Allais's Trading Process And The Dynamic Evolution of an OLG Markets Economy

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Abstract

We study a trading process for a pure exchange economy with overlapping generations. This process is based on the maximization, at each stage, of a collective benefit (or surplus). We show that this process converges to a Pareto-optimal allocation. This extends the second fundamental convergence theorem of Allais [1967] to a pure exchange economy with overlapping generations.

Key Words : Trading Process, Distributable Surplus, Benefit Functions, Overlapping generations.

JEL Classification Numbers : D000, D500.

1 Introduction

The Nobel prize winner Maurice Allais is well-known for his contributions to the economics of uncertainty. He is less known for his seminal contribution to the overlapping generations model and to the notion of economic efficiency.

Over the years, Allais devoted a considerable time working on the notion of surplus. He introduced the notion of distributable surplus to study the

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efficiency properties of a market economy. By doing so, he was led to reconstruct a general model of the economy that relies on the notion of surplus and that differs from the walrasian model.

Allais calls his new general model a markets economy. He considers well defined agents who transact at possibly specific prices (hence there is no *commissaire-priseur*[1]). There are no more supply and demand functions. Agents are supposed to make transactions that create surpluses which they share among themselves (see Allais [1994] , pages 55-56, for a comparison between the walrasian model and his model). An equilibrium notion for such an economy is an allocation at which there are no more transactions creating surpluses. Because of this, the equilibrium is called stable. Morevoer, it can be shown that, under some assumptions, Allais' equilibria are Pareto-optimal.

Allais seems to consider that his model describes quite correctly actual transactions. However, he does not give a lot of details on how agents transact and create surpluses and how a stable equilibrium could be reached. In a first text published in 1981 (see Allais [1989], page 361), Allais, following an idea of Edgeworth, seems to envisage a peculiar trading process along which agents' utilities always increase as a result of continual mutually beneficial transactions. However, there is no actual study of the process. Elsewhere (see Allais [1968]), Allais considers informally that the underlying trading process will lead to a stable equilibrium (his so-called participation principle) .

In the entry "Economic surplus and the equimarginal principle" in the Palgrave, Allais writes (see Allais, [1988] ,page 63)

> *In their essence all economic operations, whatever they may be, can be thought of as boiling down to the pursuit of surplus, realization and allocation of distributable surpluses. The corresponding model is the Allais model of the economy of markets (1967) defined by the fundamental rule that every agent tries to find one or several other agents ready to accept at specific prices a bilateral or multilateral exchange (accompanied by corresponding production decisions) which will release a positive surplus that can be shared out, and which is realized and distributed once discovered...*
>
> *Since in the evolution of an economy of markets, surpluses are constantly being realized and allocated, the preference indexes of the consumption units are never decreasing, at the same time as*

[1] In fact, Allais is not interested in the notion of price vectors that every agent should take as given. The important thing is the set of exchanges that agents decide in their interest. This view is reminiscent of the core theory (a theory of exchange without prices). For a recent attempt to explain exchanges under uncertainty by adapting the notion of core, see Koutsougeras and Yannelis [1995].

> *some are increasing. This means that for a given structure, that is to say, for given preferences, resources, and technical know-how, the working of an economy of markets tends to bring it near and near to a state of stable general economic equilibrium, hence a state of maximum efficiency....*
>
> *Naturally such evolution takes place only if sufficient information exists about the actual possibilities of realizing surpluses.*

His point of view is completed by the following assertion, page 67:

> *In fact, what is really important is not so much the knowledge of the properties of a state of maximum efficiency as the rules of the game which have been applied to the economy effectively to move nearer to a state of maximum efficiency.*
>
> *The decentralized search for surpluses is truly the dynamic principle from which a thorough and yet very simple conception of the operation of the whole economy can be derived.*

A first attempt to describe a trading process according to Allais' idea is that of Montbrial [1971]. However, the argument is still not formalized. A recent new attempt to formalize Allais' process is that of Courtault and Tallon [2000]. The key idea underlying their paper is the use of the benefit function, another name for the distributable surplus, that was introduced and rigourously studied by Luenberger $[1992, a, b]$[2].

They study the following process: Given an allocation (equal to initial endowments), one finds another allocation that maximizes the sum of individual benefits. Then given that allocation one restarts the maximization, and so on... Courtault and Tallon showed that the resulting sequence of allocations converges to a stable equilibrium at which the sum of individual benefits is nil. The stable equilibrium is Pareto-optimal. Hence, they give a proof of what Allais calls the second fundamental theorem for markets economy. This theorem states that a markets economy will converge to a situation of a "stable equilibrium"[3].

Our paper extends Courtault and Tallon's result to a dynamic setting. These authors consider an exchange economy with a finite number of goods

[2] Luenberger extends Allais'analysis and proves a series of results linking the benefit function (or the distributable surplus) to efficiency properties (see also Luenberger [1996]).

[3] Incidently, they prove what was somewhat conjectured by Luenberger who wrote (see Luenberger [1995], pages 237-238):

and agents. Hence, their model is not well suited for considering dynamic settings. To do so, we rely upon another idea of Allais, namely, the OLG model.

Indeed, in Allais [1947] one can find the first analysis involving an infinite number of overlapping generations of agents. This is the first model with both an infinite horizon and an infinite number of economic agents together with a non-trivial and realistic demographic structure. Since all economic processes we are aware of involve a finite number of goods and agents, it seems interesting to study an Allais' process in an OLG setting.

The next section describes the model and the main assumptions. In section 3, we present the notion of collective benefit which will be the criterium to be maximized along an exchange process. To ensure existence of collective benefit, we have to modify the original definition of Luenberger [1992].We also study the maximization of the collective benefit in the first stage of the process. There, we show the existence of a solution (the results of this section ensure that the exchange process will be well defined at each stage). In section 4, we study the limit allocation of the exchange process. We show that, in the long-run, the collective benefit is nil and that the limit allocation is a Pareto-optimum. In section 5, we discuss our results especially with regard to the assumptions and to the parts of the literature on economic processes which are the closest to our setting (incidently, we shall see that Allais has recently abandoned in part his idea of a trading process). Section 6 contains some concluding remarks.

2 The Model

We consider an overlapping-generations model of a pure exchange economy. Time is represented by the set of non-nil natural integers $\mathbb{N}_*$. At each date $t \in \mathbb{N}_*$ a consumer is born. Consumer's life lasts two periods. It follows that population is constant through time and equal to 2. A date 1, there are an old agent (who shall die at the end of the period and who was born at date 0) and a young agent who is just born.

Consider an exchange economy. A simple process for achieving *an equilibrium* is just to let individuals trade among themselves. An individual engages in trade only if that trade will increase its utility. Hence, the trading process monotonically increases all utility levels until no further increase can be made. The resulting point is by definition Pareto efficient, and hence, under appropriate assumptions (especially convexity), the final allocation will define an equilibrium..."

A consumer born at t gets endowments of goods in both t and $t+1$, which are denoted respectively by ω_t^t and ω_t^{t+1}, $\omega_t^t \in \mathbb{R}_{++}^d$, $\omega_t^{t+1} \in \mathbb{R}_{++}^d$. The old consumer who is alive at date 1 has an endowment $\omega_0^1 \in \mathbb{R}_{++}^d$. Hence, at each date t, the agregate endowment is equal to $\omega_{t-1}^t + \omega_t^t$.

Each agent born at date t has preferences which are represented by a utility function $u_t : \mathbb{R}_+^d \times \mathbb{R}_+^d \to \mathbb{R}$, $(x_t^t, x_t^{t+1}) \mapsto u_t(x_t^t, x_t^{t+1})$ where x_t^t is the consumption vector at t, and x_t^{t+1} is the consumption vector at $t+1$. The old agent who is alive at 0 is given a utility function $u_0 : \mathbb{R}_+^d \to \mathbb{R}$, $x_0^1 \mapsto u_0(x_0^1)$. We shall assume:

(H1) For each date $t \in \mathbb{N}_*$, $u_t : \mathbb{R}_+^d \times \mathbb{R}_+^d \to \mathbb{R}$, $(x_t^t, x_t^{t+1}) \mapsto u_t(x_t^t, x_t^{t+1})$ *is continuous, strictly monotonic (i.e.* $(x_t^t, x_t^{t+1}) \geq (y_t^t, y_t^{t+1})$, $(x_t^t, x_t^{t+1}) \neq (y_t^t, y_t^{t+1}) \Rightarrow u_t(x_t^t, x_t^{t+1}) > u_t(y_t^t, y_t^{t+1})$*) and strictly quasiconcave*

(i.e. $\forall \lambda \in (0,1)$, $(x_t^t, x_t^{t+1}) \neq (y_t^t, y_t^{t+1}) \Rightarrow u_t(\lambda(x_t^t, x_t^{t+1}) + (1-\lambda)(y_t^t, y_t^{t+1})) > \min\left\{u_t(x_t^t, x_t^{t+1}), u_t(y_t^t, y_t^{t+1})\right\}$*).*

The same properties hold for $u_0(.)$.

