{4,4} &= 659.47243 + 19434.408k_1 + 18140.083k_2 \\
&\quad + 143181.92k_1^2 + 267314.59k_1k_2 + 124766.18k_2^2.
\end{aligned}
$$

Choosing the power basis representation with the orgin as initial point and trade-off parameter $\mu = 10^{-5}$, PENBMI stops by a linesearch failure and YALMIP displays a warning. However, we obtain a feasible solution $\lambda = 150.88$ and $K = [1.4181,\ -1.6809]$. This computation requires 43 outer iterations, 433 inner iterations and 825 linesearch steps. On the other hand, in the Lagrange basis representation, the problem was solved with no error or warning, yielding $\lambda = 9.8287 \cdot 10^{-1}$, $K = [-5.0902 \cdot 10^{-2},\ -2.0985 \cdot 10^{-2}]$ with 17 outer iterations, 100 inner iterations and 159 linesearch steps.

We notice however that using the same trade-off parameter μ for both representations is not fair since H^P and H^S have significantly different scalings. If we choose $\mu = 0.1$ for the power basis representation, no problem is detected during the process and we obtain $\lambda = 150.87$, $K = [8.0929 \cdot 10^{-2},\ -1.6953 \cdot 10^{-1}]$ after 26 outer iterations, 188 inner iterations and 238 linesearch steps. So it seems that the Lagrange basis representation becomes relevant mainly for high degree systems. This is confirmed by the experiments below.

Consider the `AC7`, `AC17`, `REA3`, `UWV`, `NN5`, `NN1` and `HE1` SOF BMI problems of COMPl$_e$ib. In Table 1 we report comparative results for the power and Lagrange basis representations. As in Example 3.1, the main strategy to choose the target polynomials (and hence the interpolation points) is to mirror the open-loop stable roots, and to shift the open-loop roots to, say $-5.0000 \cdot 10^{-1}$ (any other small negative value may be suitable). We see that the behavior indicators of PENBMI are significantly better in the Lagrange basis, and the improvement is more dramatic for larger degree examples. More specifically:

- for small degree systems like `AC17` there is only a minor improvement;
- at the first attempt to solve the `REA3` example strict feasibility was not achieved in the power basis, since λ is almost zero. Therefore it was necessary to tune the μ parameter. Results of the second attempt show that the BMI problem was solved and the Lagrange basis computation was slightly less expensive than the power basis computation;
- the underwater vehicle example `UWV` has two inputs and two outputs. However, because of cancellation of higher degree terms in the characteristic polynomial, the degree of the Hermite matrix is equal to 2 and we can use PENBMI on this problem;

- on open-loop stable systems such as `UWV` or `AC17`, the improvement brought by the Lagrange basis is less significant. Since the main purpose of our optimization problem is to minimize the norm of control gain, we observe that the Lagrangian basis is still slightly better than the power basis;

- PENBMI is unable to reach a feasible point for examples `NN5`, `NN1` and `HE1`, when we choose the origin as the initial point. Indeed, local optimization techniques seek an optimal point inside the feasible set in a neighborhood of the initial point. Therefore, achievement of the solver may be very sensitive to the initial point. When the initial point is defined heuristically or randomly, the improvement is significant for system `NN5` in Lagrange basis. However, there is no improvement over `NN1` and `HE1`, when we use this simple strategy to define the target polynomial.

Table 1: PENBMI performance on SOF BMI problems

system	basis	μ	K_0	out. iter.	inn. iter.	lin. steps	K	λ
`AC7` $n=9$	pow.	1	$[0\,0]$	27	148	167	$[1.1205 \quad -3.0946\cdot 10^{-1}]$	51.640
	Lag.	10^{-5}	$[0\,0]$	15	51	67	$[5.7336 \quad 3.9995]$	$3.6356\cdot 10^{-1}$
`AC17` $n=4$	pow.	1	$[0\,0]$	14	65	173	$[1.6619\cdot 10^{-1} \quad 8.5782\cdot 10^{-1}]$	5.8306
	Lag.	1	$[0\,0]$	16	36	57	$[-1.0855\cdot 10^{-2} \quad 1.5128\cdot 10^{-1}]$	1.0459
`REA3` $n=12$	pow.	1	$[0\,0\,0]$	21	28	28	$[0 \quad -1.0435\cdot 10^{-5} \quad -2.2281\cdot 10^{-4}]$	$8.4187\cdot 10^{-13}$
		10^{-5}	$[0\,0\,0]$	46	458	2460	$[0 \quad -43711 \quad -23491]$	43787
	Lag.	10^{-2}	$[0\,0\,0]$	16	48	68	$[0 \quad -4.2556\cdot 10^{-1} \quad -8.9973\cdot 10^{-2}]$	$9.9105\cdot 10^{-1}$
`UWV` $n=8$	pow.	1	$[0\,0;0\,0]$	13	98	188	$\begin{bmatrix} -1.4319\cdot 10^{-5} & -2.6474\cdot 10^{-6} \\ -3.0817\cdot 10^{-1} & -5.6976\cdot 10^{-2} \end{bmatrix}$	27.918
	Lag.	1	$[0\,0;0\,0]$	15	65	82	$\begin{bmatrix} -1.6755\cdot 10^{-12} & -6.9006\cdot 10^{-13} \\ -3.6060\cdot 10^{-8} & -1.4851\cdot 10^{-8} \end{bmatrix}$	1.0000
`NN5` $n=7$	pow.	1	$[10\,5]$	29	162	300	$[12.382 \quad 9.0331]$	$3.9610\cdot 10^{-1}$
	Lag.	10^{-5}	$[10\,5]$	18	45	56	$[30.931 \quad 22.295]$	$1.7652\cdot 10^{-1}$
`NN1` $n=3$	pow.	10^{-3}	$[0\,30]$	15	53	59	$[7.9924 \quad 72.171]$	4.2238
	Lag.	10^{-4}	$[0\,30]$	14	49	52	$[26.936 \quad 177.20]$	4.6019
`HE1` $n=4$	pow.	1	$[1\,1]$	18	73	80	$[-1.5482 \quad -3.9063]$	34.359
	Lag.	10^{-1}	$[1\,1]$	18	80	87	$[-5.1376 \quad 11.589]$	32.168

