# ASYMPTOTICALLY MINIMUM VARIANCE ESTIMATOR IN THE SINGULAR CASE

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## ABSTRACT

This paper addresses asymptotically (in the number of measurements) minimum variance (AMV) estimators within the class of estimators based on a mixture of real and complex-valued sequence of statistics whose first covariance of its asymptotic distribution is singular. Thanks to two conditions, we extend the standard AMV estimator. We prove that these conditions are satisfied for the estimates of orthogonal projection matrices used in subspace-based algorithms. Finally, we illustrate our findings for subspace-based algorithms in the DOA estimation for complex noncircular signals.

## 1. INTRODUCTION

The methods of moments is very common in parameter estimation and have been applied successfully to a variety of problems in signal processing. To provide a benchmark for the efficiency of existing algorithms based on these moments, AMV estimators in the class of consistent estimators have been considered. Stoica *et al* [2] with their asymptotically best consistent estimator (ABC) and Porat and Friedlander [3] were the first to derive such estimators for estimating the ARMA parameters of real-valued Gaussian processes from second-order statistics. Then, this approach was extended to high-order statistics [4] and to complex noncircular signals [5], and has been used to blind channel identification and DOA estimation among many other applications. In all these cases, the derivation of the AMV estimator is supported by the assumption that the covariance (first covariance matrix for complex-valued statistics) matrix of the asymptotic distribution of the involved statistics is nonsingular. But in many applications such that the subspace-based algorithms where the involved statistics are estimates of orthogonal projection matrices, this covariance matrix is singular.

The aim of this contribution is to extend the standard AMV results to the mixture of real and complex-valued sequence of statistics when this first covariance matrix is singular. In Section 2, subspace-based algorithms for estimating DOA in the context of complex non-circular signals are presented as a motivating and illustrating example for this study. Section 3 extend the standard AMV results when the involved statistics satisfy two conditions with a special attention to projectors. Finally, Section 4 serves to illustrate our findings for subspace-based algorithms in the DOA estimation for complex noncircular signals.

## 2. MOTIVATING EXAMPLE

Let an array of $M$ sensors receive the signals emitted by $K$ narrowband sources. The observations are modelled as

$$\mathbf{y}_t = \mathbf{A}\mathbf{x}_t + \mathbf{n}_t, \qquad t = 1,\ldots,T,$$

where $(\mathbf{y}_t)_{t=1,\ldots,T}$ are i.i.d. $\mathbf{A} = [\mathbf{a}_1,\ldots,\mathbf{a}_K]$ where $\mathbf{a}_k$ is parameterized by the scalar parameter $_k$. $\mathbf{x}_t = (x_{t,1},\ldots,x_{t,K})^T$ and $\mathbf{n}_t$ model signals transmitted by sources and additive measurement noise respectively. $\mathbf{x}_t$ and $\mathbf{n}_t$ are independent, zero-mean, $\mathbf{n}_t$ is assumed Gaussian complex circular, spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = {}^2_n\mathbf{I}_M$, while $\mathbf{x}_t$ is complex noncircular, not necessarily Gaussian and possibly spatially correlated with nonsingular covariance matrices $\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^H)$ and $\mathbf{R}'_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t\mathbf{x}_t^T)$. Consequently this leads to two covariance matrices of $\mathbf{y}_t$ that convey information about $\stackrel{\text{def}}{=} ( {}_1,\ldots, {}_K)^T$:

$$\mathbf{R}_y = \mathbf{A}\mathbf{R}_x\mathbf{A}^H + {}^2_n\mathbf{I}_M \quad \text{and} \quad \mathbf{R}'_y = \mathbf{A}\mathbf{R}'_x\mathbf{A}^T \neq \mathbf{O},$$

where we suppose that is identifiable from $\mathbf{R}_y$ or $\mathbf{R}'_y$. These covariance matrices are classically estimated by $\mathbf{R}_{y,T} = \frac{1}{T}\ {}^T_{t=1}\mathbf{y}_t\mathbf{y}_t^H$ and $\mathbf{R}'_{y,T} = \frac{1}{T}\ {}^T_{t=1}\mathbf{y}_t\mathbf{y}_t^T$, respectively.

The first idea[1] to estimate from $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ is to use similar subspace-based algorithms derived from the projection matrices $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$ associated with the common noise subspace of $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$. For example, the asymptotic performance of the estimates given by the standard MUSIC algorithm and a MUSIC-like algorithm based on $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$ respectively are similar. In particular for only one source, the asymptotic variance are given by [6]:

$$C\ _1 = \frac{1}{\ _1}\left[\frac{\ ^2_n}{\ ^2_1} + \frac{1}{M}\frac{\ ^4_n}{\ ^4_1}\right] \quad \text{and} \quad C\ _1 = \frac{1}{\ _1\ ^2_1}\left[\frac{\ ^2_n}{\ ^2_1} + \frac{1}{M}\frac{\ ^4_n}{\ ^4_1}\right]$$

with $_1$ is a purely geometric factor and where $_1$ $(0 \leq\ _1 \leq 1)$ is the noncircularity rate defined by $\mathrm{E}(x^2_{t,1}) = {}_1e^{i\ _1}\mathrm{E}|x^2_{t,1}| = {}_1e^{i\ _1}\ ^2_1$. Consequently a problem crops up: how to combine the statistics $\mathbf{\Pi}_{y,T}$ and $\mathbf{\Pi}'_{y,T}$ to improve the estimate of ?

