

# Bayesian Cramer-Rao Bound for OFDM Rapidly Time-varying Channel Complex Gains Estimation

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**Abstract**—In this paper, we consider the Bayesian Cramer-Rao bound (BCRB) for the dynamical estimation of multi-path Rayleigh channel complex gains in data-aided (DA) and non-data-aided (NDA) OFDM systems. This bound is derived in an on-line and off-line high Doppler scenarios for time-varying complex gains within one OFDM symbol, assuming the availability of prior information. In NDA context, whereas this true BCRB is hard to evaluate, we present a closed-form expression of a BCRB, *i.e.*, the Modified BCRB (MBCRB). We discuss, based on the theoretical and simulation results, the benefit of using the a priori information and, the past and the future observations for the complex gains estimation.

**Index Terms**—Bayesian Cramer-Rao Bound, OFDM, Rayleigh complex gains.

## I. INTRODUCTION

In the case of wideband Orthogonal Frequency Division Multiplexing (OFDM) mobile communication systems, high speeds of terminals cause Doppler effects that result inter-sub-carrier interference (ICI) and could seriously affect the performance. In such case, dynamic channel estimation [9] [10] is a fundamental function, because the radio channel is frequency selective and time-varying. Channel estimation can be summarized to estimate certain physical propagation parameters, such as multi-path delays and multi-path complex gains. In Radio-Frequencies transmission, the delays are quasi invariant over several OFDM symbols but the complex gains may change significantly, even within one OFDM symbol. Exploiting the channel nature and assuming the availability of delay information, a lot of methods estimate the time-variations of the multi-path complex gains in OFDM [3] [4] [5] [6] [7] [8] and CDMA [12] systems.

In this context the question arises of the ultimate accuracy that can be achieved in channel estimation operations. Establishing bounds to such an accuracy is an important goal since it provides benchmarks for evaluating the performance of channel estimators. Tools to approach this problem are available from the parameters estimation theory [15] [21] in the form of Cramer-Rao Bounds (CRBs), which give fundamental lower limits to the variance of any parameter estimator. A Modified CRB (MCRB), easier to evaluate than the Standard CRB (SCRB), has been introduced in [16] [17]. The MCRB proves useful when, in addition to the parameter to be estimated, the observed data also depend on other unwanted parameters. More recently, the problem of deriving CRBs suited to time-varying parameters has been addressed throughout the Bayesian context. In [18], the authors propose

a general framework for deriving analytical expression of on-line CRBs.

In [1] [2], we have derived the expression of the on-line BCRB, in data-aided (DA) and non-data-aided (NDA) contexts, for the dynamic estimation of time-varying multi-path Rayleigh channel complex gains with slowly variations (*i.e.*, time-invariant complex gains within one OFDM symbol). In [3] [4], in order to evaluate the quality of our complex gains estimator, we have just given the expression of the on-line BCRB in DA context for the case of rapidly time-varying channels.

In this contribution we investigate the BCRB related to the estimation of rapidly time-varying Rayleigh channel complex gains with Jakes spectrum for OFDM systems (*i.e.*, time-varying complex gains within one OFDM symbol). Explicit expressions of the BCRB and its variant, MBCRB, are provided in NDA and DA contexts and, in on-line and off-line scenarios.

This paper is organized as follows: Section II sets the system model, whereas Section III recalls the general BCRB and the modified MBCRB. Section IV derives the BCRB and the MBCRB for "time-varying" multi-path complex gains estimation. Section V illustrates and interpretes different results. Finally, our conclusions are presented in Section VI.

The notations adopted are as follows: Upper (lower) bold face letters denote matrices (column vectors).  $[\mathbf{x}]_k$  denotes the  $k$ th element of the vector  $\mathbf{x}$ , and  $[\mathbf{X}]_{k,m}$  denotes the  $[k, m]$ th element of the matrix  $\mathbf{X}$ . We will use the matlab notation  $\mathbf{X}_{[k_1:k_2, m_1:m_2]}$  to extract a submatrix within  $\mathbf{X}$  from row  $k_1$  to row  $k_2$  and from column  $m_1$  to column  $m_2$ .  $\mathbf{I}_N$  is a  $N \times N$  identity matrix and  $\mathbf{0}_N$  is a  $N \times N$  matrix of zeros.  $\text{diag}\{\mathbf{x}\}$  is a diagonal matrix with  $\mathbf{x}$  on its main diagonal,  $\text{diag}\{\mathbf{X}\}$  is a vector whose elements are the elements of the main diagonal of  $\mathbf{X}$  and  $\text{blkdiag}\{\mathbf{X}, \mathbf{Y}\}$  is a block diagonal matrix with the matrices  $\mathbf{X}$  and  $\mathbf{Y}$  on its main diagonal. The superscripts  $(\cdot)^T$  and  $(\cdot)^H$  stand respectively for transpose and Hermitian operators.  $|\cdot|$ , and  $\text{Tr}(\cdot)$  are respectively the determinant and trace operations.  $\text{Re}(\cdot)$ ,  $\text{Im}(\cdot)$  and  $(\cdot)^*$  are respectively the real part, imaginary part and conjugate of a complex number or matrix.  $E_{x,y}[\cdot]$  is the expectation over  $x$  and  $y$  and  $J_0(\cdot)$  is the zeroth-order Bessel function of the first kind.  $\nabla_{\mathbf{x}}$  and  $\Delta_{\mathbf{y}}^x$  represent the first and the second-order partial derivatives operator *i.e.*,  $\nabla_{\mathbf{x}} = [\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_N}]^T$  and  $\Delta_{\mathbf{y}}^x = \nabla_{\mathbf{y}}^* \nabla_{\mathbf{x}}^T$ .