Let $l^\infty(\mathbb{N}_*, R_{++}^{2d}) \equiv \{\underline{y} \in (\mathbb{R}_{++}^{2d})^{\mathbb{N}_*} \mid \| \underline{y} \|_\infty = \sup_{t\in\mathbb{N}_*} \| (y_{t-1}^t, y_t^t) \| < \infty\}$[4].

We also assume:

(H2) $\underline{\omega} = ((\omega_{t-1}^t, \omega_t^t))_{t\in\mathbb{N}_*} \in l^\infty(\mathbb{N}_*, \mathbb{R}_{++}^{2d})$.

This assumption means that there is no unbounded growth of society's global resources.

We let $FA(\underline{\omega})$ be the set of all *feasible allocations*:

$$FA(\underline{\omega}) = \{\underline{x} = ((x_{t-1}^t, x_t^t))_{t\in\mathbb{N}_*} \in (\mathbb{R}_+^{2d})^{\mathbb{N}_*} \mid \forall t \geq 1, x_{t-1}^t + x_t^t = \omega_{t-1}^t + \omega_t^t\}^5$$

In the sequel, it will be useful to redefine all agents' preferences on $(\mathbb{R}_+^{2d})^{\mathbb{N}_*}$. Hence, for all t greater than 1, we define:

$Y_t' = \prod_{j=1}^\infty Y_j$ where $Y_j = \mathbb{R}_+^{2d}$ for $j = t$ or $t+1$ and $Y_j = \mathbb{R}^{2d}$ for other $j's$

$$\mathbf{U}_t : Y_t' \to \mathbb{R},\ \underline{x} \mapsto \mathbf{U}_t(\underline{x}) = u_t(x_t^t, x_t^{t+1})$$

Define $Y_0' = \mathbb{R}_+^{2d} \times \prod_{j=2}^\infty \mathbb{R}^{2d}$. For the old agent born at date 1, we define:

$\mathbf{U}_0 : Y_0' \to \mathbb{R}$, $\underline{x} \mapsto \mathbf{U}_0(\underline{x}) = u_0(x_0^1)$.

We set $\mathbf{U}(\underline{x}) = (\mathbf{U}_t(\underline{x}))_{t\geq 0}$.

[4] In this expression, $(\mathbb{R}_{++}^{2d})^{\mathbb{N}_*}$ is the set of sequences that starts at $t = 1$, and whose terms lie in $\mathbb{R}_{++}^{2d}$. When a variable is underlined (like $\underline{y}$), it means that it is a sequence.

[5] For simplicity, we do not allow for free disposal. However, given the monotonicity assumptions made on preferences, this is innocuous.

It will be also useful to make use of the *benefit function* introduced by Luenberger [1992] (see also Luenberger [1995], page 98 for a textbook presentation and this paper's Appendix[6]). Before applying this notion to our framework, we recall its definition.

Let $X \subset \mathbb{R}^l_+$ be a consumption set and $U : X \to \mathbb{R}$ be a utility function. Let $g \in X \setminus \{0\}$. We define the benefit function $b : (X \setminus \{0\}) \times X \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$,

$$b(g, x, \alpha) = \sup \{\beta \mid U(x - \beta g) \geq \alpha,\, x - \beta g \in X\}$$

The benefit function may take the value $-\infty$ when there is no β such that $x - \beta g \in X$ and $U(x - \beta g) \geq \alpha$[7].

This function measures how many units of g an individual would be willing to give up to move from a utility level of α to the point x. This function converts preferences into a numerical function that has a cardinal meaning.

We now apply this idea to our setting. Let $\underline{g} \in l^\infty(\mathbb{N}_*, \mathbb{R}^{2d}_{++})$, $\underline{x} \in l^\infty(\mathbb{N}_*, \mathbb{R}^{2d}_+)$, $\underline{\alpha} \in \mathbb{R}^{\mathbb{N}}$. Now for all t greater than 1, let us define:

$$\begin{array}{rcl} B_t & : & l^\infty(\mathbb{N}_*, \mathbb{R}^{2d}_{++}) \times l^\infty(\mathbb{N}_*, \mathbb{R}^{2d}_+) \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R} \cup \{-\infty\}, \\ (\underline{g}, \underline{x}, \underline{\alpha}) & \mapsto & \sup\{\beta \mid \mathbf{U}_t(\underline{x} - \beta \underline{g}) \geq \alpha_t,\ x^t_t - \beta g_t \in \mathbb{R}^d_+,\ x^{t+1}_t - \beta g_{t+1} \in \mathbb{R}^d_+\}. \\ & = & \sup\{\beta \mid u_t(x^t_t - \beta g_t, x^{t+1}_t - \beta g_{t+1}) \geq \alpha_t,\ x^t_t - \beta g_t \in \mathbb{R}^d_+,\ x^{t+1}_t - \beta g_{t+1} \in \mathbb{R}^d_+\} \end{array}$$

For $t = 0$,

$$\begin{array}{rcl} B_0 & : & l^\infty(\mathbb{N}_*, \mathbb{R}^{2d}_{++}) \times l^\infty(\mathbb{N}_*, \mathbb{R}^{2d}_+) \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R} \cup \{-\infty\}, \\ (\underline{g}, \underline{x}, \underline{\alpha}) & \mapsto & \sup\{\beta \mid \mathbf{U}_0(\underline{x} - \beta \underline{g}) \geq \alpha_0, x^1_0 - \beta g_1 \in \mathbb{R}^d_+\} \\ & = & \sup\{\beta \mid u_0(x^1_0 - \beta g_1) \geq \alpha_0, x^1_0 - \beta g_1 \in \mathbb{R}^d_+\} \end{array}$$

Notice that $B_t(\underline{g}, \underline{x}, \underline{\alpha})$ is equal to $b((g_t, g_{t+1}), (x^t_t, x^{t+1}_t), \alpha_t)$.

In the sequel of this paper, $\underline{g}$ is given once and for all so we shall write $B_t(\underline{x}, \underline{\alpha})$ instead of $B_t(\underline{g}, \underline{x}, \underline{\alpha})$.

Notice that the function B_t is not exactly the benefit function associated to the functions $\mathbf{U}_t$ since we do not impose in the definition that $\underline{x} - \beta \underline{g}$ is in

[6] This appendix aims at presenting the properties of the benefit function used in this paper. These properties have been established by Luenberger in several papers (see the references) and we simply recall them for making this paper reasonably self-contained (note, however, that we do not duplicate exactly Luenberger' results since we re-establish these in a more general setting (namely in a framework with infinite dimensions)).

[7] The supremum of the empty set is equal to $-\infty$.

$l^\infty(\mathbb{N}_*, \mathbb{R}_+^{2d})$. Indeed, one may find some values of β such that $x_t^t - \beta g_t \in \mathbb{R}_+^d$, $x_t^{t+1} - \beta g_{t+1} \in \mathbb{R}_+^d$ but for which $\underline{x} - \beta \underline{g}$ is not in $l^\infty(\mathbb{N}_*, \mathbb{R}_+^{2d})$. This is unimportant and what matters is that B_t is equal to the value of $b(.)$ for the agent born at date t (for this agent, it is too strong to impose that $\underline{x} - \beta \underline{g}$ is in $l^\infty(\mathbb{N}_*, \mathbb{R}_+^{2d})$). We have introduced the function $\mathbf{U}_t$ and B_t to simplify notations.

Finally, we define $IR(\underline{y})$ the set of individually rational allocations, *i.e.*, the feasible allocations that yield for each agent a utility level at least equal to the one he gets with the allocation $\underline{y}$. More formally, for all $\underline{y} \in l_\infty(\mathbb{N}_*, \mathbb{R}_+^{2d})$,

$$IR(\underline{y}) := \{\underline{x} \in FA(\underline{\omega}) \mid \forall t \geq 1, u_t(x_t^t, x_t^{t+1}) \geq u_t(y_t^t, y_t^{t+1}), u_0(x_0^1) \geq u_0(y_0^1)\}$$

3 A Dynamic Exchange Process *à la* Allais

In the preceding section, we have seen how to compute a *benefit* for each agent, a feasible allocation being given. Two comments should be made at this stage. On the one hand, since agents' benefits are measured in units of potentially different bundles, differentiated by their dates of disposal, it might not be sensible to agregate these benefits. On the other hand, even if agregating these benefits is meaningful, the sum runs across an infinite number of generations. Hence, the sum might be undefined, from a mathematical point of view. To avoid this defect, we assume that there is a system of intergenerational weights attached to each benefit. An interpretation of these weights will be given below.