Another idea to estimate from $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ is to use subspace-based algorithms derived from the projection matrix $\mathbf{\Pi}_{\tilde{y},T}$ associated with the noise subspace of the sample covariance matrix $\mathbf{R}_{\tilde{y},T}$ of the extended observation $\tilde{\mathbf{y}}_t \stackrel{\text{def}}{=} (\mathbf{y}_t^T, \mathbf{y}_t{}^H)^T$. Efficient subspace-based algorithms based on $\mathbf{\Pi}_{\tilde{y},T}$ have been proposed and analyzed in [6] in the particular case of uncorrelated sources with maximum noncircularity rates. However in the general case of arbitrary extended spatial covariances ($\mathbf{R}_x$ and $\mathbf{R}'_x$) of the sources, only weighted MUSIC-like algorithms seem to take benefit of the second covariance matrix $\mathbf{R}'_{y,T}$. But the asymptotic performances of these estimates are largely outperformed by those of the AMV estimator based on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$ [6]. Therefore a question arises as well: Does there exist an algorithm based

[1] Note that [1] was to the best of our knowledge, the first contribution that proposed an algorithm taking into account the noncircularity.

on the projector $\mathbf{\Pi}_{\tilde{y},T}$ whose performances approach those of the AMV estimator based on $\mathbf{R}_{y,T}$ and $\mathbf{R}'_{y,T}$?

A solution of the two aforementioned problems is to use the notion of AMV estimators respectively based on the matrix-valued statistics $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathbf{\Pi}_{\tilde{y},T}$. But to apply the standard results [4] on AMV estimators to these projectors, two conditions must be satisfied. First, the involved subspace-based algorithm considered as a mapping must be complex differentiable w.r.t. $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ [resp. $\mathbf{\Pi}_{\tilde{y},T}$] at the point $(\mathbf{\Pi}_{y}, \mathbf{\Pi}'_{y})$ [resp. $\mathbf{\Pi}_{\tilde{y}}$]. Second, the first covariance matrix $\mathbf{C}_s(\ )$ of the asymptotic distribution of $\mathbf{s}_T \stackrel{\text{def}}{=} \text{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ [resp. $\mathbf{s}_T \stackrel{\text{def}}{=} \text{vec}(\mathbf{\Pi}_{\tilde{y},T})$] must be nonsingular. While the first condition is satisfied because the projection matrices are Hermitian, it will be specified in Section 3.2, that the second is not satisfied. So we have to elaborate a little bit.

## 3. ASYMPTOTICALLY MINIMUM VARIANCE ESTIMATOR

### 3.1 Arbitrary sequence of statistics

Consider a general $N$-multidimensional mixture of real and complex-valued sequence of statistics $\mathbf{s}_T$ which is a consistent estimate of $\mathbf{s}(\ )$ for which the real-valued parameter $\ \in \mathbb{R}^K$ is identifiable from $\mathbf{s}(\ )$. We suppose that $\mathbf{s}_T$ is asymptotically zero-mean Gaussian distributed where the first covariance matrix $\mathbf{C}_s(\ )$ is possibly singular:

$$\sqrt{T}\ (\mathbf{s}_T - \mathbf{s}(\ )) \xrightarrow{\mathscr{L}} \mathscr{N}_c\left(\mathbf{0}; \mathbf{C}_s(\ ), \mathbf{C}'_s(\ )\right).$$

To consider the asymptotic performance of an algorithm based on $\mathbf{s}_T$, we adopt a functional analysis which consists in recognizing that the whole process of constructing an estimate $\ _T$ of $\ $ is equivalent to defining a functional relation linking this estimate $\ _T$ to the statistics $\mathbf{s}_T$ from which it is inferred. This functional dependence is denoted $\mathbf{s}_T \longmapsto \ _T = \text{Alg}(\mathbf{s}_T)$. Considering a mapping Alg(.) differentiable w.r.t. $(\ (\mathbf{s}),\ (\mathbf{s}))$, we prove the following theorem.

