A New Energy Efficiency Measure for Quasi-Static MIMO Channels

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ABSTRACT

In this paper, we consider the multiple input multiple output (MIMO) quasi static channel. Our objective is to study the power allocation (over the transmit antennas) problem where not only the performance with respect to (w.r.t.) the transmission reliability but also the cost in terms of the consumed power is accounted for. We first review the existing results w.r.t energy efficiency functions (benefit per cost) which focus mainly on the single input single output (SISO) case and then propose several extensions to the MIMO case. Then, we introduce a new energy efficiency metric based on the outage probability. We conjecture that there is a non-trivial solution to the proposed optimization problem. Several special cases are thoroughly analyzed and simulation results will be provided to sustain the conducted analysis.

Categories and Subject Descriptors

H.1.1. [**Information theory**]: Information theory

General Terms

Performance, Reliability

Keywords

MIMO, energy efficiency function, power allocation, outage probability

1. INTRODUCTION

We consider the single user MIMO channel where the multiple antennas available at the transmitter and the receiver are known ([1], [2]) to increase the performance of the transmission by increasing the diversity and multiplexing gains.

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The objective in this paper is to study the power allocation (PA) policy that optimizes different energy efficiency functions that accounts for both the transmission reliability in terms of packet error probability or outage probability and the power consumed to achieve this performance. Energy efficiency functions have been studied in [3] where the authors analyze the power control problem in a SISO CDMA multi-user setting assuming that the users have limited battery lifetime and wish to maximize their energy efficiency utilities, the number of reliable bits transmitted over all the carriers per Joule of energy consumed (benefit per cost function). The channel is assumed to vary slowly and perfect channel state information is assumed at both the receivers and transmitters. In [5], the authors investigate the same problem in the context of multi-user multiple access channels. In [6], the authors extend the analysis in [3] to the case of multi-carrier CDMA. The users decide how much power to transmit over each carrier to maximize their energy efficiency utilities. It is shown that, assuming linear receivers, the energy efficiency function of a user is maximized when it transmits on its best carrier (the carrier that requires the least amount of power to achieve a particular target signal-to-interference plus noise ration (SINR)). In all these references, the multi-user non-cooperative power control game is investigated but this issue is out of the scope in this paper. Here, we only focus on the single-user case and adopt two different approaches.

First, we briefly review the key results and then propose several extensions to the MIMO case where both, the receiver and transmitter are equipped with multiple antennas, n_r and n_t respectively. The power control problem becomes a power allocation one where the transmitter decides how to split its available power over the transmit antennas in order to maximize the energy efficiency function.

Second, we assume that there is no channel state information available at the transmitter but only channel distribution information. In this case, the packet error probability is no longer a good metric to measure the transmission reliability. To overcome this difficulty, we introduce a new energy efficiency metric based on the outage probability [7]. The outage probability represents the probability that the instantaneous channel does not support a target transmission rate $R > 0$. For a fixed transmission rate $R > 0$, the proposed energy efficiency function measures the number of successfully transmitted bits (benefit) per Joule of energy consumed (cost). The general power allocation is difficult to be thoroughly analyzed and, thus, we will assume the Rayleigh fading channel model and focus our analysis on several particular cases. The main difficulty is the study of the outage probability. There are very few analytical results available w.r.t. the minimal outage probability. For the Rayleigh channel model, Telatar [1] conjectures that the optimal PA policy that minimizes the outage probability is to uniformly divide the available power over a subset of transmit antennas. To the authors knowledge this conjecture is still an open problem. Assuming the MISO Rayleigh channel, the authors of [8] prove that, for any fixed target rate, there exist $n_t - 1$ SNR thresholds (where n_t is the number of transmit antennas) such that whenever the actual SNR is in between the ℓ-th and $\ell + 1$-th ordered thresholds (with $\ell \in \{1, \ldots, n_t - 1\}$) then the optimal PA policy is to uniformly divide the power over $\ell + 1$ antennas. In [9], the authors gave a thorough analytical solution for the 2×1 MISO Rayleigh channel. The main difficulty in studying the outage probability is the fact that the probability distribution function of the mutual information is generally intractable. In many papers, the outage probability is studied assuming a uniform PA policy over all the antennas and also using the Gaussian approximation of the distribution of the mutual information. This approximation is tight in the asymptotic regime in terms of the number of antennas but simulations show that even for a finite number of antennas the approximation is accurate [10], [11].

The paper is structured as follows. The system model is introduced in Sec. 2. In Sec. 3, we review the existing results w.r.t. energy efficiency functions and propose several extensions to the MIMO case. We intoduce a new energy efficiency metric based on the outage probability in Sec. 4 we introduce and several particular cases are investigated, in Sec. 5 simulation results are provided and we end by some concluding remarks (Sec; 6).