Once one has defined a meaningful notion of agregate benefits, we are in position to devise an Exchange Process along Allais' line. To do so, we follow the idea first exposed in Courtault and Tallon [2000]. Taking a feasible allocation as given (namely, the allocation that consists of agents'endowments), we look for another allocation which maximizes the agregate benefits (recall that the latter is being defined conditionally on the allocation). At this stage, we have to ensure that the problem does have a solution. Then, taking a solution of the problem as a new initial feasible allocation, we restart the process. In the next section, we show that the process converges and we study the properties of the limit allocation.

3.1 Collective Benefit

We want to give sense to the infinite sum of individual benefits, $\sum_{t=0}^{+\infty} \gamma^t B_t$, $\gamma \in (0, 1)$, which may be thought of as the *collective benefit*.

In order for this series to converge, we have introduced a sequence of intertemporal weights $(\gamma_t)_t$ where $\gamma_t = \gamma^t$. The interpretation of this assumption is that future agents count less in the process, and agents living very far in the future count for almost nothing.

The sum $\sum_{t=0}^{+\infty} \gamma^t B_t$ has an unambiguous interpretation when $\underline{x} - B_t \underline{g}$ is in $l^\infty(\mathbb{N}_*, \mathbb{R}_+^{2d})$ for all t. Indeed, in this case, all individual benefits B_t are well defined in unit of $\underline{g}$ and so is the sum $\sum_{t=0}^{+\infty} \gamma^t B_t$. In this case, the quantities B_t correspond to the benefit functions associated to the utility functions $\mathbf{U}_t$ as in Luenberger's definition.

Things are less clear when for at least one date t the vector $\underline{x} - B_t \underline{g}$ is not in $l^\infty(\mathbb{N}_*, \mathbb{R}_+^{2d})$. Then, the quantity B_t cannot be interpreted as a benefit for the function $\mathbf{U}_t$. Rather, it is a benefit B_t for the function u_t and it is expressed in unit of a bundle of goods available at date t. The bundles corresponding to each date t are not a priori comparable accross times. However, using an expression such as $\sum_{t=0}^{+\infty} \gamma^t B_t$ implies implicitely a possibility of comparison.

We shall rely on the following assumption in order to ensure that our notion of collective benefit is well defined.

(H3) $\forall \underline{x} \in IR(\underline{\omega})$, $\overline{\rho} = \sup_{t \in \mathbb{N}} B_t(\underline{x}, \mathbf{U}(0)) < +\infty$.

Under assumption *(H1)*, if all utility functions are alike, the previous assumption is always true. This may not be the case anymore, if utility functions differ across time[8].

Lemma 1. *Assume (H1)-(H3) and* $\gamma \in (0, 1)$. *Then the series* $\sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}, \mathbf{U}(\underline{\omega}))$ *converges uniformly on* $IR(\underline{\omega})$.

Proof. By assumption, for every $\underline{x} \in IR(\underline{\omega})$, $u_0(x_0^1) \geq u_0(\omega_0^1)$ and for all t greater than 1, $u_t(x_t^t, x_t^{t+1}) \geq u_t(\omega_t^t, \omega_t^{t+1})$. Clearly $B_t(\underline{x}, \mathbf{U}(\underline{\omega})) \geq 0$. Under *(H1)* u_t is monotonic. Since b_t is non-increasing with respect to its second argument (see, *e.g.* Luenberger [1992)], proposition 2 (a), see also the appendix), under *(H3)* $B_t(\underline{x}, \mathbf{U}(\underline{\omega}))$ must be upper-bounded by $\overline{\rho}$. So for every $\underline{x} \in IR(\underline{\omega})$, for every t, $\gamma^t B_t(\underline{x}, \mathbf{U}(\underline{\omega})) \leq \gamma^t \ \overline{\rho}$. So the series $\sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}, \mathbf{U}(\underline{\omega}))$ converges uniformly since the series $\sum_{t=0}^{+\infty} \gamma^t \overline{\rho}$ converges. Q.E.D.

3.2 Initialization of an Exchange Process

[8] Consider the case where $U_t(x_t^t, x_t^{t+1}) = U(\lambda^t x_t^t, \lambda^t x_t^{t+1})$ where $\lambda > 1$.

As was explained at the beginning of the section, we want to study the following problem :

$$\max_{\underline{x} \in IR(\underline{\omega})} \sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}, \mathbf{U}(\underline{\omega})) \tag{P}$$

We shall now show that there exists a solution, which we denote by $\underline{x}_1$. This will enable us to study another problem where $\underline{x}_1$ is substituted for $\underline{\omega}$ and so on...

In order to prove the existence of a solution to the maximization problem, we shall need several intermediate results. Basically, we shall use the well known result that an upper semicontinuous function defined on a compact set attains its maximum. The only difficulty here has do to with the fact that we study a problem in an infinite dimension space. Since several linear topologies are possible for such sets, it is important to precise which topology is used.

In what follows, $(\mathbb{R}^{2d})^{\mathbb{N}}$ will be endowed with the product topology. Checking compactness of the set of feasible choices and upper semicontinuity of the collective benefit in the product topology will be done in the following lemmas.

Lemma 2 *$FA(\underline{\omega})$ and $IR(\underline{\omega})$ are compact in the product topology.*

Proof. Recall that

$$FA(\underline{\omega}) = \{\underline{x} \in (\mathbb{R}^{2d}_+)^{\mathbb{N}_*} |\ \forall t \geq 1, x^t_{t-1} + x^t_t = \omega^t_{t-1} + \omega^t_t\}$$

Clearly, for all t greater that 1, x^t_{t-1} and x^t_t belong to the same compact set, say E_t. This set is just the box $\left[0, \omega^t_{t-1} + \omega^t_t\right]$. Hence $(x^t_{t-1}, x^t_t) \in E^2_t$, which is compact since it is a product of compact sets. Hence, $FA(\underline{\omega}) \subset \prod_{t=1}^{\infty} E^2_t$. By the Tychonoff Product Theorem $\prod_{t=1}^{\infty} E^2_t$ is compact in the product topology (see, *e.g.* Aliprantis and Border [1999], Theorem 2.57, page 52) . Hence $FA(\underline{\omega})$ is a subset of a compact set.

In order to show that it is itself compact , it is sufficient to prove that it is closed . Recall that a sequence $(\underline{x}^n)_n$ in $(\mathbb{R}^{2d}_+)^{\mathbb{N}_*}$ converges to $\underline{x}$ in the product topology if and only if for all t greater than 1, the sequence of coordinates $((x^t_{t-1,n}, x^t_{t,n}))_{n\in\mathbb{N}} \to (x^t_{t-1}, x^t_t)$.

So, take a sequence $(\underline{x}^n)_n \subset FA(\underline{\omega})$, that converges to $\underline{x}$ and let us show that $\underline{x}$ is in $FA(\underline{\omega})$. By definition, for all t greater than 1, $((x^t_{t-1,n}, x^t_{t,n}))_{n\in\mathbb{N}} \to (x^t_{t-1}, x^t_t)$. Also, for all $n \in \mathbb{N}$, $x^t_{t-1,n} + x^t_{t,n} = \omega^t_{t-1} + \omega^t_t$. Hence, by continuity, $x^t_{t-1} + x^t_t = \omega^t_{t-1} + \omega^t_t$. So, $\underline{x} \in FA(\underline{\omega})$. This shows that $FA(\underline{\omega})$ is closed and hence compact.

Finally, let us show that $IR(\underline{\omega})$ is closed in the product topology. Recall that:

$$IR(\underline{\omega}) := \{\underline{x} \in FA(\underline{\omega}) \mid \forall t \geq 1,\ u_t(x_t^t, x_t^{t+1}) \geq u_t(\omega_t^t, \omega_t^{t+1}), u_0(x_0^1) \geq u(y_0^1)\}.$$

Take a sequence $(\underline{x}^n)_n$ in $IR(\underline{\omega})$ that converges to $\underline{x}$. By assumption, for all t, for all n, $u_t(x_{t,n}^t, x_{t,n}^{t+1}) \geq u_t(\omega_t^t, \omega_t^{t+1})$. Since u_t is continuous by *(H1)*, $(u_t(x_{t,n}^t, x_{t,n}^{t+1}))_n \to u_t(x_t^t, x_t^{t+1})$. So $u_t(x_t^t, x_t^{t+1}) \geq u_t(\omega_t^t, \omega_t^{t+1})$. Hence, $\underline{x} \in IR(\underline{\omega})$, and $IR(\underline{\omega})$ is closed. As a closed subset of a compact set, it is itself compact. Q.E.D.

