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Thème : Computational Geometry, Geometric Graphs
Équipe-Projet Geometrica

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Abstract: This paper extends the result of Steele [6, 5] on the worst-case length of the Euclidean minimum spanning tree $EMST$ and the Euclidean minimum insertion tree $EMIT$ of a set of n points $S \subset \mathbb{R}^d$. More precisely, we show that, if the weight w of an edge e is its Euclidean length to the power of α, the following quantities $\sum_{e \in EMST} w(e)$ and $\sum_{e \in EMIT} w(e)$ are both worst-case $O(n^{1-\alpha/d})$, where d is the dimension and α, $0 < \alpha < d$, is the weight. Also, we analyze and compare the value of $\sum_{e \in T} w(e)$ for some trees T embedded in $\mathbb{R}^d$ which are of interest in (but not limited to) the point location problem [2].

Key-words: Computational Geometry, Geometric Graphs

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Sûr la Taille de Quelques Arbres Plongées dans $\mathbb{R}^d$

Résumé : Ce papier étend le résultat de Steele [6, 5] sûr la taille au pire des cas de la plus petite arbre de couverture minimal Euclidienne et l'arbre d'insertion minimal d'un ensemble de n points $S \subset \mathbb{R}^d$. Plus précisément, nous démontrons que si le poids w d'une arête e est sa longueur Euclidienne à la puissance α, les quantités suivantes $\sum_{e \in EMST} w(E)$ et $\sum_{e \in EMIT} w(E)$ valent au pire des cas $O(n^{1-\alpha/d})$, où d est la dimension et α, $0 < \alpha < d$, est le poid. Nous déterminons et comparons aussi la valeur de $\sum_{e \in T} w(E)$ pour des arbres T plongées dans $\mathbb{R}^d$, qui sont d'intèrêt au problème de la localisation des points [2].

Mots-clés : Géométrie Algorithmique, Graphes Géométriques

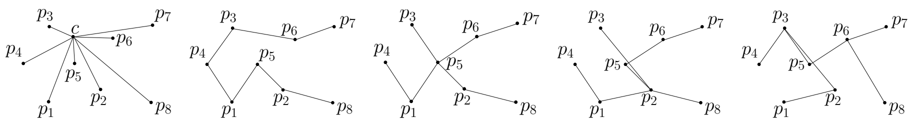
1 Introduction

This paper extends the result of Steele [6, 5] on the worst-case length of the Euclidean minimum spanning tree $EMST$ and the Euclidean minimum insertion tree $EMIT$ of a set of n points $S \subset \mathbb{R}^d$. More precisely, we show that, if the weight w of an edge e is its Euclidean length to the power α, the following quantities $\sum_{e \in EMST} w(e)$ and $\sum_{e \in EMIT} w(e)$ are both worst-case $O(n^{1-\alpha/d})$, where d is the dimension and α, $0 < \alpha < d$, is the weight. Also, we analyze and compare the value of $\sum_{e \in T} w(e)$ for some trees T embedded in $\mathbb{R}^d$ which are of interest in (but not limited to) the point location problem [2].

Let $S = \{p_i, 1 \leq i \leq n\}$ be a set of points in $\mathbb{R}^d$ and $G = (V, E)$ be the complete graph such that the vertex $v_i \in V$ is embedded on the point $p_i \in S$; the edge $e_{ij} \in E$ linking two vertices v_i and v_j is weighted by $|p_i - p_j|^\alpha$, its Euclidean length to the power of α. G is usually referred to as the *geometric graph* of S. We will denote the sum of the weight of the edges of G by $|G|_\alpha$ (or $|G|$ if $\alpha = 1$). We will also refer to $|G|_\alpha$ as the *weighted-length* of G.

Consider the following trees:
(i) A *star* is a tree having one vertex that is linked to all others (see Figure 1a).
(ii) A *path* is a tree having all vertices of degree 2 but two with degree 1 (see Figure 1b).
(iii) Among all the trees spanning S, a tree with the minimal length is called an *Euclidean minimum spanning tree* of S and denoted $EMST(S)$ (see Figure 1c).
(iv) Consider that an ordering is given by a permutation σ, vertices are inserted in the order $v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}$. We build incrementally a spanning tree T_i for $S_i = \{p_{\sigma(j)} \in S, i \leq j\}$ with $T_1 = \{v_{\sigma(1)}\}$, $T_i = T_{i-1} \cup \{v_{\sigma(i)} v_{\sigma(j)}\}$ and a fixed k, with $1 \leq k < n$, such that $v_{\sigma(i)} v_{\sigma(j)}$ has the shortest length for any $\max(1, i-k) \leq j < i$. This tree is called the *Euclidean minimum k-insertion tree*, and will be denoted by $EMIT_k(S)$ (see Figure 1e); when $k = n-1$, we will write $EMIT(S)$ (see Figure 1d). $|EMIT(S)|$ depends on σ and for some permutations it coincides with $|EMST(S)|$.

It is noteworthy that both the combinatorics of $EMST(S)$ and of $EMIT(S)$ are invariant to α (since $f(\lambda) = \lambda^\alpha$, is monotonically increasing for positive α and λ). What changes is the sum of the weights associated with the edges. Also, $EMST$ assumes that the position of all the points in S is available (static) whereas $EMIT$ clearly does not (dynamic).