**Theorem 1** *The covariance matrix* $\mathbf{C}\ $ *of the asymptotic distribution of an estimator of* $\ $ *given by an arbitrary algorithm based on* $\mathbf{s}_T$ *is bounded below by the real symmetric matrix* $\left(\mathscr{S}^H \mathbf{C}_s^{\#}(\ )\mathscr{S}\right)^{-1}$ [2]

$$\mathbf{C}\ \geq \left(\mathscr{S}^H \mathbf{C}_s^{\#}(\ )\mathscr{S}\right)^{-1} \tag{1}$$

*if the following two conditions hold*

$$\text{Span}(\mathscr{S}) \subset \text{Span}(\mathbf{C}_s(\ )) \ \text{ and } \ \mathbf{s}_T^* = \mathbf{K}\mathbf{s}_T \tag{2}$$

*where* $\mathbf{K}$ *is an arbitrary permutation matrix and* $\mathscr{S} \stackrel{\text{def}}{=} \frac{d\mathbf{s}(\ )}{d\ }$.

**Proof**: The second condition implies that $\mathbf{s}^*(\ ) = \mathbf{K}\mathbf{s}(\ )$ and proves that any mapping Alg(.) differentiable w.r.t. $(\ (\mathbf{s}),\ (\mathbf{s}))$ becomes differentiable w.r.t. $\mathbf{s}$ alone and consequently the identifiability condition implies the constraint $\mathbf{D}_s^{\text{Alg}}\mathscr{S} = \mathbf{I}_K$ for the Jacobian matrix $\mathbf{D}_s^{\text{Alg}}$ of this mapping at the point $\mathbf{s}(\ )$ of an arbitrary estimate of $\ $ based on $\mathbf{s}_T$ (see [5]). Consequently rank$(\mathscr{S}) = K$ and because $\mathbf{C}\ = \mathbf{D}_s^{\text{Alg}}\mathbf{C}_s(\ )(\mathbf{D}_s^{\text{Alg}})^H$, the proof comes down to minimizing $\mathbf{D}\mathbf{C}_s(\ )\mathbf{D}^H$ w.r.t. $\mathbf{D}$ under the constraint $\mathbf{D}\mathscr{S} = \mathbf{I}_K$. Consider the eigenvalue decomposition $\mathbf{U}\mathbf{\Sigma}\mathbf{U}^H$ of the rank-$r$ singular matrix $\mathbf{C}_s(\ )$ where $\mathbf{U}$ is an $N \times r$ unitary matrix with $r < N$. The condition Span$(\mathscr{S}) \subset$ Span$(\mathbf{C}_s(\ ))$ is equivalent to $\mathscr{S} = \mathbf{U}\mathscr{S}'$ where $\mathscr{S}'$ is a $r \times K$ matrix. And the previous minimization becomes equivalent to minimizing $\mathbf{D}'\mathbf{\Sigma}\mathbf{D}'^H$ w.r.t. $\mathbf{D}' \stackrel{\text{def}}{=} \mathbf{D}\mathbf{U}$ under the constraint $\mathbf{D}'\mathscr{S}' = \mathbf{I}_K$, where here, $\mathbf{\Sigma}$ is nonsingular. This minimization is standard for real-valued statistics (see e.g., [4]). It has been extended for a mixture of real and complex-valued statistics in [5], and the minimum is $(\mathscr{S}'^H\mathbf{\Sigma}^{-1}\mathscr{S}')^{-1} = (\mathscr{S}'^H\mathbf{U}^H\mathbf{U}\mathbf{\Sigma}^{-1}\mathbf{U}^H\mathbf{U}\mathscr{S}')^{-1} = \left(\mathscr{S}^H\mathbf{C}_s^{\#}(\ )\mathscr{S}\right)^{-1}$. ■

**Remark 1:** The second condition (2) holds for Hermitian matrix-valued statistics. For complex symmetric matrix-valued statistics, the complex conjugate associated terms must be added.

**Remark 2:** In the trivial case where there are $r$ linear relations between the components of $\mathbf{s}_T$ with $N-r$ components statistically uncorrelated, there exists an $N \times (N-r)$ matrix $\mathbf{B}$ such that $\mathbf{s}_T = \mathbf{B}\mathbf{s}'_T$ with Cov$(\mathbf{s}'_T)$ nonsingular. Consequently Span$(\mathscr{S}) \subset$ Span$(\mathbf{B})$ and Span(Cov$(\mathbf{s}_T)$) = Span$(\mathbf{B})$. Therefore first condition (2) holds.

**Remark 3:** In their discussions about the generalization of the optimal weighted subspace fitting approach, Cardoso and Moulines [7] have introduced a range space condition different from condition (2), and they have derived (1) as a lower bound to the covariance of the asymptotic distribution of weighted subspace fitting estimates.

Furthermore, under the assumptions of theorem 1, we prove that this lowest bound is asymptotically tight, i.e., there exists an algorithm Alg(.), whose covariance of the asymptotic distribution of $\ _T$ satisfies (1) with equality.