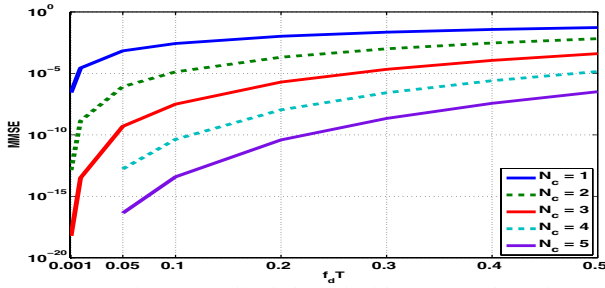


Fig. 1. MMSE for a normalized channel with  $L = 6$  paths and  $v = 144$

## II. OFDM SYSTEM AND CHANNEL MODELS

### A. OFDM System Model

Consider an OFDM system with  $N$  sub-carriers, and a cyclic prefix length  $N_g$ . The duration of an OFDM symbol is  $T = vT_s$ , where  $T_s$  is the sampling time and  $v = N + N_g$ . Let  $\mathbf{x}(n) = [x(n)[-\frac{N}{2}], x(n)[-\frac{N}{2} + 1], \dots, x(n)[\frac{N}{2} - 1]]^T$  be the  $n$ th transmitted OFDM symbol, where  $\{x(n)[b]\}$  are normalized symbols (*i.e.*,  $E[x(n)[b]x(n)[b]^*] = 1$ ). After transmission over a multi-path Rayleigh channel, the  $n$ th received OFDM symbol  $\mathbf{y}(n) = [y(n)[-\frac{N}{2}], y(n)[-\frac{N}{2} + 1], \dots, y(n)[\frac{N}{2} - 1]]^T$  is given by [5] [7] [8]:

$$\mathbf{y}(n) = \mathbf{H}(n) \mathbf{x}(n) + \mathbf{w}(n) \quad (1)$$

where  $\mathbf{w}(n) = [w(n)[-\frac{N}{2}], w(n)[-\frac{N}{2} + 1], \dots, w(n)[\frac{N}{2} - 1]]^T$  is a complex Gaussian noise vector with covariance matrix  $\sigma^2 \mathbf{I}_N$  and  $\mathbf{H}(n)$  is a  $N \times N$  channel matrix with elements given by:

$$[\mathbf{H}(n)]_{k,m} = \frac{1}{N} \sum_{l=1}^L \left[ e^{-j2\pi(\frac{m-1}{N} - \frac{1}{2})\tau_l} \sum_{q=0}^{N-1} \alpha_l^{(n)}(qT_s) e^{j2\pi\frac{m-k}{N}q} \right] \quad (2)$$

where  $L$  is the number of paths,  $\alpha_l$  is the  $l$ th complex gain of variance  $\sigma_{\alpha_l}^2$  and  $\tau_l \times T_s$  is the  $l$ th delay ( $\tau_L < N_g$ ). The  $L$  individual elements of  $\{\alpha_l^{(n)}(qT_s) = \alpha_l(qT_s + nT)\}$  are uncorrelated. They are wide-sense stationary (WSS), narrow-band complex Gaussian processes, with the so-called Jakes' power spectrum of maximum Doppler frequency  $f_d$ , *i.e.*,  $E[\alpha_l(q_1T_s)\alpha_l^*(q_2T_s)] = \sigma_{\alpha_l}^2 J_0(2\pi f_d T_s(q_1 - q_2))$  [13]. The average energy of the channel is normalized to one, *i.e.*,  $\sum_{l=1}^L \sigma_{\alpha_l}^2 = 1$ .

### B. Complex Gain Polynomial Modeling

In [10], a piece-wise linear method is used to approximate the equivalent discrete-time channel taps. In [5] [6], the authors show that the time-variation of Rayleigh channel complex gain, within  $N_c$  OFDM symbols, can be approximated by a polynomial model of  $N_c$  coefficients, chosen according to the Doppler spread  $f_d T$ .

In this section, we show that, whatever  $f_d T \leq 0.5$ , each Rayleigh channel complex gain  $\alpha_l^{(n)} = [\alpha_l^{(n)}(-N_g T_s), \dots, \alpha_l^{(n)}((N-1)T_s)]^T$  can be modeled as a polynomial time-variation of  $N_c \leq 5$  coefficients (*i.e.*, a  $(N_c - 1)$  degree polynomial), within one OFDM symbol.

The optimal polynomial  $\alpha_{\text{pol}_l}^{(n)}$ , which is least-squares fitted (linear and polynomial regression) [14] to  $\alpha_l^{(n)}$ , and its  $N_c$

coefficients  $\mathbf{c}_l^{(n)} = [c_{1,l}^{(n)}, \dots, c_{N_c,l}^{(n)}]^T$  are given by:

$$\alpha_{\text{pol}_l}^{(n)} = \mathbf{Q}^T \mathbf{c}_l^{(n)} = \mathbf{S} \alpha_l^{(n)} \quad \text{and} \quad \mathbf{c}_l^{(n)} = (\mathbf{Q}\mathbf{Q}^T)^{-1} \mathbf{Q} \alpha_l^{(n)} \quad (3)$$

where  $\mathbf{Q}$  is a  $N_c \times v$  matrix of elements  $[\mathbf{Q}]_{k,m} = (m - N_g - 1)^{(k-1)}$  and  $\mathbf{S} = \mathbf{Q}^T (\mathbf{Q}\mathbf{Q}^T)^{-1} \mathbf{Q}$  is a  $v \times v$  matrix. It provides the MMSE approximation for all polynomials containing  $N_c$  coefficients, given by:

$$\text{MMSE}_l^{(0)} = \frac{1}{v} \text{Tr}(\mathbf{MMSE}_l^{(0)}) \quad (4)$$

$$\mathbf{MMSE}_l^{(p)} = E[\xi_l^{(n)} \xi_l^{(n-p)H}] = (\mathbf{I}_v - \mathbf{S}) \mathbf{R}_{\alpha_l}^{(p)} (\mathbf{I}_v - \mathbf{S}^T)$$

with  $\xi_l^{(n)} = \alpha_l^{(n)} - \alpha_{\text{pol}_l}^{(n)}$  is the model error and  $\mathbf{R}_{\alpha_l}^{(p)} = E[\alpha_l^{(n)} \alpha_l^{(n-p)H}]$  is the  $v \times v$  correlation matrix of  $\alpha_l^{(n)}$  with elements given by:

$$[\mathbf{R}_{\alpha_l}^{(p)}]_{k,m} = \sigma_{\alpha_l}^2 J_0\left(2\pi f_d T_s (k - m + pv)\right) \quad (5)$$