(a) Star (b) Path (c) $EMST$ (d) $EMIT$ (e) $EMIT_2$

Figure 1: **Trees embedded in the plane.**

Steele proves [4] that if p_i are i.i.d. random variables with compact support, then $|EMST(S)|_\alpha = O(n^{1-\alpha/d})$ with probability 1. For the extreme case of $\alpha = d$, Aldous and Steele [1] show that $|EMST(S)|_d = O(1)$ if the variables above are evenly distributed in the unit cube. Without any dependence on probabilistic hypotheses, Steele proves [6] that the complexity of $|EMST(S)|$ is bounded by $O(n^{1-1/d})$ in the worst case. Finally, the asymptotic length of $|EMIT(S)|$ is shown to be the same as the one of $|EMST(S)|$ [5]. In other words, $|EMIT(S)| = O(n^{1-1/d})$. This result is surprising because it means that *a priori* knowledge of S does not affect the asymptotic length of trees following the greedy strategy of an $EMST$. This fact has application in the dynamic point location problem [2].

This work is presented as follow: First, in Section 2, we extend the result of Steele [5] stating that $|EMIT|$ and $|EMST|$ have the same asymptotical behavior to the case of $|EMIT|_\alpha$ and $|EMST|_\alpha$ for $0 < \alpha < d$ as well. Second, in Section 3, we obtain the expected weighted-length of some stars of interest inside the unit ball. Then in Section 4 we obtain some ratios between the expected weighted-length of such stars and random paths in the unit ball. Finally, in Section 5 we obtain bounds on the expected weighted-length of $EMIT_k$.

2 Weighted Euclidean Minimum Insertion Tree

We extend the result of Steele [5] for $|EMIT|_\alpha$ (and consequently $|EMST|_\alpha$) with the following theorem:

Theorem 1. *Let S be a sequence of n points in $[0,1]^d$, then $|EMST(S)|_\alpha \leq |EMIT(S)|_\alpha \leq \gamma_{d,\alpha} n^{1-\alpha/d}$, with $d \geq 2$ and $0 < \alpha < d$. Where, $\gamma_{d,\alpha} = 1 + \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)}$.*

The proof of Theorem 1 follows exactly the same line as Steele [5], and starts with the two lemmas below. Given a fixed sequence $S = \{p_1, p_2, \ldots, p_n\}$ of points in $\mathbb{R}^d$, then we can build a spanning tree for S by sequentially joining x_i to the tree formed by $\{p_1, p_2, \ldots, p_{i-1}\}$ for $1 < i \leq n$. Let $w_i \in \mathbb{R}$ be defined as follow:

$$w_i = \min_{1 \leq j < i} |p_i - p_j|^\alpha, \tag{1}$$

then w_i is the minimal cost of joining p_i to a vertex of a spanning tree of $\{p_1, p_2, \ldots, p_{i-1}\}$. Now, we have that $|EMIT(S)|_\alpha = \sum_{1<i\leq n} w_i$.

Lemma 2. *If $\{p_1, p_2, \ldots, p_n\} \subset [0,1]^d$ and $w_i = \min_{1\leq j<i} |p_i - p_j|^\alpha$, for $1 < i \leq n$, $d \geq 2$ and $0 < \alpha < d$, then for any $0 < \lambda < \infty$ we have*

$$\sum_{\lambda \leq w_i < 2^\alpha \lambda} w_i^{d/\alpha} \leq 8^d d^{d/2}. \tag{2}$$

Proof. Let $C = \{i : \lambda \leq w_i < 2^\alpha \lambda\}$ and for each $i \in C$ let B_i be a ball of radius $r_i = \frac{1}{4} w_i^{1/\alpha}$ with center p_i. We will argue by contradiction that $B_i \cap B_j = \emptyset$ for all $i < j$. If $B_i \cap B_j \neq \emptyset$, then the bounds $r_i \leq 2^\alpha \lambda$ and $r_j \leq 2^\alpha \lambda$ gives us

$$|p_i - p_j| \leq \frac{1}{4}({w_i}^{1/\alpha} + {w_j}^{1/\alpha}) < \lambda^{1/\alpha}. \tag{3}$$

But, by definition of w_j we have $|p_i - p_j|^\alpha \geq w_j$ for all $i < j$, which implies $|p_i - p_j| \geq w_j^{1/\alpha}$ for all $i < j$; and, by the lower bound on the summands in Eq.(2)

we have $\lambda \leq w_j$, which means $\lambda^{1/\alpha} \leq {w_j}^{1/\alpha}$, so we also see $|p_i - p_j| \geq \lambda^{1/\alpha}$. Since $|p_i - p_j| \geq \lambda^{1/\alpha}$ contradicts Eq.(3), we have $B_i \cap B_j = \emptyset$.

