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A POSTERIORI ERROR ESTIMATORS FOR THE FULLY DISCRETE TIME DEPENDENT STOKES PROBLEM WITH SOME DIFFERENT BOUNDARY CONDITIONS

FIDA EL CHAMI[†] AND TONI SAYAH[‡]

ABSTRACT. In this paper we study the time dependent Stokes problem with some different boundary conditions. We establish a decoupled variational formulation into a system of velocity and a Poisson equation for the pressure. Hence, the velocity is approximated with **curl** conforming finite elements in space and Euler scheme in time and the pressure with standard continuous elements in space and Euler scheme in time. Finally, we establish optimal a priori and a posteriori estimates

Keywords Partial differential equations, a priori and a posteriori errors, time dependent Stokes equations.

1. INTRODUCTION.

This paper is devoted to the numerical solution of the time dependent Stokes equations for an incompressible fluid

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \times]0, T[, \quad (1.1)$$

with the incompressibility condition

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega \times]0, T[, \quad (1.2)$$

the initial condition

$$\mathbf{u}(0) = \mathbf{0} \quad \text{in } \Omega, \quad (1.3)$$

the boundary conditions

$$\mathbf{u} \times \mathbf{n} = 0, \quad p = 0 \quad \text{on } \partial\Omega \times]0, T[, \quad (1.4)$$

or

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \mathbf{curl} \mathbf{u} \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \times]0, T[\quad (1.5)$$

where $]0, T[$ is a given time-interval, Ω is a convex bounded simply connected domain of $\mathbb{R}^3$ with a polyhedral connected boundary $\Gamma = \partial\Omega$ and $\mathbf{n}$ the exterior unit normal to Γ , $\mathbf{u}$ the velocity and p the pressure. All the quantities are taken at the point (x, t) where $x = (x_i)_{1 \leq i \leq 3} \in \mathbb{R}^3$ denotes the position and $t \in [0, T]$ the time. $\mathbf{f}$ denotes the external forces applied to the fluid and $\nu > 0$ is the viscosity.

These sets of boundary conditions lend themselves readily to a variational formulation where the Laplacian operator is expressed by a $(\mathbf{curl}, \mathbf{curl})$ term and the incompressibility condition by an equation of the form $(\nabla q, \mathbf{v})$. This suggests to use a partially non-confirming finite element method, where just the **curl** of the velocity is continuous at interface boundaries whereas the pressure is globally continuous.

We refer also to Girault's work [17] for a vector potential-vorticity approximation of similar Navier-Stokes type problems and to [16] for the steady-state incompressible Navier-Stokes equations with non standard boundary conditions. We also refer to [25] where Repin establishes a posteriori estimates for the velocity, stress and pressure fields for the stationary Stokes problem and where his approach is based on duality

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theory of the calculus of variations. A posteriori estimates for the Stokes problem and for some viscous flow problems were studied by a number of authors, [4], [30], [31], [9], [22] and [23]. Typically, they have been obtained in the frame of the so-called "residual method" originally proposed in [2] and [3] for the finite element approximations. For the a priori estimations of the Stokes problem, we can also cite the works of S. Repin [26], [27], [28]. Finally, in [1], we treat the same problem in stationary case.

In this work, we treat the time dependent stokes equations with some different boundary conditions. We establish a decoupled variational formulation into a system of velocity and a Poisson equation for the pressure. We introduce a fully discrete system using the **curl** conforming finite elements for the velocity and the standard continuous elements for the pressure in space, and the Euler scheme in time. Finally, we establish an optimal a priori and a posteriori error estimates.

2. DESCRIPTION AND ANALYSIS OF THE MODEL

We denote by (*Problem1*) the system of equations (1.1), (1.2), (1.3) and (1.4), and by (*Problem2*) the system of equations (1.1), (1.2) (1.3) and (1.5). In all the paper, we denote by C and c generic positive constants.