2. SYSTEM MODEL

We consider an MIMO channel where the transmitter and receiver are equipped with multiple antennas, n_t, n_r respectively. The equivalent baseband signal at the receiver can be written as:

$$\underline{y} = \mathbf{H}\underline{x} + \underline{z}, \tag{1}$$

where $\mathbf{H}$ is the $n_r \times n_t$ channel matrix whose entries are i.i.d. complex Gaussian random variables of zero mean and unit variance, $\underline{x}$ is the n_t-dimensional column vector of symbols transmitted and $\underline{z}$ is a n_r-dimensional complex Gaussian noise distributed as $\mathcal{N}(\underline{0}, \mathbf{\Sigma}_z)$. The channel under study is quasi-static meaning that $\mathbf{H}$ is a random matrix that stays constant for the whole duration of the transmission. We will denote $\mathbf{Q} = \mathbb{E}[\underline{x}\underline{x}^H]$ the $n_t \times n_t$ input covariance matrix which we assume to be constrained as follows: $\mathrm{Tr}(\mathbf{Q}) \leq \overline{P}$. The main objective is to study the PA policy that maximizes an energy-efficiency function (benefit per cost):

$$\xi(\mathbf{Q}, R) = \frac{F(\mathbf{Q}, R)}{\mathrm{Tr}(\mathbf{Q})}, \tag{2}$$

where the benefit $F(\mathbf{Q}, R)$ represents the number of bits reliably/succesfully transmitted at the rate $R > 0$ nats per second while consuming a certain amount of transmit power (the cost $\mathrm{Tr}(\mathbf{Q})$).

3. REVIEW OF EXISTING RESULTS AND SIMPLE EXTENSIONS

In this section, we assume that both the transmitter and the receiver have perfect channel state information. We first review some of the relevant references that have investigated energy efficiency functions and then propose several extensions. In [3], the authors investigated the power control problem in a SISO (i.e., $n_r = 1$, $n_t = 1$) Gaussian channel where the considered performance metric is

$$\xi(p, R) = \frac{LRf(\gamma)}{Mp}, \tag{3}$$

where L represents the information bits, M the packet size ($M > L$ after the channel coding), R the transmission rate, γ is the received signal-to-noise ratio (SNR) and a linear function of the transmit power p. The function $f(\gamma)$ represents essentially the probability of correct reception and has the following properties: $\lim_{p\to\infty} f(\gamma) = 1$ and $\lim_{p\to 0} \frac{f(\gamma)}{p} = 0$. The optimal power $p^* > 0$ is the necessary power to achieve the optimal SNR $\gamma^* \triangleq \arg\max_{\gamma \geq 0} \frac{f(\gamma)}{\gamma}$. The fact that in general the solution is non-trivial can be proved using the arguments in [4] where the authors show that for any sigmoidal-shaped function $f(x)$, the function $\frac{f(x)}{x}$ is quasi-concave and has a unique global maximum point $x^* > 0$ that can be identified geometrically (the intersection point between the curve $y = f(x)$ and its tangent that passes through the origin $(0, 0)$).

The authors in [6] have extended this analysis to the study of the PA problem in multi-carrier CDMA systems where the transmitter sends independent data flows over the orthogonal $D \geq 2$ carriers, linear receivers are assumed and the noise covariance matrix is $\mathbf{\Sigma}_z = \sigma^2 \mathbf{I}$. The energy-efficiency utility writes in this case:

$$\xi(\underline{p}, R) = \frac{RL}{M} \frac{\sum_{i=1}^{D} f(\gamma_i)}{\sum_{i=1}^{D} p_i}, \tag{4}$$

with the same notations as in (3), $\underline{p} = (p_1, \ldots, p_D)$, $p_i \geq 0$ represents the power allocated to the i-th carrier and γ_i is the receive SNR on the i-th carrier which is a linear function of p_i (from the linear receiver assumption). The authors prove that the optimal PA policy is to transmit only over the best carrier w.r.t. the receiving conditions, $\forall i \in \{1, \ldots, D\}$:

$$p_i^* = \left| \begin{array}{ll} \tilde{p}_i, & \text{if } i = k, \\ 0, & \text{otherwise,} \end{array} \right. \tag{5}$$

where $k = \arg \min_{j \in \{1, \ldots, D\}} \tilde{p}_j$ where $\tilde{p}_j$ represents the necessary power to achieve the receive SNR equal to $\gamma^* = \arg\max_{\gamma \geq 0} \frac{f(\gamma)}{\gamma}$ on carrier d. If a power constraint is considered, then the solution becomes simply $\min\{\overline{P}, p_i^*\}$, for all $i \in \{1, \ldots, D\}$.

Now, we would like to extend this result to MIMO channels. The major difficulty is that the output SNR will be strongly related to the encoding-decoding schemes implemented. For simplicity reasons we consider the two extreme cases w.r.t. the tradeoff between the diversity and multiplexing gains brought by the MIMO systems.

