

# A GENERALIZATION OF A TRACE INEQUALITY FOR POSITIVE DEFINITE MATRICES.

ELENA VERONICA BELMEGA, MARC JUNGERS, AND SAMSON LASAULCE

ABSTRACT. In this note we generalize the trace inequality derived by [1] to the case where the number of terms of the sum (denoted by  $K$ ) is arbitrary. More precisely we prove that  $\mathcal{T}_K = \text{Tr} \left\{ \sum_{k=1}^K (\mathbf{A}_k - \mathbf{B}_k) \left[ \left( \sum_{\ell=1}^k \mathbf{B}_\ell \right)^{-1} - \left( \sum_{\ell=1}^k \mathbf{A}_\ell \right)^{-1} \right] \right\} \geq 0$  for any set of positive definite matrices.

## 1. INTRODUCTION

Trace inequalities are useful in many areas like multiple input multiple output (MIMO) systems in control theory and communications. Proving the trace inequality under investigation in this note is a sufficient condition to ensure the uniqueness of a Nash equilibrium in certain MIMO communications game [2] where Rosen's diagonally strict concavity condition [3] is valid. The considered inequality has been proven by e.g., [4] for  $K = 1$  and by [1] for  $K = 2$ . Here we generalize it to  $K \in \mathbb{N}^*$ . The main result of this note is as follows.

**Theorem 1.1.** *Let  $K \in \mathbb{N}^*$ . Assume that*

- (i):  $\mathbf{A}_1 = \mathbf{A}_1^H \succ 0, \mathbf{B}_1 = \mathbf{B}_1^H \succ 0$ ;
- (ii):  $\forall k \in \{2, \dots, K\}, \mathbf{A}_k = \mathbf{A}_k^H \succeq 0$  and  $\mathbf{B}_k = \mathbf{B}_k^H \succeq 0$ .

*Then, we have that*

$$(1.1) \quad \mathcal{T}_K \triangleq \text{Tr} \left\{ \sum_{k=1}^K (\mathbf{A}_k - \mathbf{B}_k) \left[ \left( \sum_{\ell=1}^k \mathbf{B}_\ell \right)^{-1} - \left( \sum_{\ell=1}^k \mathbf{A}_\ell \right)^{-1} \right] \right\} \geq 0.$$

## 2. AUXILIARY RESULTS

In order to prove Theorem 1.1, we will use two auxiliary lemmas which are stated here for the sake of clarity.

**Lemma 2.1.** [1] *Let  $\mathbf{A}, \mathbf{B}$  be two positive definite matrices,  $\mathbf{C}, \mathbf{D}$ , two positive semidefinite matrices whereas  $\mathbf{X}$  is only assumed to be Hermitian. Then*

$$(2.1) \quad \text{Tr} \{ \mathbf{X} \mathbf{A}^{-1} \mathbf{X} \mathbf{B}^{-1} \} - \text{Tr} \{ \mathbf{X} (\mathbf{A} + \mathbf{C})^{-1} \mathbf{X} (\mathbf{B} + \mathbf{D})^{-1} \} \geq 0.$$

The proof is given in [1].

**Lemma 2.2.** *Let  $\mathbf{A}, \mathbf{B}$  be two positive definite matrices,  $\mathbf{C}, \mathbf{D}$ , two positive semi-definite matrices. Then*

$$(2.2) \quad \begin{aligned} \text{Tr} \{ (\mathbf{A} - \mathbf{B})(\mathbf{B} + \mathbf{D})^{-1}(\mathbf{C} - \mathbf{D})(\mathbf{A} + \mathbf{C})^{-1} \} &= \\ \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{B} + \mathbf{D})^{-1}(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{C})^{-1} \} &\in \mathbb{R}. \end{aligned}$$

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*Proof.* To prove the desired result, let us define  $\mathcal{E}$  by

$$(2.3) \quad \mathcal{E} = \text{Tr} \{ (\mathbf{C} - \mathbf{D}) [(\mathbf{B} + \mathbf{D})^{-1} - (\mathbf{A} + \mathbf{C})^{-1}] \}$$

and write it in two different ways.

$$(2.4) \quad \begin{aligned} \mathcal{E} &= \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{B} + \mathbf{D})^{-1}[\mathbf{A} + \mathbf{C} - \mathbf{B} - \mathbf{D}](\mathbf{A} + \mathbf{C})^{-1} \} \\ &= \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{B} + \mathbf{D})^{-1}(\mathbf{C} - \mathbf{D})(\mathbf{A} + \mathbf{C})^{-1} \} + \\ &\quad \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{B} + \mathbf{D})^{-1}(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{C})^{-1} \}. \end{aligned}$$

$$(2.5) \quad \begin{aligned} \mathcal{E} &= \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{A} + \mathbf{C})^{-1}[\mathbf{A} + \mathbf{C} - \mathbf{B} - \mathbf{D}](\mathbf{B} + \mathbf{D})^{-1} \} \\ &= \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{A} + \mathbf{C})^{-1}(\mathbf{C} - \mathbf{D})(\mathbf{B} + \mathbf{D})^{-1} \} + \\ &\quad \text{Tr} \{ (\mathbf{C} - \mathbf{D})(\mathbf{A} + \mathbf{C})^{-1}(\mathbf{A} - \mathbf{B})(\mathbf{B} + \mathbf{D})^{-1} \}. \end{aligned}$$