**Theorem 2** *The following nonlinear least square algorithm is an AMV algorithm based on* $\mathbf{s}_T$.

$$\ _T = \arg\min_{\ \in\mathbb{R}^K} [\mathbf{s}_T - \mathbf{s}(\ )]^H \mathbf{C}_s^{\#}(\ )[\mathbf{s}_T - \mathbf{s}(\ )]. \tag{3}$$

**Proof**: By a perturbation analysis, $\ _T = \ + \ _T$ is associated with $\mathbf{s}_T = \mathbf{s}(\ ) + \ \mathbf{s}_T$. If $V(\ ) \stackrel{\text{def}}{=} [\mathbf{s}(\ ) - \mathbf{s}(\ )]^H\mathbf{C}_s^{\#}(\ )[\mathbf{s}(\ ) - \mathbf{s}(\ )]$ and $V_T(\ ) \stackrel{\text{def}}{=} [\mathbf{s}_T - \mathbf{s}(\ )]^H\mathbf{C}_s^{\#}(\ )[\mathbf{s}_T - \mathbf{s}(\ )]$, we have: $\frac{dV(\ )}{d\ }|_{\ =\ } = \mathbf{0}$ and $\frac{dV_T(\ )}{d\ }|_{\ =\ +\ _T} = \mathbf{0}$. Expanding these two derivatives, we straightforwardly obtain: $\left(\mathscr{S}^H\mathbf{C}_s^{\#}(\ )\mathscr{S} + \mathscr{S}^T\mathbf{C}_s^{\#}(\ )^*\mathscr{S}^*\right)\ _T + o(\ _T) = \mathscr{S}^H\mathbf{C}_s^{\#}(\ )\ \mathbf{s}_T + \mathscr{S}^T\mathbf{C}_s^{\#}(\ )^*\ \mathbf{s}_T^* + o(\ \mathbf{s}_T)$. Consequently algorithm (3) satisfies:

$$\begin{aligned} \ _T &= \left(\mathscr{S}^H\mathbf{C}_s^{\#}(\ )\mathscr{S} + \mathscr{S}^T\mathbf{C}_s^{\#}(\ )^*\mathscr{S}^*\right)^{-1} \\ &\quad \left(\mathscr{S}^H\mathbf{C}_s^{\#}(\ ), \mathscr{S}^T\mathbf{C}_s^{\#}(\ )^*\right)\begin{pmatrix} \ \mathbf{s}_T \\ \ \mathbf{s}_T^* \end{pmatrix} + o(\ \mathbf{s}_T) \\ &= \left(\mathscr{S}^H\mathbf{C}_s^{\#}(\ )\mathscr{S}\right)^{-1}\mathscr{S}^H\mathbf{C}_s^{\#}(\ )\ \mathbf{s}_T + o(\ \mathbf{s}_T), \end{aligned}$$

by using $\mathscr{S}^* = \mathbf{K}\mathscr{S}$ and $(\mathbf{C}_s^{\#}(\ ))^* = \mathbf{K}\mathbf{C}_s^{\#}(\ )\mathbf{K}^T$ in the second equality. Consequently, the Jacobian of the mapping Alg(.) involved by (3) is

[2] The superscript # denotes the Moore Penrose inverse.

$\mathbf{D}_s^{\mathrm{Alg}} = \left(\mathscr{S}^H \mathbf{C}_s^{\#}(\ )\mathscr{S}\right)^{-1} \mathscr{S}^H \mathbf{C}_s^{\#}(\ )$ and $\mathbf{C} = \mathbf{D}_s^{\mathrm{Alg}} \mathbf{C}_s(\ )(\mathbf{D}_s^{\mathrm{Alg}})^H = (\mathscr{S}^H \mathbf{C}_s^{\#}(\ )\mathscr{S})^{-1}$. ■

### 3.2 Application to projectors

This subsection is concerned with general properties of subspace-based algorithms in the context of the generic model: $\mathbf{y}_t = \mathbf{A}(\ )\mathbf{x}_t + \mathbf{n}_t$, where $(\mathbf{y}_t)_{t=1,...,T}$ are i.i.d., $\mathbf{x}_t$ and $\mathbf{n}_t$ are zero-mean and independent, $\mathbf{n}_t$ is assumed Gaussian complex circular, spatially uncorrelated with $\mathrm{E}(\mathbf{n}_t\mathbf{n}_t^H) = \ {}^2_n\mathbf{I}_M$, while $\mathbf{x}_t$ is complex noncircular, not necessarily Gaussian with $\mathbf{R}_x$ nonsingular. We assume that $\mathrm{rank}(\mathbf{A}(\ )) = L < M$ and that the real-valued parameter $\ \in \mathbb{R}^K$ is uniquely determined by the range space of $\mathbf{A}(\ )$. Therefore is uniquely determined by the common projector $\mathbf{\Pi}_y$ onto the noise subspace associated with $\mathbf{R}_y = \mathbf{R}_s + \ {}^2_n\mathbf{I}_M$ and $\mathbf{R}'_y = \mathbf{R}'_s \neq \mathbf{O}$ as well. To the extended observation $\tilde{\mathbf{y}}_t \stackrel{\mathrm{def}}{=} (\mathbf{y}_t^T, \mathbf{y}_t{}^H)^T$, $\mathbf{R}_{\tilde{y}} \stackrel{\mathrm{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^H) = \mathbf{R}_{\tilde{s}} + \ {}^2_n\mathbf{I}_{2M}$ where we suppose here that $\mathbf{R}_{\tilde{s}}$ is nonsingular.