3.0.1 *Case of Full Multiplexing Mode*

We assume that the transmitter sends independent data flows on the n_t antennas, meaning that the covariance matrix is diagonal $\mathbf{Q} = \text{diag}(p_1, \ldots, p_{n_t})$. We observe that the multi-carrier CDMA case studied in [6] is a particular case where the channel matrix $\mathbf{H}$ is assumed diagonal, $n_r = n_t = D$ and the noise covariance matrix $\mathbf{\Sigma}_z = \sigma^2\mathbf{I}$. In our case, the received baseband signal is:

$$\underline{y} = \sum_{i=1}^{n_t} \underline{h}_i x_i + \underline{z},$$

where $\underline{h}_i$ is the i-th column of the channel matrix. The efficiency function writes similarly to (4):

$$\xi(\underline{p}, R) = \frac{RL}{M} \frac{\sum_{i=1}^{n_t} f(\gamma_i)}{\sum_{i=1}^{n_t} p_i}, \qquad (6)$$

with $f(\cdot)$ represents the packet transmission success rate and γ_i is the output SINR of the matched filter receiver for the i-th component of the transmitted signal: $\gamma_i = p_i \underline{h}_i^H \left(\mathbf{\Sigma}_z + \sum_{j\neq i} p_j \underline{h}_j \underline{h}_j^H\right)^{-1} \underline{h}_i$. We can upper bound the energy efficiency function as follows:

$$\begin{aligned} \xi(\underline{p}, R) &= \frac{RL}{M} \frac{\sum_{i=1}^{n_t} f(\gamma_i)}{\sum_{i=1}^{n_t} \gamma_i} \frac{\sum_{i=1}^{n_t} \gamma_i}{\sum_{i=1}^{n_t} p_i} \\ &\leq \frac{RLf(\gamma^*)}{M\gamma^*} \frac{\sum_{i=1}^{n_t} p_i \underline{h}_i^H \left(\mathbf{\Sigma}_z + \sum_{j\neq i} p_j \underline{h}_j \underline{h}_j^H\right)^{-1} \underline{h}_i}{\sum_{i=1}^{n_t} p_i} \\ &\leq \frac{RLf(\gamma^*)}{M\gamma^*} \underline{h}_k^H \mathbf{\Sigma}_z^{-1} \underline{h}_k, \end{aligned}$$

where $k = \arg\max_{j\in\{1,\ldots,n_t\}} \underline{h}_j^H \mathbf{\Sigma}_z^{-1} \underline{h}_j$. The first inequality follows from $\frac{\sum_{i=1}^{n_t} f(\gamma_i)}{\sum_{i=1}^{n_t} \gamma_i} \leq \frac{f(\gamma^*)}{\gamma^*}$ (see [6] for a detailed proof). The second inequality comes from the fact that

$$\begin{aligned} \underline{h}_i^H \left(\mathbf{\Sigma}_z + \sum_{j\neq i} p_j \underline{h}_j \underline{h}_j^H\right)^{-1} \underline{h}_i &\leq \underline{h}_i^H \mathbf{\Sigma}_z^{-1} \underline{h}_i \\ &\leq \underline{h}_k^H \mathbf{\Sigma}_z^{-1} \underline{h}_k, \end{aligned} \qquad (7)$$

which follows from $\mathbf{\Sigma}_z^{-1} \succeq \left(\mathbf{\Sigma}_z + \sum_{j\neq i} p_j \underline{h}_j \underline{h}_j^H\right)^{-1}$.

It turns out that using the following PA scheme:

$$p_i^* = \left| \begin{array}{ll} \min\left\{\overline{P}, \frac{\gamma^*}{\underline{h}_k^H \mathbf{\Sigma}_z^{-1} \underline{h}_k}\right\}, & \text{if } \ i = k, \\ 0, & \text{otherwise.} \end{array} \right. \qquad (8)$$

this upper bound can be achieved.

Thus, the energy efficiency function is maximized whenever the transmitter chooses to transmit only on its best channel $k = \arg\max_{j\in\{1,\ldots,n_t\}} \underline{h}_j^H \mathbf{\Sigma}_z^{-1} \underline{h}_j$ with the power that achieves the target SINR equal to γ^* (which maximizes the function $\frac{f(\gamma)}{\gamma}$). We see that this solution is very similar to the one in [6].

