

Another Look at the Brachistochrone Problem

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If A and B are two points in the plane with B lower and to the right of A , then we might be tempted to think that an object falling under the influence of gravity from A would arrive at B most rapidly if it followed the trajectory of the segment joining A to B . Galileo considered this problem and conjectured that a circular arc would give a superior result. Other scientists over a long period, for example Johann and Jakob Bernouilli, Euler and Newton, considered the problem and this eventually gave rise to the calculus of variations. A solution to this problem is called a brachistochrone. In this article we aim to consider a mathematical problem arising from this question, which we will call the mathematical brachistochrone problem or m.b.p. for short. (The origin of this problem may be found in various places, in particular, in [2].)

We will write $C([a, b])$ for the real vector space of real-valued continuous functions defined on the closed interval $[a, b]$. The expression

$$\|\gamma\| = \sup_{t \in [a, b]} |\gamma(t)|$$

defines a norm on $C([a, b])$ and, with this norm, $C([a, b])$ is a Banach space. We take two strictly positive numbers b and β and consider the following optimisation problem:

$$\min \int_0^b \left(\frac{1 + \gamma'^2(t)}{\gamma(t)} \right)^{\frac{1}{2}} dt,$$

where $\gamma \in C([0, b])$, $\gamma(0) = 0$, $\gamma(b) = \beta$ and γ is strictly positive and continuously differentiable on $(0, b]$. Often there is a constant before the integral sign; however, it plays no role in the analysis of the problem, so we can neglect it. It should be noticed that the function under the integral sign is not defined at 0 and so the integral is an improper integral. Hence we need to add the condition that the integral is defined. Although this problem is well-known, it is difficult to find a complete, rigorous discussion of it. The aim of this article is to present such a discussion.

1 Preliminaries

We will suppose that all vector spaces are real. Let E be a vector space and f a real-valued function defined on a non-empty subset X of E . Suppose that $v \in E$ and that there exists $\epsilon > 0$ such that the segment $[x - \epsilon v, x + \epsilon v]$ is contained in X . If the limit

$$\lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists, then we call this limit the directional derivative of f at x in the direction v and we write $\partial_v f(x)$ for this limit. The directional derivative is always defined for the vector 0, but not necessarily for other

vectors; however, if it is defined for a certain v , then it is also defined for λv , for any $\lambda \in [0, 1]$. The directions v for which $\partial_v f(x)$ is defined are called $(X-)$ admissible directions for f at x . If x is an extremum (minimum or maximum) of f , then $\partial_v f(x) = 0$ in all admissible directions for f at x . If E is a normed vector space, then this result is also true for local extrema.

Let f be a real-valued function defined on a subset X of a vector space E and suppose that, if x and $x + v$ belong to X , then the directional derivative $\partial_v f(x)$ is defined and

$$f(x + v) - f(x) \geq \partial_v f(x).$$

Then we say that f is convex on X . If we have equality only if $v = 0$, then f is said to be strictly convex. Clearly, if f is convex and $\partial_v f(x) = 0$, for all v such that $x + v \in X$, then x is a minimum, which is unique if f is strictly convex.

Remark Usually we define a convex function to be a real-valued function f defined on a convex set X such that

$$f(x + \lambda v) \leq (1 - \lambda)f(x) + \lambda f(x + v),$$

whenever x and $x + v$ belong to X and $\lambda \in [0, 1]$ and we say that f is strictly convex if we have equality only if $v = 0$. It can be shown that these definitions are equivalent if the directional derivative $\partial_v f(x)$ is defined when x and $x + v$ belong to X . Thus the definitions here generalize the usual definitions to sets which are not necessarily convex.

It is often difficult to determine whether a function is convex or not. The following elementary result, a proof of which may be found in [1], is very useful in this direction:

Proposition 1.1 *Let O be an open subset of \mathbb{R}^n , $X \subset O$ and $f : O \rightarrow \mathbb{R}$ of class C^2 . Then*

- **a.** *f is convex on X if and only if the Hessian matrix of f is positive for all $x \in X$;*
- **b.** *f is strictly convex on X if the Hessian matrix of f is positive definite for all $x \in X$.*

The next elementary result, due to du Bois-Reymond, is fundamental in the calculus of variations. Let us write $C^1([a, b])$ for the subspace of $C([a, b])$ composed of C^1 functions.

Theorem 1.1 *If $f \in C([a, b])$ and $\int_a^b f(t)v'(t)dt = 0$ for all functions $v \in C^1([a, b])$ such that $v(a) = v(b) = 0$, then f is a constant function.*

PROOF Let $c = \frac{1}{b-a} \int_a^b f(t)dt$ and let us set $v(s) = \int_a^s (f(t) - c)dt$. Then $v \in C^1([a, b])$, $v(a) = v(b) = 0$ and $v'(s) = f(s) - c$. Also

$$0 \leq \int_a^b (f(t) - c)^2 dt = \int_a^b (f(t) - c)v'(t)dt = \int_a^b f(t)v'(t)dt - cv(x)|_a^b = 0.$$

As $f(t) - c$ is continuous, $f(t) - c = 0$, for all t , and the result follows. □

Corollary 1.1 *If $f, g \in C([a, b])$ and*

$$\int_a^b f(t)v(t) + g(t)v'(t)dt = 0,$$

for all functions $v \in C^1([a, b])$ such that $v(a) = v(b) = 0$, then $g \in C^1([a, b])$ and $g' = f$.

