Partial Differential Equations

Fractal porous medium equation

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Abstract

We study a generalization of the porous medium equation involving nonlocal terms. In particular, the L^p decay of solutions of the Cauchy problem is proved. Explicit self-similar solutions with compact support generalizing the KZB (or Barenblatt) solutions are constructed in the case corresponding to transport equation with a nonlocal velocity.

Résumé

Équation des milieux poreux fractionnaire
Cette Note est consacrée à l'étude d'une généralisation non locale de l'équation des milieux poreux. On obtient en particulier des estimations L^p des solutions du problème de Cauchy. On exhibe aussi des formules explicites de solutions auto-similaires à support compact qui ont sensiblement la même structure que celle bien connue de KZB (or Barenblatt) dans le cas important où l'équation est de type transport avec une loi de vitesse non-locale.

Version française abrégée

Nous considérons le problème de Cauchy pour l'équation non-locale suivante

$$\partial_t u - \nabla \cdot (|u|^{m-1} \nabla^{\alpha-1} u) = 0, \tag{1}$$

avec $m \geq 1$, $x \in \mathbb{R}^d$, $\alpha \in (0, 2)$, $t > 0$, à laquelle on ajoute une condition initiale du type $u(0, x) = u_0(x)$ avec $u_0 \in L^1(\mathbb{R}^d)$. $\nabla^{\alpha-1}$ est un opérateur intégral singulier généralisant le gradient usuel ($\alpha = 2$) et lié au laplacien fractionnaire. Quand $m = 2$ et que la solution est positive, l'équation (1) est une équation de transport avec une vitesse non-locale.

Etant donné que l'équation (1) est dégénerée, une approximation parabolique ou une régularisation sont utilisées pour montrer l'existence de solutions

$$\partial_t u - \nabla \cdot (|u|^{m-1} \nabla^{\alpha-1} u) = \varepsilon \Delta u, \qquad \partial_t u - \nabla \cdot ((\varepsilon + |u|^{m-1}) \nabla^{\alpha-1} u) = 0 \tag{2}$$

avec $\varepsilon > 0$.

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Les résultats principaux de cette note sont des estimations en norme $L^p(\mathbb{R}^d)$ de la solution du problème de Cauchy avec $m \geq 1$. Ensuite, dans le cas $m = 2$, on obtient de formules explicites pour des solutions auto-similaires qui se propagent à une vitesse finie.

Théorème 1 (Le problème de Cauchy et l'asymptotique)
(i) Étant donnée une fonction $u_0 \geq 0$ telle que $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, il existe une solution du problème de Cauchy pour (1) qui est positive, globale en temps, et de masse constante : $\int u(t,x)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x$.
(ii) Les normes $L^p(\mathbb{R}^d)$ $(1 \leq p < \infty)$ de chaque solution u tendent vers 0 quand $t \to \infty$ avec le taux algébrique suivant

$$\|u(t)\|_p \leq C(d,\alpha,m,p)\|u_0\|_1^{\frac{\frac{m-1}{p}+\frac{\alpha}{d}}{m-1+\frac{\alpha}{d}}} t^{-\frac{p-1}{m-1+\frac{\alpha}{d}}}.$$

Théorème 2 (Solutions auto-similaires) *La fonction*

$$u(t,x) = Ct^{-\frac{d}{d+\alpha}} \left(1 - |x|^2 t^{-\frac{2}{d+\alpha}}\right)_+^{\frac{\alpha}{2}}$$

est une solution auto-similaire de l'équation (1) pour $m = 2$.

