

# THE $p$ -ADIC EISENSTEIN MEASURE AND SHAHIDI-TYPE $p$ -ADIC INTEGRAL FOR $SL(2)$

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Our general goal is two-fold: first, to construct  $p$ -adic Eisenstein measures on classical groups using the method of modular distributions and second, to apply Shahidi-type theory to construct certain  $p$ -adic  $L$ -functions using Fourier expansions of these series.

In the present paper we confine ourselves with the group  $SL(2)$ , and we try to explain our techniques in this case.

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## INTRODUCTION AND MOTIVATION

The theory of *complex valued* automorphic  $L$ -functions is now fairly well understood, especially the role played by the Langlands Program. Let us quickly survey (parts of) this theory.

**Conjecture 1 (Langlands).** To each reductive group  $G$  over a number field  $K$ , each automorphic (complex) representation  $\pi$  of  $G$ , and each finite dimensional representation  $r$  of the (complex) group  ${}^L G$ , there is defined an automorphic  $L$ -function  $L(s, \pi, r)$ , which enjoys an analytic continuation and functional equation generalizing the Riemann zeta function  $\pi^{s/2}\Gamma(s/2)\zeta(s)$  (or Artin's  $L$ -function  $L(s, \sigma)$ , when  $G = (1)$  when  $K$  is arbitrary,  $\pi = Id$ ,  ${}^L G = Gal(L/K)$ , and  $r$  is a  $n$ -dimensional Artin representation  $\sigma$ , etc.).

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**Some Special Cases.** Take  $G = GL(2)$  over a number field  $K$  and  ${}^L G = GL(2, \mathbf{C})$ ; take  $r$  to be the  $k + 1$  dimensional (**symmetric power**) representation on homogeneous polynomials of degree  $k$  in two variables and  $\pi$  any automorphic representation of  $G$ . Then when  $k = 1$  we have [JaLa],  $k = 2$  [Ge-Ja],  $k = 3$  [Kim-Sh],  $k = 4$  [Kim], and for  $k > 4$  the set of poles of  $L(s, \pi, r)$  is unknown.

**Conjecture 2 (Functoriality).** Suppose we are given **two** reductive groups  $G$  and  $G'$ , an analytic homomorphism  $\rho : {}^L G \rightarrow {}^L G'$ , and an automorphic representation  $\pi = \otimes \pi_v$  of  $G$ . Then there is an automorphic representation  $\pi' = \otimes \pi'_v$  of  $G'$  such that for all  $v \notin S_{\pi'}$  (i.e. unramified  $v$ ),  $t(\pi'_v)$  is the conjugacy class in  ${}^L G'$  which contains  $t(\pi_v)$ . In particular

$$L(s, \pi', r') = L(s, \pi, r' \circ \rho)$$

for each finite-dimensional representation  $r'$  of  ${}^L G'$ .

This **Functoriality** Conjecture gives us the similar conjectures made for the L-functions of number theory, and (an infinity) of new ones too. For example, the functoriality of symmetric power liftings gives us the Conjectures of **Ramanujan**, **Selberg**, and **Sato-Tate** (including Maass forms). More importantly, over a great range of subjects this Conjecture suggests to us the

”*Algebraic*”

versus

”*Analytic*”

results of number theory, a good example being **Artin’s-Langlands’** Conjecture:  $L(s, \sigma) = L(s, \pi_\sigma)$ .

One small approach towards these two Conjectures has been through various methods of defining and exploring these  $L$ -functions. These have included the method of (1) Tate, Godement-Jacquet [Go-Ja], (2) Rankin-Selberg (see, for example [Ja] ) (3) "doubling" à la Piatetski-Shapiro and Rallis [GRPS], developed also by S.Boecherer [Boe85], (4) Shimura [Shi75], and (5) Shahidi [Sha88], [GeSha].

Our (long term) goal is to study  $p$ -adic  $L$ -functions in similar ways as *complex* L-functions. In particular, we are interested in those  $p$ -adic  $L$ -functions that arise via a  $p$ -adic *analysis* of Shahidi’s method in (5).

Let us again review the *complex* analysis of (5). We start with the Eisenstein series

$$E(s, P, \varphi, g) = \sum_{\gamma \in P \backslash G} \varphi_s(\gamma g),$$

on  $G$  attached to a maximal parabolic subgroup  $P = MU^P$ , an auto-morphic representation  $\pi$  of  $M$ ; this series generalizes

$$E(z, s) = \frac{1}{2} \sum \frac{y^s}{|cz + d|^{2s}}, \quad (c, d) = 1,$$

$M$  is the Levi component of  $P$ , and  $\varphi$  belongs to the space of the induced representation  $Ind_P^G(\pi)$ ,  $\varphi_s = \varphi \otimes |\det_M(\cdot)|_{\mathbb{A}}^s$ .

Langlands in [La] (generalizing Selberg) showed that the "constant term" of  $E(s)$  was analytically continuable with a functional equation and was expressible in terms of  $L(s, \pi, r_j)$  (in the numerator and denominator). Shahidi then computed a *non-constant* Fourier coefficient which eventually yielded the analytic continuation and functional equation of almost all  $L(s, \pi, r_j)$ . Along the way, the  $\psi$ -th Fourier coefficient of  $E(s, P, \varphi, e)$  is determined:

$$E_\psi(e, \varphi, s) = \prod_{v \in S} W_v(e_v) \prod_{j=1}^m \frac{1}{L^S(1 + j s, \pi, \tilde{r}_j)},$$

where  $r_j$  are certain fundamental representations of the  $L$ -group  ${}^L M$ ,  $\tilde{r}_j$  their contragredient. What does this say about the simplest possible  $(G, M)$  pair? With  $G = SL(2)$ ,  $MN = B$ ,  $\pi = I$ , and  $\psi$  a non-trivial character of  $N$ , let

$$E_\psi(s, \varphi, e) = \int E(s, \varphi, n) \psi(n) dn,$$

the integration being over the quotient space of  $N(\mathbf{A})$  by  $N(\mathbf{Q})$ . Then

$$E_\psi(s, \varphi, e) = W_\infty(s) \frac{1}{\zeta(1 + s)}.$$

We shall see in this paper how this result carries over  *$p$ -adically*.

### 1. STATEMENT OF THE MAIN RESULT

**THEOREM 1.1.** *Assume that  $p$  is regular. Then there exists an explicitly defined  $p$ -adic measure  $\mu^*$  on  $\mathbb{Z}_p^*$  such that for all even positive integers  $k > 0$  one has*

$$\int_{\mathbb{Z}_p^*} y_p^k \mu^* = c_k(p) \zeta(1 - k)^{-1} (1 - p^{k-1})^{-1}.$$

Here  $c_k(p) = \Gamma(k) 4i^k p^{2k-1} (p-1)$ ,  $y_p^k(y) = y^k$  for all  $y \in \mathbb{Z}_p^*$ . Moreover, the measure  $\mu^*$  can be expressed through the first Fourier coefficient of  $(1)E^*$ , the one-variable measure attached to a certain two-variable measure  $E^*$ .

Note that in accordance with p.125 of [Wa82] the factor  $g(T, \theta)$  of the  $p$ -adic  $L$ -function is invertible for all primes, if  $\theta = 1$ , and if  $p$  is

regular then this statement remains true for all  $\theta$ . The factorization in question (p.127) has the form

$$A(T) = \prod g(T, \theta), A(T) = p^\mu P(T)U(T) \in \Lambda,$$

where  $U(T)$  is a unit,  $P(T)$  is a polynomial of degree  $\lambda$  (in our case  $\lambda = 0$  and  $\mu = 0$ ).

Remarks.

1) The equality of this Theorem expresses a comparison between the evaluation of the complex  $L$ -function (say, in page 81 of [GeSh]) at  $s = -k$  and the value of the  $p$ -adic integral on the left.

2) The special values on the right side of of Theorem 1.1 come from evaluation of certain  $p$ -adic integrals of the distribution  $\mu^*$  constructed in the proof. This is the reverse order of interpolation of the special values: there the  $p$ -adic analytic function is constructed from the special values, as for example in Theorem 5.11 at p.57 of [Wa82], and [Ko77], Chapter 2, p.46, not the other way around.

3) For a non regular  $p$  such a distribution can not exist, because the  $p$ -adic Mellin transform of a bounded distribution must be holomorphic. However, one could expect to construct an element of the fraction algebra of the algebra of bounded distributions having a similar interpolation property; its  $p$ -adic Mellin transform would define a meromorphic  $p$ -adic analytic function with a finite number of poles.

In order to construct the distribution  $\mu^*$ , the Eisenstein series of higher level are used.

## 2. FOURIER EXPANSION OF CLASSICAL EISENSTEIN SERIES ON $SL_2$

In this section we would like to prepare the construction of the Eisenstein measure coming from the Fourier expansion of the two types of classical Eisenstein series of weight  $k \geq 3$  [Ka76].

Let us recall definitions and some of their properties.

Let  $k \geq 3$ . Put for  $a, b \in \mathbb{Z}/N\mathbb{Z}$ ,

$$(2.1) \quad E_{k,N}(z; a, b) = \sum (cz + d)^{-k} \in \mathcal{M}_k(\Gamma(N))$$

in the space  $\mathcal{M}_k(\Gamma(N))$  of modular forms of weight  $k$  for  $\Gamma(N)$ , see [Miy], p.271, where  $(c, d) \equiv (a, b) \pmod{N}$  and  $(c, d) \neq (0, 0)$ .