PROOF For $s \in [a, b]$, let us set $F(s) = \int_a^s f(t)dt$. Then $F \in C^1([a, b])$ and $F'(s) = f(s)$. As

$$\int_a^b f(t)v(t)dt = F(t)v(t)|_a^b - \int_a^b F(t)v'(t)dt = - \int_a^b F(t)v'(t)dt,$$

we have

$$0 = \int_a^b f(t)v(t) + g(t)v'(t)dt = \int_a^b (g(t) - F(t))v'(t)dt.$$

From the theorem, there is a constant $c \in \mathbb{R}$ such that $g(t) - F(t) = c$, or $g(t) = F(t) + c$. Therefore $g = F + c \in C^1[a, b]$ and $g' = F' = f$. \square

2 Extrema of functions defined by a integral

Suppose that L is a C^1 real-valued function defined on an open subset $O \subset \mathbb{R}^2$ and that γ is a real-valued, C^1 function defined on a closed interval $\bar{I} = [a, b]$. We also assume that $(\gamma(t), \gamma'(t)) \in O$, for all $t \in \bar{I} = [a, b]$. We set

$$\mathcal{L}(\gamma) = \int_a^b L(\gamma(t), \gamma'(t))dt.$$

To simplify the notation, we will write $[\gamma(t)]$ for $(\gamma(t), \gamma'(t))$ and so we may write

$$\mathcal{L}(\gamma) = \int_a^b L[\gamma(t)]dt.$$

The function L is called a Lagrangian (function).

We now fix $\alpha, \beta \in \mathbb{R}$ and write X for the subset of $C([a, b])$ composed of those γ such that $\gamma(a) = \alpha$, $\gamma(b) = \beta$ and $(\gamma(t), \gamma'(t)) \in O$, for all $t \in \bar{I} = [a, b]$. We propose to look for a necessary condition for γ to be an extremum of \mathcal{L} on X . To do so, we first find the admissible directions v and an expression for the directional derivative $\partial \mathcal{L}_v(\gamma)$. Clearly, if v is an admissible direction, then $v \in C^1([a, b])$ and $v(a) = v(b) = 0$. In fact, all such functions v are admissible directions, as we will now see. For s small, $\gamma + sv \in X$ and

$$\lim_{s \rightarrow 0} \frac{\mathcal{L}(\gamma + sv) - \mathcal{L}(\gamma)}{s} = \frac{\partial \mathcal{L}}{\partial s}(\gamma + sv)|_{s=0}.$$

We have

$$\mathcal{L}(\gamma + sv) = \int_a^b L[(\gamma + sv)(t)]dt.$$

Given the continuity of the integrand with respect to s , the derivative $\frac{\partial \mathcal{L}}{\partial s}(\gamma + sv)$ exists for small s and so v is an admissible direction. To obtain an expression for the directional derivative $\partial_v \mathcal{L}(\gamma)$, we differentiate with respect to s :

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial s}(\gamma + sv) &= \int_a^b \frac{\partial L}{\partial s}[(\gamma + sv)(t)]dt \\ &= \int_a^b \frac{\partial L}{\partial x}[(\gamma + sv)(t)]v(t) + \frac{\partial L}{\partial y}[(\gamma + sv)(t)]v'(t)dt. \end{aligned}$$

As the integrand is continuous with respect to s , we obtain

$$\partial_v \mathcal{L}(\gamma) = \int_a^b \frac{\partial L}{\partial x}[\gamma(t)]v(t) + \frac{\partial L}{\partial y}[\gamma(t)]v'(t)dt.$$

Thus we have shown that all $v \in C^1([a, b])$ such that $v(a) = v(b) = 0$ are admissible directions and we have found an expression for the directional derivative $\partial \mathcal{L}_v(\gamma)$ for any such v .

If γ is an extremum and $v \in C^1[a, b]$ is such that $v(a) = v(b) = 0$, then $\partial_v \mathcal{L}(\gamma) = 0$, and so from Corollary 1.1, we obtain

$$\frac{\partial L}{\partial x}[\gamma(t)] = \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)], \quad (1)$$

for $t \in [a, b]$. This equation is known as the Euler-Lagrange equation. Functions which satisfy the Euler-Lagrange equation on some interval are referred to as stationary functions (or extremals). Such functions may or may not be extrema, or even local extrema.