In Table 2 we show the influence of the target polynomial on the computational cost for the `PAS` system. Open-loop poles of the system are

$$\sigma_0 = \left(0,\ 0,\ -9.5970\cdot 10^{-1},\ -36.646 \pm 523.05i\right)$$

and we choose alternative target polynomials with the following roots

$$\begin{aligned} \sigma_1 &= \left(-5.0000\cdot 10^{-2},\ -5.0000\cdot 10^{-2},\ -9.5970\cdot 10^{-1},\ -36.646 \pm 523.05i\right) \\ \sigma_2 &= \left(-1.0000\cdot 10^{-3},\ -1.0000\cdot 10^{-3},\ -9.5970\cdot 10^{-1},\ -36.646 \pm 523.05i\right) \\ \sigma_3 &= \left(0,\ -1.0000\cdot 10^{-4},\ -9.5970\cdot 10^{-1},\ -36.646 \pm 523.05i\right). \end{aligned}$$

Table 2: Influence of target polynomial on PENBMI behavior

system	PAS degree 5			
basis	power	Lagrange	Lagrange	Lagrange
μ	10^{-3}	10^{-8}	10^{-5}	10^{-2}
roots	σ_0	σ_1	σ_2	σ_3
K_0	[0 0 0]	[0 0 0]	[0 0 0]	[0 0 0]
out.iter.	11	19	17	15
inn.iter.	74	27	33	29
lin.steps	194	28	44	32
K^T	$\begin{bmatrix} -6.5390 \cdot 10^{-4} \\ -58.350 \\ -37.751 \end{bmatrix}$	$\begin{bmatrix} -8.4106 \cdot 10^{-6} \\ -3.9048 \\ -9.9675 \cdot 10^{-1} \end{bmatrix}$	$\begin{bmatrix} -3.3369 \cdot 10^{-4} \\ -20.480 \\ -1.2157 \end{bmatrix}$	$\begin{bmatrix} -8.6755 \cdot 10^{-8} \\ -4.1040 \cdot 10^{-1} \\ -1.7471 \cdot 10^{-1} \end{bmatrix}$
λ	73.2917	$1.4901 \cdot 10^{-12}$	$8.1649 \cdot 10^{-3}$	$2.7241 \cdot 10^{-3}$

One can easy to see that the computational cost is decreasing significantly when the point defining the target polynomial is getting closer to the PENBMI initial iterate.

Consider the `NN6` SOF BMI problem that was not solvable in the power basis, see [5]. Open-loop poles of the system are

$$\sigma_0 = \left(2.7303,\ 0,\ -7.2028 \cdot 10^{-2} \pm 60.804i,\ -1.0785 \cdot 10^{-1} \pm 15.677i,\ -2.6764,\ -3.3000,\ -19.694\right).$$

The strategy to define the target polynomial is to change the unstable open-loop poles into slightly stable poles (shifting the real part to a small negative value). According to this strategy, our target polynomial has the following roots

$$\begin{aligned} \sigma_1 \ = \ & \left(-1.0000 \cdot 10^{-3} \pm i,\ -7.2028 \cdot 10^{-2} \pm 60.804i,\ -1.0785 \cdot 10^{-1} \pm 15.677i,\right. \\ & \left. -2.6764,\ -3.3000,\ -19.694\right). \end{aligned}$$

The BMI SOF problem is solved with no error or warning in the Lagrange basis, yielding $\lambda = 8.8487 \cdot 10^{-1}$, $K = [1.3682, 4.8816, 44.959, 59.016]$ with 17 outer iterations, 80 inner iterations and 138 linesearch steps, using the orgin as initial point and trade-off parameter $\mu = 10^{-5}$.