We recall the following definition:

Let X be a topological space. A function $f : X \to \overline{\mathbb{R}}$, is said to be upper semicontinuous if $(x_n)_n \to x$ in X implies $f(x) \geq \limsup_{n\to+\infty} f(x_n)$.

Lemma 3. *Under the assumptions of Lemma 1, the function* $F : IR(\underline{\omega}) \to \mathbb{R}$, $\underline{x} \mapsto \sum_{t=0}^{\infty} \gamma^t B_t(\underline{x}, \mathbf{U}(\underline{\omega}))$ *is upper semicontinuous* .

Proof.

The proof proceeds along two steps. In the first step we show that B_t is upper semi continuous and in the second step we show that collective benefit is itself upper semi continuous.

Notice that $B_t(\underline{x}, \mathbf{U}(\underline{\omega}))$ is equal to $b((x_t^t, x_t^{t+1}), u_t(\omega_t^t, \omega_t^{t+1}))$, where $b(.,.)$ is the usual benefit function as defined in finite dimension by Luenberger [1992].

• We have from Proposition 3 of Luenberger [1992] that $b(.,.)$ is upper semicontinuous with respect to all its arguments (for the euclidian topology). Hence $B_t(.,.)$ is upper semicontinuous in the product topology ($(\underline{x}_n, \mathbf{U}^n)_n \to (\underline{x}, \mathbf{U})$ implies $B_t(\underline{x}, \mathbf{U}) \geq \limsup_{n\to+\infty} B_t(\underline{x}_n, \mathbf{U}^n)$) since it is the composition of the upper semicontinuous function $b(.,.)$ with projections). So $B_t(., \mathbf{U}(\underline{\omega}))$ is upper semicontinuous.

• Under *(H3)*, for all t, for all $\underline{x}$ in $IR(\underline{\omega})$, $\gamma^t(B_t(\underline{x}, \mathbf{U}(\underline{\omega})) - \overline{\rho}) \leq 0$ and for all t, $\gamma^t \left[B_t(., \mathbf{U}(\underline{\omega})) - \overline{\rho}\right]$ is upper semicontinuous as the sum of two upper semicontinuous functions. Hence the series

$$\sum_{t=0}^{\infty} \gamma^t \left[B_t(., \mathbf{U}(\underline{\omega})) - \overline{\rho}\right]$$

is upper semicontinuous (see, *e.g.*, Becker and Boyd [1997], page 46).

So

$$\sum_{t=0}^{\infty} \gamma^t(B_t(., \mathbf{U}(\underline{\omega})) = \sum_{t=0}^{\infty} \gamma^t \left[B_t(., \mathbf{U}(\underline{\omega})) - \overline{\rho}\right] + \sum_{t=0}^{\infty} \gamma^t \overline{\rho}$$

is upper semicontinuous as the sum of two upper semicontinuous functions. Q.E.D.

We are now in position to prove the main result of this section.

Proposition 1. *Under the assumptions of Lemma 1, problem **(P)** has a solution.*

Proof. Since $IR(\underline{\omega})$ is compact in the product topology (Lemma 2) and $F : IR(\underline{\omega}) \to \mathbb{R}$, $\underline{x} \mapsto \sum_{t=0}^{\infty} \gamma^t B_t(\underline{x}, \mathbf{U}(\underline{\omega}))$ is upper semicontinuous on $IR(\underline{\omega})$ (Lemma 3) , the problem does have a solution by Weierstrass Theorem (see *e.g.*, Becker and Boyd [1997] page 114). Q.E.D.

Remark: A priori, the solution of the above problem is not unique (under our assumptions, the benefit function is concave (but not necessarily strictly concave)). However it does not matter which solution is chosen for the following stages of the exchange process.

Let us precise what we call an exchange process:

Definition: *An exchange or a trading process from an endowment $\underline{\omega}$ is a sequence $(\underline{x}_n)_n$ starting at $\underline{x}_0 = \underline{\omega}$ which is obtained when one solves recursively for each $n \geq 1$ the problem (P) where $\mathbf{U}(\underline{x}_{n-1})$ is substituted for $\mathbf{U}(\underline{\omega})$*

$$\max_{\underline{x} \in IR(\underline{x}_{n-1})} \sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}, \mathbf{U}(\underline{x}_{n-1}))$$

Note that this sequence is well defined since for each n one may use the arguments of Proposition 1 and Lemmas 2 and 3 to show that the problem has a solution. Again, there may be several solutions, but we only need to use one of them.

4 The Limit of an Exchange Process

4.1 Existence of a Limit Allocation

In the preceding sections, we have defined an exchange process from a given endowment, *i.e.*, the sequence of feasible allocations in $IR(\underline{\omega})$ that obtains when one solves recursively problem (P). As will be shown below, this sequence converges. Moreover, in the limit, the collective benefit is nil. The limit allocation can be thought of as being the outcome of the trading process (since there are no more possible gains to trade).

Proposition 2. *An exchange process $(\underline{x}_n)_n$ converges in $IR(\underline{\omega})$ and the limit of the collective benefit is nil.*

Proof. The proof is by contradiction and involves several steps.

Let $(\underline{x}_n)_n$ be an *exchange process.*

• Let us consider the sequence $(H_n)_{n\in\mathbb{N}}$ defined by:

$$\forall n \in \mathbb{N}, H_n = \sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}_n, \mathbf{U}(\underline{x}_{n-1}))$$

By definition of the benefit function (see the appendix) we have for all t, and for all n:

$$\mathbf{U}_t(\underline{x}_n - B_t(\underline{x}_n, \mathbf{U}(\underline{x}_{n-1}))\underline{g}) \geq \mathbf{U}_t(\underline{x}_{n-1})$$

Moreover for all n , $\underline{x}_n \in IR(\underline{x}_{n-1})$ we have:

$$\forall t, \forall n, \mathbf{U}_t(\underline{x}_n) \geq \mathbf{U}_t(\underline{x}_{n-1})$$

Hence, for all agents, utility increases along the process.

• Clearly, since $\mathbf{U}_t(.)$ is continuous in the product topology and since $IR(\underline{\omega})$ is compact, so is $\mathbf{U}_t(IR(\underline{\omega}))$. Hence, for each t, the sequence $(\mathbf{U}_t(\underline{x}_n))_n$ is convergent (since it is increasing and upper-bounded).

• Since for each given $\underline{x}$, $b_t(\underline{x}, .)$ is non-increasing with respect to its second argument, it follows that for all t for all n :

$$B_t(\underline{x}, \mathbf{U}(\underline{x}_n)) \leq B_t(\underline{x}, \mathbf{U}(\underline{x}_{n-1}))$$

In particular for $\underline{x} = \underline{x}_{n+1}$:

$$\sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}_{n+1}, \mathbf{U}(\underline{x}_n)) \leq \sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}_{n+1}, \mathbf{U}(\underline{x}_{n-1})) \leq \sum_{t=0}^{+\infty} \gamma^t B_t(\underline{x}_n, \mathbf{U}(\underline{x}_{n-1}))$$

The last inequality holds since $\underline{x}_n$ maximizes $\sum_{t=0}^{+\infty} \gamma^t B_t(., \mathbf{U}(\underline{x}_{n-1}))$. The preceding inequalities imply that:

$$\forall n, H_{n+1} \leq H_n$$

It follows that $(H_n)_n$ is a non-increasing sequence. From the monotonicity assumption in *(H1)* again, $(H_n)_n$ is minored by 0 (since the benefits are lower bounded by 0). Hence $(H_n)_n \to H \geq 0$ (the sequence $(H_n)_n$ being non-increasing and lower bounded).

• The sequence $(\underline{x}_n)_n$ lies in $IR(\underline{\omega})$, which is compact. Hence, there is a subsequence $(\underline{x}_{n_k})_{k\in\mathbb{N}}$ which converges to $\underline{x}' \in IR(\underline{\omega})$. Clearly the subsequence $(H_{n_k})_k$ converges to $H$and for all t, the subsequence $(\mathbf{U}_t(\underline{x}_{n_k}))_k$ converges to the limit of $(\mathbf{U}_t(\underline{x}_n))_n$ which is therefore equal to $\mathbf{U}_t(\underline{x}')$. Moreover for all t, the subsequence $(\mathbf{U}_t(\underline{x}_{n_k-1}))_k$ converges to $\mathbf{U}_t(\underline{x}')$.