3 Extrema of functions defined by an improper integral

In the previous section, we supposed that the pair $(\gamma(t), \gamma'(t))$ was defined for all $t \in \bar{I}$ and that $(\gamma(t), \gamma'(t)) \in O$, the domain of L , for all $t \in \bar{I}$. These assumptions are too restrictive to handle the problem which interests us. However, if we slightly relax the conditions, we still obtain the Euler-Lagrange equation for an extremum.

Let I and J be open intervals of \mathbb{R} , where $I = (c, d)$ with $c \in \mathbb{R}$, and $O = I \times J$. We suppose that L is a C^1 real-valued function defined on O and that $\gamma \in C([a, b])$ is continuously differentiable on $(a, b]$, with $(\gamma(t), \gamma'(t)) \in O$, for all $t \in (a, b]$. If we set

$$\mathcal{L}(\gamma) = \int_a^b L(\gamma(t), \gamma'(t)) dt = \int_a^b L[\gamma(t)] dt,$$

then $\mathcal{L}(\gamma)$ is an improper integral which may or may not be defined. Let $\alpha, \beta \in \mathbb{R}$. We will write X for the subset of $C([a, b])$ composed of those γ which are continuously differentiable on $(a, b]$, with $(\gamma(t), \gamma'(t)) \in O$, for all $t \in (a, b]$, and such that $\gamma(a) = \alpha$, $\gamma(b) = \beta$ and $L(\gamma)$ is defined. (Notice that the m.b.p. is of this form, with $(c, d) = (0, \infty)$, $J = \mathbb{R}$ and $a = \alpha = 0$.)

Theorem 3.1 *If γ is an extremum of \mathcal{L} on X , then γ satisfies the Euler-Lagrange equation on $(a, b]$.*

PROOF Suppose that $v \in C^1([a, b])$ and $v(b) = 0$. In addition suppose that there exists $c \in (a, b)$ such that v vanishes on $[a, c]$. Then it is easy to see that v is an admissible direction of \mathcal{L} at γ , for any $\gamma \in X$, and

$$\partial_v \mathcal{L}(\gamma) = \int_c^b \frac{\partial L}{\partial x}[\gamma(t)] v(t) + \frac{\partial L}{\partial y}[\gamma(t)] v'(t) dt.$$

The restriction of v to $[c, b]$ belongs to $C^1([c, b])$. Therefore, if γ is an extremum and $u \in C^1([c, b])$ is the restriction of v to $[c, b]$, then we have

$$\int_c^b \frac{\partial L}{\partial x}[\gamma(t)] u(t) + \frac{\partial L}{\partial y}[\gamma(t)] u'(t) dt = 0.$$

We would like to show that this is the case for all elements of $C^1([c, b])$, with $u(c) = u(b) = 0$. However, not all members u of $C^1([c, b])$, with $u(c) = u(b) = 0$, are such restrictions. This will be the case if and only if $u'(c) = 0$. Nevertheless the equality does apply in other cases.

Let $u \in C^1([c, b])$, with $u(c) = u(b) = 0$, and suppose that $u'(c) = \delta > 0$. We take $\epsilon \in (0, 1]$ such that $d = c - \epsilon > a$ and define a real-valued function g on $[a, c]$ in the following way: g has the value 0 on $[a, d]$, g restricted to $[d, d + \frac{\epsilon}{2}]$ is a "hat" function with slopes δ and $-\delta$, and g restricted to $[d + \frac{\epsilon}{2}, c]$ is an "inverted hat" function with slopes $-\delta$ and δ . If we set

$$v(t) = \begin{cases} \int_a^t g(s) ds & t \in [a, c] \\ u(t) & t \in [c, b], \end{cases}$$

then v is a C^1 function extending u to $[a, b]$, such that v has the value 0 on the interval $[a, d]$; hence v is an admissible direction for \mathcal{L} at γ . In addition, on the interval $[d, c]$, $|v'(t)| = \delta$ and $|v(t)| \leq \epsilon \delta \leq \delta$ and so

$$\left| \int_d^c \frac{\partial L}{\partial x}[\gamma(t)] v(t) + \frac{\partial L}{\partial y}[\gamma(t)] v'(t) dt \right| \leq \delta \int_d^c \left| \frac{\partial L}{\partial x}[\gamma(t)] \right| + \left| \frac{\partial L}{\partial y}[\gamma(t)] \right| dt,$$

which converges to 0, when ϵ converges to 0. It now follows that

$$\int_c^b \frac{\partial L}{\partial x}[\gamma(t)]u(t) + \frac{\partial L}{\partial y}[\gamma(t)]u'(t)dt = 0.$$

If $u'(c) < 0$, then we can use an analogous argument to obtain the same result. If we now apply Corollary 1.1, we see that γ satisfies the Euler-Lagrange equation on $[c, b]$. As c was chosen arbitrarily in the interval (a, b) , γ satisfies the Euler-Lagrange equation on (a, b) . \square