The action of  $\sigma = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$  is given by

$$(2.2) \quad [E_{k,N}(z; a, b)]_k \sigma = E_{k,N}(z; a_1, b_1)$$

with  $(a_1, b_1) = (a, b)\sigma$  using the equality (2.1); the action of the involution  $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$  is given by

$$(2.3) \quad [E_{k,N}(z; a, b)]|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} = N^{k/2} E_{k,N}(Nz; b, -a).$$

We write the Fourier expansion of the series (2.1) using Proposition 2.2 in [PaTV], §2.

**Proposition 2.1.** *Suppose that  $k \geq 3$ . Then there is the following Fourier expansion:*

$$E_{k,N}(z; a, b) = \delta\left(\frac{a}{N}\right) [\zeta(k; b, N) + (-1)^k \zeta(k; -b, N)] + \frac{(-2\pi i)^k}{N^k \Gamma(k)} \times \left( \sum_{\substack{dd' > 0 \\ d' \equiv a \pmod N}} \text{sgn}(d) d^{k-1} e\left(\frac{db}{N}\right) e\left(\frac{dd'z}{N}\right) \right),$$

where  $e(x) = \exp(2\pi i x)$ ,  $\delta(x) = 1$  if  $x$  is an integer,  $\delta(x) = 0$  otherwise, and  $\zeta(k; a, N) = \sum_{0 < n \equiv a \pmod N} n^{-k}$  denotes the value of the partial Riemann zeta function.

*Proof.* See [PaTV], §2, with  $s = 0$  and  $k \geq 3$ .

Next we define

$$(2.4) \quad E_{k,N}^*(z; a, b) = \sum_{\substack{(c,d) \equiv (a,b) \pmod N \\ (c,d)=1}} (cz + d)^{-k}$$

where  $(c, d) \equiv (a, b) \pmod N$  and  $(c, d) = 1$ .

One can express the series (2.4) in terms of the series (2.1) using Hecke's method (compare with Hecke's formula (8) in [He27]):

$$(2.5) \quad \begin{aligned} E_{k,N}^*(z; a, b) &= \sum_{(c,d) \equiv (a,b) \pmod N} \sum_{\delta | (c,d)} \mu(\delta) (cz + d)^{-k} \\ &= \sum_{\delta=1}^{\infty} \mu(\delta) \delta^{-k} \sum_{(c',d') \equiv (a',b') \pmod{N'}} (c'z + d')^{-k} \\ &= \sum_{\delta=1}^{\infty} \mu(\delta) \delta^{-k} E_{k,N'}(z; a', b') = \\ &= \sum_{\delta' | N} \mu(\delta') \delta'^{-k} \sum_{\delta''=1}^{\infty} \mu(\delta'') \delta''^{-k} E_{k,N'}(z; a', b') \\ &= \sum_{\substack{t \pmod N \\ (t,N)=1}} c_t E_{k,N}(z; ta, tb) \in \mathcal{M}_k(\Gamma(N)), \end{aligned}$$

where we write

$\delta' = (p^m, \delta)$ ,  $N' = p^m/\delta'$ ,  $(a', b') = (a/\delta', b/\delta')$ ,  $\delta = \delta'\delta''$ , and

$$c_t = \begin{cases} \sum_{tn \equiv 1 \pmod{N}} \frac{\mu(n)}{n^k}, & \text{if } (t, N) = 1 \\ 0, & \text{if } (t, N) > 1 \end{cases}$$

(here  $\mu$  denotes the Moebius function!).

### 3. METHOD OF MODULAR DISTRIBUTIONS

In this section we describe the method of modular distributions as developed in [PaTV] and apply it to the series  $E^*$  in order to construct the  $p$ -adic analytic families of Fourier expansions from  $E^*$ .

We start with the ‘‘Igusa tower’’.

Let  $A$  be an algebraic extension  $K$  of  $\mathbb{Q}_p$  or its ring of integers  $\mathcal{O}_K$ . Let us fix an embedding  $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$  (where  $\mathbb{C}_p$  is the Tate field, i.e. the completion of an algebraic closure of  $\mathbb{Q}_p$ ) and let  $\mathcal{M}_k(\Gamma_1(p^m); A)$ ,  $\mathcal{M}_k(\Gamma_0(p^m), \psi; A)$  be the submodules of  $A[[q]]$  generated by the  $q$ -expansions of the form

$$i_p(f) = \sum_{n \geq 0} i_p(a_n(f))q^n \in A[[q]],$$

where  $f = \sum_{n \geq 0} a_n(f)q^n \in \mathcal{M}_k(\Gamma_1(p^m), \overline{\mathbb{Q}})$  is a classical modular form with algebraic Fourier coefficients  $a_n(f) \in \overline{\mathbb{Q}}$  in  $i_p^{-1}(A)$ . One puts

$$\mathcal{M}_k = \bigcup_{m \geq 0} \mathcal{M}_k(p^m),$$

where  $\mathcal{M}_k(p^m) = \mathcal{M}_k(\Gamma_1(p^m); A)$ .

We call  $\mathcal{M}_k$  the modular *Igusa tower* (see [Ig] and also section 8.4 of [Hi04]).

Now consider the commutative profinite group

$$\mathbb{Z}_p^* = \varprojlim_m (\mathbb{Z}/p^m\mathbb{Z})^*$$

and its group  $X = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^\times)$  of (continuous)  $p$ -adic characters (this is a  $\mathbb{C}_p$ -analytic Lie group analogous to  $\text{Hom}_{\text{cont}}(\mathbb{R}_+^\times, \mathbb{C}^\times) \cong \mathbb{C}$  (by  $s \mapsto (y \mapsto y^s)$ ). The group  $X$  is isomorphic to a finite union of discs  $U = \{z \in \mathbb{C}_p \mid |z|_p < 1\}$ :  $\text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =$

$$\text{Hom}_{\text{cont}}(\mu_{p-1}^*, \mathbb{C}_p^*) \times \text{Hom}_{\text{cont}}((1 + p\mathbb{Z}_p), \mathbb{C}_p^*) \quad (1 + p \mapsto 1 + z \in U),$$

so it has a natural  $p$ -adic analytic structure.

The  $p$ -adic  $L$ -function  $L : X \rightarrow \mathbb{C}_p$  is a meromorphic function on  $X$  coming from a  $p$ -adic measure on  $\mathbb{Z}_p^*$ . A distribution on  $\mathbb{Z}_p^*$  with values

in any  $A$ -module  $M$  (or an " $M$ -valued distribution") is a morphism of  $A$ -modules

$$\Phi : \text{Step}(\mathbb{Z}_p^*, A) \rightarrow M, \quad \varphi \mapsto \Phi(\varphi) =: \int_Y \varphi(y) \Phi,$$

where  $A$  is a normed topological ring containing  $\mathbb{Z}_p$ . In particular, an  $A$ -valued distribution can be constructed from  $\Phi$  by applying an  $A$ -linear mapping  $\ell : M \rightarrow A$ .

In the case  $M = \mathcal{M}_k$  we call a distribution  $\Phi$  with values in  $\mathcal{M}$  a *modular* distribution.

**Examples of distributions with values in  $\mathcal{M}_k$ .** Let  $\Phi : \text{Step}(\mathbb{Z}_p^*, \mathbb{C}_p) \rightarrow \mathcal{M}_k$  be a distribution on  $\mathbb{Z}_p^*$  with values in  $\mathcal{M}_k$ .

a) *Eisenstein distributions.* For an arbitrary  $N$ , let us consider the following Eisenstein distributions: put

$$\begin{aligned} E_{k,N}(z, a, b) &:= \frac{N^{k-1} \Gamma(k)}{(-2\pi i)^k} \sum_{x \bmod N} e(-ax/N) E_{k,N}(Nz; x, b) \\ (3.1) \quad &= \delta\left(\frac{b}{N}\right) \zeta(1-k; a, N) + \sum_{0 < dd', d \equiv a, d' \equiv b \pmod N} \text{sgn } d \cdot d^{k-1} e(dd'z) \in \mathcal{M}_k(N^2), \end{aligned}$$

(compare with the  $q$ -expansions in (2.14) in [PaTV], §2, and in Proposition 2.1). These series are holomorphic modular forms and these series produce distributions on  $Y \times Y$  with values in  $\mathcal{M}_k$ :

$$\begin{aligned} \Phi((a + (p^m) \times (b + (p^m))) &= \\ E_k((a + (p^m) \times (b + (p^m))) &:= E_{k,p^m}(z, a, b) \in \mathcal{M}_k(p^{2m}). \end{aligned}$$

b) *Partial modular forms.* This example is not directly used in the proof of our Main theorem, however, it clarifies the nature of the modular distributions. For any  $f = \sum_{n \geq 0} a_n(f) q^n \in \mathcal{M}_k(\text{SL}_2(\mathbb{Z}))$  one puts

$$\Phi_f(a + (p^m)) := \sum_{\substack{n \geq 0 \\ n \equiv a \pmod{p^m}}} a_n(f) q^n \in \mathcal{M}_k(p^{2m}).$$

c) *Partial theta series* (also with a spherical polynomial, see [Hi85] and §5 of [PaIAS]). Also, this nice example is not directly used in the proof of our Main theorem, and we reserve it for future applications.