• *Suppose now that* $(\underline{x}_n)_n$ *does not converge to* $\underline{x}'$. Hence: there is a neighborhood of $\underline{x}'$,$V(\underline{x}')$, such that $\forall j \in \mathbb{N}, \exists\ n_j > j$, such that $\underline{x}_{nj} \notin V(\underline{x}')$.

This subsequence $((\underline{x})_{n_j})_j$ lies in the compact $IR(\underline{\omega})$. Hence, one can find a subsequence of $((\underline{x})_{n_j})_j$ that converges. By relabelling, we also denote this sub-subsequence by $(\underline{x}_{n_j})_j$. Let $\underline{x}$" be its limit. By construction $\underline{x}' \neq \underline{x}$".

Clearly the subsequence $(H_{n_j})_j$ converges to H and for all t, the subsequence $(\mathbf{U}_t(\underline{x}_{n_j}))_j$ converges to the limit of $(\mathbf{U}_t(\underline{x}_n))_n$. So this limit $\mathbf{U}_t(\underline{x}$") is also equal to $\mathbf{U}_t(\underline{x}')$. And for all t, the subsequence $(\mathbf{U}_t(\underline{x}_{n_j-1}))_j$ converges to $\mathbf{U}_t(\underline{x}$") .

• Since $\sum_{t=0}^{\infty}\gamma^t B_t(.,.)$ is upper semicontinuous (the proof is similar to that of Lemma 3), one has:

$$\sum_{t=0}^{\infty}\gamma^t B_t(\underline{x}', \mathbf{U}(\underline{x}')) \geq \limsup_{k\to+\infty}\sum_{t=0}^{\infty}\gamma^t B_t(\underline{x}_{n_k}, \mathbf{U}(\underline{x}_{n_k-1})) = H$$

For the same reason, one also has:

$$\sum_{t=0}^{\infty}\gamma^t B_t(\underline{x}'', \mathbf{U}(\underline{x}'')) \geq \limsup_{j\to+\infty}\sum_{t=0}^{\infty}\gamma^t B_t(\underline{x}_{n_j}, \mathbf{U}(\underline{x}_{n_{j-1}})) = H$$

• Under *(H1)*, for all t, u_t is monotone so, for all $\alpha > 0$, and (x_t^t, x_t^{t+1}) in $\mathbb{R}_+^d \times \mathbb{R}_+^d$, $u_t(x_t^t + \alpha g_t, x_t^{t+1} + \alpha g_{t+1}) > u_t(x_t^t, x_t^{t+1})$. Hence, by Proposition 1(a) of Luenberger [1992] (see also the appendix), one has both : $\sum_{t=0}^{\infty}\gamma^t B_t(\underline{x}', \mathbf{U}(\underline{x}')) = 0$ and $\sum_{t=0}^{\infty}\gamma^t B_t(\underline{x}'', \mathbf{U}(\underline{x}'')) = 0$. Hence, $H = 0$. This proves that the limit of the collective benefit is nil.

• By definition, $\underline{x}'$ and $\underline{x}'' \in FA(\underline{w})$ which is convex and $\underline{x}' \neq \underline{x}''$. For some t $u_t(\lambda\underline{x}' + (1-\lambda)\underline{x}'') > \min\{u_t(\underline{x}'), u_t(\underline{x}'')\} = u_t(\underline{x}')$ since by *(H1)*, $u_t(.,.)$ is strictly quasi-concave and for others $u_t(\lambda\underline{x}' + (1-\lambda)\underline{x}'' \geq \min\{u_t(\underline{x}'), u_t(\underline{x}'')\} = u_t(\underline{x}')$. If $\lambda(x_t'^t, x_t'^{t+1}) + (1-\lambda)(x_t''^t, x_t''^{t+1})$ is in the interior of $E_t \times E_{t+1}$, one has: $B_t(\lambda\underline{x}' + (1-\lambda)\underline{x}'', \mathbf{U}(\underline{x}')) > 0$[9]. If not, this means that at least one component of $\lambda(x_t'^t, x_t'^{t+1}) + (1-\lambda)(x_t''^t, x_t''^{t+1})$ is nil. Then there is at least one good that is not consumed by the agent born at t. In this case, using an argument proposed by Luenberger (see Luenberger [1995], page 191), one may devise a scheme such that: 1) the agent born at t will receive a positive amount of each good and enjoy a utility level strictly higher than $\mathbf{U}_t(\underline{x}')$, 2) the utilities of the consumers (born at $t-1$ or $t+1$) are still higher than the ones they get with $\underline{x}'$.

• Let us call $\underline{z}$ the new resulting (feasible allocation). By construction, (z_t^t, z_t^{t+1}) is interior, and hence: $B_t(\underline{z}, \mathbf{U}(\underline{x}')) > 0$. For all other agents, the

[9]Indeed, by continuity of the utility function, it is always possible to decrease the utility by substracting some units of $(g_t^t,\ g_t^{t+1})$ while still being in the consumption set.

benefit is non-negative. Hence, we have found a feasible allocation such that: $\sum_{t=0}^{\infty} \gamma^t B_t(\underline{z}, \mathbf{U}(\underline{x}')) \equiv \widetilde{H} > 0$.

• We know that $(\sum_{t=0}^{\infty} \gamma^t B_t(\underline{x}_{n_k}, \mathbf{U}(\underline{x}_{n_k-1})))_k \to 0$. Then: $\exists K, \forall k \geq K$, $\sum_{t=0}^{\infty} \gamma^t B_t(\underline{x}_{n_k}, \mathbf{U}(\underline{x}_{n_k-1})) < \frac{H_\epsilon}{2}$. We have $\underline{x}' \in \bigcap_{k \in \mathbb{N}} IR(\underline{x}_{n_k-1})$ and $\underline{z} \in IR(\underline{x}')$. Hence, $\underline{z} \in \bigcap_{k \in \mathbb{N}} IR(\underline{x}_{n_k-1})$

But for $k > K$, since the b_t are decreasing with respect to their second argument, one has: $\sum_{t=0}^{\infty} \gamma^t B_t(\underline{z}, \mathbf{U}(\underline{x}_{n_k-1})) \geq \sum_{t=0}^{\infty} \gamma^t B_t(\underline{z}, \mathbf{U}(\underline{x}')) = \widetilde{H}$.

Hence

$$\sum_{t=0}^{\infty} \gamma^t B_t(\underline{z}, \mathbf{U}(\underline{x}_{n_k-1})) > \frac{\widetilde{H}}{2} \geq \sum_{t=0}^{\infty} \gamma^t B_t(\underline{x}_{n_k}, \mathbf{U}(\underline{x}_{n_k-1}))$$

So $\underline{x}_{n_k}$ does not maximize $\sum_{t=0}^{\infty} \gamma^t B_t(., \mathbf{U}(\underline{x}_{n_k-1}))$.

This contradicts the definition of $\underline{x}_{n_k}$.

So the sequence $(\underline{x}_n) \to \underline{x}'$ and the exchange process converges. Q.E.D.

4.2 Pareto-optimality of the Limit Allocation

In the preceding proposition, we have seen that the limit collective benefit is nil. There are no more tradeoffs. It should come as no suprise that the limit allocation is Pareto-optimal (recall the quotation of Luenberger in the introduction).

Proposition 3. *Let $\underline{x}$ be the limit of the allocations of an Exchange Process. Then $\underline{x}$ is a Pareto optimum .*

Proof. Suppose not. Then: $\exists \underline{x}' \in FA(\underline{\omega})$: $\exists t' \in N, \mathbf{U}_{t'}(\underline{x}') > \mathbf{U}_{t'}(\underline{x})$, and $\forall t \in \mathbb{N}$, $\mathbf{U}_t(\underline{x}') \geq \mathbf{U}_t(\underline{x})$. Hence $\underline{x}' \in IR(\underline{\omega})$. Then, one may construct a similar allocation to that used at the end of the proof of Proposition 2 and which yields a contradiction[10]. Q.E.D.

Interestingly, an exchange process converges to a feasible allocation which is Pareto-optimal and for which the collective benefit is nil (there are no more gains to trade both intra and intergenerationnaly). Hence, this is a version of the second zero-maximum theorem of Luenberger according to which an allocation that zero-maximizes the distributable surplus is Pareto-optimal (see, for instance, Luenberger [1995]).

[10] This allocation can be constructed with $\underline{x}'' = \lambda \underline{x}' + (1-\lambda)\underline{x}$, $\lambda \in (0,1)$ and using the fact that $B_t(\underline{x}, \mathbf{U}(\underline{x})) = 0$.