The above result gives us a necessary condition for γ to be a minimum, but not a sufficient condition. However, if we add some assumptions, then this condition becomes sufficient. Suppose first that L is convex. As L is of class C^1 , for any point $x \in O$, the differential $L'(x)$ is defined and therefore the directional derivative in all directions $h \in \mathbb{R}^2$:

$$\partial_h L(x) = L'(x)h = \frac{\partial L}{\partial x_1}(x)h_1 + \frac{\partial L}{\partial x_2}(x)h_2.$$

As L is convex, if x and $x + h$ are in O , then

$$L(x + h) - L(x) \geq \frac{\partial L}{\partial x_1}(x)h_1 + \frac{\partial L}{\partial x_2}(x)h_2.$$

Suppose now that v is continuous, of class C^1 on $(a, b]$ and such that $\int_a^b [(\gamma + v)](t)dt$ is defined. If $c \in (a, b)$, and $t \in [c, b]$, then $((\gamma + v)(t), (\gamma + v)'(t)) \in O$ and so

$$\begin{aligned} \int_c^b L[(\gamma + v)(t)]dt - \int_c^b L[\gamma(t)] &\geq \int_c^b \frac{\partial L}{\partial x}[\gamma(t)]v(t) + \frac{\partial L}{\partial y}[\gamma(t)]v'(t)dt \\ &= \int_c^b \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)]v(t) + \frac{\partial L}{\partial y}[\gamma(t)]v'(t)dt \\ &= \int_c^b \frac{d}{dt} \left(\frac{\partial L}{\partial y}[\gamma(t)]v(t) \right) dt \\ &= \frac{\partial L}{\partial y}[\gamma(t)]v(t)|_c^b. \end{aligned}$$

If we now suppose that $\frac{\partial L}{\partial y}$ is bounded and $\gamma + v \in X$, then

$$\mathcal{L}(\gamma + v) - \mathcal{L}(\gamma) = \int_a^b L[(\gamma + v)(t)]dt - \int_a^b L[\gamma(t)] \geq 0,$$

because $v(a) = v(b) = 0$. Therefore γ is a minimum. If L is strictly convex, then an analogous reasoning shows that γ is a unique minimum. To sum up, we have the following result:

Proposition 3.1 *Suppose that L is convex (resp. strictly convex) on O and that $\gamma \in X$ satisfies the Euler-Lagrange equation on $(a, b]$. If $\frac{\partial L}{\partial y}$ is bounded, then γ is a minimum (resp. unique minimum) of \mathcal{L} on X .*

4 Lagrangians of class C^2

We suppose that L , O and X are defined as in one of the two previous sections. Our present object is to consider the case where the Lagrangian L is of class C^2 .

Theorem 4.1 *Suppose that L is of class C^2 and is such that the partial derivative $\frac{\partial^2 L}{\partial y^2}$ does not vanish on O . If $\gamma \in X$ satisfies the Euler-Lagrange equation, then γ is of class C^2 on $I = (a, b)$.*

PROOF Let us take $t_0 \in I$ and set $x_0 = \gamma(t_0)$ and $y_0 = \gamma'(t_0)$. We consider the mapping

$$\Phi : O \longrightarrow \mathbb{R} \times \mathbb{R}, (x, y) \longmapsto \left(x, \frac{\partial L}{\partial y}(x, y)\right).$$

As $\frac{\partial^2 L}{\partial y^2}(x_0, y_0) \neq 0$, the Jacobian of Φ at (x_0, y_0) is non-zero. It follows from the inverse mapping theorem that there is a neighbourhood U of (x_0, y_0) and a neighbourhood V of (x_0, z_0) , where $z_0 = \frac{\partial L}{\partial y}(x_0, y_0)$, such that $\Phi : U \longrightarrow V$ is a C^1 diffeomorphism. We can write

$$\Phi^{-1}(x, z) = (x, h(x, z)),$$

where h is a mapping of class C^1 . We now define a vector field $X : V \longrightarrow \mathbb{R} \times \mathbb{R}$ by

$$X(x, z) = \left(h(x, z), \frac{\partial L}{\partial x}(x, h(x, z))\right).$$

X is of class C^1 , so there is a maximal integral curve $\phi(t) = (x(t), z(t))$ of X , such that $\phi(t_0) = (x_0, z_0)$, defined on an open interval J containing t_0 . This integral curve is of class C^1 . In addition, $x'(t) = h(x(t), z(t))$ and so $x'(t)$ is of class C^1 . It follows that $x(t)$ is of class C^2 . Let us now set

$$\psi(t) = \left(\gamma(t), \frac{\partial L}{\partial y}[\gamma(t)]\right).$$

For t close to t_0 we have

$$h\left(\gamma(t), \frac{\partial L}{\partial y}[\gamma(t)]\right) = \gamma'(t)$$

and

$$\frac{\partial L}{\partial x}\left(\gamma(t), h\left(\gamma(t), \frac{\partial L}{\partial y}[\gamma(t)]\right)\right) = \frac{\partial L}{\partial x}[\gamma(t)] = \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)].$$