1. Introduction

We study a nonlocal generalization of the porous medium equation

$$\partial_t u - \nabla \cdot (|u|^{m-1}\nabla^{\alpha-1}u) = 0, \tag{1}$$

where $m \geq 1$, $\alpha \in (0,2)$, $x \in \mathbb{R}^d$, $t > 0$, supplemented with the following initial condition in $L^1(\mathbb{R}^d)$

$$u(0,x) = u_0(x). \tag{2}$$

The pseudodifferential (vector-valued) operator ∇^β in (1) is defined via the Fourier transform as $\nabla^\beta u = \mathcal{F}^{-1}(i\xi|\xi|^{\beta-1}\mathcal{F}u)$. This definition is consistent with the usual gradient: $\nabla^1 = \nabla$; the components of ∇^0 are the Riesz transforms; moreover we have $\nabla \cdot \nabla^{\alpha-1} = \nabla^{\frac{\alpha}{2}} \cdot \nabla^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$, where $(-\Delta)^{\frac{\alpha}{2}}$ denotes the fractional Laplacian operator: $(-\Delta)^{\frac{\alpha}{2}}u = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F}u)$. It can also be defined as follows $\nabla^{\alpha-1}u = \nabla I_{2-\alpha}u$, where I_β for $\beta \in (0,2)$ is the integral smoothing operator

$$I_\beta(u)(x) = -C_\beta \int \frac{u(x+z) - u(x)}{|z|^{d-\beta}}\,\mathrm{d}z$$

with some $C_\beta > 0$.

Notice that for $\alpha = 2$ we recover the Boussinesq equation ($m = 2$), and the porous media equation ($m > 1$): $\quad \partial_t u = \nabla \cdot (|u|^{m-1}\nabla u), \quad t > 0,\ x \in \mathbb{R}^d$.

In the case $m = 2$ and for nonnegative initial data u_0, the equation (1) is of transport type $\partial_t u + \nabla\cdot(u\mathbf{v}) = 0$ with a nonlocal velocity $\mathbf{v} = -\nabla I_{2-\alpha}u$.

Related works and results. Recently, L. Caffarelli and J. L. Vázquez [3] studied (1) in the case $m = 2$. They proved the existence of weak solutions for non-negative bounded integrable initial data (with proper decay at infinity). They also treat the case of bounded and compactly supported initial data, which propagate with finite speed. By this paper, we contribute to those results showing optimal decay L^p-estimates of solutions and constructing explicit compactly supported self-similar solutions.

Equation (1) is also a multidimensional generalization of the one studied in [2, (2.12)] when $d = 1$ and $m = 2$ (for the integral of u) as a model of the dynamics of dislocations in crystals. The structure of (1) suggests that it should enjoy the conservation of mass and nice comparison properties as was shown in [2].

To some extent, (1) is also comparable to either the classical granular media equation or the aggregation equation without viscosity $\partial_t u = \nabla \cdot (u(\nabla K * u))$ studied in, e.g., [9]. Indeed, the Fourier multiplier ∇^β has an integral form which is quite similar to $\nabla K*$. Note that, however, rather restrictive assumptions on K are made in [9], assumptions which do not permit one to use the diffusive character of the equation (1); moreover, restrictive smoothness assumptions are made on u_0.

A related equation $\partial_t u = \nabla \cdot (u^3 \nabla (-\Delta)^{1/2} u)$ is studied in [6]. It formally corresponds to (1) with $\alpha = m = 3$; it can also be seen as the fractal version of the classical thin film equation. Such a nonlocal higher order equation appears in the modeling of propagation of fractures in rocks. For this reason, it is expected to enjoy a finite propagation speed property as the classical one. Since this equation is of order 3, the mathematical analysis is more involved and additional technical efforts are necessary to prove the existence of global in time nonnegative solutions.

2. The Cauchy problem and asymptotics

Our first main result states that good L^p estimates for solutions of (1)–(2) can be obtained.

Theorem 2.1 (Existence and decay of solutions for the Cauchy problem)
(i) Suppose that $u_0 \geq 0$ is such that $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
There then exists a unique global in time nonnegative solution u of (1)–(2) such that $\int u(t,x)\,\mathrm{d}x = \int u_0(x)\,\mathrm{d}x$.
(ii) the $L^p(\mathbb{R}^d)$ norms ($1 \leq p < \infty$) of each solution u decay algebraically as $t \to \infty$

$$\|u(t)\|_p \leq C(d,\alpha,m,p)\|u_0\|_1^{\frac{\frac{m-1}{p}+\frac{\alpha}{d}}{m-1+\frac{\alpha}{d}}} t^{-\frac{p-1}{m-1+\frac{\alpha}{d}}}.$$