Consequently, is also determined by the orthogonal projector $\mathbf{\Pi}_{\tilde{y}}$ onto the $2L$-dimentional noise subspace of $\tilde{\mathbf{y}}_t$. Thus we can consider the orthogonal projector $\mathbf{\Pi}_{y,T}$, $\mathbf{\Pi}'_{y,T}$ and $\mathbf{\Pi}_{\tilde{y},T}$ onto the noise subspace of the sample covariance matrices $\mathbf{R}_{y,T}$, $\mathbf{R}'_{y,T}$ and $\mathbf{R}_{\tilde{y},T}$ respectively.

To prove that the first covariance matrices $\mathbf{C}_s$ of the asymptotic distribution of the statistics $\mathbf{s}_T = \mathrm{vec}(\mathbf{\Pi}_{y,T})$, $\mathrm{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$ are singular, we need the following lemma proved in [6]:

**Lemma 1** *The covariance* $\mathbf{C}$ *and* $\mathbf{C}_{\tilde{}}$ *are given by*

$$\mathbf{C} = (\mathbf{\Pi}_y^* \otimes \mathbf{U}) + (\mathbf{U}^* \otimes \mathbf{\Pi}_y) \quad (4)$$

$$\mathbf{C}_{\tilde{}} = (\mathbf{I} + \mathbf{K}_{2M}(\mathbf{J}\otimes\mathbf{J}))\left((\mathbf{\Pi}_{\tilde{y}}^* \otimes \tilde{\mathbf{U}}) + (\tilde{\mathbf{U}}^* \otimes \mathbf{\Pi}_{\tilde{y}})\right) \quad (5)$$

*with* $\mathbf{U} \stackrel{\mathrm{def}}{=} \ {}^2_n\mathbf{R}_s^{\#}\mathbf{R}_y\mathbf{R}_s^{\#}$ *and* $\tilde{\mathbf{U}} \stackrel{\mathrm{def}}{=} \ {}^2_n\mathbf{R}_{\tilde{s}}^{\#}\mathbf{R}_{\tilde{y}}\mathbf{R}_{\tilde{s}}^{\#}$*, where* $\mathbf{K}_L$ *is the vec-permutation matrix which transforms* $\mathrm{vec}(.)$ *to* $\mathrm{vec}(.^T)$ *for any* $L\times L$ *square matrix and* $\mathbf{J} = \begin{pmatrix} \mathbf{O} & \mathbf{I}_M \\ \mathbf{I}_M & \mathbf{O} \end{pmatrix}$.

Consequently, if we consider the eigenvalue decompositions $\sum_{l=1}^{M} \ _l\mathbf{u}_l\mathbf{u}_l^H$ and $\sum_{l=1}^{2M} \tilde{}_l\tilde{\mathbf{u}}_l\tilde{\mathbf{u}}_l^H$ of respectively $\mathbf{R}_y$ and $\mathbf{R}_{\tilde{y}}$,

$$\mathbf{C} = \sum_{l',l''\in\mathscr{L}} \ _{l',l''}(\mathbf{u}_{l'}^* \otimes \mathbf{u}_{l''})(\mathbf{u}_{l'}^T \otimes \mathbf{u}_{l''}^H)$$

$$\mathbf{C}_{\tilde{}} = \sum_{l',l''\in\tilde{\mathscr{L}}} \tilde{}_{l',l''}(\tilde{\mathbf{u}}_{l'}^* \otimes \tilde{\mathbf{u}}_{l''})(\tilde{\mathbf{u}}_{l'}^T \otimes \tilde{\mathbf{u}}_{l''}^H + \tilde{\mathbf{u}}_{l''}^H\mathbf{J} \otimes \tilde{\mathbf{u}}_{l'}^T\mathbf{J})$$

where $\mathscr{L}$ [resp. $\tilde{\mathscr{L}}$] is the set $\{(l',l'')\ |1 \leq l' \leq L < l'' \leq M \bigcup 1 \leq l'' \leq L < l' \leq M\}$ [resp. $\{(l',l'')\ |1 \leq l' \leq 2L < l'' \leq 2M \cup 1 \leq l'' \leq 2L < l' \leq 2M\}$] and the values of $\ _{l',l''} \neq 0$ [resp. $\tilde{}_{l',l''} \neq 0$] are irrelevant.