*Remarks.* i) For any Dirichlet character  $\chi \bmod p^m$  viewed as a function on  $\mathbb{Z}_p^*$  with values in  $i_p(\mathbb{Q}^{\text{ab}})$ , the integral

$$\int_Y \chi(y) d\Phi_f = \Phi_f(\chi) = \sum_{n \geq 0} \chi(n) a_n(f) q^n \in \mathcal{M}_k(N^2 p^{2m})$$

coincides then with the twisted modular form  $f_\chi$ . Notice that this integral is just a finite sum of partial modular forms.

- ii) The distributions a), b), c) are bounded (after a regularisation of the constant term in a)) with respect to the  $p$ -adic norm on  $\mathcal{M}_k = \cup_{p^m} \mathcal{M}_k(\Gamma_1(p^m), A) \subset A[[q]]$  given by  $|g|_p = \sup_n |a(n, g)|_p$  for  $g = \sum_{n \geq 0} a(n, g)q^n \in \mathcal{M}_k(\Gamma_1(p^m), A)$ ).
- iii) Starting from distributions a), b), c) one can construct many other distributions, for example, using the operation of convolution on  $\mathbb{Z}_p^*$  (as in [Hi85], where the case of the convolution of a theta distribution with an Eisenstein distribution was considered).

**A new construction.** It provides a rather simple method which attaches to a distribution  $\Phi$  on  $\mathbb{Z}_p^*$  with values in a suitable vector space  $\mathcal{M}_k$  of modular forms, a family  $\mu_{\alpha, \Phi, f}$  of  $p$ -adic measures on  $\mathbb{Z}_p^*$  parametrized by non-zero eigenvalues  $\alpha$  associated with primitive eigenforms  $f$ .

The idea is to use the operator  $U = U_p$  of Atkin-Lehner which acts on  $\mathcal{M}_k$  by  $g \mapsto U = \sum_{n \geq 0} a(pn, g)q^n$ , where  $g = \sum_{n \geq 0} a(n, g)q^n \in \mathcal{M}_k \subset A[[q]]$ ,  $a(n, g) \in A$ .

**Definition 3.1.** a) For an  $\alpha \in A$  put  $\mathcal{M}_k^{(\alpha)} = \text{Ker}(U - \alpha I)$  the  $A$ -submodule of  $\mathcal{M}_k$  of eigenfunctions of the  $A$ -linear operator  $U$  (of the eigenvalue  $\alpha$ ).

b) Put  $\mathcal{M}_k^\alpha = \cup_{n \geq 1} \text{Ker}(U - \alpha I)^n$  the  $\alpha$ -primary (characteristic)  $A$ -submodule of  $\mathcal{M}_k$ . Let us define  $U^\alpha$  as the restriction of  $U$  to  $\mathcal{M}_k^\alpha$ .

c) Put  $\mathcal{M}_k^\alpha(p^m) = \mathcal{M}_k^\alpha \cap \mathcal{M}_k(p^m)$ ,  $\mathcal{M}_k^{(\alpha)}(p^m) = \mathcal{M}_k^{(\alpha)} \cap \mathcal{M}_k(p^m)$ .

**Proposition 3.2.** Let  $A = \overline{\mathbb{Q}}_p$ . If  $N_0 = Np$ , then  $U^m(\mathcal{M}_k(N_0p^m)) \subset \mathcal{M}_k(N_0)$ .

*Proof* follows from a known formula [Se73],

$$U^m = p^{m(k/2-1)} W_{N_0p^m} \text{Tr}_{N_0}^{N_0p^m} W_{N_0},$$

where  $g|_k W_N(z) = (\sqrt{N}z)^{-k} g(-1/Nz) : \mathcal{M}_k(N) \rightarrow \mathcal{M}_k(N)$  the main involution of level  $N$  (over the complex numbers).

**Proposition 3.3.** Let  $A = \overline{\mathbb{Q}}_p$  and let  $\alpha$  be a non-zero element of  $A$ ; then

a)  $(U^\alpha)^m : \mathcal{M}_k^\alpha(N_0p^m) \xrightarrow{\sim} \mathcal{M}_k^\alpha(N_0p^m)$  is an invertible  $\overline{\mathbb{Q}}_p$ -linear operator.

b) The  $\overline{\mathbb{Q}}_p$ -vector subspace  $\mathcal{M}_k^\alpha(N_0p^m) = \mathcal{M}_k^\alpha(N_0)$  is independent of  $m$  (a "control theorem").

c) Let  $\pi_{\alpha, m} : \mathcal{M}_k(N_0p^m) \rightarrow \mathcal{M}_k^\alpha(N_0p^m)$  be the canonical projector onto the  $\alpha$ -primary subspace of  $U$  (of the kernel

$$\text{Ker } \pi_{\alpha, m} = \bigcap_{n \geq 1} \text{Im}(U - \alpha I)^n = \oplus_{\beta \neq \alpha} \mathcal{M}_k^\beta(N_0p^m);$$

then the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M}_k(N_0 p^m) & \xrightarrow{\pi_{\alpha,m}} & \mathcal{M}_k^\alpha(N_0 p^m) \\ U^m \downarrow & & \downarrow \wr(U^\alpha)^m \\ \mathcal{M}_k(N_0) & \xrightarrow{\pi_{\alpha,0}} & \mathcal{M}_k^\alpha(N_0) \end{array}$$

*Proof.* Due to the reduction theory of endomorphisms in a finite dimensional subspace over a field  $K$ , the projector  $\pi_{\alpha,m}$  onto the  $\alpha$ -primary subspace  $\bigcup_{n \geq 1} \text{Ker}(U - \alpha I)^n$  has the kernel  $\bigcap_{n \geq 1} \text{Im}(U - \alpha I)^n$  and it can be expressed as a polynomial of  $U$  with coefficients in  $K$ , hence  $\pi_{\alpha,m}$  commutes with  $U$ . On the other hand, the restriction of  $\pi_{\alpha,m}$  on  $\mathcal{M}_k(N_0)$  coincides with  $\pi_{\alpha,0} : \mathcal{M}_k(N_0) \rightarrow \mathcal{M}_k^\alpha(N_0)$  because its image is

$$\bigcup_{n \geq 1} \text{Ker}(U - \alpha I)^n \cap \mathcal{M}_k(N_0) = \bigcup_{n \geq 1} \text{Ker}(U|_{\mathcal{M}_k(N_0)} - \alpha I)^n,$$

and the kernel is

$$\bigcap_{n \geq 1} \text{Im}(U - \alpha I)^n \cap \mathcal{M}_k(N_0) = \bigcap_{n \geq 1} \text{Im}(U|_{\mathcal{M}_k(N_0)} - \alpha I)^n.$$

Define the  $\alpha$ -primary part  $\pi_\alpha(\Phi) = \Phi^\alpha$  of  $\Phi$  by the equality

$$(3.2) \quad \Phi^\alpha(a + (p^m)) = (U^\alpha)^{-m'} \left[ \pi_{\alpha,0}(\Phi(a + (p^m)) | U^{m'}) \right]$$

valid for all sufficiently big  $m' \gg 0$ .

Our next task will be to apply this method to the one dimensional Eisenstein distribution  ${}^{(1)}E^*$ , and given by a natural summation procedure (we shall see in (3.5) that this distribution has a geometric meaning as a certain distribution on the orbit space related to the following action of the group  $\mathbb{Z}_p^*$  on the product  $\mathbb{Z}_p \times \mathbb{Z}_p$  by the formula  $(t, (a, b)) \mapsto (at^{-1}, tb)$ ):

$$(3.3) \quad \begin{aligned} & {}^{(1)}E^*(a + (p^m)) = E_{k,p^m}^*(a) := (p^m)^{k-1} \Gamma(k) \times \\ & \sum_{x,b \bmod p^m} e\left(-\frac{ax}{p^m}\right) E_{k,p^m}^*(p^m z; x, b) \in \mathcal{M}_k. \end{aligned}$$

Here for a complex number  $k \geq 3$  and  $a, b \bmod p^m$  we use the series

$$\begin{aligned} E_{k,p^m}^*(z; a, b) &= \sum (cz + d)^{-k} \\ & ((c, d) \text{ coprime}, (c, d) \equiv (a, b) \bmod p^m). \end{aligned}$$

Notice that we consider the Eisenstein series of one complex variable; however, the Eisenstein distribution depends on two  $p$ -adic variables, denoted by  $a$  and  $b$ ,  $(a, b) \in \mathbb{Z}_p^2$ .

Notice that there exists also a two-dimensional distribution

$$(3.4) \quad \begin{aligned} E^*(a + (p^m), b + (p^m)) &= E_{k,p^m}^*(a, b) := (p^m)^{k-1} \Gamma(k) \times \\ &\sum_{x \bmod p^m} e\left(-\frac{ax}{p^m}\right) E_{k,p^m}^*(p^m z; x, b) \in \mathcal{M}_k. \end{aligned}$$

The use of the Möbius transform shows that one has (compare again with Hecke's formula at page 476 in [He27])

$$E_{k,p^m}^*(z; a, b) = \sum_{t \bmod p^m} E_{k,p^m}(z; ta, tb) c_t \in \mathcal{M}_k.$$

Let us substitute this identity into (3.4) giving

$$(3.5) \quad \begin{aligned} E^*(a + (p^m), b + (p^m)) &= E_{k,p^m}^*(a, b) := (p^m)^{k-1} \Gamma(k) \times \\ &\sum_{x \bmod p^m} e\left(-\frac{ax}{p^m}\right) \sum_{t \bmod p^m} c_t E_{k,p^m}(p^m z; tx, tb) \\ &= (p^m)^{k-1} \Gamma(k) \times \\ &\sum_{t \bmod p^m} \sum_{x' \bmod p^m} e\left(-\frac{at^{-1}x'}{p^m}\right) c_t E_{k,p^m}(p^m z; x', tb) \\ &= (p^m)^{k-1} \Gamma(k) \sum_{t \bmod p^m} c_t E_{k,p^m}(at^{-1}, tb) \in \mathcal{M}_k, \end{aligned}$$

where we have used the previously defined Eisenstein distribution (3.1).