The strategy of the proof of existence of solutions is to consider either solutions $u = u_\varepsilon$ of

$$\partial_t u - \nabla \cdot (|u|^{m-1}\nabla^{\alpha-1}u) = \varepsilon \Delta u, \qquad \text{if } \alpha \in (0,1] \tag{3}$$

or

$$\partial_t u - \nabla \cdot ((\varepsilon + |u|^{m-1})\nabla^{\alpha-1}u) = 0, \qquad \text{if } \alpha \in (1,2) \tag{4}$$

and then pass to the limit $\varepsilon \searrow 0$. Solutions of the approximating equation (3) exist because the term $\varepsilon\Delta u$ is strong enough to regularize (1) when $0 < \alpha \leq 1$, to be compared with the construction achieved in [2, Sec. 4, 5] via viscosity solutions. For the regularized equation (4) with $\alpha \in (1,2)$, first one studies the time discretized problem whose solutions are obtained via theory of pseudomonotone operators. We follow here [6], keeping in mind that our problem is simpler since it is of order $\alpha \in (0,2)$. In particular, it is easier than in [6] to pass to the limit in the nonlinear term since the operator $I_{2-\alpha}$ is regularizing. The necessary estimates on the uniform boundedness of approximating solutions u_ε will follow easily when the *a priori* $L^p(\mathbb{R}^d)$ decay estimates are derived, see below.

In order to prove the announced nonnegativity property as well as the L^p estimates of solutions (similar to those for degenerated partial differential equations like the porous medium equation in [4, Ch. 2]), we recall the generalized Kato inequalities

$$\int (-\Delta)^{\frac{\alpha}{2}} w \operatorname{sgn} w \,\mathrm{d}x \geq 0, \qquad \int (-\Delta)^{\frac{\alpha}{2}} w\, w^+ \,\mathrm{d}x \geq 0, \qquad \int (-\Delta)^{\frac{\alpha}{2}} w\, w^- \,\mathrm{d}x \leq 0, \tag{5}$$

where $w^+ = \max\{0, w\}$, $w^- = \max\{0, -w\}$, and the Stroock–Varopoulos inequality

$$\int (-\Delta)^{\frac{\alpha}{2}} w |w|^{p-2} w \,\mathrm{d}x \geq \frac{4(p-1)}{p^2} \int \left|\nabla^{\frac{\alpha}{2}} |w|^{\frac{p}{2}}\right|^2 \mathrm{d}x \tag{6}$$

valid for each $w \in L^p(\mathbb{R}^d)$ such that $(-\Delta)^{\frac{\alpha}{2}} w \in L^p(\mathbb{R}^d)$. Note that the constant in (6) is the same as for the usual Laplacian operator $-\Delta$ ($\alpha = 2$). The proof is given, e.g., in [10, Prop. 1.6] and [10, Th. 2.1, combined with the Beurling–Deny condition (1.7)].

Inequalities (5) are used to prove the nonnegativity property of solutions of (1) with nonnegative data (2). Note that $\|u(t)\|_1 = \|u_0\|_1$ for nonnegative solutions of (1)–(2).

We give below a formal argument involving (6) which, in fact, can be applied to nonnegative solutions of each of the approximating equations (3), (4), and leads to the L^p decay estimates. Multiply (1) by u^{p-1} with $p > 1$ and integrate by parts to get

$$
\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p\,\mathrm{d}x &= -(p-1)\int u^{m-1}u^{p-2}\nabla^{\alpha-1}u\cdot\nabla u\,\mathrm{d}x\\
&\le -\frac{p-1}{p+m-2}\int u^{p+m-2}(-\Delta)^{\frac{\alpha}{2}}u\\
&\le -\frac{4p(p-1)}{(p+m-1)^2}\left\|\nabla^{\frac{\alpha}{2}}\left(u^{\frac{p+m-1}{2}}\right)\right\|_2^2
\end{aligned}
\tag{7}
$$

after applying the Stroock–Varopoulos inequality (6). To estimate the right hand side of the above inequality, we use the Nash inequality