Now, since all of the balls B_i are disjoint and contained in a sphere with radius $2\sqrt{d}$, the sum of their volumes is bounded by the volume of the sphere of radius $2\sqrt{d}$. Thus, if ω_d denotes the volume of the unit ball in $\mathbb{R}^d$, we have the bound $\sum_{i \in C} \omega_d w_i^{d/\alpha} 4^{-d} \leq \omega_d 2^d d^{d/2}$ from which Eq.(2) follows. □

Lemma 3. *Let* Ψ *be a positive and non-increasing function on the interval* $\left(0, \sqrt{d}\right]$*, then for any* $0 < a < b \leq \sqrt{d}$*, with* $d \geq 2$ *and* $0 < \alpha < d$*,*

$$\sum_{a \leq w_i \leq b} {w_i}^{(d/\alpha)+1} \Psi(w_i) \leq \frac{2^\alpha}{2^\alpha - 1} \cdot 8^d d^{d/2} \int_{a/2^\alpha}^{b} \Psi(\lambda) \mathrm{d}\lambda. \tag{4}$$

Proof. By Lemma 2 we have for any $0 < \lambda < \infty$,

$$\sum_{a \leq w_i < b} {w_i}^{d/\alpha} I(\lambda \leq w_i < 2^\alpha \lambda) \leq 8^d d^{d/2},$$

where

$$I(\lambda \leq w_i < 2^\alpha \lambda) = I(\frac{1}{2^\alpha} w_i \leq \lambda < w_i)$$

is the indicator function. If we multiply by $\Psi(\lambda)$ and integrate over $[\frac{1}{2^\alpha} a, b]$, we find

$$\sum_{a \leq w_i \leq b} {w_i}^{(d/\alpha)} \int_{w_i/2^\alpha}^{w_i} \Psi(\lambda) \mathrm{d}\lambda \leq 8^d d^{d/2} \int_{a/2^\alpha}^{b} \Psi(\lambda) \mathrm{d}\lambda. \tag{5}$$

Since Ψ is non-increasing, the integrand on the left-hand side of Eq.(5) is bounded from below by $\Psi(w_i)$, so $\Psi(w_i) w_i (1 - \frac{1}{2^\alpha}) \leq \int_{w_i/2^\alpha}^{w_i} \Psi(\lambda) \mathrm{d}\lambda$, and Eq.(4) follows from Eq(5). □

Proof of Theorem 1. Divide the set $\{w_2, w_3, \ldots, w_n\}$ in two sets $R_1 = \{w_i : w_i \leq n^{-\alpha/d}\}$ and $R_2 = \{w_i : w_i > n^{-\alpha/d}\}$. We have the trivial bound $\sum_{w_i \in R_1} w_i = \sum_{w_i \leq n^{-\alpha/d}} w_i \leq n \cdot n^{-\alpha/d} = n^{1-\alpha/d}$. Now, let $[a, b] = \left[n^{-\alpha/d}, \sqrt{d}\right]$, $\Psi(\lambda) = \lambda^{-d/\alpha}$ in Eq.(4), then we have:

$$\sum_{w_i \geq n^{-\alpha/d}} w_i \leq \frac{2^\alpha}{2^\alpha - 1} \cdot 8^d d^{d/2} (d/\alpha - 1)^{-1} \cdot \left(2^{d-\alpha} n^{1-\alpha/d} - d^{-(d/\alpha - 1)/2}\right).$$

And hence, $\sum_{w_i \in R_2} w_i \leq \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)} \cdot n^{1-\alpha/d}$. Now, we have that

$$\sum_{i=2}^{n} w_i = \sum_{w_i \in R_1} w_i + \sum_{w_i \in R_2} w_i \leq \left(1 + \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)}\right) \cdot n^{1-\alpha/d},$$

from which Theorem 1 follows. The constant is $\gamma_{d,\alpha} = 1 + \frac{2^{4d} d^{d/2}}{(2^\alpha - 1)(d/\alpha - 1)}$. □

3 Two Stars Embedded in $\mathbb{R}^d$

Assume a set of points $S = \{p_i, 1 \leq i \leq n\}$, evenly distributed inside the unit ball $\mathcal{B}$. Let c be a point in $\mathcal{B}$, and $E(|cp|^\alpha, p \in \mathcal{B})$ be the expected value of the distance between c and a point inside the unit ball to the power of α, with $\alpha > 0$, as $n \to \infty$. The following theorem distinguish two stars of particular interest.

Theorem 4. *The shortest and largest expected weighted-length stars inside the ball are respectively: the star centered at O, the center of the ball; and the star centered at Ω, a point on the boundary of the ball, denoted by $\mathcal{H}$.*

We have that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{|p_i - c|^\alpha}{n} = E(|cp|^\alpha, p \in \mathcal{B}).$$

In other words, $\lim\limits_{n\to\infty} E(|cp|^\alpha, p \in \mathcal{B})$ is the weighted-length of an edge of the star centered in c as $n \to \infty$. Now, let us turn the attention to the so-called *weighted Steiner star* of S, the star with the smallest weighted-length we can imagine of S. Consider that the weighted Steiner star of the first n points is centered at c_n^*, then we have

$$\sum_{i=1}^{n} |p_i - c_n^*|^\alpha \leq \sum_{i=1}^{n} |p_i - c|^\alpha, \tag{6}$$

for any $n > 0$ and $c \in \mathcal{B}$. Let $A_n(c) = \sum_{i=1}^{n} |p_i - c|^\alpha / n$. The proof of Theorem 4 is divided into several lemmas. First, we show that $A_n(c_n^*)$ converges. Then we show that there is a point $\bar{c} \in \mathcal{B}$ such that $A_n(\bar{c})$ and $A_n(c_n^*)$ converge to the same limit. Finally, we argue that $\bar{c}$ must be the center of the ball, the point O.