It follows from their definition that the numbers

$$c_t = \begin{cases} \sum_{tn \equiv 1 \pmod{p^m}} \frac{\mu(n)}{n^k}, & \text{if } (t, p^m) = 1 \\ 0, & \text{if } (t, p^m) > 1 \end{cases}$$

form themselves a complex-valued distribution (i.e. with values in  $A = \mathbb{C}$ )  $\mu_k$  such that

$$\int_{\mathbb{Z}_p^*} \chi(y) \mu_k(y) = L(k, \bar{\chi})^{-1} (1 - \bar{\chi}(p)p^{-k})^{-1} (1 + \bar{\chi}(-1)(-1)^k).$$

Let us explain in more detail: indeed,

$$\mu_k(t + (p^m)) = \begin{cases} \sum_{tn \equiv 1 \pmod{p^m}} \frac{\mu(n)}{n^k}, & \text{if } (t, p^m) = 1 \\ 0, & \text{if } (t, p^m) > 1 \end{cases}$$

is a partial Dirichlet series, so that the finite-additivity clearly holds in the absolutely convergent case (like in [Co-PeRi], §1). Moreover

$$\begin{aligned} \chi(tn) = 1 &\Rightarrow \chi(t) = \bar{\chi}(n), \\ \int_{\mathbb{Z}_p^*} \chi(y)\mu_k(y) &= \sum_{t \bmod p^m} \chi(t)\mu_k(t + (p^m)) = \sum_{(n,p)=1} \mu(n)\bar{\chi}(n)n^{-k} \\ &= L(k, \bar{\chi})^{-1}(1 - \bar{\chi}(p)p^{-k})^{-1}(1 + \bar{\chi}(-1)(-1)^k), \end{aligned}$$

that is, a Dirichlet series with the Euler  $p$ -factor removed from its Euler product.

Hence the formula (3.5) defines a two-variable Eisenstein distribution  $E_{k,p^m}(a, b)$  and of the one-variable distribution  $\mu_k$ . This definition will be used with  $k \geq 3$ .

A *geometric meaning* of this construction is related to the action of the group  $\mathbb{Z}_p^*$  on the product  $\mathbb{Z}_p \times \mathbb{Z}_p$  by the formula

$$(t, (a, b)) \mapsto (at^{-1}, tb)$$

and follows from the formula (3.5). This action produces a natural (two-variable) distribution  $E_{k,p^m}^*(a, b)$  on the product  $\mathbb{Z}_p \times \mathbb{Z}_p$  out of the two distributions  $E_{k,p^m}(a, b)$  and  $\mu_k(t)$ . Then the one variable distribution

$${}^{(1)}E_{k,p^m}^*(a) = \sum_{b \bmod p^m} E_{k,p^m}^*(a, b)$$

given by (3.3) can be viewed as the direct image (the integration along the fibers) of the two-variable distribution  $E_{k,p^m}^*(a, b)$  with respect to the projection of  $\mathbb{Z}_p \times \mathbb{Z}_p$  onto the first component.

In order to give another explanation of the distribution property of the series (3.3) one can also use the following Eisenstein series with a Dirichlet character  $\chi \bmod p^m$ :

$$\begin{aligned} E_k(\chi) &= \sum_{a,b \in \mathbb{Z}/p^m\mathbb{Z}} \chi(a)E_k(a, b; p^m) = \\ &\sum_{a,b \in \mathbb{Z}/p^m\mathbb{Z}} \chi(a)E_{k,p^m}(a, b) = (p^m)^{k-1}\Gamma(k) \\ &\times \sum_{a,b \in \mathbb{Z}/p^m\mathbb{Z}} \chi(a) \sum_{x \bmod p^m} e\left(-\frac{ax}{p^m}\right) E_{k,p^m}(p^m z; x, b) \\ &= L(1 - k, \chi) + \\ &\sum_{a,b \in \mathbb{Z}/p^m\mathbb{Z}} \chi(a) \sum_{0 < dd', d \equiv a, d' \equiv b \bmod p^m} \text{sgn } d \cdot d^{k-1} e(dd'z) \in \mathcal{M}_k(p^m), \end{aligned}$$

where

$$L(1-k, \chi) = \sum_{a, b \in \mathbb{Z}/p^m \mathbb{Z}} \chi(a) \delta\left(\frac{b}{p^m}\right) \zeta(1-k; a, p^m)$$

Let us substitute this identity into (3.3) giving

$$\begin{aligned} (3.6) \quad & {}^{(1)}E^*(a + (p^m)) = E_{k, p^m}^*(a) \\ & := (p^m)^{k-1} \Gamma(k) \times \sum_{x, b \bmod p^m} e\left(-\frac{ax}{p^m}\right) \\ & = \sum_{t \bmod p^m} E_{k, p^m}(p^m z; tx, tb) c_t \\ & = (p^m)^{k-1} \Gamma(k) \times \\ & \quad \sum_{t \bmod p^m} \sum_{x', b \bmod p^m} e\left(-\frac{at^{-1}x'}{p^m}\right) E_{k, p^m}(p^m z; x', tb) c_t \\ & (p^m)^{k-1} \Gamma(k) \sum_{t \bmod p^m} c_t E_{k, p^m}(at^{-1}, tb) \in \mathcal{M}_k. \end{aligned}$$

where  $x' = tx \bmod p^m$  (notice that  $t \in (\mathbb{Z}/p^m \mathbb{Z})^*$  is invertible).

We obtain again the geometric meaning of this construction related to the action of the group  $\mathbb{Z}_p^*$  on the product  $\mathbb{Z}_p \times \mathbb{Z}_p$  by the formula

$$(t, (a, b)) \mapsto (at^{-1}, tb)$$

which will also follow from the formula (3.6) (to be discussed below). This action produces a natural (two-variable) distribution  $E_{k, p^m}^*(a, b)$  on the orbit space out of the two distributions  $E_{k, p^m}(a, b)$  and  $\mu_k(t)$ . Then the one variable distribution (3.3)

$${}^{(1)}E_{k, p^m}^*(a) = \sum_{b \bmod p^m} E_{k, p^m}^*(a, b)$$

is obtained by "integration along the fibers".

Also we can evaluate the integrals  $\int_{\mathbb{Z}_p^*} \chi E_{k, p^m}^*$

$$\begin{aligned} (3.7) \quad & = \sum_{a \bmod p^m} \chi(a) E_{k, p^m}^*(a) \sum_{t \bmod p^m} \sum_{a, b \bmod p^m} c_t E_{k, p^m}(at^{-1}, tb) \\ & = \sum_{t \bmod p^m} \sum_{a, b \bmod p^m} \chi(t) \chi(at^{-1}) c_t E_{k, p^m}(at^{-1}, tb) \\ & = \sum_{t \bmod p^m} \chi(t) c_t \sum_{a', b' \bmod p^m} \chi(a') E_{k, p^m}(a', b') \\ & = L(k, \bar{\chi})^{-1} (1 - \bar{\chi}(p) p^{-k})^{-1} \cdot E_k(\chi), \end{aligned}$$

using the complex-valued distribution  $\mu_k$ , such that

$$\int_{\mathbb{Z}_p^*} \chi(y)\mu_k(y) = L(k, \bar{\chi})^{-1}(1 - \bar{\chi}(p)p^{-k})^{-1}(1 + \bar{\chi}(-1)(-1)^k).$$

Let  $k \geq 3$  be an integer. Then we consider the Eisenstein distributions  $\Phi_k = {}^{(1)}E_k^*$  of weight  $k$ , and take  $\alpha = 1$ .

**Definition 3.4.** *The numerically-valued distributions  $\mu_k^*$  is given by the equality*

$$\mu_k^*(a + (p^m)) = \ell(\pi_1({}^{(1)}E_k^*(a + (p^m)))),$$

where  $\pi_1 : \mathcal{M}_k \rightarrow \mathcal{M}_k^1 \subset \mathcal{M}_k(Np)$  is the canonical projection (3.2) to  $\mathcal{M}_k^1$  with  $\alpha = 1$ .

Recall that this Proposition assures that different  $\pi_{\alpha,m}$  can be glued together using the commutative diagram, so we do not have to worry about the exact level of this Eisenstein series defining the Eisenstein distribution. Also,  $\ell : \mathcal{M}_k^{(1)} \rightarrow A$  is given by  $\ell(\sum_n a_n q^n) = a_1$ .

We shall establish the boundedness of each distribution  $\mu_k^*$  under the assumptions of the Theorem 1.1 in Section 1.

In Section 4 we explain how to obtain the inverse of the Kubota-Leopoldt zeta-function from the distributions  $\mu_k^*$  of the definition 3.4.