$$\|v\|_2^{2(1+\frac{\alpha}{d})} \le C_N\|\nabla^{\frac{\alpha}{2}}v\|_2^2\|v\|_1^{\frac{2\alpha}{d}} \tag{8}$$

valid for all functions v with $v \in L^1(\mathbb{R}^d)$, $\nabla^{\frac{\alpha}{2}}v \in L^2(\mathbb{R}^d)$ with a constant $C_N = C(d,\alpha)$. The proof of (8) for $d = 1$ can be found in, e.g., [7, Lemma 2.2], and this extends easily to the general case $d \ge 1$. Moreover, we will need for $1 < p \le r \le 2p$ and $r = p + m - 1$ (i.e. $m \ge 1$, $p \ge m - 1$), the Gagliardo–Nirenberg type inequality

$$\|u\|_p^a \le C_N\left\|\nabla^{\frac{\alpha}{2}}|u|^{\frac{r}{2}}\right\|_2^2\|u\|_1^b \tag{9}$$

with $a = \frac{p}{p-1}\frac{d(r-1)+\alpha}{d}$ and $b = \frac{p\alpha+d(r-p)}{d(p-1)}$. This inequality is a consequence of the Nash inequality (8) written for $v = |u|^{\frac{r}{2}}$, i.e. $\|u\|_r^{r(1+\frac{\alpha}{d})} \le C_N\left\|\nabla^{\frac{\alpha}{2}}|u|^{\frac{r}{2}}\right\|_2^2\|u\|_{\frac{r}{2}}^{\frac{r\alpha}{d}}$, and two Hölder inequalities for the L^q norms:

$$\|u\|_p \le \|u\|_r^{\gamma}\|u\|_1^{1-\gamma}, \qquad \|u\|_{\frac{r}{2}} \le \|u\|_p^{\delta}\|u\|_1^{1-\delta},$$

which hold with $\gamma = \frac{p-1}{r-1}\frac{r}{p}$ and $\delta = \frac{r-2}{p-1}\frac{p}{r}$. Combining the above three inequalities, we get (9). This permits us to write

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^p\,\mathrm{d}x \le -K\|u\|_p^a\|u\|_1^{-b}$$

with some positive constant K. The above inequality leads to the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t) \le -KM^{-b}f(t)^{\frac{a}{p}}$$

for the function $f(t) = \|u(t)\|_p^p$, $M = \|u_0\|_1$, and $a/p > 1$, which immediately gives the algebraic decay of the L^p norms, $p \ge m-1$, of all the approximating solutions $u = u_\varepsilon$ of either (3) or (4) with the bounds independent of $\varepsilon > 0$: $f(t) \le \left(K\left(\frac{a}{p}-1\right)M^{-b}\,t\right)^{-\frac{1}{\frac{a}{p}-1}}$. Of course, this is sufficient to get the conclusion of Theorem 2.1 iii) for all the solutions of (1) and all $1 \le p < \infty$, since $\frac{1}{\frac{a}{p}-1} = \frac{d(p-1)}{d(m-1)+\alpha}$, remember that $r = p + m - 1$, and finally interpolate between $L^1(\mathbb{R}^d)$ and $L^p(\mathbb{R}^d)$, with p sufficiently large.

3. Self-similar solutions

The next step is to construct nonnegative self-similar solutions in the case $m = 2$. We look for solutions that are invariant under some scalings and with a prescribed mass. A classical dimensional analysis permits us to conclude that solutions should be of the following form

$$u(t,x) = \frac{1}{t^{d\lambda}}\Phi_{\alpha,m}\left(\frac{x}{t^\lambda}\right), \quad \lambda = \frac{1}{(m-1)d+\alpha}, \quad y = \frac{x}{t^\lambda}, \tag{10}$$

for some function $\Phi_{\alpha,m} : \mathbb{R}^d \to \mathbb{R}^+$ satisfying the following elliptic equation $\mathbb{R}^d$

$$-\lambda\nabla \cdot (y\Phi_{\alpha,m}) = \nabla \cdot \left(\Phi_{\alpha,m}^{m-1}\nabla^{\alpha-1}\Phi_{\alpha,m}\right). \tag{11}$$

In the case $m \geq 1$, this equation reduces to some Dirichlet problem; it is worth mentioning that this is not true anymore for fast diffusions, that is to say when $m < 1$.

