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**Quantum transfer operators and chaotic scattering**

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Consider a symplectic diffeomorphism $T$ on $T^*\mathbb{R}^d$, which can be generated near the origin by a single function $W(x_1, \xi_0)$, in the sense that the dynamics $(x_1, \xi_1) = T(x_0, \xi_0)$ is the implicit solution of the two equations $\xi_1 = \partial_{x_1} W(x_1, \xi_0)$, $x_0 = \partial_{\xi_0} W(x_1, \xi_0)$. One can associate to $T$ a family of *quantum transfer operators* $M(T, h)$ acting on $L^2(\mathbb{R}^d)$, of the form:

$$[M(T,h)\psi](x_1) = \int a(x_1, \xi_0)\, e^{\frac{i}{h}(W(x_1,\xi_0)-\langle \xi_0, x_0\rangle)}\, \psi(x_0)\, \frac{dx_0\, d\xi_0}{(2\pi h)^d}\,. \tag{1}$$

Here $a \in C^\infty(T^*\mathbb{R}^d)$ is called the *symbol* of the operator. The "small parameter" $h > 0$ is the typical wavelength on which the integral kernel of the operator oscillates; it is often called "Planck's constant", due to the appearance of such operators in quantum mechanics.

The operator $M(T, h)$ (understood as a family $(M(T, h))_{h\in(0,1]}$) can be interpreted as a "quantization" of the symplectic map $T$, for the following reason. Consider a phase space point $(x_0, \xi_0) \in T^*\mathbb{R}^d$. There exist wavefunctions (*quantum states*) $\psi_{x_0,\xi_0,h} \in L^2(\mathbb{R}^d)$ which are localized near the position $x_0 \in \mathbb{R}^d$, and whose $h$-Fourier transform is localized near the momentum $\xi_0 \in \mathbb{R}^d$ (equivalently, the usual Fourier transform is localized near $h^{-1}\xi_0$). Such wavefunctions are said to be *microlocalized* near $(x_0, \xi_0)$; in some sense, they represents the best quantum approximation of a "point particle" at $(x_0, \xi_0)$. In the semiclassical limit $h \to 0$, the application of stationary phase expansions to the integral ((1)) shows that the image state $M(T, h)\psi_{x_0,\xi_0,h}$ is microlocalized near the point $(x_1, \xi_1) = T(x_0, \xi_0)$; that is, this operator transports the quantum mass at the point $(x_0, \xi_0)$ to the point $T(x_0, \xi_0)$.

Similar families of operators have appeared in the theory of linear PDEs in the 1960s: the "Fourier integral operators" invented by Hörmander. A modern account (closer to the above definition) can be found in the recent lecture notes of C.Evans & M.Zworski [1]. We are using these operators as nice models for "quantum chaos", that is the study of quantum systems, the classical limits of which are "chaotic". In this framework, these operators (sometimes called "quantum maps") generate a quantum dynamical system:

$$L^2(\mathbb{R}^d) \ni \psi \mapsto M(T, h)\psi\,. \tag{2}$$

These quantum maps provide a discrete time generalization of the Schrödinger flow $U^t(h) = \exp(-itP(h)/h)$ associated with the Schrödinger equation $ih\partial_t\psi = P(h)\,\psi$, where $P(h)$ is a selfadjoint operator, e.g. of the form $P(h) = -\frac{h^2\Delta}{2} + V(x)$; in that case, the classical evolution is the Hamilton flow $\phi^t_p$ generated by the classical Hamiltonian $p(x, \xi) = \frac{|\xi|^2}{2} + V(x)$ on $T^*\mathbb{R}^d$.

As usual in dynamics, one is mostly interested in the long time properties of the dynamical system (2). For such a linear dynamics, these properties are encoded in the *spectrum* of $M(T, h)$. Therefore, a major focus of investigation concerns the spectral properties of the operators $M(T, h)$, especially in the semiclassical limit

$h \to 0$, where the connection to the classical map is most effective. Quantum maps have mostly be studied in cases where $M(T,h)$ is replaced by a unitary operator on some $N$-dimensional Hilbert space, with $N \sim h^{-1}$. This is the case if $T$ is a symplectomorphism on a compact symplectic manifold, like the 2-torus [4]. More recently, one has got interested in operators $M(T,h)$ which act unitarily on states microlocalized inside a certain domain of $T^*\mathbb{R}^d$, but "semiclassically kill" states microlocalized outside a larger bounded domain (these properties depend on the choice of the symbol $a(x_1,\xi_0)$). As a result, the spectrum of $M(T,h)$ is contained in the unit disk, and its effective rank is $\leq Ch^{-d}$ (according to the handwaving argument that one quantum state occupies a volume $\sim h^d$ in phase space). Such operators have been called "open quantum maps".