5 Discussions of the Model and Comparison With Previous Work on Economic Processes

5.1 Discussions of the Assumptions

Several assumptions have been used for proving the results of this paper. Most of these were set in order to simplify the analysis (and perhaps stronger than necessary). Others are more important.

First of all, we assumed a constant population. This assumption is in fact innocuous. One could adapt our argument to the case of a growing population (it will suffice to assume a high enough rate of discount of generations' benefits (if not, we are back to the case where the discount rate is nil, see below)). The same comments apply to the case where there is some heterogeneity across agents (say intragenerationnaly). The fact that we had worked within the framework of a pure exchange economy could be thought of as being more restrictive a priori. This is not so (see Ghiglino and Tvede [2000] for an exposition of how transforming a Production and Exchange OLG Economy into an OLG Exchange Economy).

As was alluded to above, there is a potential technical difficulty that arises when one does not allow for discounting generations' benefits. Then the collective benefit may be no more well defined (since the series may not have a limit, or if it has one, the limit could be infinite). In this case, though we did not investigate it thoroughly, one may think that most of our results still hold if one uses the overtaking criterium as the criterium to be maximized (for instance, see Boyd and Becker [1997]).

Also difficult seems to be the issue of uniqueness of the solution of problem (P). This is important since the maximum collective benefit may be reached at different candidate solutions (and, accordingly, may yield different benefits for a given individual). Such an issue also arises in OLG of the Allais [1947]-Diamond [1965]-Samuelson [1958] type. A potential solution in this context is to choose at each date an equilibrium that delivers the highest utility for the initial old agents (if there is still a multiplicity of solutions, one could choose the best allocation for the initial young, and so on...) . By the way, indeterminacy also pledges the notion of Walrasian equilbrium (though local uniqueness may be obtained). Another solution would amount to use a selection from the correspondence defined by the set of solutions. It would be nice if one could show the existence of a continuous selection (however, a crucial assumption for this, is that the correspondence be lower-hemi continuous, a property which does not seem to be easy to state (see Moore [1999], for an introduction).

A third and important difficulty has to do with the interpretation of the process. Does it describe the actual dynamics of the real economic process, or is it only an interpretation of the issue of individual interactions? The first interpretation is sometimes adopted by certain authors. This is the case of Allais [1989], and Luenberger [1995], page 238, who explains why what we call the limit of the exchange process might differ from a walrasian equilibrium:

> *This simple process does lead to an equilibrium, but not necessarily to the Walras equilibrium specified by the original endowments. It is likely that during the process trades will be made with terms that are not in accord with Walras equilibrium prices, and therefore one party, although achieving a utility increase, will lose relative to what could be achieved using these prices. For this reason the final equilibrium may differ from the Walras equilibrium.*

But one could stick to a somewhat more orthodox interpretation. Modern economic analysis differs from physics since it mainly concentrates on the equilibrium of individual interactions and not on the process that leads to this equilibrium. In this perspective, the trading process is just a mean to define an equilibrium (*i.e.* the limit of the process). In so doing, one does insist on the fact that at an equilibrium no more beneficial trade is possible (otherwise, agents would engage in mutually beneficial transactions).

This last interpretation is especially more interesting given the temporal aspect of the model. One could perhaps interpret the trading process as something reflecting actual transactions occuring within a period. But it is hard to extend this interpretation to a setting involving simultaneously generations living at very distant dates. Of course, the maximization of the collective benefit obeys the maximum principle, so that current generations maximize the sum of current benefits taking into account the future optimal value of the collective benefit. But it would be more interesting to adapt to our setting something like the temporary equilibrium notion used in standard OLG models like those of Allais [1947]-Diamond [1965].

5.2 Discussion and Comparison With The Literature On Economic Processes

5.2.1 A General View on Exchange Process

In order to simplify the analysis, let us consider an exchange economy with a finite horizon and a finite number I of agents, and l goods. Each agent $i = 1, .., I$, is endowed with a utility function U_i defined on his consumption set X_i, $x_i \in X_i \mapsto \mathbb{R}$. For convenience we take X_i as being $\mathbb{R}^l_+$, and we also endow each agent i with a vector of goods ω_i in $\mathbb{R}^l_+$.

We then define a general exchange process as a sequence of feasible allocations, *i.e.* such that: $\sum_{i=1}^{I} x_i^n = \sum_{i=1}^{I} \omega_i$ for each n.

One may wonder what properties should enjoy an exchange process. If exchanges are voluntary, they should be individually rational. More precisely: an individually rational exchange process is a sequence of feasible allocations $((x_i^n)_{i \in I})_n$ such that for all $n \in \mathbb{N}$, for all $i = 1, ..., I$: $U_i(x_i^{n+1}) \geq U_i(x_i^n)$.

Naturally, one may wonder if the sequence of feasible allocations of an exchange process does converge. Before giving an answer to this question, let us ask if the induced sequence of utility levels along an exchange process does converge.

If individual utility functions are continuous, one may show that the above sequence of utility levels does indeed converge[11]. Hence, there is a feasible allocation that generates the limit of the utility levels. Let us call a limit allocation such an allocation. Note that the convergence of the induced utility levels does not imply that the sequence of allocations of the exchange process does converge to a limit allocation. Moreover, there are several limit allocations (unless the utility functions are surjective).

Another natural question is the stability of the exchange process. More precisely, is the limit allocation Pareto-optimal ? In so far as many exchanges are allowed across agents, it should be natural to impose that the limit allocation be Pareto-optimal.

In view of all these properties, it is of interest to recall the result of Courtault and Tallon [2000].

Proposition (Courtault and Tallon [2000]). *Suppose that all the utility functions are continuous, strictly increasing and strictly quasiconcave. Let $(x^n)_n$ be a sequence of feasible allocations such that for all n in $\mathbb{N}^*$, x^n is a solution to:*

$$\max_{x^n \in IR(x^{n-1})} \sum_{i=1}^{I} b_i(x_i^n, U_i(x_i^{n-1}))$$

where $IR(x^{n-1})$ is the set of feasible allocations x such that for i, $U_i(x_i) \geq U_i(x_i^{n-1})$ and x^1 solves:

[11] To do this, three steps are involved. In the first step, one shows that the set of individually rational feasible allocations is compact. In the second step, one shows that the image of this set by the utility functions is also a compact set. The third step amounts to notice that an increasing bounded sequence of utility values converges.

$$\max_{x^n \in IR(\omega)} \sum_{i=1}^{I} b_i(x_i^1, U_i(\omega_i))$$

Then $(x^n)_n$ *converges to a feasible allocation which is Pareto-optimal.*

In this respect, notice that in this paper, we have generalized this result to the case of an OLG exchange economy[12].

5.2.2 Edgeworth Barter Exchange Process

Economic Processes have been studied formally along several ways. Perhaps the most important strand is the literature on the walrassian and non-walrasian tâtonnement (a short and interesting review is presented in Takayama, [1985], pages 339-347, see also Bryant [2000], Fisher [1983], [1987], Hahn [1982]). Among the alternative to the walrasian tâtonnement, the more interesting to us is the Edgeworth processes (studied among notably by Uzawa [1962], Hahn [1962], Ruppert and Russel [1972]).

These processes originate in the ideas developped in 1891 by Edgeworth and published in his *Papers Relating to Political Economy*, Vol. II, London, Macmillan). The process studied by Edgeworth relies on the idea that when individuals participate in the process, the utility of the stock of commodities held by each of them increases over time as a result of exchanges. As Takayama puts it: "When the process reaches a Pareto optimal point, it cannot move any further by definition; hence it is an equilibrium point"[13].

There are two variants of an Edgeworth Process. In the first variant, contrary to the second variant, there is no price system. Recall that in the Allais ' process, there is no price system. This is however the unique feature which is common to the Allais ' process and those of the first variant. Indeed, the Allais' process involves an optimization stage which is absent of the process of the first variant (see, *e.g.*, Hahn [1982], pages 772-777). This is why we shall uniquely concentrate on the processes of the second variant.

From a formal point of view, an Edgeworth process with a price system may be described as follows (we follow mostly Takayama's presentation, see Takayama [1985], page 344, together with that of Arrow and Hahn [1971],

[12]The proof is however somewhat different from that of Courtault and Tallon [2000]. They first show that the sequence of the utility levels $(U_i(x_i^{n-1}))_i$ converges. Then, they show that there exists a unique feasible allocation that yields the limit utility values and toward which the sequence of $((x_i^n)_i)_n$ converges. Finally, they show that the limiting allocation is Pareto-optimal.