It can be checked that the analysis in this subsection extends easily to any linear receiver such that $p_i \frac{\partial \gamma_i}{\partial p_i} = \gamma_i$, $\forall i \in \{1, \ldots, n_t\}$. The optimal solution is again beamforming in the best direction such that $k = \arg\max_{j\in\{1,\ldots,n_t\}} \frac{\partial \gamma_j}{\partial p_j}$:

$$p_i^* = \left| \begin{array}{ll} \min\left\{\overline{P}, \frac{\gamma^*}{\frac{\partial \gamma_i}{\partial p_i}}\right\}, & \text{if } \ i = k, \\ 0, & \text{otherwise.} \end{array} \right. \qquad (9)$$

In conclusion, when independent information is sent over the transmit antennas and assuming a linear receiver, the optimal PA policy that maximizes the energy efficiency function in (6) is beamforming in the direction that requires the minimal power to achieve the target SINR that maximizes the function $\frac{f(\gamma)}{\gamma}$.

3.0.2 *Case of Full Diversity Mode*

Now, we assume that the transmitter wants to maximize the diversity gain and, thus, sends the same information over all the transmit antennas such that $x_i = \sqrt{p_i} x_0$, for all $i \in \{1, \ldots, n_t\}$. The baseband received signal can be written as:

$$\underline{y} = \underline{\tilde{h}} x_0 + \underline{z},$$

where $\underline{\tilde{h}} = \sum_{i=1}^{n_t} \sqrt{p_i} \underline{h}_i$ and $\underline{h}_i$ is the i-th column of the channel matrix $\mathbf{H}$. The received SNR at the output of the matched filter (or the MRC receiver) is:

$$\begin{aligned} \gamma_{\text{MRC}} &= \underline{\tilde{h}}^H \mathbf{\Sigma}_z^{-1} \underline{\tilde{h}} && (10) \\ &= \sum_{i=1}^{n_t} \sum_{j=1}^{n_t} \sqrt{p_i} \sqrt{p_j} \underline{h}_i^H \mathbf{\Sigma}_z^{-1} \underline{h}_j. && (11) \end{aligned}$$

The energy efficiency function we want to maximize is $\xi(\underline{p}, R) = \frac{RL}{M} \frac{f(\gamma_{\text{MRC}})}{\sum_{i=1}^{n_t} p_i}$, under the power constraint $\sum_{i=1}^{n_t} p_i \leq \overline{P}$. The problem is difficult to be solved analytically except for a particular case where $n_r = n_t = n$, $\mathbf{H} = \text{diag}(h_{11}, \ldots, h_{nn})$ and $\mathbf{\Sigma}_z = \text{diag}(\sigma_1^2, \ldots, \sigma_n^2)$. We observe that this is the dual case of the one studied in [6] (where the transmitter sends independent information over the parallel sub-channels). The SNR at the output of the matched filter is: $\gamma_{\text{MRC}} = \sum_{i=1}^{n} \frac{|h_{ii}|^2 p_i}{\sigma_i^2}$ and the objective function becomes:

$$\begin{aligned} \xi(\underline{p}, R) &= \frac{RLf(\gamma_{\text{MRC}})}{M\gamma_{\text{MRC}}} \frac{\sum_{i=1}^{n} \frac{|h_{ii}|^2 p_i}{\sigma_i^2}}{\sum_{i=1}^{n_t} p_i} \\ &\leq \frac{RLf(\gamma^*)}{M\gamma^*} \frac{|h_{kk}|^2}{\sigma_k^2}, \end{aligned} \qquad (12)$$

where $k = \arg\max_{j\in\{1,\ldots,n\}} \frac{|h_{jj}|^2}{\sigma_j^2}$. The following PA scheme achieves the given upper bound:

$$p_i^* = \left| \begin{array}{ll} \min\left\{n_t\overline{P}, \frac{\gamma^* \sigma_k^2}{|h_{kk}|^2}\right\}, & \text{if } \ i = k, \\ 0, & \text{otherwise,} \end{array} \right. \qquad (13)$$

which corresponds to choosing the link with the best output SNR and to transmit with a power that achieves γ^* which is exactly the same solution as the one obtained in the dual case.

4. NEW ENERGY EFFICIENCY FUNCTION

In this section, we assume that only the receiver has perfect channel state information and that the noise covariance matrix is proportional to the identity matrix $\mathbf{\Sigma}_z = \sigma^2\mathbf{I}$ unless otherwise specified.

4.1 Arbitrary Number of Antennas

In what follows we propose a information theoretical performance metric based on the outage probability [7]. The probability of an outage is defined as the event that the mutual information of the instantaneous channel does not support a target data rate R.

We propose the following energy efficiency function:

$$\xi(\mathbf{Q},R) = \frac{RL}{M}\frac{1-\mathrm{P}_{\mathrm{out}}(\mathbf{Q},R)}{\mathrm{Tr}(\mathbf{Q})}, \quad (14)$$

where $\mathrm{P}_{\mathrm{out}}(\mathbf{Q},R) = \Pr\left[\log|\mathbf{I}+\rho\mathbf{H}\mathbf{Q}\mathbf{H}^H| < R\right]$ is the outage probability. The objective is to find the optimal covariance matrix $\mathbf{Q}$ that maximizes the energy efficiency function $F(\mathbf{Q},R)$. Similarly $\xi(\mathbf{Q},R)$ represents the number of successfully transmitted nats per Joules of consumed energy.