Fig. 1 gives the MMSE in terms of  $f_d T$  for different value of  $N_c$ . As can be seen, for  $f_d T \leq 0.5$  and  $N_c = 5$ , we have  $\text{MMSE} < 4 \cdot 10^{-7}$ . This proves that, for high values of  $f_d T$ ,  $\alpha_l^{(n)}$  can be represented by a polynomial model of  $N_c \leq 5$  coefficients.

$\mathbf{c}_l^{(n)}$  are correlated complex Gaussian variables with zero-means and correlation matrix given by:

$$\mathbf{R}_{\mathbf{c}_l}^{(p)} = E[\mathbf{c}_l^{(n)} \mathbf{c}_l^{(n-p)H}] = (\mathbf{Q}\mathbf{Q}^T)^{-1} \mathbf{Q} \mathbf{R}_{\alpha_l}^{(p)} \mathbf{Q}^T (\mathbf{Q}\mathbf{Q}^T)^{-1} \quad (6)$$

It should be noted that the last coefficients are very small. So, it is very difficult to find an estimator that can give a good estimation of the small coefficients in presence of noise. In the sequel, we will study the performance of the coefficients estimator in terms of  $N_c$  and  $f_d T$ .

Under this polynomial approximation, the observation model in (1) for the  $n$ th OFDM symbol can be rewritten as:

$$\mathbf{y}(n) = \mathcal{K}(n) \mathbf{c}(n) + \mathbf{w}(n) \quad (7)$$

where  $\mathbf{c}(n) = [\mathbf{c}_1^{(n)T}, \dots, \mathbf{c}_L^{(n)T}]^T$  is a  $LN_c \times 1$  vector,  $\mathcal{K}(n) = \frac{1}{N} [\mathbf{Z}_1^{(n)}, \dots, \mathbf{Z}_L^{(n)}]$  is a  $N \times LN_c$  matrix and  $\mathbf{Z}_l^{(n)} = [\mathbf{M}_1 \text{diag}\{\mathbf{x}(n)\} \mathbf{f}_l, \dots, \mathbf{M}_{N_c} \text{diag}\{\mathbf{x}(n)\} \mathbf{f}_l]$  is a  $N \times N_c$  matrix, where  $\mathbf{f}_l$  is the  $l$ th column of the  $N \times L$  Fourier matrix  $\mathbf{F}$  and  $\mathbf{M}_d$  is a  $N \times N$  matrix given by:

$$[\mathbf{F}]_{k,l} = e^{-j2\pi(\frac{k-1}{N} - \frac{1}{2})\tau_l} \quad \text{and} \quad [\mathbf{M}_d]_{k,m} = \sum_{q=0}^{N-1} q^{d-1} e^{j2\pi\frac{m-k}{N}q} \quad (8)$$

## III. CRAMER-RAO BOUNDS (CRBs)

In this section, we present the family of Cramer-Rao Bounds (CRBs). The CRBs provide a lower bound on the Mean Square Error (MSE) achievable by any unbiased estimator. We give the general expression of the Bayesian CRB (BCRB) and its Modified Version (MBCRB). The BCRB is particularly suited for problems where a priori information is available.

Let  $\hat{\mathbf{c}}(\mathbf{y})$  denotes an unbiased estimator of  $\mathbf{c}$  using the set of measurements  $\mathbf{y}$ . The estimation of  $\mathbf{c}$  can be considered following two main scenarios off-line and on-line. In the off-line scenario, the receiver waits until the whole observation frame, *i.e.*,  $\mathbf{y} = [\mathbf{y}_{(1)}^T, \dots, \mathbf{y}_{(K)}^T]^T$ , has been received in order to estimate  $\mathbf{c} = [\mathbf{c}_{(1)}^T, \dots, \mathbf{c}_{(K)}^T]^T$ . In the on-line scenario, the receiver estimates  $\mathbf{c}_{(n)}$  based on the current and previous observations only, *i.e.*,  $\mathbf{y} = [\mathbf{y}_{(1)}^T, \dots, \mathbf{y}_{(n)}^T]^T$ . In the sequel, the BCRB will be considered within the context of both the off-line and the on-line scenarios. The BCRB has been proposed in [15] such that:

$$E_{\mathbf{y}, \mathbf{c}} \left[ (\hat{\mathbf{c}}(\mathbf{y}) - \mathbf{c})(\hat{\mathbf{c}}(\mathbf{y}) - \mathbf{c})^H \right] \geq \mathbf{BCRB}(\mathbf{c}) \quad (9)$$