In order to write the variational formulation of the previous problems, we introduce some spaces:

$$\begin{aligned} W^{m,p}(\Omega) &= \{v \in L^p(\Omega), \partial^\alpha v \in L^p(\Omega), \quad \forall |\alpha| \leq m\}, \\ H^m(\Omega) &= W^{m,2}(\Omega), \end{aligned}$$

equipped with the following semi-norm and norm :

$$|v|_{m,p,\Omega} = \left\{ \sum_{|\alpha|=m} \int_{\Omega} |\partial^\alpha v(x)|^p dx \right\}^{1/p} \quad \text{and} \quad \|v\|_{m,p,\Omega} = \left\{ \sum_{k \leq m} |v|_{k,p,\Omega}^p \right\}^{1/p}$$

As usual, we shall omit p when $p = 2$ and denote by $(\cdot, \cdot)$ the scalar product of $L^2(\Omega)$. Also, recall the familiar notation :

$$H_0^1(\Omega) = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma\},$$

with the Poincaré inequality

$$\forall v \in H_0^1(\Omega); \quad \|v\|_{0,\Omega} \leq C \|v\|_{1,\Omega}. \quad (2.1)$$

Finally, we introduce the spaces :

$$\begin{aligned} H(\text{div}, \Omega) &= \{\mathbf{v} \in L^2(\Omega)^3, \text{div } \mathbf{v} \in L^2(\Omega)\}; \quad H_0(\text{div}, \Omega) = \{\mathbf{v} \in H(\text{div}, \Omega), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\} \\ H(\mathbf{curl}, \Omega) &= \{\mathbf{v} \in L^2(\Omega)^3, \mathbf{curl } \mathbf{v} \in L^2(\Omega)^3\}; \quad H_0(\mathbf{curl}, \Omega) = \{\mathbf{v} \in H(\mathbf{curl}, \Omega), \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma\} \end{aligned}$$

normed respectively by :

$$\|\mathbf{v}\|_{H(\text{div}, \Omega)} = \{\|\mathbf{v}\|_{0,\Omega}^2 + \|\text{div } \mathbf{v}\|_{0,\Omega}^2\}^{1/2},$$

and

$$\|\mathbf{v}\|_{H(\mathbf{curl}, \Omega)} = \{\|\mathbf{v}\|_{0,\Omega}^2 + \|\mathbf{curl } \mathbf{v}\|_{0,\Omega}^2\}^{1/2}.$$

For the theoretical foundations of this spaces, we can refer to Duvaut & Lions [13] and for the following regularity theorems, we refer to Bernardi [8], Dauge [11], Girault & Raviart [15], Grisvard [18] and Nedelec [21].

Lemma 2.1. *For each $\mathbf{g} \in L^2(\Omega)$, there exists a unique solution $\mathbf{w}$ in $H^1(\Omega)/\mathbb{R}$ (resp. $H_0^1(\Omega)$) such that :*

$$(\nabla \mathbf{w}, \nabla \mathbf{v}) = (\mathbf{g}, \nabla \mathbf{v}) \quad \forall \mathbf{v} \in H^1(\Omega)/\mathbb{R} \quad (\text{resp } H_0^1(\Omega))$$

and there exists a positive constant C such that :

$$\|\mathbf{w}\|_{1,\Omega} \leq C\|\mathbf{g}\|_{0,\Omega}.$$

Theorem 2.2. All functions $\mathbf{v} \in L^2(\Omega)^3$ satisfying :

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{curl} \mathbf{v} \in L^2(\Omega)^3, \quad \mathbf{v} \cdot \mathbf{n} = 0 \text{ or } \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma$$

belong to $H^1(\Omega)$ and we have

$$\|\mathbf{v}\|_{1,\Omega} \leq C\|\operatorname{curl} \mathbf{v}\|_{0,\Omega}.$$

As usual, for handling time-dependent problems, it is convenient to consider functions defined on a time interval $]a, b[$ with values in a functional space, say X . More precisely, let $\|\cdot\|_X$ denote the norm of X ; then for any r , $1 \leq r \leq \infty$, we define

$$L^r(a, b; X) = \left\{ f \text{ measurable in }]a, b[; \int_a^b \|f(t)\|_X^r dt < \infty \right\}$$

equipped with the norm

$$\|f\|_{L^r(a,b;X)} = \left(\int_a^b \|f(t)\|_X^r dt \right)^{1/r},$$

with the usual modifications if $r = \infty$. It is a Banach space if X is a Banach space.

In view of the relation :

$$-\Delta \mathbf{u} = \operatorname{curl} \operatorname{curl} \mathbf{u} \quad (\text{as we have } \operatorname{div} \mathbf{u} = 0)$$

we can easily prove the following theorems.

Theorem 2.3. Let $\mathbf{f} \in L^\infty(0, T; L^2(\Omega)^3)$.