Let us now assume that the map $T$ has chaotic properties: the nonwandering set $\Gamma$ is a fractal set included inside $B(0,R)$, and $T$ is uniformly hyperbolic on $\Gamma$. We may then expect this dynamical structure to imply some form of *quantum decay*: indeed, a quantum state cannot be localized on a ball of radius smaller than $\sqrt{h}$, and such a ball is not fully contained in $\Gamma$, so most of the ball will escape to infinity through the map $T$. On the other hand, quantum mechanics involves *interefence effects*, which may balance this purely classical decay. Following old works of M.Ikawa [3] and P.Gaspard & S.Rice [2] in the framework of Euclidean obstacle scattering, one is lead to the following condition for quantum decay:

**Theorem 1.** *For any $(x,\xi) \in \Gamma$, call $\varphi^u(x,\xi) = -\log|\det DT_{|E^u(x,\xi)}|$ the unstable Jacobian of $T$ at $(x,\xi)$, and consider the corresponding topological pressure $\mathcal{P}(\frac{1}{2}\varphi^u)$.*

*If that pressure is negative, then for any $1 > \gamma > \exp\{\mathcal{P}(\frac{1}{2}\varphi^u)\}$, and any small enough $h > 0$, the operator $M(T,h)$ has a spectral radius $\leq \gamma$.*

In dimension $d = 1$ (that is, when $T$ acts on $T^*R$), the negativity of that pressure is equivalent with the fact that the Hausdorff dimension $d_H(\Gamma) < 1$. This equivalence breaks down in higher dimension, but a negative pressure is still correlated with $\Gamma$ being a "thin" set.

The above theorem has been obtained by M.Zworski and myself in the framework of Euclidean scattering by smooth potentials [6]. The extension to quantum maps $M(T,h)$ is straightforward, and should be part of a work in preparation with J.Sjöstrand and M.Zworski. In general we do not expect the above to be optimal. Following a recent work of V.Petkov & L.Stoyanov [8], one should be able to compare $M(T,h)$ with classical transfer operators of the form $\mathcal{L}_{\frac{1}{2}\varphi^u+i/h}$, apply Dolgopyat's method to the latter to get a spectral radius $\gamma = \exp\{\mathcal{P}(\frac{1}{2}\varphi^u) - \epsilon_1\}$ for the classical and the quantum operators.

Most of the $\mathcal{O}(h^{-d})$ eigenvalues of $M(T,h)$ can be very close to the origin when $h \to 0$. Indeed, the fractal character of the trapped set has a strong influence on the semiclassical density of eigenvalues: any point situated at distance $\gg h^{1/2}$ from $\Gamma$ will be pushed out of $B(0,R_1)$ through the classical dynamics (either in the past or in the future), within a time $|n| \leq C\log(1/h)$, where semiclassical methods still apply. As a result, the eigenstates of $M(T,h)$ associated with nonnegligible eigenvalues must be "supported" by the tubular neighbourhood of $\Gamma$ of radius $\sqrt{h}$. A direct volume estimate of this neighbourhood, and the above-mentioned argument on the volume occupied by a quantum states, lead to the following upper bound for the density of eigenvalues:

**Theorem 2.** *Assume that the hyperbolic trapped set $\Gamma \subset T^*\mathbb{R}^d$ has upper Minkowski dimension $d_M > 0$. Then, for any small $\epsilon, \epsilon' > 0$ and any small enough $h > 0$, one has*

$$\#\{\lambda \in \mathrm{Spec}(M(T,h)),\ |\lambda| > \epsilon\} \leq C_{\epsilon,\epsilon'}\, h^{-d_M/2-\epsilon'}\,. \tag{3}$$

*(eigenvalues are counted with multiplicities.)*

A similar result has been first obtained by J.Sjöstrand in the case of Euclidean scattering by smooth potentials [10], and has then been refined and generalized to various settings. The case of quantum maps should also appear in the forthcoming work with J.Sjöstrand and M.Zworski.

The "fractal upper bound" ((3)) is actually conjectured to be an asymptotics. This has been shown numerically in various cases, including hyperbolic scattering [5] as well as quantum maps [9]. This asymptotics has been proved only for a very specific quantum maps [7], and represents an interesting challenge for more realistic systems.

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