It follows that ψ is an integral curve of X . However $\psi(t_0) = (x_0, z_0)$ and so $\psi(t) = \phi(t)$ on a neighbourhood of t_0 . Therefore $\gamma(t) = x(t)$ and so γ is of class C^2 on a neighbourhood of t_0 . We have shown what we set out to show, namely that γ is of class C^2 on the interval I . \square

Suppose now that γ is of class C^2 , as for example under the conditions of the theorem. Then we may derive from the Euler-Lagrange equation another equation, which is often easier to use. We have, for $t \in (a, b)$,

$$\begin{aligned} \frac{d}{dt}L[\gamma(t)] &= \frac{\partial L}{\partial x}[\gamma(t)]\gamma'(t) + \frac{\partial L}{\partial y}[\gamma(t)]\gamma''(t) \\ &= \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)]\gamma'(t) + \frac{\partial L}{\partial y}[\gamma(t)]\gamma''(t) \\ &= \frac{d}{dt} \left(\frac{\partial L}{\partial y}[\gamma(t)]\gamma'(t) \right) \end{aligned}$$

and it follows that there is a constant c such that

$$L[\gamma(t)] - \frac{\partial L}{\partial y}[\gamma(t)]\gamma'(t) = c. \quad (2)$$

The equation we have just found is called the Beltrami equation.

If a function γ satisfies the equation (2), it is not necessarily a stationary function; however, if γ' does not vanish on an interval, then the Euler-Lagrange equation is satisfied on the interval. Here is a proof. Suppose that $\gamma' \neq 0$ on an interval I and that γ satisfies the equation (2). First, we have

$$\frac{d}{dt}L[\gamma(t)] = \frac{\partial L}{\partial x}[\gamma(t)]\gamma'(t) + \frac{\partial L}{\partial y}[\gamma(t)]\gamma''(t)$$

and, from equation (2),

$$\frac{d}{dt}L[\gamma(t)] = \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)]\gamma'(t) + \frac{\partial L}{\partial y}[\gamma(t)]\gamma''(t).$$

Therefore

$$\frac{\partial L}{\partial x}[\gamma(t)]\gamma'(t) = \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)]\gamma'(t)$$

As $\gamma'(t) \neq 0$, we have

$$\frac{\partial L}{\partial x}[\gamma(t)] = \frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)].$$

5 The m.b.p. and possible solutions

In this section we will apply the previous development to the mathematical brachistochrone problem and establish certain properties that a solution must have. For $(x, y) \in O = \mathbb{R}_+^* \times \mathbb{R}$, let

$$L(x, y) = \left(\frac{1 + y^2}{x} \right)^{\frac{1}{2}}.$$

As

$$\frac{\partial L}{\partial x} = -\frac{1}{2} \left(\frac{1 + y^2}{x^3} \right)^{\frac{1}{2}} \quad \text{and} \quad \frac{\partial L}{\partial y} = \frac{y}{(x(1 + y^2))^{\frac{1}{2}}},$$

L is of class C^1 . We fix $b > 0$. As in Section 3, for $\gamma \in C([0, b])$ continuously differentiable on $(0, b]$, with $(\gamma(t), \gamma'(t)) \in O$, for all $t \in (0, b]$, we set

$$\mathcal{L}(\gamma) = \int_0^b L(\gamma(t), \gamma'(t)) dt = \int_0^b L[\gamma(t)] dt.$$

The improper integral $\mathcal{L}(\gamma)$ may or may not be defined. We now take $\beta > 0$ and write X for the subset of $C([0, b])$ composed of those γ such that $L(\gamma)$ is defined, $\gamma(0) = 0$ and $\gamma(b) = \beta$. It is easy to check that, if $\gamma(t) = \frac{\beta}{b}t$, then $\gamma \in X$ and so X is not empty. The mathematical brachistochrone problem (m.b.s) is to minimize \mathcal{L} over X .

There is no difficulty in seeing that the second partial derivatives of L are defined and continuous and so L is of class C^2 . In particular,

$$\frac{\partial^2 L}{\partial y^2} = \frac{1}{x^{\frac{1}{2}}(1 + y^2)^{\frac{3}{2}}} > 0.$$

As L is of class C^2 and $\frac{\partial^2 L}{\partial y^2} \neq 0$, from Theorem 4.1, a stationary function γ is of class C^2 , and we may use equation (2). We have

$$L[\gamma(t)] - \frac{\partial L}{\partial y}[\gamma(t)]\gamma'(t) = c,$$

i.e.

$$\left(\frac{1 + \gamma'^2(t)}{\gamma(t)} \right)^{\frac{1}{2}} - \frac{\gamma'^2(t)}{\gamma(t)^{\frac{1}{2}}(1 + \gamma'^2(t))^{\frac{1}{2}}} = c,$$

from which we derive

$$\frac{1}{\gamma(t)^{\frac{1}{2}}(1 + \gamma'^2(t))^{\frac{1}{2}}} = c > 0.$$

Finally we obtain

$$\gamma(t)(1 + \gamma'^2(t)) = k,$$

where $k = \frac{1}{c^2}$. Any solution of the m.b.p. must satisfy such a differential equation on the interval $(0, b)$. Using the Euler-Lagrange equation (1), we can obtain more information.