Lemma 5. *Let $A_n(c) = \sum_{i=1}^{n} |p_i - c|^\alpha / n$ and c_n^* be the center of the weighted Steiner star of the first n points $A_n(c_n^*)$. Then $A_n(c_n^*)$ converges.*

Proof. As points are evenly distributed in $\mathcal{B}$, c_n^* is a non-constant sequence with probability 1, and thus we can assume that $0 < A_n(c_n^*) < 2^\alpha$ for any $n > 1$ and $d \geq 1$. We have from Eq.(6)

$$\begin{aligned} A_{n+1}(c_{n+1}^*) &\leq A_{n+1}(c_n^*) = \sum_{i=1}^{n} \frac{|p_i - c_n^*|^\alpha}{n+1} + \frac{|p_{n+1} - c_n^*|^\alpha}{n+1} \\ &\leq \sum_{i=1}^{n} \frac{|p_i - c_n^*|^\alpha}{n} + \frac{|p_{n+1} - c_n^*|^\alpha}{n}. \end{aligned}$$

But $|p_{n+1} - c_n^*| \leq 2$, and thus $A_{n+1}(c_{n+1}^*) \leq A_n(c_n^*) + 2^\alpha/n$, which means that $\lim\limits_{n\to\infty} \dfrac{A_{n+1}(c_{n+1}^*)}{A_n(c_n^*)} \leq 1$. Analogously, we have, from the fact that $|p_{n+1} - c_{n+1}^*| \leq 2$ and Eq.(6), $A_n(c_n^*) \leq \left(\frac{n+1}{n}\right) A_{n+1}(c_{n+1}^*) - \frac{2^\alpha}{n}$. This means that $\lim\limits_{n\to\infty} \dfrac{A_{n+1}(c_{n+1}^*)}{A_n(c_n^*)} \geq 1$, and thus $\lim\limits_{n\to\infty} \dfrac{A_{n+1}(c_{n+1}^*)}{A_n(c_n^*)} = 1$. As $0 < A_n(c_n^*) < 2^\alpha$ for any $n > 1$ and $d \geq 1$, $A_n(c_n^*)$ converges. $\square$

Now, we need the following easy lemma.

Lemma 6. *Assume* $a, b \in \mathbb{R}$, $0 \leq a, b \leq 2$, *for each* $\alpha > 0$ *there is a function* $f_\alpha(b)$ *such that* $(a+b)^\alpha \leq a^\alpha + f_\alpha(b)$ *and* $\lim_{b\to 0} f_\alpha(b) = 0$.

Proof. For $\alpha \leq 1$, we have $f_\alpha(b) = b^\alpha$, as $(a+b)^\alpha \leq a^\alpha + b^\alpha$.
For $\alpha \geq 2$ and $\alpha \in \mathbb{N}$, we have $f_\alpha(b) = 2^{2\alpha-1}b$, as

$$(a+b)^\alpha = a^\alpha + b\sum_{i=1}^{\alpha}\binom{\alpha}{i}a^{\alpha-i}b^{i-1} \leq a^\alpha + 2^{2\alpha-1}b.$$

Let, $\{x\}$ signify $x - \lfloor x \rfloor$, then for $\alpha > 1$ and $\alpha \notin \mathbb{N}$, we have $f_\alpha(b) = 2^{2\lfloor\alpha\rfloor}b^{\{\alpha\}} + 2^{2\alpha}b$, as

$$\begin{aligned}(a+b)^\alpha &= (a+b)^{\lfloor\alpha\rfloor}(a+b)^{\{\alpha\}} \\ &\leq \left(a^{\lfloor\alpha\rfloor} + 2^{2\lfloor\alpha\rfloor-1}b\right)\left(a^{\{\alpha\}} + b^{\{\alpha\}}\right) \\ &\leq a^\alpha + 2^{2\lfloor\alpha\rfloor}b^{\{\alpha\}} + 2^{2\alpha}b.\end{aligned}$$

□

Lemma 7. *There is a point* $\overline{c} \in \mathcal{B}$ *such that* $A_n(\overline{c})$ *and* $A_n(c_n^*)$ *converge to the same limit.*

Proof. If a topological space X is compact, then every infinite subset of X has an accumulation point. Assume $\overline{c}$ is an accumulation point of the sequence $\{c_i^*\}_{i=1,2,\ldots,\infty}$, then we have a subsequence of indices $\zeta : \mathbb{N}^* \to \mathbb{N}^*$ such that $\{c_{\zeta(i)}^*\}_{i=1,2,\ldots,\infty}$ converges to $\overline{c}$. Because of the triangulation inequality and by direct application of Lemma 6, for any $i > 0$, we have that

$$A_{\zeta(i)}(\overline{c}) \leq A_{\zeta(i)}(c_{\zeta(i)}^*) + f_\alpha(|\overline{c}c_{\zeta(i)}^*|).$$

As $|\overline{c}c_{\zeta(i)}^*|$ converges to 0, then $f_\alpha(|\overline{c}c_{\zeta(i)}^*|)$ also converges to 0 (see Lemma 6). And thus $\lim_{i\to\infty} A_{\zeta(i)}(\overline{c}) = \lim_{i\to\infty} A_{\zeta(i)}(c_{\zeta(i)}^*)$. Therefore, as $A_n(c_n^*)$ converges, $\lim_{i\to\infty} A_{\zeta(i)}(c_{\zeta(i)}^*) = \lim_{n\to\infty} A_n(c_n^*)$ and $\lim_{n\to\infty} A_n(\overline{c}) = \lim_{i\to\infty} A_{\zeta(i)}(\overline{c}) = \lim_{n\to\infty} A_n(c_n^*)$. □