By using the commutation property of the trace and the two expressions of  $\mathcal{E}$ , we find the desired result. The only thing which needs to be proven is that  $\mathcal{E}$  is real. For this purpose, observe that if we denote by  $\mathbf{M} = (\mathbf{C} - \mathbf{D})(\mathbf{B} + \mathbf{D})^{-1}(\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{C})^{-1}$  then  $\mathbf{M}^H = (\mathbf{A} + \mathbf{C})^{-1}(\mathbf{A} - \mathbf{B})(\mathbf{B} + \mathbf{D})^{-1}(\mathbf{C} - \mathbf{D})$  and from the result just proven we obtain that  $\text{Tr}(\mathbf{M}^H) = \text{Tr}(\mathbf{M})$  and thus we have that  $\text{Tr}(\mathbf{M}) \in \mathbb{R}$ .  $\square$

### 3. PROOF OF THEOREM 1.1

Define  $\mathbf{X}_k = \sum_{i=1}^k \mathbf{A}_i$ ,  $\mathbf{Y}_k = \sum_{i=1}^k \mathbf{B}_i$ , for all  $k \geq 1$  which are both positive definite matrices. Notice that  $\mathcal{T}_K$  can be re-written recursively as follows:

$$(3.1) \quad \begin{cases} \mathcal{T}_1 &= \text{Tr} \{ (\mathbf{A}_1 - \mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1 - \mathbf{B}_1)\mathbf{X}_1^{-1} \} \\ \mathcal{T}_K &= \mathcal{T}_{K-1} + \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{A}_K - \mathbf{B}_K)\mathbf{X}_K^{-1} \} + \\ &\quad \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_{K-1} - \mathbf{Y}_{K-1})\mathbf{X}_K^{-1} \} \end{cases}$$

We proceed in two steps. First, we find a lower bound for  $\mathcal{T}_K$  and then we prove that this bound is positive. First, let us prove that, for all  $K \geq 1$ :

$$(3.2) \quad \mathcal{T}_K \geq \frac{1}{2} \sum_{i=1}^K \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i - \mathbf{B}_i)\mathbf{X}_i^{-1} \} + \frac{1}{2} \text{Tr} \{ (\mathbf{X}_K - \mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K - \mathbf{Y}_K)\mathbf{X}_K^{-1} \}$$

To this end we proceed by induction. For all  $K \in \mathbb{N}^*$ , define the proposition:

$$(3.3) \quad \mathcal{P}_K : \mathcal{T}_K \geq \frac{1}{2} \sum_{i=1}^K \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i)\mathbf{Y}_i^{-1}(\mathbf{A}_i - \mathbf{B}_i)\mathbf{X}_i^{-1} \} + \frac{1}{2} \text{Tr} \{ (\mathbf{X}_K - \mathbf{Y}_K)\mathbf{Y}_K^{-1}(\mathbf{X}_K - \mathbf{Y}_K)\mathbf{X}_K^{-1} \}.$$

It is easy to check that, for  $K = 1$ ,  $\mathcal{P}_1$  is true:

$$(3.4) \quad \begin{aligned} \mathcal{T}_1 &= \text{Tr} \{ (\mathbf{A}_1 - \mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1 - \mathbf{B}_1)\mathbf{X}_1^{-1} \} \\ &= \frac{1}{2} \text{Tr} \{ (\mathbf{A}_1 - \mathbf{B}_1)\mathbf{Y}_1^{-1}(\mathbf{A}_1 - \mathbf{B}_1)\mathbf{X}_1^{-1} \} + \frac{1}{2} \text{Tr} \{ (\mathbf{X}_1 - \mathbf{Y}_1)\mathbf{Y}_1^{-1}(\mathbf{X}_1 - \mathbf{Y}_1)\mathbf{X}_1^{-1} \}. \end{aligned}$$

Now, let us assume that  $\mathcal{P}_{K-1}$  is true and then prove that this implies that  $\mathcal{P}_K$  is also true. We have that:

$$(3.5) \quad \begin{aligned} \mathcal{T}_{K-1} &\geq \frac{1}{2} \sum_{i=1}^{K-1} \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_{K-1}^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_{K-1}^{-1} \}. \end{aligned}$$