Proposition 5.1 *Let γ be a solution of the brachistochrone problem. Then*

- **a.** $\lim_{t \rightarrow 0} \gamma'(t) = \infty$;
- **b.** γ is not constant on an interval;
- **c.** γ has at most one critical point, which is a maximum;
- **d.** γ is either strictly increasing, or is strictly increasing, reaches a maximum and then is strictly decreasing;
- **e.** γ' is strictly decreasing on $(0, b)$.

PROOF **a.** It is sufficient to notice that $\lim_{t \rightarrow 0} \gamma(t) = 0$.

b. There exist continuous functions a and b such that

$$\frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)] = \frac{a(t)\gamma''(t) - \gamma'(t)b(t)}{\gamma(t)(1 + \gamma'^2(t))}.$$

If γ is constant on an interval, then $\frac{d}{dt} \frac{\partial L}{\partial y}[\gamma(t)]$ vanishes on the interval. However, the expression $\frac{\partial L}{\partial x}[\gamma(t)]$ does not vanish. It follows that γ is not constant on an interval.

c. The function γ is bounded by k and reaches the value k at a point t_0 , if and only if t_0 is a critical point. Suppose that t_0 and t_1 are both critical points. As γ is not constant on the interval $[t_0, t_1]$, there is a point t in the interval such that $\gamma(t) < k$. However, γ is continuous on the compact interval $[t_0, t_1]$ and so reaches a minimum at some point t_2 . As $\gamma(t_2) < k$ and $\gamma'(t_2) = 0$, we have a contradiction. Hence there can be at most one critical point, which is clearly a maximum.

d. Suppose that γ has no critical point or has a critical point at b , then γ has no critical point in the interval $(0, b)$. If there exist points s and t such that $\gamma(s) = \gamma(t)$, then, from Rolle's theorem, there exists $r \in (s, t)$, such that $\gamma'(r) = 0$, a contradiction. On the other hand, if there exist s and t such that $\gamma(s) > \gamma(t)$, then from the mean value theorem, there exists $v \in (s, t)$ such that $\gamma'(v) < 0$. However, as $\gamma(0) = 0$ and $\gamma(t) > 0$, for $t \in (0, b]$, there exists $u \in (0, v)$ such that $\gamma'(u) > 0$. From the intermediate value theorem, there exists $r \in (u, v)$, such that $\gamma'(r) = 0$, a contradiction. Thus, γ is strictly increasing. Suppose now that γ has a critical point t' in the interval $(0, b)$. Applying arguments analogous to those which we have just used, we see that γ is strictly increasing on the interval $[0, t']$ and strictly decreasing on the interval $[t', b]$.

e. This follows directly from **d.** and the differential equation satisfied by γ . □

6 Parametric representation of possible solutions

In this section we will give a parametric representation of a possible solution γ on the interval $(0, b]$ of the m.b.p. and thus learn more about such a possible solution. We set

$$h(t) = \begin{cases} 2 \arctan \frac{1}{\gamma'(t)} & t \in (0, b] \text{ and } \gamma'(t) > 0 \\ \pi & t \in (0, b] \text{ and } \gamma'(t) = 0 \\ 2(\pi + \arctan \frac{1}{\gamma'(t)}) & t \in (0, b] \text{ and } \gamma'(t) < 0 \end{cases}.$$

(If γ does not reach a maximum (resp. reaches a maximum at b), then we ignore the second and third parts (resp. the third part) of the definition.) It is easy to see that h is continuous and of class C^1 , at least when $\gamma'(t) \neq 0$, and, from Proposition 5.1, h is strictly increasing. It follows that the image of h is

an interval $(0, \theta_1) \subset (0, 2\pi)$ and $h(b) = \theta_1$. Let us set $I_1 = (0, \theta_1) \cap (0, \pi)$ and $I_2 = (0, \theta_1) \cap (\pi, 2\pi)$. (I_2 may be empty.) For $\gamma'(t) \neq 0$, we have

$$\gamma'(t) = \cot \frac{h(t)}{2} \implies 1 + \gamma'^2(t) = 1 + \cot^2 \frac{h(t)}{2} = \frac{1}{\sin^2 \frac{h(t)}{2}}.$$

Therefore

$$\gamma(t) = k \sin^2 \frac{h(t)}{2} = \frac{k}{2}(1 - \cos h(t)).$$

Also

$$\gamma(t) = \frac{k}{2}(1 - \cos h(t)) \implies \gamma'(t) = \frac{k}{2}(\sin h(t))h'(t) \implies \cot \frac{h(t)}{2} = \frac{k}{2}(\sin h(t))h'(t).$$