#### 4. $p$ -ADIC $L$ -FUNCTIONS AND MELLIN TRANSFORMS

Let us recall with the notation  $\zeta_{(p)}^{(c)}(-k) = (1 - p^k)(1 - c^{k+1})\zeta(-k)$  (for  $c > 1$  coprime to  $p$ ) the following theorem:

**THEOREM 4.1 (Kummer).** *For any polynomial  $h(x) = \sum_{i=0}^n \alpha_i x^i \in \mathbb{Z}_p[x]$  over  $\mathbb{Z}_p$  such that  $x \in \mathbb{Z}_p \implies h(x) \in p^m \mathbb{Z}_p$  one has*

$$\sum_{i=0}^n \alpha_i \zeta_{(p)}^{(c)}(-i) \in p^m \mathbb{Z}_p.$$

The result of Kubota and Leopoldt has a very natural interpretation in the framework of the theory of non-Archimedean integration (due to B. Mazur): there exists a  $p$ -adic measure  $\mu^{(c)}$  on  $\mathbb{Z}_p^\times$  with values in  $\mathbb{Z}_p$  such that  $\int_{\mathbb{Z}_p^\times} x^k \mu^{(c)} = \zeta_{(p)}^{(c)}(-k)$ .

Indeed, Mazur's result can be explained as follows: if we integrate  $h(x)$  over  $\mathbb{Z}_p^\times$  we exactly get the above congruence (see Theorem 4.1). On the other hand, in order to define a measure  $\mu^{(c)}$  satisfying the above condition it suffices for any continuous function  $\phi : \mathbb{Z}_p^\times \rightarrow \mathbb{Z}_p$  to define its integral  $\int_{\mathbb{Z}_p^\times} \phi(x)\mu^{(c)}$ . For this purpose we approximate  $\phi(x)$  by a polynomial (for which the integral is already defined), and then pass to the limit.

A Dirichlet character  $\chi : (\mathbb{Z}/p^m\mathbb{Z})^\times \rightarrow \overline{\mathbb{Q}}^\times$  is an element of the torsion subgroup

$$X_p^{\text{tors}} \subset X_p = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$$

and the above equality also holds for the special values  $L(-k, \chi)$  of the Dirichlet  $L$ -series

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)n^{-s} = \prod_p (1 - \chi(p)p^{-s})^{-1},$$

so that we have

$$\zeta_p(\chi x_p^k) = i_p [(1 - \chi(p)p^k)L(-k, \chi)] \quad (k \geq 1, k \in \mathbb{Z}, \chi \in X_p^{\text{tors}}).$$

The construction of Kubota and Leopoldt is equivalent to the existence of a  $p$ -adic analytic function  $\zeta_p : X_p \rightarrow \mathbb{C}_p$  with a single pole at the point  $x = x_p^{-1}$ , which becomes a bounded holomorphic function on  $X_p$  after multiplication by the elementary factor  $(x_p x - 1)$ , ( $x \in X_p$ ), and is uniquely determined by the condition

$$\zeta_p(x_p^k) = (1 - p^k)\zeta(-k) \quad (k \geq 1).$$

This original construction was successfully used by K. Iwasawa for the description of the class groups of cyclotomic fields. The Main conjecture by Iwasawa, relating zeroes of the  $p$ -adic zeta-function and the class groups of cyclotomic fields was proved by Mazur and Wiles in [MW84], implying that for a regular prime  $p$ , the Kubota-Leopoldt zeta function has no zeroes.

**Remark on  $\mu_c$  and the pole of  $\zeta_p$ .** The single simple pole of  $\zeta_p(x)$  at the point  $x = x_p^{-1}$  can be removed by the regularizing factor

$$(c^{k+1} - 1) = (x(c)c - 1) \text{ for } x = x_p^k.$$

This regularizing factor comes directly from the Kummer theorem. However such a factor would not be needed for a regular prime  $p$  and the measure  $\mu^*$  because in this case the function  $\zeta_p^{-1}$  is holomorphic on  $X_p$  with a single simple zero. Also, notice that for a non regular prime the construction of  $\mu^*$  would require a certain regularizing factor with a finite number of zeroes which correspond to zeroes of  $\zeta_p(x)$ , due to the theorem of Mazur-Wiles (see [MW84]).

The main result says that there exists a  $p$ -adic measure  $\mu^*$  which provides an interpolation of the inverse special values

$$\zeta(1 - k)^{-1}(1 - p^{k-1})^{-1}$$

that is, for all  $k \geq 3 \pmod{p^m}$  one has

$$\int_{\mathbb{Z}_p^*} x_p^k(y)\mu^* = \mu_k^*(\mathbb{Z}_p^*).$$

In order to clarify the relation between  $\mu^*$  and  $\mu_k^*$  notice that

$$\int_{\mathbb{Z}_p^*} \varphi(y)x_p^k(y)\mu^* = \int_{\mathbb{Z}_p^*} \varphi(y)\mu_k^*, \text{ where } x_p^k(y) = y^k,$$

for any continuous function  $\varphi(y)$ .

In other words, the Mellin transform of the distributions  $\mu^*$  coincides essentially with the inverse Kubota-Leopoldt zeta-function  $\zeta_p^{-1}(x)$ , that is, up to normalization,

$$\int_{\mathbb{Z}_p^*} x(y)\mu^*(y) = \zeta_p^{-1}(x),$$

where  $x \in X_p$ . Recall that  $X_p$  is the following  $p$ -adic analytic Lie group:

$$X_p = \text{Hom}_{cont}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times).$$

In particular, for any  $k \geq 3$ , and  $x = x_p^{k-1}$  one has

$$\zeta_p^{-1}(x) = \int_{\mathbb{Z}_p^*} x_p^k \mu^* = (1 - p^{k-1})^{-1} \zeta(1 - k)^{-1} \quad (k \geq 3 \text{ even}).$$

### 5. AXIOMATIC CHARACTERIZATION OF THE PROPERTIES OF $\mu^*$

Recall (see Definition 3.4): for any  $k \geq 3$  we put  $\Phi_k = E_k^*$ ,  $\alpha = 1$ , and

$$\mu_k^*(a + (p^m)) = \ell(\pi_1({}^{(1)}E_k^*(a + (p^m)))),$$

where

$${}^{(1)}E_k^*(a + (p^m)) = \sum_n a_{n,k}(a + (p^m))q^n,$$

$\pi_1 : \mathcal{M}_k \rightarrow \mathcal{M}_k^1 \subset \mathcal{M}_k(Np)$  is defined using Proposition 3.3, which assures that different  $\pi_{\alpha,m}$  can be glued together using the commutative diagram, so we do not have to worry about the exact level of this Eisenstein series defining the Eisenstein distribution; also,  $\ell : \mathcal{M}_k^1 \rightarrow A$  is given by

$$\ell\left(\sum_n a_n q^n\right) = a_1.$$

#### Remarks.

1) We establish the boundedness of  $\mu^*$  under the assumptions of the Theorem 1.1 in two steps. First, we give an axiomatic characterization of this situation, and then we check that this characterization is actually valid in our case.

By (3.2) one has

$$\pi_1({}^{(1)}E_k^*(a + (p^m))) = (U_p^1)^{-m'} (\pi_{1,0}({}^{(1)}E_k^*(a + (p^m))))|U_p^{m'}$$

for any sufficiently large  $m'$ . Keeping in mind that  $\pi_{1,0}$  is fixed and  $U_p^1$  is invertible, we see that it suffices to prove that

$$\beta({}^{(1)}E_k^*(a + (p^m))|U_p^{m'}) = \beta \sum_n a_{np^{m'},k}(a + (p^m))q^n \in \mathcal{O}_p[[q]]$$

for some  $\beta \in \mathbb{Q}_p^*$  independent of  $m'$  (i.e. the Fourier coefficients in the RHS are  $p$ -bounded).

2) Evaluation of the linear form

$$\mu_k^*(a + (p^m)) = \ell(\pi_1({}^{(1)}E_k^*(a + (p^m))))$$

makes it possible to compute all the integrals

$$\sum_{a \bmod p^m} \chi(a) \mu_k^*(a + (p^m)) = \ell(\pi_1({}^{(1)}E_k^*(\chi)))$$

in an explicit form.

*Proof of 1):* we may and will suppose that  $n = 1$ , which is only needed for our purposes, and  $m' \gg 0$  (in view of the distribution property for  $E^*$ ,

-First step: to substitute formulas of Proposition 2.1 into (2.5) (and to carry out the summation over  $d, d', \delta, \delta'$ )

-Second step: to compute the Fourier coefficients of the series  ${}^{(1)}E^*$

-Third step: to evaluate the particular Fourier coefficient  $a_{p^{m'}}$

Recall that we assume  $k \geq 3$  (that is, assuming the absolute convergence and holomorphy).

Let us use the identity (2.5) and compute the Fourier expansion of the series (3.3) with  $k \geq 3$ :

$$(5.1) \quad \begin{aligned} {}^{(1)}E_k^*(a + (p^m)) &= {}^{(1)}E_{k,p^m}^*(a + (p^m)) = E_{k,p^m}^*(a) \\ &:= (p^m)^{k-1} \Gamma(k) \sum_{x,b \bmod p^m} e(-ax/(p^m)) E_{k,p^m}^*(p^m z; x, b) \end{aligned}$$

where

$$(5.2) \quad E_{k,p^m}^*(p^m z; x, b) = \sum_{t \bmod p^m} c_t E_{k,p^m}(p^m z; tx, tb),$$

and

$$c_t = \begin{cases} \sum_{tn \equiv 1 \pmod{p^m}} \frac{\mu(n)}{n^k}, & \text{if } (t, p^m) = 1 \\ 0, & \text{if } (t, p^m) > 1 \end{cases}$$

(here  $\mu$  denotes again the Moebius function!).