Therefore $\mathbf{C}$ , $\mathbf{C}_{,}$ $\stackrel{\mathrm{def}}{=} \begin{pmatrix} \mathbf{C} & \mathbf{C}_{,\ '} \\ \mathbf{C}^H_{,} & \mathbf{C}_{'} \end{pmatrix}$ and $\mathbf{C}_{\tilde{}}$ respectively first covariance matrices $\mathbf{C}_s$ of the asymptotic distribution of the statistics $\mathbf{s}_T = \mathrm{vec}(\mathbf{\Pi}_{y,T})$, $\mathrm{vec}(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ and $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y},T})$ are singular.

Thanks to lemma 1 and the following lemma which is proved in the appendix:

**Lemma 2** *The covariance* $\mathbf{C}_{,}$ *is given by*

$$\mathbf{C}_{,} = \begin{pmatrix} \mathbf{K}_M & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_M \end{pmatrix}\left(\left(\begin{pmatrix} \mathbf{U} & \mathbf{U}'' \\ \mathbf{U}''^H & \mathbf{U}' \end{pmatrix} \otimes \mathbf{\Pi}_y^*\right)\right.$$
$$\left.\begin{pmatrix} \mathbf{K}_M & \mathbf{O} \\ \mathbf{O} & \mathbf{K}_M \end{pmatrix} + \begin{pmatrix} \mathbf{U}^* & \mathbf{U}''^* \\ \mathbf{U}''^T & \mathbf{U}'^* \end{pmatrix} \otimes \mathbf{\Pi}_y.\right) \quad (6)$$

*with* $\mathbf{U}' \stackrel{\mathrm{def}}{=} \ {}^2_n\mathbf{R}_s'^{*\#}\mathbf{R}_y^*\mathbf{R}_s'^{\#}$ *and* $\mathbf{U}'' \stackrel{\mathrm{def}}{=} \ {}^2_n\mathbf{R}_s^{\#}\mathbf{R}'_y\mathbf{R}_s'^{\#}$.

We note that expressions (4), (6) and (5) of $\mathbf{C}$ , $\mathbf{C}_{,}$ and $\mathbf{C}_{\tilde{}}$ respectively, do not depend on the fourth-order moments of the sources, consequently we have proved the following:

**Theorem 3** *The asymptotic performance given by an arbitrary subspace-based algorithm built from* $\mathbf{R}_{y,T}$*,* $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ *or* $\mathbf{R}_{\tilde{y},T}$ *depends on the distribution of* $\mathbf{x}_t$ *through its second-order moments only. Furthermore, for subspace-based algorithms built from* $\mathbf{R}_{y,T}$*, this asymptotic performance depends only on the first covariance matrix* $\mathbf{R}_x$*.*

Using these results we prove the following:

**Theorem 4** *The covariance matrix* $\mathbf{C}$ *of the asymptotic distribution of an estimator of given by an arbitrary subspace algorithm based on the statistics* $\mathbf{\Pi}_{y,T}$*,* $(\mathbf{\Pi}_{y,T}, \mathbf{\Pi}'_{y,T})$ *or* $\mathbf{\Pi}_{\tilde{y},T}$ *is bounded below by the real symmetric matrix* $\left(\mathscr{S}^H\mathbf{C}_s^{\#}\mathscr{S}\right)^{-1}$ *where* $\mathscr{S} \stackrel{\mathrm{def}}{=} \frac{d\mathbf{s}(\ )}{d}$ *with* $\mathbf{s}(\ )$ *is respectively* $\mathrm{vec}(\mathbf{\Pi}_y)$*,* $\mathrm{vec}(\mathbf{\Pi}_y, \mathbf{\Pi}'_y)$ *or* $\mathrm{vec}(\mathbf{\Pi}_{\tilde{y}})$*.*

*Furthermore, the following nonlinear least square algorithm is an AMV subspace-based algorithm:*

$$_T = \arg\min_{\in\mathbb{R}^K} [\mathbf{s}_T - \mathbf{s}(\ )]^H \mathbf{W}_T [\mathbf{s}_T - \mathbf{s}(\ )] \quad (7)$$

*where a consistent estimate* $\mathbf{W}_T$ *of* $\mathbf{C}_s^{\#}$ *is available from* $\mathbf{R}_{y,T}$*,* $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ *or* $\mathbf{R}_{\tilde{y},T}$ *respectively.*