Proof of Theorem 4. By symmetry, $\lim_{n\to\infty} A_n(O) \leq \lim_{n\to\infty} A_n(c)$ for any $c \in \mathcal{B}$, and from Eq.(6) and Lemma 7, we have that $\lim_{n\to\infty} A_n(O) \geq \lim_{n\to\infty} A_n(c_n^*) = \lim_{n\to\infty} A_n(\overline{c}) \geq \lim_{n\to\infty} A_n(O)$. Therefore, at the limit, we have that the length of the weighted Steiner star is equivalent to the length of the star centered at O.

With analogous arguments, we have that the largest weighted-star and a star centered at the boundary of the ball have equivalent length. □

Denote the shortest and largest expected weighted-length stars inside the ball by $\mathcal{S}$ and $\mathcal{H}$ respectively. Let $E(|Op|^\alpha, p \in \mathcal{B})$ and $E(|\Omega p|^\alpha, p \in \mathcal{B})$ be the expected value of an edge of $\mathcal{S}$ and $\mathcal{H}$ respectively, then as $n \to \infty$ the size of $\mathcal{S}$ and $\mathcal{H}$ are given accordingly by $n \cdot E(|Op|^\alpha, p \in \mathcal{B})$ and $n \cdot E(|\Omega p|^\alpha, p \in \mathcal{B})$.

We analyze in the sequel the values of $E(|Op|^\alpha, p \in \mathcal{B})$ and $E(|\Omega p|^\alpha, p \in \mathcal{B})$.

Theorem 8. *When points are uniformly i.i.d in a ball, the expected size $E(|Op|^{\alpha}, p \in \mathcal{B})$ of and edge of the star centered at the center of the unit ball, with positive α, is given by:*

$$\left(\frac{d}{d+\alpha}\right).$$

Proof. Let $\mathcal{B}_l$ be a ball with radius l centered at the origin, we have

$$\begin{aligned} E(|Op|^{\alpha}, p \in \mathcal{B}) &= \int_0^1 l^{\alpha} Prob(p \in \mathcal{B}_{l+\mathrm{d}l} \setminus \mathcal{B}_l) = \int_0^1 l^{\alpha} \frac{(\mathrm{d}V_d(l)/l)\mathrm{d}l}{V_d(1)} \\ &= \int_0^1 dl^{d-1+\alpha}\mathrm{d}l = \frac{d}{d+\alpha}, \end{aligned}$$

where $V_d(l)$ is the volume of a ball of radius l (and $\mathrm{d}V_d(l)/l$ is its area). □

Theorem 9. *When points are uniformly i.i.d in a ball, the expected size $E(|\Omega p|^{\alpha}, p \in \mathcal{B})$ of and edge of the star centered at the boundary of the unit ball, with positive α, is given by:*

$$2^{d+\alpha}\left(\frac{2d+\alpha}{2d+2\alpha}\right)\frac{B\left(\frac{d}{2}+\frac{1}{2}, \frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}{B\left(\frac{d}{2}+\frac{1}{2}, \frac{1}{2}\right)},$$

where $B(x,y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\mathrm{d}\lambda$ is the so-called Beta function.

The computation of the average is more involved than in Theorem 8, and we split the computation into several lemmas.

Lemma 10. *Consider the spherical cap $\mathcal{H}_h$ formed by crossing a ball $\mathcal{B}_R$ with radius R centered at the origin, with the plane $x = R - h$. Denote h the height of the cap. The volume of $\mathcal{H}_h$ is the volume of the intersection between the half-space $x \geq R - h$ and $\mathcal{B}_R$. This volume is given by:*

$$R^d \frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d+1}{2})} \int_0^{\arccos\left(\frac{R-h}{R}\right)} \sin^d(\lambda)\mathrm{d}\lambda. \tag{7}$$

Proof. The volume $V_d(r)$ of a ball with radius r in dimension d is given by $r^d \cdot \pi^{\frac{d}{2}}/\Gamma(1+\frac{d}{2})$. Each cross-section $x = R - h + \delta$, $0 \leq \delta \leq h$ is a $(d-1)$-dimensional ball. If we integrate all of those balls along the x axis, we have $\int_{R-h}^{R} V_{d-1}(\sqrt{R^2-t^2})\mathrm{d}t$. Eq.(7) follows from replacing t by $\lambda = R\cos(t)$. □

Lemma 11. *Let Ω be a point on the boundary of the unit ball $\mathcal{B}_{unit}$, and $P_{\mathcal{H}}(l) = Prob(|\Omega p| \leq l\,; p \in \mathcal{B}_{unit})$ be the cumulative distribution function of distances between an uniformly distributed random point inside $\mathcal{B}_{unit}$ and Ω, then*

$$P_{\mathcal{H}}(l) = \frac{1}{B\left(\frac{d}{2}+\frac{1}{2}, \frac{1}{2}\right)}\left(\int_0^{\arccos(1-l^2/2)} \sin^d(\lambda)\mathrm{d}\lambda + l^d \int_0^{\arccos(l/2)} \sin^d(\lambda)\mathrm{d}\lambda\right),$$

where $B(x,y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\mathrm{d}\lambda$ is the Beta function.