[13]Again, see the quotation of Allais in the introduction.

chapter 13). In a setting with a finite number of agents, say N, and m goods, we consider the following problem, for a given vector of endowments $\widehat{x}^n$:

$$\max \sum_{i=1}^{N} \alpha_i U_i(x_i)$$

$$\begin{aligned} s.t. \sum_{i=1}^{N} x_{ij} &= \sum_{i=1}^{N} \widehat{x}_{ij}^n, \text{ for all } j = 1, ..., m \\ \sum_j p_j^n x_{ij} &= p^n \widehat{x}_i^n, \text{ for all } i = 1, ..., n \\ U_i(x_i) &\geq U_i(\widehat{x}_i^n), \text{ for all } i = 1, ..., n \end{aligned}$$

Here, the coefficients α_i represent the relative weights of agents in the process. Also, p^n is a given price vector. Let $\widehat{x}^{n+1}$ be the solution of the problem. There is an additionnal constraint: agents must exchange at the same relative prices (something that Allais seeks to avoid).

The process restarts with $\widehat{x}^{n+1}$ substituted for $\widehat{x}^n$. The price vector changes according to the following rule: $p_j^{n+1} = \sum_{i=1}^{N} x_{ij}^n - \sum_{i=1}^{N} \widehat{x}_{ij}^n$ where for all i, x_i^n is the (walrasian) demand of agent i, *i.e.*, namely, x_i^n solves the problem:

$$\max_{x \in \left\{x : \left\langle p^n, x - \widehat{x}_i^n \right\rangle = 0\right\}} U_i(x)$$

And so on[14]. In the literature, one is particularly interested in the stability of the process (not the existence per se), and in particular, in the convergence of the sequence (x^n, p^n) to a walrasian equilibrium. Arrow and Hahn [1971], page 334, Theorem 4, proves such convergence under mild assumptions.

The above process has much in common with the Allais' process studied in this paper. Clearly, the fact that utility increases along the process is a common feature. However, there are two points which make the process very different. On the one hand, the Edgeworth process still involves a price vector, which is absent of the Allais' process. On the other hand, what is maximized is the sum of agents utilities instead of agents benefits. The trouble with this is that utility functions are then given a cardinal interpretation, wich is disputable, whereas benefit functions are cardinal *per se*. To be more precise, the outcome of the Edgeworth process depends clearly on the choice of a particular utility function which is not the case in the Allais' process.

[14] Here, contrary to Takayama, the time is not chosen to be represented by an interval (so, we use difference equations instead of differential equations).

However, as was seen in section 3, the definition of the collective benefit raises some similar difficulties when there is an infinite number of agents living at different instants of time.

5.2.3 New Views of Allais on Trading Process

In the last revision of the *Traité d'économie Pure* [1994], Allais seems to reconsider the idea of the trading process. He no longer retains the idea that utilities of all agents should increase during the process. However, he is still willing to show that the final outcome of the process is Pareto-optimal.

More specifically, he claims to have proven the following result. Consider an economy with M agents and Q goods, each agents being given a utility function $U_i(.)$ defined on the $\mathbb{R}_+^Q$ (presumably), and an endowment $\omega_i \in \mathbb{R}_+^Q$. Let g be a vector in $\mathbb{R}_+^Q$. Let us call ε_1 the allocation that one obtains when each agent consumes one's endowment (for all agent i, $x_1^i = \omega_i$). Next, consider a new (feasible) allocation $\varepsilon_2 = \varepsilon_1 + \delta\varepsilon_1$. And so on, $\varepsilon_{n+1} = \varepsilon_n + \delta\varepsilon_n$. At each new allocation corresponds a "surplus" $\delta\sigma_n = \sum_{i=1}^{M} \delta\sigma_n^i$ where the quantities $\delta\sigma_n^i$ are defined implicitely as:

$$U_i(x_{n+1}^i - \delta\sigma_n^i g) = U_i(x_n^i)$$

Allais assumes that $g = (1, 0, ..., 0)$, and that $\delta\sigma_n$ is positive for all n. He notes that this implies that the $\delta\varepsilon_n$ should be fairly small. Finally, he defines a global distributable surplus $\triangle\sigma_n = \sum_{i=1}^{M} \triangle\sigma_n^i$ where the $\triangle\sigma_n^i$ are implicitely defined as follows:

$$U_i(x_{n+1}^i - \triangle\sigma_n^i g) = U_i(\omega_i)$$

Then Allais shows that if $\delta\sigma_{n+1}$is positive for all n, one has $\triangle\sigma_{n+1} > \triangle\sigma_n > 0$ for all n[15]. Finally, he proves that the sequence $(\triangle\sigma_n)_n$ converges and that the sequence $(\varepsilon_n)_n$ converges to an allocation at which the surplus is nil (which is then Pareto-optimal).

The important point in Allais' presentation is that he no more considers a process along which each individual surplus $\delta\sigma_n^i$ is non-negative. He views this new process as being more realistic (see pages 92 (note 19) and 138 in the introduction of Allais [1994]). Along this process certain agents may face a decrease in welfare. But what remains important is that in the long-run economic efficiency is achieved. In Allais' view, his new process is well suited

[15] The assumption that the changes in allocations are small is important. Indeed Allais presents a counter-example that shows that when the changes are "important" one may have positive increasing sequence $(\delta\sigma_n)_n$ without having for all dates $\triangle\sigma_{n+1} > \triangle\sigma_n$.

for describing international trade. Allais claims that free trade may harm certain countries but lead to worldwide efficiency.

The new view of Allais is interesting but two points should be made. On the one hand, from a technical point of view, one should note that the conditions under which Allais'result holds true must be precised (in particular, differentiability is assumed but does not seem necessary). On the other hand, there remains an indetermination with regard to the way the allocations are changed along the process (the sequence is given and not determined).

6 Conclusion

This paper was devoted to a study of an Allais exchange process for a dynamic economy with overlapping generations.

We relied upon a study by Courtault and Tallon [2000] who consider an economy with a finite number of agents and of goods. In this paper, the process of exchange consists in maximizing a weigthed sum of benefits under the constraints that the allocations yield for each agent a utility level at least equal to a reference allocation. The next round of the process takes as given the solution of the problem. We have shown that the problem is well defined, *i.e.*, there is a solution, and that the sequence of allocations converges to a stable equilibrium wich is a Pareto-optimum (the equilibrium is stable because, it zero-maximizes the sum of individual benefits : there are no more surpluses to be gained by making new exchanges).

We have already stressed a number of critics as to the relevence of the process and as to the assumptions used in the paper. There are at least three ways to extend the results of this paper.

First of all, it would be interesting to impose more desirable properties of a stable equilibrium. It should not only be a Pareto-optimum but also in the core. But this implies studying core allocations from a benefit point of view. A first step toward this aim has been done recently, see Courtault, Crettez and Hayek [2001].

Second, and a little bit differently, it would be interesting to study a process from a benefit point of view but whose outcome would only be pairwise optimal (this would weaken the amount of informations needed to describe the process).

Third, there is natural extension within an OLG setup. In applying the idea of Courtault and Tallon to the entire dynamic economy, we have imposed an implicit coordination across an infinite number of generations. However, a more natural concept of (walrasian) equilibrium for an OLG set up is perhaps the notion of temporary equilibrium. It requires generations to make (not necessarily exact) expectations about future transactions and to undertake exchanges based upon these expectations. An interesting extension of the present paper would amount to build an Allais's process in a temporary equilibrium setup. More precisely, at each date, only two generations would make exchanges and the young generation would do this on the basis of her expectations of future exchanges (this would be in the spirit of Allard, Bronsard and Richelle [1989]).

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APPENDIX
Study of the benefit function in the infinite dimension case

Let $\mathbb{E}$ be a partially ordered topological vector space, $\mathbb{E}_+ = \{x \in \mathbb{E}, x \geq 0\}$ be the positive cone of $\mathbb{E}$. The set $\mathbb{E}_+$ is convex. We suppose that it is closed (for example in a normed Riesz space the positive cone is always closed).

Notice that instead of $\mathbb{E}_+$ we could work with a closed convex subset of $\mathbb{E}$ with a lower bound.

We consider now an exchange economy inhabited by I agents. Each agent $i \in \mathcal{I} = \{1, 2, ..., I\}$ is given the same consumption set, namely $\mathbb{E}_+$, a continuous utility function $U_i : \mathbb{E}_+ \to \mathbb{R}$ and a vector of endowments $\omega_i \in \mathbb{E}_+$.