The BCRB<sup>1</sup> is the inverse of the Bayesian Information Matrix (BIM), which can be written as:

$$\mathbf{B} = E_{\mathbf{c}}[\mathbf{F}(\mathbf{c})] + E_{\mathbf{c}}[-\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{c}))] \quad (10)$$

where  $p(\mathbf{c})$  is the prior distribution and  $\mathbf{F}(\mathbf{c})$  is the Fisher Information Matrix (FIM) defined as:

$$\mathbf{F}(\mathbf{c}) = E_{\mathbf{y}|\mathbf{c}}[-\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{y}|\mathbf{c}))] \quad (11)$$

where  $p(\mathbf{y}|\mathbf{c})$  is the conditional probability density function of  $\mathbf{y}$  given  $\mathbf{c}$ . Unfortunately, in most cases of NDA context, the computation of  $\mathbf{F}(\mathbf{c})$  is generally quite tedious because the  $p(\mathbf{y}|\mathbf{c})$  cannot be carried out analytically due to the nuisance parameters  $\mathbf{x} = [\mathbf{x}_{(1)}^T, \dots, \mathbf{x}_{(K)}^T]^T$ , which are OFDM symbols in our case. In order to circumvent this problem, a Modified BCRB (MBCRB) has been proposed in [19]. This MBCRB is the inverse of the following information matrix:

$$\mathbf{C} = E_{\mathbf{c}}[\mathbf{G}(\mathbf{c})] + E_{\mathbf{c}}[-\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{c}))] \quad (12)$$

where  $\mathbf{G}(\mathbf{c})$  is the modified FIM defined as:

$$\mathbf{G}(\mathbf{c}) = E_{\mathbf{x}} E_{\mathbf{y}|\mathbf{x}, \mathbf{c}}[-\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{y}|\mathbf{x}, \mathbf{c}))] \quad (13)$$

It should be noted that the MBCRB in NDA context is equal to the BCRB in DA context (*i.e.* the nuisance parameters  $\mathbf{x}$  are a priori known).

In our objective, we are interested in the estimation of the complex gains  $\boldsymbol{\alpha} = [\boldsymbol{\alpha}_{(1)}^T, \dots, \boldsymbol{\alpha}_{(K)}^T]^T$ , where  $\boldsymbol{\alpha}_{(n)} = [\boldsymbol{\alpha}_{1}^{(n)T}, \dots, \boldsymbol{\alpha}_{L}^{(n)T}]^T$ . Actually,  $\boldsymbol{\alpha}$  is related to  $\mathbf{c}$  as:

$$\boldsymbol{\alpha} = \mathbf{Q}\mathbf{c} + \boldsymbol{\xi} \quad (14)$$

where  $\mathbf{Q} = \text{blkdiag}\{\mathbf{Q}^T, \dots, \mathbf{Q}^T\}$  is a  $KLv \times KLn_c$  matrix and  $\boldsymbol{\xi} = [\boldsymbol{\xi}_{(1)}^T, \dots, \boldsymbol{\xi}_{(K)}^T]^T$  with  $\boldsymbol{\xi}_{(n)} = [\boldsymbol{\xi}_{1}^{(n)T}, \dots, \boldsymbol{\xi}_{L}^{(n)T}]^T$ . Hence the estimation of  $\boldsymbol{\alpha}$  can be stated by:  $\hat{\boldsymbol{\alpha}} = \mathbf{Q}\hat{\mathbf{c}}$ . By neglecting the cross-covariance terms between the errors  $\boldsymbol{\alpha}_{\text{pol}} - \hat{\boldsymbol{\alpha}}$  and  $\boldsymbol{\xi}$ , we can write:

$$E \left[ (\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha})^H \right] = E \left[ (\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_{\text{pol}})(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_{\text{pol}})^H \right] + E[\boldsymbol{\xi}\boldsymbol{\xi}^H] \quad (15)$$

<sup>1</sup>We recall that Standard Cramer-Rao Bound (SCRb) is the inverse of the Fisher Information Matrix (FIM) (the a priori information is not used).

where  $\boldsymbol{\alpha}_{\text{pol}} = \mathbf{Q}\mathbf{c}$ . So, using the transformation of parameters property defined in [21], we obtain the BCRB for the estimation of  $\boldsymbol{\alpha}$  from the BCRB for  $\mathbf{c}$  as:

$$\begin{aligned} \mathbf{BCRB}(\boldsymbol{\alpha}) &= (\nabla_{\mathbf{c}} \boldsymbol{\alpha}_{\text{pol}}) \mathbf{BCRB}(\mathbf{c}) (\nabla_{\mathbf{c}} \boldsymbol{\alpha}_{\text{pol}})^T + E[\boldsymbol{\xi}\boldsymbol{\xi}^H] \\ &= \mathbf{Q} \mathbf{BCRB}(\mathbf{c}) \mathbf{Q}^T + \mathbf{MMSE} \end{aligned} \quad (16)$$

where the  $KLv \times KLv$  matrix  $\mathbf{MMSE}$  is given by:

$$\mathbf{MMSE}_{[i(l,p), i(l,p')]} = \mathbf{MMSE}_l^{(p-p')} \text{ for } l \in [1, L] \text{ } p, p' \in [0, K-1] \quad (17)$$

with  $i(l, p) = 1 + (l-1)v + pLv : lv + pLv$  and  $\mathbf{MMSE}_l^{(p)}$  is the correlation matrix of the model error  $\boldsymbol{\xi}_l^{(n)}$  defined in (4). Notice that there are zero matrices between the block matrices  $\mathbf{MMSE}_l^{(p)}$  since the  $L$  complex gains are uncorrelated. For  $K = L = 2$ ,  $\mathbf{MMSE}$  is given by:

$$\mathbf{MMSE} = \begin{bmatrix} \mathbf{MMSE}_1^{(0)} & \mathbf{0}_v & \mathbf{MMSE}_1^{(-1)} & \mathbf{0}_v \\ \mathbf{0}_v & \mathbf{MMSE}_2^{(0)} & \mathbf{0}_v & \mathbf{MMSE}_2^{(-1)} \\ \mathbf{MMSE}_1^{(1)} & \mathbf{0}_v & \mathbf{MMSE}_1^{(0)} & \mathbf{0}_v \\ \mathbf{0}_v & \mathbf{MMSE}_2^{(1)} & \mathbf{0}_v & \mathbf{MMSE}_2^{(0)} \end{bmatrix}$$