Let us consider the spectral decomposition of the covariance matrix, $\mathbf{Q} = \mathbf{U}\mathbf{P}\mathbf{U}^H$ ($\mathbf{U}$ is a unitary matrix the columns of which are the eigenvectors of $\mathbf{Q}$ and $\mathbf{P}$ is a diagonal matrix the entries of which are the eigenvalues of $\mathbf{Q}$). We start by studying the optimal choice of eigenvectors and we obtain the following result.

THEOREM 4.1. There is no loss of optimality in the maximization problem: $\max_{\mathbf{Q}:\mathrm{Tr}(\mathbf{Q})\leq\overline{P}} \xi(\mathbf{Q},R)$, with $\xi(\mathbf{Q},R)$ given in (14) by choosing the eigenvectors matrix $\mathbf{U} = \mathbf{I}$.

PROOF. From [1] we know that the distribution of $\mathbf{HU}$ remains unchanged. Knowing also that $\mathrm{Tr}(\mathbf{Q}) = \mathrm{Tr}(\mathbf{P})$ the objective function $\xi(\mathbf{Q},R) = \xi(\mathbf{P},R)$ and the constraint $\mathrm{Tr}(\mathbf{P}) \leq n_t\overline{P}$ do not depend on the eigenvector basis $\mathbf{U}$. We can then choose $\mathbf{U} = \mathbf{I}$ without loss of optimality. □

The objective function becomes:

$$\xi(\mathbf{P},R) = \frac{RL}{M}\frac{1-\Pr\left[\log|\mathbf{I}+\rho\mathbf{H}\mathbf{P}\mathbf{H}^H| < R\right]}{\mathrm{Tr}(\mathbf{P})}, \quad (15)$$

which we want to maximize w.r.t. $\mathbf{P} = \mathrm{diag}(p_1,\ldots,p_{n_t}) = \mathrm{diag}(\underline{p})$ under the power constraint $\mathrm{Tr}(\mathbf{P}) \leq \overline{P}$. The general optimization of $\xi(\mathbf{P},R)$ w.r.t. $\mathbf{P}$ is difficult. However, under the assumption that $\mathbf{P}$ is not proportional to the identity matrix, we can prove that the optimal solution is different than the trivial one $\mathbf{P}^* \neq \mathbf{0}$. Even the problem of finding the optimal diagonal matrix $\mathbf{P}$ that minimizes the outage probability is still an open problem. In [1], Telatar conjectures that, for a Rayleigh fading channel, the optimal PA policy that minimize the outage probability is the uniform power policy, but only over a certain number of transmit dimensions.

CONJECTURE 4.2. There exists a non trivial solution ($\mathbf{P}^* \neq \mathbf{0}$) to the aforementioned energy efficiency optimization problem in (15).

In what follows, we study some particular cases where the distribution of the mutual information is known and explicit expressions of the outage probability can be derived.

4.2 Particular Case: $n_r = n_t = 1$ (SISO)

In this section, we assume that the terminals have a single antenna. We want to check whether the maximization of $\xi(p,R) = \frac{1-\Pr[\log(1+\rho|h|^2p)<R]}{p}$, under the power constraint $p \leq \overline{P}$ leads to the trivial solution $p^* = 0$. Knowing that $h \sim \mathcal{CN}(0,1)$ then $|h|^2 \sim \mathrm{expon}(1)$ is an exponential distributed random variable and the outage probability is easily calculated in this case:

$$\xi(p,R) = \frac{RL}{M}\frac{\exp(-\frac{e^R-1}{\rho p})}{p}.$$

First we study the function $\xi(p,R)$ for $p \in (0,+\infty)$, without any constraints. We denote by $c = \frac{e^R-1}{\rho} > 0$ (assuming $R > 0$). We have $\lim_{p\to 0}\xi(p,R) = 0$, $\lim_{p\to\infty}\xi(p,R) = 0$. We can evaluate the first derivative $\frac{\partial\xi}{\partial p}(p,R) = \frac{RL}{M}\frac{(c-p)\exp(-\frac{c}{p})}{p^3}$. It is straightforward to observe that the function $\xi(p,R)$ is increasing until $p^* = c$ and then is decreasing and the maximum is achieved for $p^* = c$. Considering the power constraint the optimal transmission power is $p^* = \min\{\frac{e^R-1}{\rho}, \overline{P}\}$.

4.3 Particular Case: $n_r = 1$, $n_t = 2$

In this particular case also, the outage probability $\Pr[\log(1+\rho|h_1|^2p_1+\rho|h_2|^2p_2) < R]$ can be calculated explicitly (see [9]) knowing that $|h_1|^2 \sim \mathrm{expon}(1)$, $|h_2|^2 \sim \mathrm{expon}(1)$.