Proof. If we denote $\mathcal{B}_l$ the ball of radius l centered in Ω, the desired probability is clearly $volume(\mathcal{B}_l \cap \mathcal{B}_{unit})/volume(\mathcal{B}_{unit})$. $\mathcal{B}_l \cap \mathcal{B}_{unit}$ is the union of two spherical caps limited by the plane $x = 1 - l^2/2$ which can be computed using Lemma 10. □

Proof of Theorem 9. The theorem follows from:

$$\begin{aligned}
E(|\Omega p|^\alpha, p \in \mathcal{B}) &= \int_0^2 l^\alpha P'_{\mathcal{H}}(l)\mathrm{d}l \\
&= \int_0^2 l^\alpha \left(\frac{\frac{1}{2}l^d \left(1-\frac{l^2}{4}\right)^{\frac{d-1}{2}}}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)} + dl^{d-1}\frac{\int_0^{\arccos(l/2)} \sin^d(\lambda)\mathrm{d}\lambda}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)} \right)\mathrm{d}l \\
&= \frac{1}{2}\int_0^2 \frac{l^{d+\alpha}\left(1-\frac{l^2}{4}\right)^{\frac{d-1}{2}}}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}\mathrm{d}l + \frac{1}{2}\int_0^2 2dl^{d-1+\alpha}\frac{\int_0^{\arccos(l/2)} \sin^d(\lambda)\mathrm{d}\lambda}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}\mathrm{d}l
\end{aligned}$$

The right part of the expression above corresponds exactly to the expected value of l^α where l is the length of a random segment determined by two evenly distributed points in the unit ball [3, 7]. Its value is given by:

$$\int_0^2 2dl^{d-1+\alpha}\frac{\int_0^{\arccos(l/2)} \sin^d(\lambda)d\lambda}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}\mathrm{d}l = 2^{d+\alpha}\left(\frac{d}{d+\alpha}\right)\frac{B\left(\frac{d}{2}+\frac{1}{2},\frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}.$$

The left part of the expression can be obtained as follows:

$$\begin{aligned}
\int_0^2 \frac{l^{d+\alpha}\left(1-\frac{l^2}{4}\right)^{\frac{d-1}{2}}}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}\mathrm{d}l &= 2\int_0^1 \frac{2^{d+\alpha}y^{d+\alpha}\left(1-y^2\right)^{\frac{d-1}{2}}\mathrm{d}y}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)} \\
&= \int_0^1 \frac{2^{d+\alpha}z^{\frac{d}{2}+\frac{\alpha}{2}-\frac{1}{2}}\left(1-z\right)^{\frac{d-1}{2}}\mathrm{d}z}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)} \\
&= 2^{d+\alpha}\frac{B\left(\frac{d}{2}+\frac{1}{2},\frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}.
\end{aligned}$$

Finally, we have

$$E(|\Omega p|^\alpha, p \in \mathcal{B}) = 2^{d+\alpha}\left(\frac{2d+\alpha}{2d+2\alpha}\right)\frac{B\left(\frac{d}{2}+\frac{1}{2},\frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)}.$$

□

4 Some Ratios

We may ask now what is the value of the ratio $\rho(d,\alpha)$ between $E(|\Omega p|^\alpha, p \in \mathcal{B})$ and $E(|Op|^\alpha, p \in \mathcal{B})$. It is an easy exercise to verify that $\rho(1,\alpha) = 2^\alpha$. In Corollary 12, we compute $\lim_{d\to\infty} \rho(d,\alpha)$.

Corollary 12. *The ratio* $\rho(d,\alpha) = E(|\Omega p|^\alpha, p \in \mathcal{B})/E(|Op|^\alpha, p \in \mathcal{B})$ *when* $d \to \infty$ *is given by* $2^{\alpha/2}$.

Proof. Computing $\rho(d,\alpha)$ with Theorems 8 and 9 gives:

$$\rho(d,\alpha) = 2^{d+\alpha}\left(\frac{2d+\alpha}{2d}\right)\frac{B\left(\frac{d}{2}+\frac{1}{2},\frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}{B\left(\frac{d}{2}+\frac{1}{2},\frac{1}{2}\right)} \tag{8}$$

Using the Stirling's identities:

$$B(a,b) \sim \sqrt{2\pi}\frac{a^{a-\frac{1}{2}}b^{b-\frac{1}{2}}}{(a+b)^{a+b-\frac{1}{2}}}, a,b \gg 0, \tag{9}$$

$$B(a,b) \sim \Gamma(b)a^{-b}, a \gg b > 0, \tag{10}$$

we have:

$$\begin{aligned}
\lim_{d\to\infty} \rho(d,\alpha) &= \lim_{d\to\infty} \left\{ 2^{d+\alpha} \left(\frac{2d+\alpha}{2d}\right) \frac{B\left(\frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}, \frac{d}{2}+\frac{1}{2}\right)}{B\left(\frac{d+1}{2}, \frac{1}{2}\right)} \right\} \\
&= \lim_{d\to\infty} \left\{ \frac{2^{d+\alpha}\sqrt{2\pi}\left(\frac{d}{2}+\frac{1}{2}\right)^{\frac{d}{2}}\left(\frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)^{\frac{d}{2}+\frac{\alpha}{2}}}{\sqrt{\pi}\left(d+1+\frac{\alpha}{2}\right)^{d+\frac{1}{2}+\frac{\alpha}{2}}\left(\frac{d}{2}+\frac{1}{2}\right)^{-\frac{1}{2}}} \right\} \\
&= 2^{\alpha/2} \cdot \lim_{d\to\infty} \left\{ \frac{(d+1)^{\frac{d+1}{2}}(d+1+\alpha)^{\frac{d+\alpha}{2}}}{\left(d+1+\frac{\alpha}{2}\right)^{\frac{d+1}{2}}\left(d+1+\frac{\alpha}{2}\right)^{\frac{d+\alpha}{2}}} \right\} \\
&= 2^{\alpha/2} \cdot e^{-\alpha/4} \cdot e^{\alpha/4} = 2^{\alpha/2}
\end{aligned}$$

□

If we consider a tree which is a random path in the unit ball, then the average size of its edge is given by the expected weighted-length of a random segment determined by two evenly distributed points in the unit ball. This is given by:

$$2^{d+\alpha}\left(\frac{d}{d+\alpha}\right)\frac{B\left(\frac{d}{2}+\frac{1}{2}, \frac{d}{2}+\frac{1}{2}+\frac{\alpha}{2}\right)}{B\left(\frac{d}{2}+\frac{1}{2}, \frac{1}{2}\right)} \tag{11}$$

The reader may refer to Tu and Fischbach [7] for a proof.

From Theorem 8, Corollary 12 and Eq.(11) we obtain the following corollary:

Corollary 13. *The ratio between the weighted-length of a random path in the unit ball and the weighed-length of a star centered at the center of the unit ball is* $\frac{2^\alpha}{1+\alpha/2}$ *when* $d = 1$ *and* $2^{\alpha/2}$ *when* $d \to \infty$.

From Theorem 9 and Eq.(11) we obtain the following corollary:

Corollary 14. *The ratio between the weighted-length of a star centered at the boundary of the unit ball and the weighted-length of a random path in the unit ball is given by* $\frac{2d+\alpha}{2d}$.

5 Weighted Euclidean Minimum k-Insertion Tree for Random Points

Now, we will compute a bound on the expected weighted-length of an edge of $|EMIT_k|_\alpha$ for points evenly distributed in the unit ball.

Theorem 15. *When points are uniformly i.i.d in a ball, the expected length* $E(length)$ *of an edge of* $|EMIT_k|_\alpha$*, with positive* α*, verifies:*

$$\left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right) \leq E(length) \leq 2^\alpha \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right), \tag{12}$$

where $B(x,y) = \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}\mathrm{d}\lambda$ *is the Beta function.*

First, we will evaluate the weighted-distance between the origin and the closest amongst k points $\{p_1, p_2, \ldots, p_k\}$ evenly distributed in the unit ball. This provides the lower-bound. Then, we will find an upper-bound on the weighted-distance between any point inside the ball and the closest amongst k points $\{p_1, p_2, \ldots, p_k\}$ evenly distributed in the unit ball.

Lemma 16. *Let c be a point inside the unit ball, $Prob\,(|cp| \leq l) = P_c(l)$ be the probability that the distance between a point $p \in \mathcal{B}$ and c is less or equal to l, and $P_{c,k}(l) = Prob\,(\min(|cp_j|)_{1\leq j\leq k} \leq l)$ be the cumulative distribution function of the minimum distance among k points following a uniformly i.i.d inside the unit ball and c, then*

$$P_{c,k}(l) = 1 - (1 - P_c(l))^k\,.$$

Proof.

$$\begin{aligned} P_{c,k}(l) &= Prob\,(\min(|cp_j|)_{1\leq j\leq k} \leq l) \\ &= 1 - Prob\,(|cp_j| > l\,, 1 \leq j \leq k) \\ &= 1 - Prob\,(|cp_1| > l)^k \\ &= 1 - (1 - P_c(l))^k\,. \end{aligned}$$

□

A direct consequence of Lemma 16 is the following corollary.

Corollary 17. *Let $P_{\mathcal{B},k}(l) = Prob\,(\min(|Op_j|)_{1\leq j\leq k} \leq l)$ be the cumulative distribution function of the minimum distance among k points following a uniformly i.i.d inside the unit ball, and the center of the unit ball, then*

$$P_{\mathcal{B},k}(l) = 1 - (1 - l^d)^k.$$

Lemma 18. *The expected value $E\,(\min(|Op_j|^\alpha)_{1\leq j\leq k})$ of the minimum weighted-distance among k points following a uniformly i.i.d inside the unit ball and the center of the unit ball, with positive α, is given by*

$$E\,(\min(|Op_j|^\alpha)_{1\leq j\leq k}) = \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right).$$

Proof. Using Corollary 17, we have:

$$\begin{aligned} E\,(\min(|Op_j|^\alpha)_{1\leq j\leq k}) &= \int_0^1 l^\alpha P'_{\mathcal{B},k}(l)\mathrm{d}l = kd\int_0^1 l^{d-1+\alpha}\left(1-l^d\right)^{k-1}\mathrm{d}l \\ &= k\int_0^1 \lambda^{\alpha/d}\,(1-\lambda)^{k-1}\,\mathrm{d}\lambda = kB\left(k, 1+\frac{\alpha}{d}\right) \\ &= \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right). \end{aligned}$$

□

Now, we shall obtain the upper-bound, which is more involved. First we will obtain a general expression for the expected value of $\min(|cp_j|^\alpha)_{1\leq j\leq k}$. Assume $\delta(c) = 1 + |Oc|$ in what follows.