Let $g \in \mathbb{E}_+ \setminus \{0\}$. For every $i \in \mathcal{I}$, we define the benefit function b_i : $\mathbb{E}_+ \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$,

$$b_i(x_i, \alpha_i) = \sup \{\beta,\ U_i(x_i - \beta g) \geq \alpha_i,\ x_i - \beta g \in \mathbb{E}_+\} \tag{1}$$

Set $A_i(x_i, \alpha_i) = \{\beta,\ U_i(x_i - \beta g) \geq \alpha_i,\ x_i - \beta g \in \mathbb{E}_+\}$. Notice that if $A_i(x_i, \alpha_i)$ is empty then $b_i(x_i, \alpha_i) = -\infty$. We shall show in the following proposition that if $A_i(x_i, \alpha_i)$ is not empty then

$b_i(x_i, \alpha_i) = \max \{\beta,\ U_i(x_i - \beta g) \geq \alpha_i,\ x_i - \beta g \in \mathbb{E}_+\}$ exists.

For this, note that what matters is only the upper-semi continuity of $U_i(.)$.

Proposition A1. *For every $i \in \mathcal{I}$, if U_i is upper-semi-continuous and $A_i(x_i, \alpha_i)$ is non-empty then $b_i(x_i, \alpha_i) = \max A_i(x_i, \alpha_i)$ exists.*

Proof. Fix $i \in \mathcal{I}$. First let us show that $A_i(x_i, \alpha_i)$ has an upper bound. We must show that there exists $c \in \mathbb{R}$ such that for every $\beta \in A_i(x_i, \alpha_i), \beta \leq c$. The proof is by contradiction.

Suppose that for every $c \in \mathbb{R}$, there exists $\beta \in A_i(x_i, \alpha_i)$,such that $\beta > c$. So, in particular, for every $n \in N$, there exists $\beta_n \in A_i(x_i, \alpha_i)$ such that $\beta_n > n$. We obtain a sequence $(\beta_n)_n$ such that for every n, $\beta_n > 0$, $x_i - \beta_n g \in \mathbb{E}_+$, and $\beta_n \to +\infty$ as $n \to +\infty$. Hence for every n we have $(1 \backslash \beta_n)\ x_i - g\ \in \mathbb{E}_+$ and $(1 \backslash \beta_n)\ x_i - g \to -g$ as $n \to +\infty$. But since $\mathbb{E}_+$ is closed, this implies $-g \in \mathbb{E}_+$ which is a contradiction.

Now we show that $A_i(x_i, \alpha_i)$ is closed in $\mathbb{R}$. Consider a sequence $(\beta_n)_n$ such that for every n, $\beta_n \in A_i(x_i, \alpha_i)$ and $\beta_n \to \beta$ as $n \to +\infty$. So for every $\beta_n \in \mathbb{R}, U_i(x_i - \beta_n g) \geq \alpha_i$ and we have $x_i - \beta_n g \to x_i - \beta g$ as $n \to +\infty$. Since $\mathbb{E}_+$ is closed , $x_i - \beta g \in \mathbb{E}_+$. Moreover since u_i is upper-semi-continuous $U_i(x_i - \beta g) \geq \limsup_{n \to +\infty} U_i(x_i - \beta_n g)$ hence $U_i(x_i - \beta g) \geq \alpha_i$, so $\beta \in A_i(x_i, \alpha_i)$ which means that $A_i(x_i, \alpha_i)$ is closed in $\mathbb{R}$.

Finally $A_i(x_i, \alpha_i)$ being non-empty and upper bounded in $\mathbb{R}$, it admits a least upper bound ($\sup A_i(x_i, \alpha_i)$ exists). Since $A_i(x_i, \alpha_i)$ is closed in $\mathbb{R}$, $\max A_i(x_i, \alpha_i) = \sup\ A_i(x_i, \alpha_i)$. ■

Next, we prove several properties of the benefit functions.

Proposition A2. *The following assertions hold true:*

a) $b_i(x_i, .)$ *is non-increasing with respect to its second argument.*

b) If U_i *is quasiconcave then* $b_i(x_i, \alpha)$ *is concave with respect to its first argument.*

c) If g *is good, i.e., for all positive* α*, for all* x *in* $\mathbb{E}_+$*:* $U_i(x + \alpha g) > U_i(x)$*, then: for all* x *in* $\mathbb{E}_+$*,* $b_i(x, U_i(x)) = 0$*.*

d) The function: $b_i : \mathbb{E}_+^2 \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *defined by*

$$b_i(g, x, \alpha) = \sup \{\beta,\ U_i(x - \beta g) \geq \alpha,\ x - \beta g \in \mathbb{E}_+\}$$

is upper semicontinuous.

Proof.

a) $b_i(x_i, .)$ is non-increasing with respect to its second argument.

Let $\alpha > \alpha'$. Suppose first that $A_i(x_i, \alpha') = \emptyset$. Then $A_i(x_i, \alpha') = \emptyset$ and

$$b_i(x_i, \alpha_i) = \sup A_i(x_i, \alpha) = b_i(x_i, \alpha_i') = \sup A_i(x_i, \alpha) = -\infty$$

Suppose that $A_i(x_i, \alpha') \neq \emptyset$. One has: $A_i(x_i, \alpha) \subset A_i(x_i, \alpha')$. From wich, one has: $b_i(x_i, \alpha_i) = \sup A_i(x_i, \alpha) \leq b_i(x_i, \alpha_i') = \sup A_i(x_i, \alpha)$.

b) $b_i(x_i, \alpha)$ is concave with respect to its first argument.

Let x and y two different vectors in $\mathbb{E}_+$. Let $\lambda \in [0, 1]$. We want to show that for all λ in $[0, 1]$, $b_i(\lambda x + (1-\lambda), \alpha) \geq \lambda b_i(x, \alpha) + (1-\lambda) b_i(y, \alpha_i)$. Clearly, this is true if either $b_i(x, \alpha)$ or $b_i(y, \alpha)$ or both equal $-\infty$. Suppose then that $b_i(x, \alpha) > 0$ and $b_i(y, \alpha) > 0$.

From proposition A1, one has:

$$\begin{aligned} U_i(x - b_i(x, \alpha) g) &\geq \alpha,\ x - b_i(x, \alpha) g \in \mathbb{E}_+ \\ U_i(y - b_i(y, \alpha) g) &\geq \alpha,\ y - b_i(y, \alpha) g \in \mathbb{E}_+ \end{aligned}$$

Since $\mathbb{E}_+$ is convex, one has:

$$\lambda x + (1 - \lambda) y - (\lambda b_i(x, \alpha) + (1 - \lambda) b_i(y, \alpha)) g \in \mathbb{E}_+$$

By quasiconcavity of $U_i(.)$, one has then:

$$U_i(\lambda x + (1 - \lambda) y - (\lambda b_i(x, \alpha) + (1 - \lambda) b_i(y, \alpha)) g) \geq \alpha$$

$>$From this, it follows that:

$$b_i(\lambda x + (1 - \lambda) y, \alpha) \geq \lambda b_i(x, \alpha) + (1 - \lambda) b_i(y, \alpha)$$

c) If g is *good*, then: $b_i(x, U_i(x)) = 0$.

Clearly, $0 \in A_i(x, U_i(x))$. Suppose that $b_i(x, U_i(x)) > 0$. Then, $U_i(x - b_i(x, U_i(x))g) \geq U_i(x)$. Since g is good, it follows that: $U_i(x) > U_i(x)$. Contradiction.

d) We now study the function: $b_i : \mathbb{E}_+^2 \times \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$b_i(g, x, \alpha) = \sup \{\beta,\ U_i(x - \beta g) \geq \alpha,\ x - \beta g \in \mathbb{E}_+\}$$

Let us show that it is upper semi-continuous. Let $(g_n, x_n, \alpha_n)_n$ a sequence in $\mathbb{E}_+^2 \times \mathbb{R}$ that converges to (g, x, α). We have to show that the set:

$$A(c) = \left\{(g, x, \alpha) \in \mathbb{E}_+^2 \times \mathbb{R} : b_i(g, x, \alpha) \geq c\right\}$$

is closed for all c (see Aliprantis and Border [1999] page 42).

Let $((g_n, x_n, \alpha_n))_n$ be a converging sequence in $A(c)$.

Hence for all n:

$$b_i(g_n, x_n, \alpha_n) \geq c$$

This implies that for all n:

$$U_i(x_n - g_n c) \geq \alpha_n \text{ and } x_n - g_n c \in \mathbb{E}_+$$

Indeed, by Proposition A1, $x_n - g_n b_i(g_n, x_n, \alpha_n) \in \mathbb{E}_+$. Now, since $\mathbb{E}_+$ is closed, one has: $x - gc \in \mathbb{E}_+$. By continuity of $U_i(.)$ one has:

$$U_i(x - gc) \geq \alpha$$

It then follows that: $b_i(g, x, \alpha) \geq c$. ■