6 Conclusion

The Hermite matrix arising in the symmetric formulation of the polynomial stability criterion is typically ill-scaled when expressed in the standard power basis. As a consequence, a nonlinear semidefinite programming solver such as PENNON may experience convergence problems when applied on polynomial matrix inequalities (PMIs) coming from benchmark static output feedback (SOF) problems. In this paper we reformulated Hermite's SOF PMI in a Lagrange polynomial basis. We slightly extended the results of [12] to use polynomial interpolation on possibly complex and repeated nodes to construct the Hermite matrix, bypassing potential numerical issues connected with Vandermonde matrices. In our control application, a natural choice of Lagrange nodes are the roots of a target polynomial, the desired closed-loop characteristic polynomial.

The idea of using the Lagrange polynomial basis to address numerical problems which are typically ill-scaled when formulated in the power basis has already proven successful in other contexts. For example, in [2] it was shown that roots of extremely ill-scaled polynomials (such as a degree 200 Wilkinson polynomial) can be found at machine precision using eigenvalue computation of generalized companion matrices obtained by an iterative choice of Lagrange interpolation nodes. In [13] the fast Fourier transform (a particular interpolation technique) was used to perform spectral factorization of polynomials of degree up to one million. Another example of successful use of alternative bases and high-degree polynomial interpolation to address various problems of scientific computating is the `chebfun` Matlab package, see [1]. Even though our computational results on SOF PMI problems are less dramatic, we believe that the use of alternative bases and interpolation can be instrumental to addressing various other control problems formulated in a polynomial setting.

Appendix: Matlab implementation

A Matlab implementation of the method described in this paper is available at

`homepages.laas.fr/henrion/software/hermitesof.m`

Our implementation uses the Symbolic Math Toolbox and the YALMIP interface. It is not optimized for efficiency, and therefore it can be time-consuming already for medium-size examples.

Let us use function `hermitesof` with its default tunings:

```
>> [A,B1,B,C1,C] = COMPleib('NN1');
>> A,B,C
A =
     0     1     0
     0     0     1
     0    13     0
B =
     0
     0
     1
C =

     0     5    -1
    -1    -1     0
>> [H,K] = hermitesof(A,B,C)
Quadratic matrix variable 3x3 (symmetric, real, 2 variables)
```

```
Linear matrix variable 1x2 (full, real, 2 variables)
```

Here are some sample entries of the resulting Hermite matrix

```
>> sdisplay(H(1,1))
-0.6168744435*K(2)-0.2372594014*K(1)*K(2)+0.04745188027*K(2)^2
>> sdisplay(H(3,2))
0.3019687672*K(1)+0.01984184931*K(2)-0.009656748637*K(1)*K(2)
+0.0003260141644*K(2)^2+0.04013338907*K(1)^2
```

For this example, the Hermite matrix is quadratic in feedback matrix `K`. This Hermite matrix is expressed in Lagrange basis, with Lagrange nodes chosen as the roots of the imaginary part of a random target polynomial, see the online help of function `hermitesof` for more information. In particular, it means that each call to `hermitesof` produces different coefficients. However these coefficients have comparable magnitudes:

```
>> [H,K]=hermitesof(A,B,C);
>> sdisplay(H(1,1))
-0.9592151361*K(2)-0.3689288985*K(1)*K(2)+0.0737857797*K(2)^2
>> sdisplay(H(3,2))
6.702455704*K(1)+0.5150092145*K(2)-0.3440108908*K(1)*K(2)
+0.0184651235*K(2)^2+1.258426367*K(1)^2
```

The output of function `hermitesof` is reproducible if the user provides the roots of the target polynomial:

```
>> opt = []; opt.roots = [-1 -2 -3];
>> [H,K]=hermitesof(A,B,C,opt);
>> sdisplay(H(1,1))
-0.196969697*K(2)-0.07575757576*K(1)*K(2)+0.01515151515*K(2)^2
>> sdisplay(H(3,2))
0.2*K(1)-0.01818181818*K(2)-0.01212121212*K(1)*K(2)
+0.0007575757576*K(2)^2+0.04166666667*K(1)^2
```

The Hermite matrix can also be provided in the power basis:

```
>> opt = []; opt.basis = 'p';
>> [H,K]=hermitesof(A,B,C,opt);
>> sdisplay(H(1,1))
-13*K(2)-5*K(1)*K(2)+K(2)^2
>> sdisplay(H(3,2))
0
```

For more complicated examples, the Hermite matrix `H` is not necessarily quadratic in `K`:

```
>> [A,B1,B,C1,C] = COMPleib('NN1');
>> size(B), size(C)
ans =
     5     3
ans =
     3     5
>> [H,K]=hermitesof(A,B,C)
Polynomial matrix variable 5x5 (symmetric, real, 9 variables)
Linear matrix variable 3x3 (full, real, 9 variables)
>> degree(H)
ans =
     5
```


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