Then the direct substitution  $a := at, b := bt, N := p^m, z := p^m z$  into the Fourier expansion of Proposition 2.1 gives:

$${}^{(1)}E_k^*(a + (p^m)) = E_{k,p^m}^*(a) = (p^m)^{k-1} \Gamma(k)$$

$$\begin{aligned} & \times \sum_{x,b \bmod p^m} e(-ax/(p^m)) \sum_{t \bmod p^m} c_t E_{k,p^m}(p^m z; tx, tb) \\ & = (p^m)^{k-1} \Gamma(k) \sum_{x,b \bmod p^m} e(-ax/(p^m)) \sum_{t \bmod p^m} \times \\ & \delta\left(\frac{xt}{p^m}\right) [\zeta(k; bt, p^m) + (-1)^k \zeta(k; -bt, p^m)] c_t \\ & + c_t \frac{(-2\pi i)^k}{(p^m)^k \Gamma(k)} \sum_{\substack{dd' > 0 \\ d' \equiv xt \bmod N}} \text{sgn}(d) d^{k-1} e\left(\frac{dbt}{p^m}\right) e(dd'z). \end{aligned}$$

Recall that we wish to compute only the Fourier coefficient  $a_{p^{m'},k}$ , that is, we put  $dd' = p^{m'}$ , giving the formula

$$(5.3) \quad \begin{aligned} a_{p^{m'},k}({}^{(1)}E_k^*(a + (p^m))) & = (p^m)^{-1} (-2\pi i)^k \\ & \sum_{x,b,t \bmod p^m} e(-ax/(p^m)) \sum_{\substack{dd' = p^{m'} \\ d' \equiv xt \bmod N}} c_t \text{sgn}(d) d^{k-1} e\left(\frac{dbt}{p^m}\right). \end{aligned}$$

In the summation over  $d, d'$  in (5.3) one has  $d = \pm p^{m_d}, d' = \mp p^{m' - m_d}$ , where  $m_d = 0, \dots, m'$ . Thus,

$$(5.4) \quad \begin{aligned} a_{p^{m'},k}({}^{(1)}E_k^*(a + (p^m))) & = (p^m)^{-1} (-2\pi i)^k \times \\ & \sum_{x,b,t \bmod p^m} e(-ax/(p^m)) \sum_{tn \equiv 1 \bmod p^m} \frac{\mu(n)}{n^k} \\ & \left( \sum_{\substack{m_d=0,\dots,m' \\ p^{m'-m_d} \equiv xt \bmod N}} d^{k-1} e\left(\frac{dbt}{p^m}\right) - \sum_{\substack{m_d=0,\dots,m' \\ p^{m'-m_d} \equiv -xt \bmod N}} d^{k-1} e\left(\frac{dbt}{p^m}\right) \right). \end{aligned}$$

Let us evaluate next Hecke's coefficients

$$c_t = \begin{cases} \sum_{tn \equiv 1 \bmod p^m} \frac{\mu(n)}{n^k}, & \text{if } (t, p^m) = 1 \\ 0, & \text{if } (t, p^m) > 1. \end{cases},$$

in order to relate them with the value  $\zeta(1 - k)^{-1}$  of the Theorem 1.1 (here  $\mu$  denotes again the Moebius function!).

Notice that for  $k \geq 2$  one has

$$\sum_{n \geq 1} \frac{\mu(n)}{n^k} = \zeta(k)^{-1}, \quad \sum_{n \geq 1} \chi(n) \frac{\mu(n)}{n^k} = L(k, \chi)^{-1},$$

$$(5.5) \quad \sum_{tn \equiv 1 \pmod{p^m}} \frac{\mu(n)}{n^k} = \frac{1}{\varphi(p^m)} \sum_{\chi \pmod{p^m}} \sum_{(tn, \pmod{p^m})=1} \frac{\chi(tn)\mu(n)}{n^k}$$

$$= \frac{1}{\varphi(p^m)} \sum_{\chi \pmod{p^m}} \chi(t) L(k, \bar{\chi})^{-1},$$

in view of the orthogonality relations. Recall that the numbers

$$c_t = \begin{cases} \sum_{tn \equiv 1 \pmod{p^m}} \frac{\mu(n)}{n^k}, & \text{if } (t, p^m) = 1 \\ 0, & \text{if } (t, p^m) > 1 \end{cases}$$

form themselves a complex-valued distribution (i.e. with values in  $A = \mathbb{C}$ )  $\mu_k$  such that for  $N = 1$

$$\int_{\mathbb{Z}_p^*} \chi(y) \mu_k(y) = L(k, \bar{\chi})^{-1} (1 - \bar{\chi}(p)p^{-k})^{-1} (1 + \bar{\chi}(-1)(-1)^k).$$

Let us rewrite (5.4) using (5.5). It follows that

$$(5.6) \quad a_{p^{m'}, k}({}^{(1)}E_k^*(a + (p^m))) = (p^m)^{-1} (-2\pi i)^k \frac{1}{\varphi(p^m)} \times$$

$$\sum_{x, t \pmod{p^m}} \sum_{\chi \pmod{p^m}} e(-ax/(p^m)) \chi(t) L(k, \bar{\chi})^{-1} \frac{(1 + \bar{\chi}(-1)(-1)^k)}{(1 - \bar{\chi}(p)p^{-k})}$$

$$\left( \sum_{\substack{m_d=0, \dots, m' \\ p^{m'} - m_d \equiv xt \pmod{p^m}}} d^{k-1} e\left(\frac{dbt}{p^m}\right) - \sum_{\substack{m_d=0, \dots, m' \\ p^{m'} - m_d \equiv -xt \pmod{p^m}}} d^{k-1} e\left(\frac{dbt}{p^m}\right) \right).$$

Let us recall the abstract Kummer congruences giving the following useful criterion for the existence of a measure with given properties.

**Proposition 5.1** (The abstract Kummer congruences (see [Ka78])).

Let  $Y = \mathbb{Z}_p^*$ , and  $\{f_i\}$  be a system of continuous functions  $f_i \in \mathcal{C}(Y, \mathcal{O}_p)$  in the ring  $\mathcal{C}(Y, \mathcal{O}_p)$  of all continuous functions on the compact totally disconnected group  $Y$  with values in the ring of integers  $\mathcal{O}_p$  of  $\mathbb{C}_p$  such that  $\mathbb{C}_p$ -linear span of  $\{f_i\}$  is dense in  $\mathcal{C}(Y, \mathcal{O})$ . Let also  $\{a_i\}$  be any system of elements  $a_i \in \mathcal{O}_p$ . Then the existence of an  $\mathcal{O}_p$ -valued measure  $\nu$  on  $\mathbb{Z}_p^*$  with the property

$$\int_Y f_i d\nu = a_i$$

is equivalent to the following congruences: for an arbitrary choice of elements  $b_i \in \mathcal{C}_p$ , almost all of which vanish,

$$(5.7) \quad \sum_i b_i f_i(y) \in p^m \mathcal{O}_p \text{ for all } y \in Y \text{ implies } \sum_i b_i a_i \in p^m \mathcal{O}_p.$$

In order to prove the abstract Kummer congruences under the assumptions of our theorem, we need to show that the algebraic number

(5.6) has bounded denominator, that is, for a fixed non-zero integer  $C$  independent of  $\chi$  and  $k$ , the number

$$(5.8) \quad C \cdot \sum_{x,b,t \bmod p^m} \sum_{\chi \bmod p^m} e(-ax/(p^m)) \chi(t) \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})}$$

$$\left( \sum_{\substack{m_d=0, \dots, m' \\ p^{m'-m_d} \equiv xt \bmod p^m}} d^{k-1} e\left(\frac{dbt}{p^m}\right) - \sum_{\substack{m_d=0, \dots, m' \\ p^{m'-m_d} \equiv -xt \bmod p^m}} d^{k-1} e\left(\frac{dbt}{p^m}\right) \right)$$

is divisible by  $(p^m)^2$ .

Note that the role of the dense family  $f_i$  of Proposition 5.1 here is played by the family of all Dirichlet's characters, so that the boundedness condition of this proposition becomes the main congruence

$$(5.9) \quad \forall y \in \mathbb{Z}_p^*, \sum_i b_i \chi_i(y) \equiv 0 \bmod p^m \Rightarrow C \sum_i b_i \mu_k^*(\chi_i) \equiv 0 \bmod p^m;$$

see also [Bö-Sch], §9, [Co-Sch], p.134 and Theorem 3.14, [KMS], [Schm01], §9, [CourPa], §4.6.5 "Main congruences for the Fourier expansions of regularized distributions", [PaMMJ2], §5.2 "Main congruence for the Fourier expansions", and [PaTV], congruence (4.9).

Note that for a Dirichlet character  $\chi = f_i$ , we specify the coefficient  $a_i$  as the right hand side of the linear combination of characters given by (5.8).