Lemma 19. *The expected value* $E\left(\min(|cp_j|^\alpha)_{1\leq j\leq k}\right)$ *of the minimum weighted-distance among k points following a uniformly i.i.d inside the unit ball and c, with positive* α*, is given by*

$$E\left(\min(|cp_j|^\alpha)_{1\leq j\leq k}\right) = \int_0^{\delta(c)} \alpha l^{\alpha-1}(1-P_c(l))^k \mathrm{d}l.$$

Proof. As $P_c(0) = 0$ and $P_c(\delta(c)) = 1$, integration by parts gives us the following identity:

$$\int_0^{\delta(c)} l^\alpha P_c'(l) P_c^{i-1}(l) = \frac{\delta(c)^\alpha - \int_0^{\delta(c)} \alpha l^{\alpha-1} P_c^i(l)}{i}, i > 0. \tag{13}$$

From Lemma 16, we also have the following expression for $E\left(\min(|cp_j|^\alpha)_{1\leq j\leq k}\right)$:

$$\begin{aligned} E\left(\min(|cp_j|^\alpha)_{1\leq j\leq k}\right) &= \int_0^{\delta(c)} k l^\alpha P_c'(l)(1-P_c(l))^{k-1} \\ &= \sum_{i=0}^{k-1}(-1)^i \binom{k-1}{i} \int_0^{\delta(c)} k l^\alpha P_c'(l) P_c^i(l)\mathrm{d}l. \end{aligned} \tag{14}$$

Replacing Eq.(13) in Eq.(14) leads to:

$$\begin{aligned} \sum_{i=0}^{k-1}(-1)^i \binom{k-1}{i} \int_0^{\delta(c)} k l^\alpha P_c'(l) P_c^i(l)\mathrm{d}l &= \sum_{i=0}^{k}(-1)^i \binom{k}{i} \int_0^{\delta(c)} \alpha l^{\alpha-1} P_c^i(l)\mathrm{d}l \\ &= \int_0^{\delta(c)} \alpha l^{\alpha-1} \sum_{i=0}^{k}(-1)^i \binom{k}{i} P_c^i(l)\mathrm{d}l \\ &= \int_0^{\delta(c)} \alpha l^{\alpha-1}(1-P_c(l))^k \mathrm{d}l. \end{aligned}$$

□

Proof of Theorem 15. Lemma 18 gives us the lower-bound in Theorem 15. Now, if we take a function $\Psi(l)$ such that $\Psi(l) \leq P_c(l)$ for $0 \leq l \leq \delta(c)$, it upper-bounds the integral in Lemma 19. Take $\Psi(l) = (l/\delta(c))^d$, then we have:

$$\begin{aligned} E\left(\min(|cp_j|^\alpha)_{1\leq j\leq k}\right) &= \int_0^{\delta(c)} \alpha l^{\alpha-1}(1-P_c(l))^k \mathrm{d}l \\ &\leq \int_0^{\delta(c)} \alpha l^{\alpha-1}(1-\Psi(l))^k \mathrm{d}l \\ &= \int_0^{\delta(c)} \alpha l^{\alpha-1}\left(1-\frac{l^d}{\delta(c)^d}\right)^k \mathrm{d}l \\ &= \int_0^{1} \alpha\delta(c)^\alpha \lambda^{\alpha-1}(1-\lambda^d)^k \mathrm{d}\lambda \\ &= \delta(c)^\alpha \left(\frac{\alpha}{d}\right) \int_0^1 y^{(\alpha/d)-1}(1-y)^k \mathrm{d}y \\ &= \delta(c)^\alpha \left(\frac{\alpha}{d}\right) B\left(k+1, \frac{\alpha}{d}\right). \end{aligned}$$

And thus, $\delta(c) = 2$ (c on the boundary) maximizes the value above. This completes the proof. □

Remark: The results obtained here for evenly distributed points in the unit ball (expected values, ratios, bounds) so far are valid for any positive α. Then, we have that the expected weighted-length of the i-th step s_i of $|EMIT|_d$ in the unit ball for evenly distributed point is

$$\frac{1}{i+1} = B(i+1,1) \leq s_i \leq 2^d B(i+1,1) = \frac{2^d}{i+1}.$$

Evaluating for n steps leads to

$$\Omega(\log n) = \sum_{i=2}^{n+1} \frac{1}{i} \leq |EMIT|_d \leq 2^d \sum_{i=2}^{n+1} \frac{1}{i} = O(\log n).$$

Unlike $|EMST|_d$ of points evenly distributed inside the unit cube, which converges to a constant as $n \to \infty$ [1], the expected value of $|EMIT|_d$ of points evenly distributed inside the unit ball is $\Theta(\log n)$. With the same argument, for $k > 0$ we have that the expected $|EMIT_k|_d = \Theta\left(\log k + \frac{n}{k}\right)$.

■

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