The computation of the off-line BCRB associated to the estimation of  $\boldsymbol{\alpha}_{(n)}$  is given by:

$$\mathbf{BCRB}(\boldsymbol{\alpha}_{(n)})_{\text{offline}} = \text{Tr}(\mathbf{BCRB}(\boldsymbol{\alpha})_{[i(n), i(n)]}) \quad (18)$$

where the sequence of indices  $i(n) = 1 + (n-1)Lv : nLv$ , with  $n \in [1, K]$ . The on-line BCRB associated to the observation vector  $\mathbf{y} = [\mathbf{y}_{(1)}^T, \dots, \mathbf{y}_{(K)}^T]^T$  is given by:

$$\mathbf{BCRB}(\boldsymbol{\alpha}_{(K)})_{\text{online}} = \text{Tr}(\mathbf{BCRB}(\boldsymbol{\alpha})_{[i(K), i(K)]}) \quad (19)$$

The definitions in (16), (18) and (19) will stand for the closed form of BCRB, *i.e.*, MBCRB and ABCRB.

#### IV. BCRB FOR TIME-VARYING COMPLEX GAINS ESTIMATION

In this section, we present a closed-form expression for a BCRB related to the estimation of the polynomial coefficients  $\mathbf{c}_{(n)}$  of the multi-path complex gains in NDA OFDM systems. This bound is derived for time-varying complex gains within one OFDM symbol. In DA context, we deduce the computation of the true BCRB from the computation of the MBCRB in NDA.

**Computation of  $E_{\mathbf{c}}[\mathbf{F}(\mathbf{c})]$ :** The observation model is presented in (7). Using the whiteness of the noise  $\mathbf{w} = [\mathbf{w}_{(1)}^T, \dots, \mathbf{w}_{(K)}^T]^T$  and the independence of the transmitted OFDM symbols  $\mathbf{x}$ , we then obtain that:

$$\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{y}|\mathbf{c})) = \sum_{n=1}^K \Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{y}_{(n)}|\mathbf{c}_{(n)})) \quad (20)$$

It is important to note that each term of the summation (20) is a  $KL N_c \times KL N_c$  block diagonal matrix with only one nonzero  $LN_c \times LN_c$  block matrix, namely:

$$\Delta_{\mathbf{c}}^{\mathbf{c}(n)} \ln(p(\mathbf{y}(n)|\mathbf{c}(n)))_{[i'(n),i'(n)]} = \Delta_{\mathbf{c}(n)}^{\mathbf{c}(n)} \ln(p(\mathbf{y}(n)|\mathbf{c}(n))) \quad (21)$$

where  $i'(n) = 1 + (n-1)LN_c : nLN_c$  with  $n \in [1, K]$ . As a direct consequence,  $\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{y}|\mathbf{c}))$  is a block diagonal matrix with the  $n$ th diagonal block given by (21). Moreover, because of the circularity of the observation noise, the expectation of (21) with respect to  $\mathbf{y}(n)$  and  $\mathbf{c}(n)$  does not depend on  $\mathbf{c}(n)$ . We then obtain:

$$\mathbf{E}_{\mathbf{c}}[\mathbf{F}(\mathbf{c})] = \text{blkdiag}\{\mathbf{J}, \mathbf{J}, \dots, \mathbf{J}\} \quad (22)$$

where  $\mathbf{J}$  is a  $LN_c \times LN_c$  matrix defined as:

$$\mathbf{J} = \mathbf{E}_{\mathbf{y}, \mathbf{c}}[-\Delta_{\mathbf{c}(n)}^{\mathbf{c}(n)} \ln(p(\mathbf{y}(n)|\mathbf{c}(n)))] \quad (23)$$

The log-likelihood function in (23) can be expanded as:

$$\ln(p(\mathbf{y}(n)|\mathbf{c}(n))) = \ln\left(\sum_{\mathbf{x}(n)} p(\mathbf{y}(n)|\mathbf{x}(n), \mathbf{c}(n))p(\mathbf{x}(n))\right) \quad (24)$$

The vector  $\mathbf{y}(n)$  for given  $\mathbf{x}(n)$  and  $\mathbf{c}(n)$  is a complex Gaussian vector with mean vector  $\mathbf{m}(n) = \mathcal{K}(n)\mathbf{c}(n)$  and covariance matrix  $\sigma^2\mathbf{I}_N$ . Thus,  $p(\mathbf{y}(n)|\mathbf{x}(n), \mathbf{c}(n))$  is defined as:

$$p(\mathbf{y}(n)|\mathbf{x}(n), \mathbf{c}(n)) = \frac{1}{|\pi\sigma^2\mathbf{I}_N|} e^{-\frac{1}{\sigma^2}(\mathbf{y}(n)-\mathbf{m}(n))^H(\mathbf{y}(n)-\mathbf{m}(n))} \quad (25)$$

Since each element of the vector  $\mathbf{m}(n)$  depends on all components of  $\mathbf{x}(n)$  then, the computation of  $\mathbf{J}$  is a demanding task. Hence, we resort to compute the MBCRB. Following the same reasoning as before, we have:

$$\mathbf{E}_{\mathbf{c}}[\mathbf{G}(\mathbf{c})] = \text{blkdiag}\{\mathbf{J}_m, \mathbf{J}_m, \dots, \mathbf{J}_m\} \quad (26)$$

where  $\mathbf{J}_m$  is a  $LN_c \times LN_c$  matrix defined as:

$$\mathbf{J}_m = \mathbf{E}_{\mathbf{y}, \mathbf{x}, \mathbf{c}}[-\Delta_{\mathbf{c}(n)}^{\mathbf{c}(n)} \ln(p(\mathbf{y}(n)|\mathbf{x}(n), \mathbf{c}(n)))] \quad (27)$$