4.3.1 Uniform Power Allocation Problem

We assume the uniform power allocation policy, $p_1 = p_2 = p$ the energy efficiency function writes as:

$$\xi(p,R) = \frac{RL}{M}\left(1+\frac{e^R-1}{\rho p}\right)e^{-\frac{e^R-1}{\rho p}}\frac{1}{2p}.$$

We proceed in a similar manner as in the previous case. We have $\lim_{p\to 0}\xi(p,R) = 0$, $\lim_{p\to\infty}\xi(p,R) = 0$. We can evaluate the first derivative $\frac{\partial\xi}{\partial p}(p,R) = \frac{RL}{M}\frac{1}{p^2}\left(-1-\frac{c}{p}+\frac{c^2}{p^2}\right)\exp(-\frac{c}{p})$ where c is the same constant. Thus, the function $\xi(p,R)$ is increasing in the interval $(0,p*]$ and decreasing in $(p^*,+\infty)$ for $p^* = \frac{2c}{1+\sqrt{5}}$ and then is decreasing and the maximum is achieved for $p^* = \frac{2c}{1+\sqrt{5}}$. Considering the power constraint the optimal transmission power is $p^* = \min\{\frac{2c}{1+\sqrt{5}}, \frac{\overline{P}}{2}\}$. Note that the function $f(p) = \left(1+\frac{c}{p}\right)\exp(-\frac{c}{p})$ has a unique inflexion point $\hat{p} = \frac{c}{3}$ and is a sigmoidal-shaped function w.r.t. p. Thus, in this particular case ($n_r = 1$, $n_t = 2$) also, the proposed proof for Conjecture 4.2 is valid.

4.3.2 General Power Allocation Problem

In this subsection, we no longer assume the uniform power allocation policy. The outage probability can be expressed in this case also [9] and the energy efficiency function in (15) writes as:

$$\xi(p,R) = \frac{RL}{M}\frac{f(p_1,p_2)}{p_1+p_2},$$

where $f(p_1,p_2)$ is a continuous function expressed by:

$$f(p_1,p_2) = \left| \begin{array}{ll} \frac{p_1\exp(-\frac{c}{p_1})-p_2\exp(-\frac{c}{p_2})}{p_1-p_2}, & \text{if } p_1 \neq p_2, \\ (1+\frac{c}{p})\exp(-\frac{c}{p}), & \text{if } p_1 = p_2 = p, \end{array} \right. \quad (16)$$

where c is the same parameter as the one in the previous cases.

In [9], the authors solved the conjecture of Telatar [1] assuming $n_r = 1$ and $n_t = 2$ and the noise variance $\sigma^2 = \frac{1}{\rho} = 1$. This result can be straightforwardly extended to the case where the noise variance is arbitrary to obtain the explicit solution of the minimization of the outage probability:

$$(p_1^*,p_2^*) = \left\{ \begin{array}{ll} \frac{\overline{P}}{2}(1,1), & \text{if } \eta \leq \eta_0, \\ \overline{P}(1,0), & \text{otherwise}, \end{array} \right. \quad (17)$$

where $\eta = \frac{c}{\overline{P}}$ and $\eta_0 \simeq 1.2564$. For fixed R and ρ, we observe that in the low power regime ($\overline{P} \leq \frac{c}{\eta_0}$) the beamforming solution is optimal and in the high power regime ($\overline{P} > \frac{c}{\eta_0}$) the uniform power allocation is optimal. In the case, where the beamforming solution is optimal and because of the problem symmetry, the transmitter can choose to transmit on either of its antennas ($\overline{P}(1,0)$ or $\overline{P}(0,1)$).

Extending this result to the optimization of the energy efficiency function in (16) is very difficult. When minimizing the outage probability saturating the power constraint is optimal [9] and the problem simplifies to a single parameter optimization. In our case, even for the SISO case saturating the power constraint is always not optimal and, thus, we cannot simplify the problem by choosing $p_2 = \overline{P} - p_1$. We conjecture the following result that has been validated through many simulations:

Conjecture 4.3. For any rate R and SNR ρ and power constraint $\overline{P}$ the PA policy that maximizes the energy efficiency function in (16) is given by:

$$(p_1^*, p_2^*) = \left| \begin{array}{ll} \min\{c, \overline{P}\}(1,0) & \text{if } \eta \geq \eta_0 \\ \min\left\{\frac{2c}{1+\sqrt{5}}, \frac{\overline{P}}{2}\right\}(1,1) & \text{otherwise} \end{array} \right. \quad (18)$$

Again we observe that in low power regime $\eta \geq \eta_0$ then beamforming with the optimal transmission power that maximizes the energy efficiency function in the SISO case (Sec. 4.2) is optimal. If $\eta \leq \eta_1$, the uniform power allocation given in the previous subsection (Sec. 4.3.1) is optimal. One of the main differences with the optimal power allocation policy w.r.t. the outage probability only in [9] is the fact that in order to maximize the energy-efficiency function it is not always optimal to saturate the power constraint.