Now let us set $h(t) = \theta$. Differentiating h^{-1} on I_1 and on I_2 , if not empty, we obtain

$$\frac{d}{d\theta}(h^{-1})(\theta) = \frac{k \sin \theta}{2 \cot \frac{\theta}{2}} = k \sin^2 \frac{\theta}{2} = \frac{k}{2}(1 - \cos \theta).$$

It follows that on the interval I_1 (resp. I_2 , if not empty), there is a constant c_1 (resp. c_2), such that

$$t = \frac{k}{2}(\theta - \sin \theta) + c_i.$$

As $\lim_{t \rightarrow 0} h(t) = 0$, $c_1 = 0$ and so on the interval I_1 the graph of γ has the parametric representation (P):

$$\begin{cases} t = \frac{k}{2}(\theta - \sin \theta) \\ \gamma(t) = \frac{k}{2}(1 - \cos \theta), \end{cases}$$

where $\theta \in (0, \pi)$. If we now let θ converge to π , we obtain $c_2 = 0$ and so the parametric representation (P) is valid for the whole graph of γ . Thus the graph of the function γ may be considered as lying on a cycloid.

This parametric representation enables us to obtain more information about γ . Let us consider the function α defined on $(0, 2\pi)$ as follows:

$$\alpha(\theta) = \frac{1 - \cos \theta}{\theta - \sin \theta}.$$

Clearly $\lim_{\theta \rightarrow 2\pi} \alpha(\theta) = 0$. Also

$$\theta - \sin \theta = \frac{\theta^3}{6} + o(\theta^3) \quad \text{and} \quad 1 - \cos \theta = \frac{\theta^2}{2} + o(\theta^3),$$

therefore $\lim_{\theta \rightarrow 0} \alpha(\theta) = \infty$. A simple calculation shows that

$$\alpha'(\theta) = \frac{\theta \sin \theta - 2 + 2 \cos \theta}{(\theta - \sin \theta)^2}.$$

A careful analysis of the numerator of α' shows that it is strictly negative on $(0, 2\pi)$ and hence so is α' . Therefore α is strictly decreasing on $(0, 2\pi)$. This means that there is a unique $\tilde{\theta}$ such that $\frac{\beta}{b} = \alpha(\tilde{\theta})$. However $\alpha(\theta_1) = \frac{\beta}{b}$ and so $\theta_1 = \tilde{\theta}$. Therefore, from the value of $\tilde{\theta} = \alpha^{-1}(\frac{\beta}{b})$, we may determine whether γ is strictly increasing without a critical point ($\tilde{\theta} < \pi$), strictly increasing with a critical point ($\tilde{\theta} = \pi$) or strictly increasing and then strictly decreasing ($\tilde{\theta} > \pi$). In addition, from one of the equations

$$b = \frac{k}{2}(\tilde{\theta} - \sin \tilde{\theta}) \quad \text{or} \quad \beta = \frac{k}{2}(1 - \cos \tilde{\theta}),$$

we may find k . We have shown that if a minimum γ of the m.b.p. exists, then its graph has a particular form: if

$$\tilde{\theta} = \alpha^{-1}\left(\frac{\beta}{b}\right) \quad \text{and} \quad k = \frac{b}{2(1 - \cos \tilde{\theta})},$$

then the graph of γ has the parametric representation

$$\begin{cases} t = \frac{k}{2}(\theta - \sin \theta) \\ \gamma(t) = \frac{k}{2}(1 - \cos \theta), \end{cases}$$

for $\theta \in [0, \tilde{\theta}]$. It is easy to check that such a curve lies in X . However, we have not shown it is a minimum. In the next section we will look at this question.

7 The unique minimum of the m.b.p.

From now on we will write γ_0 for the particular function we defined parametrically at the end of the last section. We aim to show that γ_0 is the unique minimum of the brachistochrone problem. We would like to use the criterion developed in Proposition 3.1. However, using Proposition 1.1, we see that the function

$$L(x, y) = \left(\frac{1 + y^2}{x} \right)^{\frac{1}{2}}$$

is not convex. We get around this difficulty by introducing another minimization problem. For $(x, y) \in O = \mathbb{R}_+^* \times \mathbb{R}$, let

$$M(x, y) = (x^{-2} + y^2)^{\frac{1}{2}}.$$

As

$$\frac{\partial M}{\partial x} = -x^{-3}(x^{-2} + y^2)^{-\frac{1}{2}} \quad \text{and} \quad \frac{\partial M}{\partial y} = y(x^{-2} + y^2)^{-\frac{1}{2}},$$

M is of class C^1 . For $\delta \in C[0, b]$ continuously differentiable on $(0, b]$, with $(\delta(t), \delta'(t)) \in O$, for all $t \in (0, b]$, we set

$$\mathcal{M}(\delta) = \int_0^b M[\delta(t)] dt.$$

The improper integral $\mathcal{M}(\delta)$ may or may not be defined. We write Y for the subset of $C([0, b])$ composed of those δ such that $\delta(0) = 0$, $\delta(b) = (2\beta)^{\frac{1}{2}}$ and $M(\delta)$ is defined. If $\gamma \in X$ and we set $\delta = (2\gamma)^{\frac{1}{2}}$, then

$$\gamma = \frac{\delta^2}{2} \quad \text{and} \quad \gamma' = \delta\delta'.$$

It is now easy to check that $\delta \in Y$ if and only if $\delta = (2\gamma)^{\frac{1}{2}}$, for some $\gamma \in X$ and, in this case, $\mathcal{L}(\gamma) = 2^{\frac{1}{2}}\mathcal{M}(\delta)$. Let us set $\delta_0 = (2\gamma_0)^{\frac{1}{2}}$.