Theorem 3.1 (Self-similar solutions) *For $m = 2$, the function $u : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}^+$ defined by (10), with*

$$\Phi_{\alpha,2}(y) = C(1-|y|^2)_+^{\frac{\alpha}{2}},$$

i.e.

$$u(t,x) = Ct^{-\frac{d}{d+\alpha}}\left(1-|x|^2 t^{-\frac{2}{d+\alpha}}\right)_+^{\frac{\alpha}{2}}$$

and a positive constant C, is a ($\frac{\alpha}{2}$-Hölder continuous) solution of (1), in the pointwise sense for $|x| \neq 1$.

When $\alpha = 2$, we recover the classical KZB (or Barenblatt) formulas, see, e.g., [12].

Theorem 3.1 implies that the decay rate in Theorem 2.1, iii) is optimal. Indeed, it exactly corresponds to that for self-similar solutions.

Proof of Theorem 3.1. As explained above, the problem of determining self-similar solutions reduces to the study of the elliptic equation (11). First, it is natural to look for solutions Φ such that $-\lambda y\Phi = \Phi^{m-1}\nabla^{\alpha-1}\Phi$. Now, since we want to construct a compactly supported solution, we look for nonnegative solutions Φ vanishing outside the unit ball $B_1 \subset \mathbb{R}^d$, and such that $-\lambda y = \Phi^{m-2}\nabla^{\alpha-1}\Phi$ in B_1.

We also emphasize that the homogeneous Dirichlet condition here should be understood in the form $\Phi \equiv 0$ outside B_1, and not $\Phi = 0$ only on ∂B_1. See [1] and also [2] for more details.

The proof of Theorem 3.1 is reduced to the following fundamental technical lemma.

Lemma 3.2 *For all $\beta \in (-1,1)$ and $\gamma \in (0,1)$, we have for each $y \in B_1$*

$$I_\beta\left((1-|y|^2)_+^{\frac{\gamma}{2}}\right) = C_{\gamma,\beta,d} \times {}_2F_1\left(\frac{d-\beta}{2}, -\frac{\gamma+\beta}{2}, \frac{d}{2}, |y|^2\right)$$

with $C_{\gamma,\beta,d} = 2^{-\beta}\frac{\Gamma(\frac{\gamma}{2}+1)\Gamma(\frac{d-\beta}{2})}{\Gamma(\frac{d}{2})\Gamma(\frac{\beta+\gamma}{2}+1)}$, and where ${}_2F_1$ denotes the hypergeometric function.

This formula is obtained through a tedious computation involving the Weber–Schafheitlin integrals, [11, p. 99]). Remark next that when $\beta + \gamma = 2$, it is known that ${}_2F_1\left(\frac{d-\beta}{2}, -1, \frac{d}{2}, z\right) = 1 - \frac{d-\beta}{d}z$.

Corollaries of Lemma 3.2 We would like to shed some light on the fact that two more pieces of information can be derived from Lemma 3.2.

• First, when $\beta + \gamma < 2$ (here $\beta = 2-\alpha$, $\gamma = \frac{\alpha}{m-1}$, $m \neq 2$), the previous computation permits one to prove that if self-similar solutions exist, they are certainly *not* of the form $C(1-|y|^2)_+^{\frac{\gamma}{2}}$.

• Second, we can deduce from Lemma 3.2 the following corollary which was proved by Getoor [5], see also [8, App.].

Corollary 3.3 *For all $\alpha \in (0,2]$, the identity*

$$K_{\alpha,d}(-\Delta)^{\frac{\alpha}{2}}(1-|y|^2)^{\frac{\alpha}{2}} = -1 \quad \text{in} \quad B_1$$

holds with $K_{\alpha,d} = \frac{\Gamma(\frac{d}{2})}{2^\alpha\Gamma(1+\frac{\alpha}{2})\Gamma(\frac{d+\alpha}{2})}$.

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