4.4 Particular Case: $n_r = n_t = 2$

In this subsection, we assume the uniform PA scheme at the transmitter ($p_1 = p_2 = p$) and we exploit the fact that the distribution of the mutual information can be well approximated by a Gaussian random variable even in the case of low number of antennas to try to prove In [10], the authors give the analytical expression of the moment generating function of the mutual information for the Rayleigh channel model and assuming uniform power allocation over the transmit antennas. They show that the exact distribution can be well approximated by a Gaussian distribution with the same mean and variance, even for a finite number of antennas. The general case is analytically intractable and, thus, for simplicity reasons, we assume here that $n_r = n_t = 2$.

$$\log\left|I + \rho p \mathbf{H}\mathbf{H}^H\right| \rightarrow \mathcal{N}(\mu_I, \sigma_I^2),$$

where $\mu_I = \int_0^\infty \log(1+\rho p\lambda)\, e^{-\lambda}[L_0^2(\lambda)+L_1^2(\lambda)]\mathrm{d}\lambda$, and $L_0(\lambda) = 1$, $L_1(\lambda) = 1-\lambda$ are the Laguerre polynomials of order 0 and 1. The variance is given by

$$\sigma_I^2 = \int_0^\infty \log^2(1+\rho p\lambda)\, e^{-\lambda}[L_0^2(\lambda) + L_1^2(\lambda)]\mathrm{d}\lambda$$
$$- \int_0^\infty \int_0^\infty \log(1+\lambda_1\rho p)\log(1+\lambda_2\rho p)$$
$$e^{-(\lambda_1+\lambda_2)}\left[L_0(\lambda_1)L_0(\lambda_2) + L_1(\lambda_1)L_1(\lambda_2)\right]^2 \mathrm{d}\lambda_1\mathrm{d}\lambda_2.$$

We further obtain:

$$\begin{array}{rcl} \mu_I & = & 2J_0 - 2J_1 + J_2, \\ \sigma_I^2 & = & 2K_0 - 2K_1 + K_2 + 4J_0^2 + 6J_1^2 + J_2^2 \\ & & -8J_0J_1 + 2J_0J_2 - 4J_2J_1, \end{array} \quad (19)$$

where, for all $i \geq 0$, $J_i = \int_0^{+\infty} \log(1+\rho p\lambda)\, e^{-\lambda}\lambda^i \mathrm{d}\lambda$ and $K_i = \int_0^{+\infty} \log^2(1+\rho p\lambda)\, e^{-\lambda}\lambda^i\mathrm{d}\lambda$.

In this case, the mean and variance can be written in function of the exponential integral $\mathrm{Ei}(x) = \int_{-\infty}^{x} \frac{e^t}{t}\mathrm{d}t$ and of the generalized exponential integral $\mathrm{E}_1^{(2)}(x) = \int_1^\infty \frac{e^{-xt}\log(t)}{t}\mathrm{d}t$ as follows. It can be checked that $J_0 = -e^{1/(\rho p)}\mathrm{Ei}\left(-\frac{1}{\rho p}\right)$, $J_1 = 1 + \left(1-\frac{1}{\rho p}\right)J_0$, $J_2 = 2J_1 + 1 - \frac{1}{\rho p} + \left(\frac{1}{\rho p}\right)^2 J_0$. Also, we have $K_0 = 2e^{1/(\rho p)}\mathrm{E}_1^{(2)}\left(-\frac{1}{\rho p}\right)$, $K_1 = K_0\left(1-\frac{1}{\rho p}\right) + 2J_0$, $K_2 = 2K_1 + 2J_1 - 2\frac{1}{\rho p}J_0 + \left(\frac{1}{\rho p}\right)^2 K_0$. We obtain $\mu_I = 1 - \frac{1}{\rho p} - \left(2 + \frac{1}{(\rho p)^2}\right)\mathrm{Ei}\left(-\frac{1}{\rho p}\right)e^{1/(\rho p)}$. Having assumed the Gaussian approximation of the mutual information, the outage probability can be explicitly derived and the energy efficiency function in (15) becomes $\xi(p,R) = \frac{RL}{M}\frac{Q\left(\frac{R-\mu_I}{\sigma_I}\right)}{2p}$, where $Q(\cdot)$ is the error function defined as $Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{+\infty} e^{-t^2/2}\mathrm{d}t$.

Conjecture 4.4. The function $Q\left(\frac{R-\mu_I}{\sigma_I}\right)$, where μ_I and σ_I^2 are given in (19) is a sigmoidal shaped function of p.