By taking the second derivative of the natural logarithm ( $\ln$ ) of (25) with respect to  $\mathbf{c}(n)$ , we simply obtain that:

$$\Delta_{\mathbf{c}(n)}^{\mathbf{c}(n)} \ln(p(\mathbf{y}(n)|\mathbf{x}(n), \mathbf{c}(n))) = -\frac{1}{\sigma^2}\mathcal{K}(n)^H\mathcal{K}(n) \quad (28)$$

Consequently, we obtain that (see Appendix A):

$$\mathbf{J}_m = \frac{1}{\sigma^2}\mathbf{E}_{\mathbf{x}}[\mathcal{K}(n)^H\mathcal{K}(n)] = \frac{1}{N^2\sigma^2}\mathcal{F}^H\mathcal{M}\mathcal{F} \quad (29)$$

where  $\mathcal{M}$  and  $\mathcal{F}$  are a  $NN_c \times NN_c$  and a  $NN_c \times LN_c$  matrices, respectively, defined as:

$$\mathcal{M} = \begin{bmatrix} \mathcal{M}_{1,1} & \cdots & \mathcal{M}_{1,N_c} \\ \vdots & \ddots & \vdots \\ \mathcal{M}_{N_c,1} & \cdots & \mathcal{M}_{N_c,N_c} \end{bmatrix} \quad (30)$$

$$\mathcal{F} = [\mathcal{F}_1 \cdots \mathcal{F}_L] \quad (31)$$

where  $\mathcal{M}_{d,d'}$  and  $\mathcal{F}_l$  are a  $N \times N$  and a  $NN_c \times N_c$  matrices, respectively, defined as:

$$\mathcal{M}_{d,d'} = \text{diag}\{\text{diag}\{\mathbf{M}_d^H\mathbf{M}_{d'}\}\} \quad (32)$$

$$\mathcal{F}_l = \text{blkdiag}\{\mathbf{f}_l, \mathbf{f}_l, \dots, \mathbf{f}_l\} \quad (33)$$

**Computation of  $\mathbf{E}_{\mathbf{c}}[-\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{c}))]$ :**  $\mathbf{c}$  is a complex Gaussian vector with zero mean and covariance matrix  $\mathbf{R}_{\mathbf{c}}$  of size  $KL N_c \times KL N_c$  defined as:

$$\mathbf{R}_{\mathbf{c}[i'(l,p),i'(l,p)']} = \mathbf{R}_{\mathbf{c}_l}^{(p-p')} \text{ for } l \in [1, L] \text{ } p, p' \in [0, K-1] \quad (34)$$

where  $i'(l, p) = 1 + (l-1)N_c + pLN_c : LN_c + pLN_c$  and  $\mathbf{R}_{\mathbf{c}_l}^{(p)}$  is the correlation matrix of  $\mathbf{c}_l^{(n)}$  defined in (6). Thus, the probability density function  $p(\mathbf{c})$  is defined as:

$$p(\mathbf{c}) = \frac{1}{|\pi\mathbf{R}_{\mathbf{c}}|} e^{-\mathbf{c}^H\mathbf{R}_{\mathbf{c}}^{-1}\mathbf{c}} \quad (35)$$

Taking the second derivative of the natural logarithm ( $\ln$ ) of (35) with respect to  $\mathbf{c}$  and making the expectation over  $\mathbf{c}$ , we simply obtain that:

$$\mathbf{E}_{\mathbf{c}}[-\Delta_{\mathbf{c}}^{\mathbf{c}} \ln(p(\mathbf{c}))] = \mathbf{R}_{\mathbf{c}}^{-1} \quad (36)$$

The MBCRB for the estimation of  $\mathbf{c}$  is given by:

$$\mathbf{MBCRB}(\mathbf{c}) = \left(\text{blkdiag}\{\mathbf{J}_m, \mathbf{J}_m, \dots, \mathbf{J}_m\} + \mathbf{R}_{\mathbf{c}}^{-1}\right)^{-1} \quad (37)$$

Notice that the MBCRB is usually looser than the BCRB. As in (16), the MBCRB for the estimation of  $\alpha$  is given by:

$$\mathbf{MBCRB}(\alpha) = \mathcal{Q}\mathbf{MBCRB}(\mathbf{c})\mathcal{Q}^T + \mathbf{MMSE} \quad (38)$$

In data-aided (DA) context, the transmitted data symbols  $\mathbf{x}(n)$  are known at the receiver. Hence, the matrix  $\mathbf{J}$  is computed like  $\mathbf{J}_m$ , but without averaging over the data symbols  $\mathbf{x}(n)$ , and consequently it depends on the  $n$ th transmitted OFDM symbol. Thus,  $\mathbf{J}(n)$  is given by:

$$\mathbf{J}(n) = \frac{1}{\sigma^2}\mathcal{K}(n)^H\mathcal{K}(n) = \frac{1}{N^2\sigma^2}\mathcal{F}(n)^H\mathcal{M}\mathcal{F}(n) \quad (39)$$

where the matrix  $\mathcal{F}(n)$  is computed like  $\mathcal{F}$  but by replacing  $\mathbf{f}_l$  in equation (33) by  $\text{diag}\{\mathbf{x}(n)\}\mathbf{f}_l$ . The BCRB for the estimation of  $\mathbf{c}$  in DA context is given by:

$$\mathbf{BCRB}(\mathbf{c}) = \left(\text{blkdiag}\{\mathbf{J}(1), \mathbf{J}(2), \dots, \mathbf{J}(K)\} + \mathbf{R}_{\mathbf{c}}^{-1}\right)^{-1} \quad (40)$$

and consequently the BCRB for the estimation of  $\alpha$  as (16). It should be noted that BCRB for the estimation of  $\alpha$  in DA context depends on the transmitted data sequence  $\mathbf{x}$ .