Moreover, each continuous function can be approximated by linear combinations of delta functions  $\delta_U(y)$  of open subsets:

$$\delta_U(y) = \begin{cases} 1, & \text{if } y \in U, \\ 0, & \text{sinon.} \end{cases} \quad \text{It suffices therefore to treat only the those linear combinations which produce delta functions } \delta_U(y) \text{ of open subsets in terms of Dirichlet characters:}$$

$$\delta(a + (p^m))(y) = \frac{1}{\varphi(p^m)} \sum_{\chi \bmod p^m} \bar{\chi}(a) \chi(y).$$

In order to establish the divisibility in (5.9) for such linear combinations, let us change the order of summation in the multiple sum (5.8)

as follows:

$$\begin{aligned}
 & \sum_{m_d=0, \dots, m'} d^{k-1} \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} \\
 & \times \sum_{\substack{x, b, t \bmod p^m \\ p^{m'} - m_d \equiv xt \bmod p^m}} \chi(t) e\left(\frac{dbt - ax}{p^m}\right) \\
 (5.10) \quad & - \sum_{m_d=0, \dots, m'} d^{k-1} \sum_{\chi \bmod p^m} \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} \\
 & \times \sum_{\substack{x, b, t \bmod p^m \\ p^{m'} - m_d \equiv -xt \bmod p^m}} \chi(t) e\left(\frac{dbt - ax}{p^m}\right).
 \end{aligned}$$

Let us prove that for  $k \geq 3$  and for any Dirichlet character  $\chi \bmod p^m$  of conductor  $C_\chi | p^m$  the term in the sum over  $\chi \bmod p^m$  in (5.10) is divisible by  $C_\chi^2 \times (p^m/C_\chi)^2 = (p^m)^2$ , that is, the inner sum is divisible by  $(p^m/C_\chi)^2$ , and

$$(5.11) \quad \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} \text{ is divisible by } C_\chi^2.$$

In order to prove the divisibility by (5.11), let's use the functional equation for the Dirichlet  $L$ -functions for the primitive characters, see [Wa82], p.29:

$$(5.12) \quad \Gamma(s) \cos\left(\frac{\pi(s - \delta)}{2}\right) L(s, \bar{\chi}) = \frac{G(\chi)}{2i^\delta} \left(\frac{2\pi}{C_\chi}\right)^s L(1 - s, \chi),$$

where  $G(\chi) = \sum_{u \bmod C_\chi} \chi(u) e\left(\frac{u}{C_\chi}\right)$  is the Gauss sum. Substitution  $s = k$  gives

$$(5.13) \quad \frac{(-2\pi i)^k}{L(k, \bar{\chi})} = \Gamma(k) \cos\left(\frac{\pi(k - \delta)}{2}\right) \frac{2i^\delta}{G(\chi)} C_\chi^k L(1 - k, \chi)^{-1}.$$

Let us take into account the fact that for a regular prime  $p$  the  $p$ -adic  $L$ -function does not vanish (due to the theorem by Mazur-Wiles, see section 4), so that its inverse values

$$(5.14) \quad L(1 - k, \chi)^{-1} (1 - \chi(p)p^{k-1})^{-1}$$

are all  $p$ -adically integral after multiplying by a constant  $c'$  independent of  $\chi$  and  $k$ .

Since  $\frac{C_\chi}{G(\bar{\chi})} = \chi(-1)G(\chi)$  is always an algebraic integer, and  $k \geq 3$ , we obtain the divisibility by  $C_\chi^2$  of the value

$$(5.15) \quad \frac{(-2\pi i)^k}{L(k, \bar{\chi})} = \Gamma(k) \cos\left(\frac{\pi(k - \delta)}{2}\right) \frac{2i^\delta C_\chi}{G(\bar{\chi})} C_\chi^{k-1} L(1 - k, \chi)^{-1}.$$

Thus, our problem is reduced to study of the divisibility of the sum

$$(5.16) \quad \sum_{\substack{x,b,t \bmod p^m \\ p^{m'}-m_d \equiv \pm xt \bmod p^m}} \chi(t) e\left(\frac{dbt - ax}{p^m}\right)$$

for any fixed  $d = \pm p^{m_d}$  taking into account that  $\chi(t)$  depends only on  $t \bmod C_\chi$ , that is  $t \equiv t_0 \bmod C_\chi$ ,  $t = t_0 + C_\chi t'$ , where  $t'$  runs through classes  $\bmod p^m/C_\chi$ .

A direct substitution of  $t = t_0 + C_\chi t'$  into (5.16) transforms this sum as follows:

$$(5.17) \quad \sum_{\substack{x,b,t \bmod p^m \\ p^{m'}-m_d \equiv \pm xt \bmod p^m}} \chi(t) e\left(\frac{dbt - ax}{p^m}\right) =$$

$$\sum_{t_0 \bmod C_\chi} \chi(t_0) \sum_{\substack{x,b \bmod p^m \\ p^{m'}-m_d \equiv \pm xt \bmod p^m}} \sum_{t' \bmod p^m/C_\chi} e\left(\frac{db(t_0 + C_\chi t') - ax}{p^m}\right)$$

$$\sum_{t_0 \bmod C_\chi} \chi(t_0) e\left(\frac{dbt_0}{p^m}\right) \sum_{\substack{x,b \bmod p^m \\ p^{m'}-m_d \equiv \pm xt \bmod p^m}} \sum_{t' \bmod p^m/C_\chi} e\left(\frac{dbC_\chi t' - ax}{p^m}\right).$$

A direct computation of the last exponential sum provides its divisibility by  $(p^m/C_\chi)^2$  as claimed. This finishes the proof of the boundedness of  $\mu^*$  (as well as that of  $\mu_k^*$ ).

Note that the Kummer congruences are used here in the equivalent form of the boundedness of a certain distribution.

This, *by Proposition (5.1)*, finishes the proof of boundedness of  $\mu^*$  (as well as that of  $\mu_k^*$ ).

For the statement 2), we need to evaluate the Fourier coefficients (5.6)(and the integrals of the related numerically valued distributions):

$$(5.18) \quad a_{p^{m'},k}({}^{(1)}E_k^*(a + (p^m))) = (p^m)^{-1}(-2\pi i)^k \frac{1}{\varphi(p^m)} \times$$

$$\sum_{x,t \bmod p^m} \sum_{\chi \bmod p^m} e(-ax/(p^m)) \chi(t) L(k, \bar{\chi})^{-1} \frac{(1 + \bar{\chi}(-1)(-1)^k)}{(1 - \bar{\chi}(p)p^{-k})} \times$$

$$\left( \sum_{\substack{m_d=0,\dots,m' \\ p^{m'}-m_d \equiv xt \bmod p^m}} d^{k-1} e\left(\frac{dbt}{p^m}\right) - \sum_{\substack{m_d=0,\dots,m' \\ p^{m'}-m_d \equiv -xt \bmod p^m}} d^{k-1} e\left(\frac{dbt}{p^m}\right) \right)$$

$$= \frac{1}{\varphi(p^m)p^m} \sum_{m_d=0,\dots,m'} d^{k-1} \sum_{\chi \bmod p^m} \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})}$$

$$\begin{aligned}
& \times \sum_{t_0 \bmod C_\chi} \chi(t_0) e\left(\frac{dbt_0}{p^m}\right) \sum_{\substack{x, b \bmod p^m \\ p^{m'} - md \equiv xt \bmod p^m}} \sum_{t' \bmod p^m / C_\chi} e\left(\frac{dbC_\chi t' - ax}{p^m}\right) \\
& - \frac{1}{\varphi(p^m)p^m} \sum_{m_d=0, \dots, m'} d^{k-1} \sum_{\chi \bmod p^m} \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} \\
& \times \sum_{t_0 \bmod C_\chi} \chi(t_0) e\left(\frac{dbt_0}{p^m}\right) \sum_{\substack{x, b \bmod p^m \\ p^{m'} - md \equiv -xt \bmod p^m}} \sum_{t' \bmod p^m / C_\chi} e\left(\frac{dbC_\chi t' - ax}{p^m}\right).
\end{aligned}$$

Finally, let's compute the integrals  $\mu_k^*(\chi)$  from the expression (5.18) by carrying out the summations  $a \bmod p^m$ . This can be done using the formula (5.5).

Let's rewrite our formulae as follows:

$$\begin{aligned}
(5.19) \quad & \int_{\mathbb{Z}_p^*} \mu_k^* = \sum_{\substack{a \bmod p^m \\ (a, p)=1}} a_{p^{m'}, k}(E_k^*(a + (p^m))) = \\
& \frac{1}{\varphi(p^m)p^m} \sum_{m_d=0, \dots, m'} d^{k-1} \sum_{\chi \bmod p^m} \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} \\
& \sum_{t_0 \bmod C_\chi} \chi(t_0) e\left(\frac{dbt_0}{p^m}\right) \times \sum_{\substack{a, x, b \bmod p^m \\ (a, p)=1, p^{m'} - md \equiv xt \bmod p^m}} \sum_{t' \bmod p^m / C_\chi} e\left(\frac{dbC_\chi t' - ax}{p^m}\right) \\
& - \frac{1}{\varphi(p^m)p^m} \sum_{m_d=0, \dots, m'} d^{k-1} \sum_{\chi \bmod p^m} \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} \\
& \sum_{t_0 \bmod C_\chi} \chi(t_0) e\left(\frac{dbt_0}{p^m}\right) \times \sum_{\substack{a, x, b \bmod p^m \\ (a, p)=1, p^{m'} - md \equiv -xt \bmod p^m}} \sum_{t' \bmod p^m / C_\chi} e\left(\frac{dbC_\chi t' - ax}{p^m}\right).
\end{aligned}$$

Next let's notice that the main term in the RHS in (5.19) is given by (5.13):

$$\frac{(-2\pi i)^k}{L(k, \bar{\chi})} = \Gamma(k) \cos\left(\frac{\pi(k - \delta)}{2}\right) \frac{2i^\delta}{G(\chi)} C_\chi^k L(1 - k, \chi)^{-1}.$$

In order to obtain the RHS of Theorem 1.1 it suffices to substitute the last expression to (5.19) and to carry out all the summations.