**Proof**: Because these matrix-valued statistics are Hermitian, the second condition of (2) is satisfied.

Considering the first condition of (2) for the first and third statistics, $\mathrm{Span}(\mathbf{C}\ (\ )) = \mathrm{Span}\{\mathbf{u}_{l'}^* \otimes \mathbf{u}_{l''}\ |l',l'' \in \mathscr{L}\}$ and $\mathrm{Span}(\mathbf{C}_{\tilde{}}(\ )) = \mathrm{Span}\{\tilde{\mathbf{u}}_{l'}^* \otimes \tilde{\mathbf{u}}_{l''}\ |l',l'' \in \tilde{\mathscr{L}}\}$. Therefore for the first statistic, this condition is equivalent to

$$\frac{d\mathrm{vec}(\ _y(\ ))}{d\ _k} = \sum_{l=K+1}^{M}\left(\mathbf{u}_l^* \otimes \frac{d\mathbf{u}_l}{d\ _k} + \frac{d\mathbf{u}_l^*}{d\ _k} \otimes \mathbf{u}_l\right)$$
$$\perp \{\mathbf{u}_{l'}^* \otimes \mathbf{u}_{l''} \mid 1 \leq l',l'' \leq L \text{ or } L < l',l'' \leq M\}$$

for $k = 1,...,K$ and noting that $\mathbf{u}_1,\ldots,\mathbf{u}_M$ are orthonormal, condition (2) is straightforwardly proved for the first statistic. This condition is proved in the same way for the third statistic.

For the second statistic, using the singular value decompositions of $\mathbf{U}$, $\mathbf{U}'$ and $\mathbf{U}''$ and noting that $\mathrm{Span}(\mathbf{U}) = \mathrm{Span}(\mathbf{U}') = \mathrm{Span}(\mathbf{U}'')$, it is straightforward to prove that $\text{Null space}(\mathbf{C}_{,}) = \mathrm{Span}\{\begin{matrix} \mathbf{u}_{l'}^* \otimes \mathbf{u}_{l''} \\ \mathbf{0} \end{matrix}, \begin{matrix} \mathbf{0} \\ \mathbf{u}_{l'}^* \otimes \mathbf{u}_{l''} \end{matrix} \mid 1 \leq l',l'' \leq L \text{ or } L < l',l'' \leq M\}\}$ and consequently $(\frac{d^T\mathrm{vec}(\ _y(\ ))}{d\ _k}, \frac{d^T\mathrm{vec}(\ _y(\ ))}{d\ _k})^T \perp \text{Null space}(\mathbf{C}_{,})$ and the first part of condition (2) is satisfied for the second statistic.

Regarding the proof of the second part of theorem 3, it is straightforward to show that the Jacobian $\mathbf{D}_s^{\text{Alg}} = \left(\mathscr{S}^H \mathbf{C}_s^{\#}(\ )\mathscr{S}\right)^{-1} \mathscr{S}^H \mathbf{C}^{\#}_{\tilde{}}(\ )$ of the mapping Alg(.) involved by (7) is preserved by following a perturbation analysis similar to that of the proof of theorem 2 where $\mathbf{W}_T = \mathbf{C}_s^{\#}(\ ) + o(\mathbf{s}_T - \mathbf{s}(\ ))$.

Issued from the singular value decompositions of $\mathbf{R}_{y,T}$, $\mathbf{R}'_{y,T}$ and $\mathbf{R}_{\tilde{y},T}$, consistent estimates of $^2$, $\mathbf{\Pi}_y$, $\mathbf{\Pi}_{\tilde{y}}$, $\mathbf{R}_s$, $\mathbf{R}'_s$, $\mathbf{R}_{\tilde{s}}$ are available and consequently, consistent estimates of $\mathbf{C}^{\#}$, $\mathbf{C}^{\#}_{,'}$ and $\mathbf{C}^{\#}_{\tilde{}}$ can be derived as well. ■

## 4. ILLUSTRATIVE EXAMPLES

We consider throughout this section two uncorrelated equipowered filtered or unfiltered BPSK modulated signals with identical non-circularity rate ( $\stackrel{\text{def}}{=}$ $_1 =$ $_2$) with phases of circularity $_1 =$ $/2$ and $_2 =$ $/3$. These signals impinge on a uniform linear array with $M = 6$ sensors for which $\mathbf{a}_k = (1, e^{i\ _k}, \ldots, e^{i(M-1)\ _k})^T$.

First we note that the subspace-based algorithms cannot take the a priori information about the signal uncorrelation into account. Considering AMV estimators based on $\mathbf{R}_{y,T}$ or $(\mathbf{R}_{y,T}, \mathbf{R}'_{y,T})$ derived in [5] and if no a priori information is available on $\mathbf{R}_x$ and $\mathbf{R}'_x$, we numerically find that:

$$\begin{aligned} \mathbf{C}^{\text{AMV}(\ )} &= \mathbf{C}^{\text{AMV(R)}} \\ \mathbf{C}^{\text{AMV}(\tilde{\ })} &= \mathbf{C}^{\text{AMV}(\ ,\ ')} = \mathbf{C}^{\text{AMV(R,R}')}. \end{aligned}$$

This property has been confirmed in the case of dependent sources thanks to many experiments as well. Furthermore, we find that these bounds coincide with the stochastic Cramer-Rao bounds under the circular or non-circular Gaussian distribution respectively. But we have not succeeded in proving these different properties analytically.