## V. DISCUSSION

In this section, we bring to the fore the behavior of the previous bounds, namely the off-line and the on-line MBCRBs (BCRB in DA) for the complex gains estimation. A normalized 4QAM OFDM system,  $N = 128$  subcarriers,  $N_g = \frac{N}{8}$  subcarriers is used (note that  $\text{SNR} = \frac{1}{\sigma^2}$ ). The normalized channel model is Rayleigh with  $L = 6$  paths of parameters given in [5] [6] [7] [8]. We consider the case of high Doppler speed with  $0.05 \leq f_d T \leq 0.5$ , and  $2 \leq N_c \leq 5$  for the polynomial modeling. In the following figures, the term MSE stands for Mean Square Error.

Fig. 2 superimposes versus time index, the on-line MBCRB and the off-line MBCRB for  $f_d T = 0.1$ ,  $N_c = 2$  and different block-observation lengths  $K$  at  $\text{SNR} = 10\text{dB}$ . In the off-line context, we can see that the best complex gains estimation is achieved at the midblock, whereas the estimates are likely to be poorer at the block border. This stems

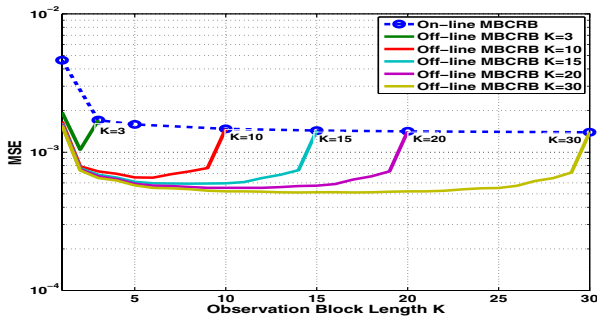


Fig. 2. BCRBs vs  $K$ , for  $\text{SNR}=10\text{dB}$ ,  $f_dT=0.1$  and  $N_c=2$

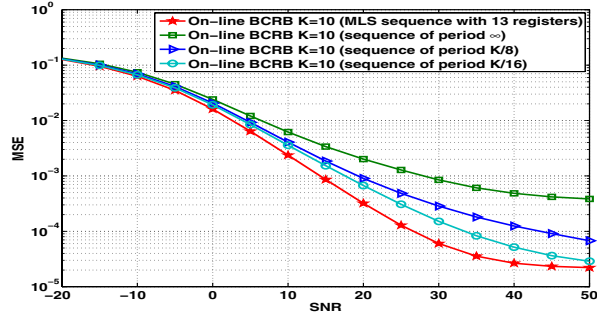


Fig. 3. BCRBs vs SNR for different data sequences with  $f_dT=0.3$  and  $N_c=3$

from the fact that in the center position of the polynomial coefficients vector  $\mathbf{c}$  we have more adjacent (past or future) and strongly correlated variables than at the border of the vector  $\mathbf{c}$ . Concerning the online bound, at the beginning when the number of observations increases, the estimator takes into account more and more information and the estimation is improved; the bound thus decreases and converges to an asymptote. The estimation performance is then limited by the observation noise independently of the number of observations taken into account. However, in order to reach the asymptote, it is sufficient to use 3 past OFDM symbols for  $f_dT=0.1$ .

As we have seen in DA context, the BCRB depends on the transmitted data sequence. We now study the DA bound behavior for different data sequences. Fig. 3 gives the DA BCRB versus SNR for  $f_dT=0.3$  and  $N_c=3$ . We can observe that there is bad sequences (period infinity, *i.e.*, all the bits are equals) and good sequences (Maximum length sequence MLS [22] with 13 shift registers). We have observed via experimentations that sequences with good deterministic auto-correlation (*i.e.*, near Dirac impulse, as for exemple sequences of type MLS) are good sequences. In such case, the bounds are very similar. That is why in the paper we have given only one example (13 registers with feedback polynomial [20033] MLS [22]) of these sequences. However, if we use sequences with bad deterministic auto-correlation, like the sequences of Walsh-Hadamard (period  $K/8$ : bit = '0' during  $K.T/16$  and bit = '1' during  $K.T/16$ ), then the performance will be degraded as shown in this figure.

We now study the bound behavior versus the polynomial coefficients  $N_c$  and SNR over a block of  $K=10$ . Fig. 4 gives the MBCRBs for  $f_dT=0.5$  in terms of SNR in (a) and in

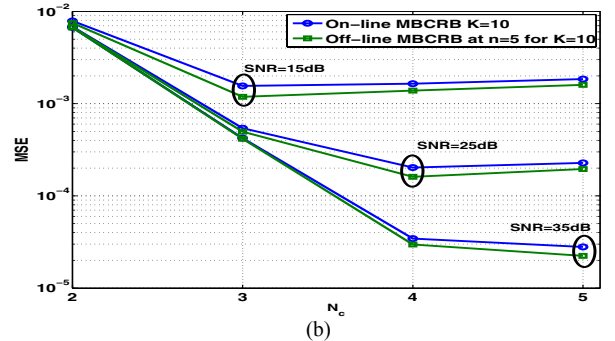
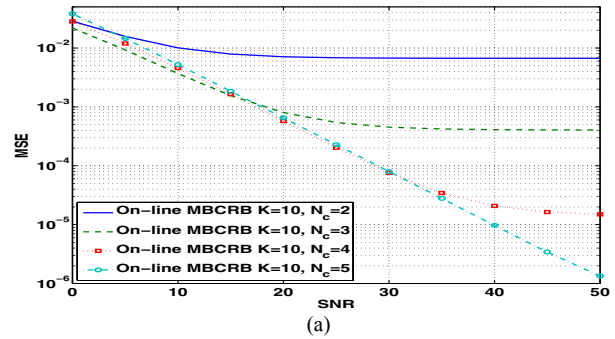


Fig. 4. (a) BCRBs vs SNR for  $f_dT=0.5$  and  $N_c=2$  to 5 ; (b) BCRBs vs  $N_c$  for  $f_dT=0.5$  and  $\text{SNR}=15\text{dB}$ ,  $25\text{dB}$  and  $35\text{dB}$