The first summation is

$$\sum_{t_0 \bmod C_\chi} \chi(t_0) e\left(\frac{dbt_0}{p^m}\right) = \begin{cases} (\varphi(p^m)/\varphi(C_\chi))\bar{\chi}(db)G(\chi), & \text{if } (db, C_\chi) = 1 \\ 0, & \text{otherwise} \end{cases}$$

where  $G(\chi) = \sum_{u \bmod C_\chi} \chi(u) e\left(\frac{u}{C_\chi}\right)$  is the Gauss sum.

The second summation is

$$\sum_{t' \bmod p^m/C_\chi} e\left(\frac{dbC_\chi t'}{p^m}\right) = \begin{cases} p^m/C_\chi, & \text{if } (p^m/C_\chi) | db, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, the only non-zero terms occur when  $d = 1$ ,  $(b, C_\chi) = 1$ . In this case  $m_d = 0$ , and

$$\sum_{\substack{a, x \bmod p^m, (a,p)=1, \\ p^{m'-m}d \equiv -xt \bmod p^m}} e\left(-\frac{ax}{p^m}\right) = \varphi(p^m) \text{ for } m' > 2m.$$

The expression (5.19) simplifies to the following:

$$\begin{aligned} \int_{\mathbb{Z}_p^*} \mu_k^* &= \sum_{\substack{a \bmod p^m \\ (a,p)=1}} a_{p^{m'},k}(E_k^*(a + (p^m))) = \\ (5.20) \quad &(p^m/C_\chi)^2 \sum_{b \bmod p^m, (b,p)=1} \sum_{\chi \bmod p^m} \bar{\chi}(b) \frac{(-2\pi i)^k (1 + \bar{\chi}(-1)(-1)^k)}{L(k, \bar{\chi}) (1 - \bar{\chi}(p)p^{-k})} G(\chi) = \\ &\Gamma(k) \sum_{b \bmod p^m, (b,p)=1} \sum_{\chi \bmod p^m} \bar{\chi}(b) \cos\left(\frac{\pi(k-\delta)}{2}\right) 2i^\delta C_\chi^k L(1-k, \chi)^{-1} \\ &\times \frac{(1 + \bar{\chi}(-1)(-1)^k)}{(1 - \bar{\chi}(p)p^{-k})}. \end{aligned}$$

Let us use

$$L(1-k, \chi)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \Big|_{s=1-k}$$

in order to simplify the summation in (5.20) as follows:

$$\begin{aligned} \Gamma(k) \sum_{b \bmod p^m, (b,p)=1} \sum_{\chi \bmod p^m} \bar{\chi}(b) \cos\left(\frac{\pi(k-\delta)}{2}\right) 2i^\delta C_\chi^k \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s} \Big|_{s=1-k} \\ \times \frac{(1 + \bar{\chi}(-1)(-1)^k)}{(1 - \bar{\chi}(p)p^{-k})} = \Gamma(k) \sum_{n=1}^{\infty} \sum_{b \bmod p^m, (b,p)=1} \cos\left(\frac{\pi(k-\delta)}{2}\right) 2i^\delta \\ (5.21) \quad \times \sum_{m'=0}^m p^{km'} \sum_{\substack{\chi \bmod p^m \\ \text{cond}(\chi)=p^{m'}}} \frac{(1 + \bar{\chi}(-1)(-1)^k)}{(1 - \bar{\chi}(p)p^{-k})} \bar{\chi}(b)\chi(n) \frac{\mu(n)}{n^s} \Big|_{s=1-k}. \end{aligned}$$

It suffices to treat the case  $m = 1$ . Then (5.21) becomes

$$\begin{aligned}
& \Gamma(k)p^k \sum_{\substack{\chi \bmod p \\ \text{cond}(\chi)=p}} \cos\left(\frac{\pi(k-\delta)}{2}\right) 2i^\delta \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \Big|_{s=1-k} \sum_{b \bmod p, (b,p)=1} \\
(5.22) \quad & \frac{(1 + \bar{\chi}(-1)(-1)^k)}{(1 - \bar{\chi}(p)p^{-k})} \bar{\chi}(b)\chi(n) + (1 + (-1)^{k+1}) \sum_{n=1}^{\infty} \mu(n)n^{-s} \Big|_{s=1-k}.
\end{aligned}$$

Let us take into account that  $k$  is even, and  $1 - k$  is odd. The only non-zero terms in the first sum correspond to  $\chi(-1) = (-1)^k$ , in which case  $\delta = 0$ ,  $\cos\left(\frac{\pi(k-\delta)}{2}\right) = (-1)^{k/2} = i^k$ . Moreover  $\chi(-p) = 0$  for non-trivial  $\chi$ , and  $(1 + (-1)^{k+1}) = 0$  for even  $k$ .

Then (5.22) transforms to

$$(5.23) \quad \Gamma(k)4(ip)^k \sum_{n=1}^{\infty} \sum_{b \bmod p, (b,p)=1} \sum_{\substack{\chi \bmod p \\ \text{cond}(\chi)=p}} \bar{\chi}(b)\chi(n) \frac{\mu(n)}{n^s} \Big|_{s=1-k}.$$

Let us take into account that

$$\sum_{\substack{\chi \bmod p \\ \text{cond}(\chi)=p}} \bar{\chi}(b)\chi(n) = \begin{cases} p-2, & \text{if } b \equiv n \pmod p \\ -1, & \text{otherwise} \end{cases} = (p-1)\delta_{b \bmod p}(n) - 1.$$

More precisely, let us substitute  $(p-1)\delta_{b \bmod p}(n) - 1$  in place of

$$\sum_{\substack{\chi \bmod p \\ \text{cond}(\chi)=p}} \bar{\chi}(b)\chi(n)$$

in (5.23), giving

$$\begin{aligned}
(5.24) \quad & \Gamma(k)4(ip)^k \sum_{n=1}^{\infty} \sum_{b \bmod p, (b,p)=1} ((p-1)\delta_{b \bmod p}(n) - 1) \frac{\mu(n)}{n^s} \Big|_{s=1-k} \\
& = \Gamma(k)4(ip)^k (p-1)\zeta(1-k)^{-1}((1-p^{k-1})^{-1} - 1).
\end{aligned}$$

Here we took into account that

$$\begin{aligned}
& \sum_{b \bmod p, (b,p)=1} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \Big|_{s=1-k} = (p-1)\zeta(1-k)^{-1} \text{ and} \\
& \sum_{\substack{b \bmod p \\ (b,p)=1}} (p-1) \sum_{n=1}^{\infty} \delta_{b \bmod p}(n) \frac{\mu(n)}{n^s} \Big|_{s=1-k} = (p-1) \sum_{\substack{b \bmod p \\ (b,p)=1}} \sum_{\substack{n=1 \\ n \equiv b \pmod p}}^{\infty} \frac{\mu(n)}{n^s} \Big|_{s=1-k} \\
& = (p-1) \sum_{\substack{n=1 \\ (n,p)=1}}^{\infty} \frac{\mu(n)}{n^s} \Big|_{s=1-k} = (p-1)\zeta(1-k)^{-1}(1-p^{k-1})^{-1}.
\end{aligned}$$

Then (5.24) transforms to

$$(5.25) \quad \begin{aligned} & \Gamma(k)4(ip)^k(p-1)\zeta(1-k)^{-1}p^{k-1}(1-p^{k-1})^{-1} \\ & = \Gamma(k)4i^k p^{2k-1}(p-1)\zeta(1-k)^{-1}(1-p^{k-1})^{-1}. \end{aligned}$$

This computation finishes the proof of Theorem 1.1.

## 6. FOURIER EXPANSION OF EISENSTEIN SERIES ON CLASSICAL GROUPS

Different approaches to Eisenstein series, their families and related distributions will be developed for other classical groups.

The fact that the construction of the above Eisenstein distribution  $E^*$  has a geometric meaning related to the action of the group  $\mathbb{Z}_p^*$  on the product  $\mathbb{Z}_p \times \mathbb{Z}_p$  by the formula

$$(t, (a, b)) \mapsto (at^{-1}, tb)$$

gives also a hint for the future development.

Note that a similar construction of the Eisenstein distributions is contained in [DarDas], p.330. It relates Eisenstein distributions on the space  $\mathbb{X} \subset \mathbb{Z}_p \times \mathbb{Z}_p$  of all primitive  $p$ -adic pairs, to the Eisenstein distributions on the  $p$ -adic projective line  $\mathbb{P}^1(\mathbb{Q}_p)$ .

We plan to use the methods of [GeSha], [Sha78], [Sha81], [Sha88], and the Eisenstein measure in [HLiSk] and [HLiSk-Ra].

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