From the recursive formula (3.1) we obtain:

$$(3.6) \quad \begin{aligned} \mathcal{T}_K &\geq \frac{1}{2} \sum_{i=1}^{K-1} \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_{K-1}^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_{K-1}^{-1} \} + \\ &\quad \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{A}_K - \mathbf{B}_K) \mathbf{X}_K^{-1} \} + \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} \\ &\geq \frac{1}{2} \sum_{i=1}^{K-1} \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} + \\ &\quad \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{A}_K - \mathbf{B}_K) \mathbf{X}_K^{-1} \} + \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} \\ &= \frac{1}{2} \sum_{i=1}^{K-1} \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} + \\ &\quad \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{A}_K - \mathbf{B}_K) \mathbf{X}_K^{-1} \} + \frac{1}{2} \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_K^{-1} (\mathbf{A}_K - \mathbf{B}_K) \mathbf{X}_K^{-1} \} \\ &= \frac{1}{2} \sum_{i=1}^K \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{A}_K - \mathbf{B}_K) \mathbf{X}_K^{-1} \} + \frac{1}{2} \text{Tr} \{ (\mathbf{A}_K - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{X}_K^{-1} \} + \\ &\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} - \mathbf{Y}_{K-1}) \mathbf{Y}_K^{-1} (\mathbf{A}_K - \mathbf{B}_K) \mathbf{X}_K^{-1} \} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i=1}^K \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \\
&\quad \frac{1}{2} \text{Tr} \{ (\mathbf{X}_{K-1} + \mathbf{A}_K - \mathbf{Y}_{K-1} - \mathbf{B}_K) \mathbf{Y}_K^{-1} (\mathbf{X}_{K-1} + \mathbf{A}_K - \mathbf{Y}_{K-1} - \mathbf{B}_K) \mathbf{X}_K^{-1} \} \\
&= \frac{1}{2} \sum_{i=1}^K \text{Tr} \{ (\mathbf{A}_i - \mathbf{B}_i) \mathbf{Y}_i^{-1} (\mathbf{A}_i - \mathbf{B}_i) \mathbf{X}_i^{-1} \} + \frac{1}{2} \text{Tr} \{ (\mathbf{X}_K - \mathbf{Y}_K) \mathbf{Y}_K^{-1} (\mathbf{X}_K - \mathbf{Y}_K) \mathbf{X}_K^{-1} \}
\end{aligned}$$

The second inequality follows from applying Lemma 2.1 for the second term on the right and considering that  $\mathbf{X}_K = \mathbf{X}_{K-1} + \mathbf{A}_K$ ,  $\mathbf{Y}_K = \mathbf{Y}_{K-1} + \mathbf{B}_K$ . The third equality follows from Lemma 2.2. Thus, we have proven the desired result.

The second step of the proof is straightforward. From (3.2), it is easy to check that  $\mathcal{T}_K \geq 0$  (all the terms of the form  $\text{Tr} \{ \mathbf{X} \mathbf{B}^{-1} \mathbf{X} \mathbf{A}^{-1} \}$  with  $\mathbf{X} = \mathbf{X}^H$ ,  $\mathbf{A} \succ \mathbf{0}$ ,  $\mathbf{B} \succ \mathbf{0}$  can be re-written as  $\text{Tr}(\mathbf{N} \mathbf{N}^H) \geq 0$  with  $\mathbf{N} = \mathbf{A}^{-1/2} \mathbf{X} \mathbf{B}^{-1/2}$ ).

#### REFERENCES

- [1] E. V. BELMEGA, S. LASAULCE, and M. DEBBAH, “A Trace Inequality for Positive Definite Matrices”, *Journal of inequalities in pure and applied mathematics (JIPAM)*, Vol. 10, No. 1, Article 5, 4 pp., (2009).
- [2] E. V. BELMEGA, S. LASAULCE, M. DEBBAH, M. JUNGERS, and J. DUMONT, “Power allocation games in wireless networks of multi-antenna terminals”, *Springer Telecommunications Systems Journal, VALUETOOLS special issue*, revised (minor revisions), (2009).
- [3] J. ROSEN, “Existence and uniqueness of equilibrium points for concave n-person games”, *Econometrica*, Vol. 33, pp. 520–534, (1965).
- [4] K. M. ABADIR AND J. R. MAGNUS, “Matrix Algebra”, *Cambridge University Press, New York, USA*, pp. 329, pp. 338, (2005).

UNIVERSITÉ PARIS-SUD XI, SUPELEC, LABORATOIRE DES SIGNAUX ET SYSTÈMES, GIF-SUR-YVETTE, FRANCE.

*E-mail address:* belmega@lss.supelec.fr

*URL:* <http://veronica.belmega.lss.supelec.fr>

CNRS, ENSEM, CRAN, VANDOEUVRE, FRANCE.

*E-mail address:* marc.jungers@cran.uhp-nancy.fr

*URL:* <http://perso.ensem.inpl-nancy.fr/Marc.Jungers/>

CNRS, SUPELEC, LABORATOIRE DES SIGNAUX ET SYSTÈMES, GIF-SUR-YVETTE, FRANCE.

*E-mail address:* lasaulce@lss.supelec.fr

*URL:* <http://samson.lasaulce.lss.supelec.fr>