TABLE I  
THE MINIMUM OF THE BCRB( $\alpha$ ) FOR GSM CHANNEL [5]

$f_dT$ \ SNR(dB)	0	21	27	37	40
0.05	$N_c=2$	$N_c=2$	$N_c=2$	$N_c=2$	$N_c=2$
0.1	$N_c=3$	$N_c=3$	$N_c=3$	$N_c=3$	$N_c=3$
0.2	$N_c=3$	$N_c=3$	$N_c=3$	$N_c=3$	$N_c=4$
0.3	$N_c=3$	$N_c=3$	$N_c=3$	$N_c=4$	$N_c=4$
0.4	$N_c=3$	$N_c=4$	$N_c=4$	$N_c=4$	$N_c=4$

terms of  $N_c$  in (b). We observe in (a) that, whatever the SNR, the bound is not always decreasing in terms of  $N_c$  but at high SNR, the bound converges to the MMSE (the model error). As we see in (b), for  $\text{SNR}=15\text{dB}$ ,  $25\text{dB}$  and  $35\text{dB}$ , the minimum of the bound is obtained at  $N_c=3$ ,  $4$  and  $5$  polynomial coefficients, respectively. This is due to the last coefficients which will be poorly estimated in presence of noise. Indeed, they are negligible compared to the noise level. Hence, in order to have a good estimation of the complex gains time-variation, we have to choose  $N_c$  according to SNR and  $f_dT$ . The Table I shows how to select  $N_c$ , for realistic values of SNR and different values of  $f_dT$ , such that the bound is minimal. For example if  $f_dT=0.3$ , we choose  $N_c=3$  and  $4$  for  $\text{SNR} \in [0; 27]$  and  $\text{SNR} \in [27; 40]$ , respectively. Therefore, we introduce a New BCRB (NBCRB) which is independant of  $N_c$ , defined as:

$$\text{NBCRB}(\alpha) = \min_{N_c} (\text{BCRB}(\alpha)) \quad (41)$$

where  $\min_{N_c}(\cdot)$  is the minimum over  $N_c$ . This definition in (41) will stand for the MBCRB in case of NDA.

We now analyse the bound behavior versus  $f_dT$ . Fig. 5 gives the NSCRB and the NMBCRB versus  $f_dT$  for  $\text{SNR}=20\text{dB}$  and  $K=10$ . We notice that the NMBCRB increases in

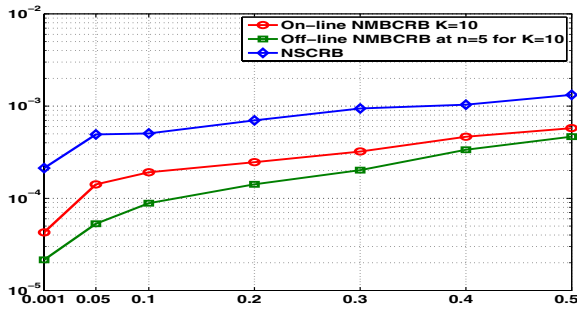


Fig. 5. NBCRBs vs  $f_d T$  for SNR = 20dB

terms of  $f_d T$ . This is because the correlation between variables becomes stronger when  $f_d T$  decreases. So, the estimation gain for slow channel variations is more significant.

## VI. CONCLUSION

In this contribution, we have derived an analytical expression of a BCRB for the estimation of Rayleigh channel complex gains with time variations within one OFDM symbol. We have introduced a New BCRB (NBCRB) and we have shown that a good estimation of the complex gains time variation can be obtained by choosing the number of polynomial coefficients according to the noise level and the Doppler spread. These bounds are useful when analyzing the performance of complex gains estimators in DA and NDA contexts and in on-line and off-line scenarios. Moreover, we have shown the benefit of using the past and the future OFDM symbols in channel estimation process, whereas most methods use only the current symbol.

## APPENDIX A EVALUATION OF $\mathbf{J}_m$

In this Appendix, we detail the calculus to obtain the expression of  $\mathbf{J}_m$  defined in (29). Using the definition of  $\mathcal{K}_{(n)}$  in section II, we have:

$$\mathbf{A} = \mathcal{K}_{(n)}^H \mathcal{K}_{(n)} = \frac{1}{N^2} \begin{bmatrix} \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,L} \\ \vdots & \ddots & \vdots \\ \mathbf{A}_{L,1} & \cdots & \mathbf{A}_{L,L} \end{bmatrix} \quad (42)$$

where  $\mathbf{A}_{l,l'} = \mathbf{Z}_l^{(n)H} \mathbf{Z}_{l'}^{(n)}$  is a  $N_c \times N_c$  matrix with elements given by:

$$[\mathbf{A}_{l,l'}]_{d,d'} = \mathbf{f}_l^H \text{diag}\{\mathbf{x}_{(n)}^H\} \mathbf{M}_d^H \mathbf{M}_{d'} \text{diag}\{\mathbf{x}_{(n)}\} \mathbf{f}_{l'} \quad (43)$$

Taking the expectation of (43) over  $\mathbf{x}$ , we obtain:

$$\mathbb{E}_{\mathbf{x}} \left[ [\mathbf{A}_{l,l'}]_{d,d'} \right] = \mathbf{f}_l^H \mathcal{M}_{d,d'} \mathbf{f}_{l'} \quad (44)$$

since the symbols are normalized uncorrelated with respect to each other. Consequently, we obtain that:

$$\mathbb{E}_{\mathbf{x}} \left[ \mathbf{A}_{l,l'} \right] = \mathcal{F}_l^H \mathcal{M}_{d,d'} \mathcal{F}_{l'} \quad (45)$$

and finally we obtain the expression of  $\mathbf{J}_m$  defined in (29).

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