Once the conjecture is proven using [4] we obtain directly the existence of a non trivial solution $p^* > 0$.

5. NUMERICAL RESULTS

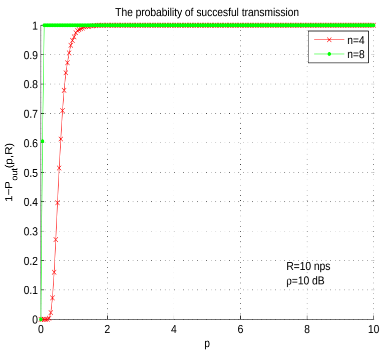
In this section, we assume that $L = M$ and thus no error correcting code is used.

We start by assuming the uniform power allocation policy and $n_r = n_t = n$. In Fig. 1(a), we trace the probability of successful transmission, and, in Fig. 1(b), the energy efficiency function for the scenario: $n_r = n_t = n$, $R = 10$nps (nats per second), $\rho = 10$dB. We observe that the outage probability is a sigmoidal-shaped function for both $n \in \{4, 8\}$ and, thus, there exist non trivial solution to the maximization of the energy efficiency function. The Conjecture 4.2 has been validated through simulations in many other scenarios.

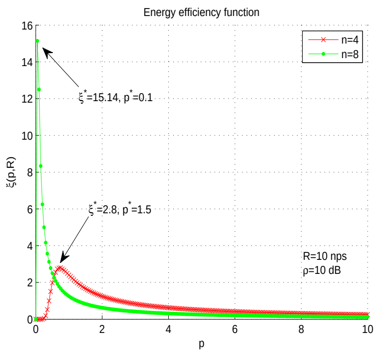
Let us now consider the general power allocation problem for the case where $n_r = 1$ and $n_t = 2$ (see Sec. 4.3). In Fig. 2(a) we plot the probability of successful transmission and in Fig. 2(b) the energy efficiency function for the scenario: $R = 5$nps, $\rho = 10$dB in function of the power constraint at the transmitter. We compare the optimal PA policy, the uniform PA policy and the beamforming PA policy for both metrics. We observe that in the low power regime ($\overline{P} \leq \frac{c}{\eta_0}$) both metrics are optimized by using beamforming PA policies and in the high power regime ($\overline{P} > \frac{c}{\eta_0}$) both metrics are optimized by using uniform PA policies. Thus, Conjecture 4.3 is verified by this simulation but also by many others. The main difference between the optimal PA policies that maximize the two metrics is that when optimizing the probability of successful transmission the power constraint is saturated while when optimizing the energy efficiency metric this is not always the case.

6. CONCLUSION

In this paper, the quasi-static MIMO Rayleigh channel was considered and the optimal way in which the transmitter shares its available power among its antennas in order

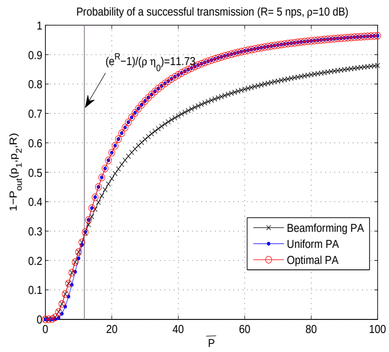


(a)

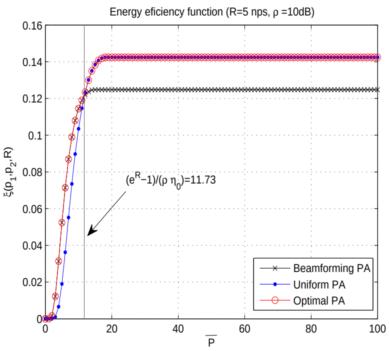


(b)

Figure 1: Uniform PA $n_r = nt = n$ for the scenario $R =$ 10nps, $\rho = 10$dB. (a) The function $1 - Pout(p, R)$ is a sigmoidal-shaped function w.r.t. p. (b) There is a non trivial solution $p^* > 0$.



(a)



(b)

Figure 2: Optimal PA vs. Uniform PA vs. Beamforming PA, $n_r = 1$, $n_t = 2$ for the scenario $n_r = 1$, $n_t = 2$. The success probability (a) and the energy efficiency function (b) w.r.t. the available power $\overline{P}$.

to maximize the energy efficiency was studied. We first proposed and solved several extensions of the known results to the MIMO channel and then introduced a new energy efficiency metric based on the outage probability. It turns out that the eigenvectors of the covariance matrix that maximizes the proposed metric are equal to the canonic vectors. In what the eigenvalues of the optimal covariance matrix are concerned, we conjectured that a non-trivial solution exists. For several particular but interesting cases a detailed analysis of this conjecture proofs was also provided.

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