Proposition 7.1 δ_0 is the unique minimum of \mathcal{M} on Y .

PROOF The second partial derivatives of M are defined and continuous and so M is of class C^2 . Using Proposition 1.1, we see that M is strictly convex. In addition, $|\frac{\partial M}{\partial y}| \leq 1$. To simplify the notation, let us write δ for δ_0 and γ for γ_0 . We have

$$\gamma(1 + \gamma'^2) = k \implies \frac{\delta^2}{2}(1 + \delta^2\delta'^2) = k$$

and

$$\begin{aligned} M[\delta(t)] - \frac{\partial M}{\partial y}[\delta(t)] &= (\delta^{-2}(t) + \delta'^2(t))^{\frac{1}{2}} - (\delta^{-2}(t) + \delta'^2(t))^{-\frac{1}{2}}\delta'^2(t) \\ &= \delta^{-1}(t)(1 + \delta^2(t)\delta'^2(t))^{\frac{1}{2}} - \delta(t)(1 + \delta^2(t)\delta'^2(t))^{-\frac{1}{2}}\delta'^2(t) \\ &= \frac{1}{(2k)^{\frac{1}{2}}}(1 + \delta^2(t)\delta'^2(t)) - \frac{1}{(2k)^{\frac{1}{2}}}\delta^2(t)\delta'^2(t) = \frac{1}{(2k)^{\frac{1}{2}}}. \end{aligned}$$

On the interval (resp. two intervals) where $\delta'(t) \neq 0$, δ satisfies the Euler-Lagrange equation, i.e.

$$\frac{\partial M}{\partial x}[\delta(t)] = \frac{d}{dt} \frac{\partial M}{\partial y}[\delta(t)].$$

If δ has a critical point in the interior of the interval $(0, b)$, then, by continuity, the Euler-Lagrange equation is also satisfied at this point. Therefore the Euler-Lagrange equation is satisfied on $(0, b)$. Applying Proposition 3.1, we obtain the result. \square

We are now in a position to show that γ_0 is the unique solution of the m.b.p.

Theorem 7.1 γ_0 is the unique minimum of \mathcal{L} on X .

PROOF For $\gamma \in X$, with $\gamma \neq \gamma_0$, we have

$$\mathcal{L}(\gamma) = 2^{\frac{1}{2}} \mathcal{M}((2\gamma)^{\frac{1}{2}}) > 2^{\frac{1}{2}} \mathcal{M}(\delta_0) = \mathcal{L}(\gamma_0).$$

This ends the proof. \square

It should be noticed that, for distinct pairs (b_1, β_1) and (b_2, β_2) , the corresponding solutions γ_1 and γ_2 of the m.b.p. are distincts. Let us see why this is so. If $b_1 \neq b_2$, then $\gamma_1 \neq \gamma_2$, because γ_i is defined on the interval $[0, b_i]$, for $i = 1, 2$. Suppose now that the pairs (b, β_1) and (b, β_2) , with $\beta_1 < \beta_2$, produce the same solution γ . Then $\frac{\beta_1}{b} < \frac{\beta_2}{b}$ and it follows that $\tilde{\theta}_1 > \tilde{\theta}_2$, where $\theta_i = \alpha^{-1}(\frac{\beta_i}{b})$, for $i = 1, 2$. As

$$\frac{b}{2(1 - \cos \tilde{\theta}_1)} = k = \frac{b}{2(1 - \cos \tilde{\theta}_2)},$$

we have

$$\cos \tilde{\theta}_1 = \cos \tilde{\theta}_2 \implies \tilde{\theta}_1 = 2\pi - \tilde{\theta}_2.$$

Given that $\tilde{\theta}_1 > \tilde{\theta}_2$, we must have $\tilde{\theta}_1 > \pi$ and $\tilde{\theta}_2 < \pi$, which implies that $\gamma_1 \neq \gamma_2$, a contradiction. Therefore, in this case too, the solutions γ_1 and γ_2 are different.

Although distinct pairs (b, β) produce distinct solutions, it may be possible for different solutions to lie on the same cycloid. If we fix β and choose b_1 and b_2 such that $\tilde{\theta}_1 = 2\pi - \tilde{\theta}_2$, then in both cases we find the same value of k and so the corresponding solutions of the m.b.p. lie on the same cycloid.

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