In Fig.1, we realize the benefits due to the second covariance matrix $\mathbf{R}'_{y,T}$ through subspace-based algorithms for non-circular signals, particularly for low DOA separations.

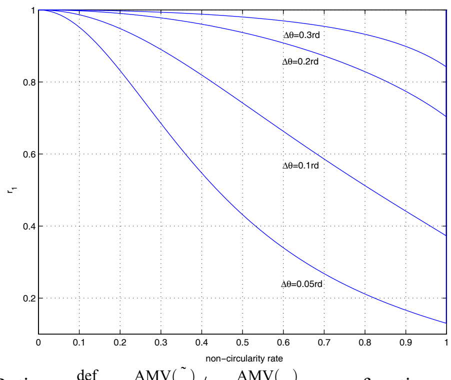


Fig.1 Ratio $r_1 \stackrel{\text{def}}{=} var_1^{\text{AMV}(\tilde{\ })}/var_1^{\text{AMV}(\ )}$ as a function of the non-circularity rate for different DOA separations for SNR = 5dB and $=$ $/6$rd.

If this uncorrelation a priori information is taken into account, Fig.2 shows the better expected benefits due to the non-circularity, particularly for low DOA separations. Consequently, the subspace-based algorithms lose their good efficiency in these circumstances.

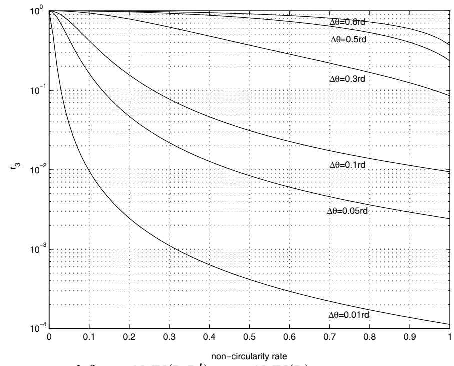


Fig.2 Ratio $r_2 \stackrel{\text{def}}{=} var_1^{\text{AMV(R,R}')}/var_1^{\text{AMV(R)}}$ as a function of the non-circularity rate for different DOA separations for SNR = 5dB and $=$ $/6$rd.

## A. APPENDIX: PROOF OF LEMMA 2

Using the following lemma proved in [6]

**Lemma 3** *The perturbations* $\mathbf{R}_{y,T} = \mathbf{R}_y + \ \mathbf{R}_{y,T}$, $\mathbf{R}'_{y,T} = \mathbf{R}'_y + \ \mathbf{R}'_{y,T}$ *and* $\mathbf{\Pi}_{y,T} = \mathbf{\Pi}_y + \ \mathbf{\Pi}_{y,T}$, $\mathbf{\Pi}'_{y,T} = \mathbf{\Pi}'_y + \ \mathbf{\Pi}'_{y,T}$ *are related by the following expressions:*

$$\begin{aligned} \text{vec}(\ \mathbf{\Pi}_T) &= -((\mathbf{\Pi}_y^* \otimes \mathbf{R}_s^{\#}) + (\mathbf{R}_s^{*\#} \otimes \mathbf{\Pi}_y))\text{vec}(\ \mathbf{R}_{y,T}) \\ &+ o(\text{vec}(\ \mathbf{R}_{y,T})) \\ \text{vec}(\ \mathbf{\Pi}'_T) &= (\mathbf{\Pi}_y^* \otimes \mathbf{R}_s^{'*\#})\text{vec}(\ \mathbf{R}_{y,T}^{'*}) \\ &- (\mathbf{R}_s^{'\#} \otimes \mathbf{\Pi}_y)\text{vec}(\ \mathbf{R}'_{y,T}) + o(\text{vec}(\ \mathbf{R}'_{y,T})), \end{aligned}$$

and the expression of the covariance matrices $\mathbf{C}_{R'_y}$, $\mathbf{C}'_{R'_y}$, $\mathbf{C}_{R_y,R'_y}$ and $\mathbf{C}'_{R_y,R'_y}$ given in [5], the standard theorem (see e.g., [8, p. 122]) on regular functions of asymptotically Gaussian statistics applies and we straightforward obtain:

$$\begin{aligned} \mathbf{C}_{\ ,\ '} &= (\mathbf{\Pi}_y^* \otimes \mathbf{U}') + (\mathbf{U}'^{*} \otimes \mathbf{\Pi}_y) \\ \mathbf{C}_{\ ,\ '} &= (\mathbf{\Pi}_y^* \otimes \mathbf{U}'') + (\mathbf{U}''^{*} \otimes \mathbf{\Pi}_y). \end{aligned}$$

Consequently, expression (6) of $\mathbf{C}_{\ '} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{C} & \mathbf{C}_{\ ,\ '} \\ \mathbf{C}^H_{\ ,\ '} & \mathbf{C}_{\ '} \end{pmatrix}$ follows. ■

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