(Problem1) has the following weak variational formulation :

Find $\mathbf{u}(t) \in H_0(\operatorname{curl}, \Omega)$ and $p(t) \in H_0^1(\Omega)$ such that

$$\left(\frac{d}{dt} \mathbf{u}(t), \mathbf{v} \right) + \nu (\operatorname{curl} \mathbf{u}(t), \operatorname{curl} \mathbf{v}) + (\nabla p(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\operatorname{curl}, \Omega) \quad (2.2)$$

$$(\nabla q, \mathbf{u}(t)) = 0 \quad \forall q \in H_0^1(\Omega) \quad (2.3)$$

$$\mathbf{u}(0) = 0 \quad (2.4)$$

and (Problem2) has the following weak variational formulation :

Find $\mathbf{u}(t) \in H(\operatorname{curl}, \Omega)$ and $p(t) \in H^1(\Omega)/\mathbb{R}$ such that

$$\left(\frac{d}{dt} \mathbf{u}(t), \mathbf{v} \right) + \nu (\operatorname{curl} \mathbf{u}(t), \operatorname{curl} \mathbf{v}) + (\nabla p(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \quad \forall \mathbf{v} \in H(\operatorname{curl}, \Omega) \quad (2.5)$$

$$(\nabla q, \mathbf{u}(t)) = 0 \quad \forall q \in H^1(\Omega) \quad (2.6)$$

$$\mathbf{u}(0) = 0 \quad (2.7)$$

Where $\mathbf{u}(t) = \mathbf{u}(t, \cdot)$ and $p(t) = p(t, \cdot)$.

Each variational formulation is splitted into a system for the velocity and a Poisson equation for the pressure.

Let us introduce the spaces :

$$V_0 = \{ \mathbf{v} \in H_0(\operatorname{curl}, \Omega); (\nabla q, \mathbf{v}) = 0 \quad \forall q \in H_0^1(\Omega) \}$$

and

$$U = \{ \mathbf{v} \in H(\operatorname{curl}, \Omega); (\nabla q, \mathbf{v}) = 0 \quad \forall q \in H^1(\Omega) \}.$$

The lemma (2.1) and the theorems (2.2) and (2.3) allow us to establish the following theorem:

Theorem 2.4. *The problem (2.2)-(2.3) is equivalent to the problem :*

Find $\mathbf{u}(t) \in V_0$ such that

$$\left(\frac{d}{dt}\mathbf{u}(t), \mathbf{v}\right) + \nu(\mathbf{curl} \mathbf{u}(t), \mathbf{curl} \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \quad \forall \mathbf{v} \in V_0 \quad (2.8)$$

Find $p(t) \in H_0^1(\Omega)$ such that

$$(\nabla p(t), \nabla q) = (\mathbf{f}(t), \nabla q) \quad \forall q \in H_0^1(\Omega). \quad (2.9)$$

The problem (2.5)-(2.6) is equivalent to the problem :

Find $\mathbf{u}(t) \in U$ such that

$$\left(\frac{d}{dt}\mathbf{u}(t), \mathbf{v}\right) + \nu(\mathbf{curl} \mathbf{u}(t), \mathbf{curl} \mathbf{v}) = (\mathbf{f}(t), \mathbf{v}) \quad \forall \mathbf{v} \in U \quad (2.10)$$

Find $p(t) \in H^1(\Omega)/\mathbb{R}$ such that

$$(\nabla p(t), \nabla q) = (\mathbf{f}(t), \nabla q) \quad \forall q \in H^1(\Omega). \quad (2.11)$$

Furthermore, the first problem has one and only one solution $(\mathbf{u}, p) \in (L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; V_0) \times L^\infty(0, T; H_0^1(\Omega)))$ and the second problem has one and only one solution $(\mathbf{u}, p) \in (L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; U) \times L^\infty(0, T; H^1(\Omega)/\mathbb{R}))$. In both cases, we have the following bounds :

$$\begin{aligned} \|p\|_{L^\infty(0, T; H_0^1(\Omega))} &\leq \|\mathbf{f}\|_{L^\infty(0, T; L^2(\Omega)^3)} \\ \|\mathbf{u}\|_{L^\infty(0, T; L^2(\Omega)^3)} + \|\mathbf{curl} \mathbf{u}\|_{L^2(0, T; L^2(\Omega)^3)} &\leq C_1 \|\mathbf{f}\|_{L^\infty(0, T; L^2(\Omega)^3)} \end{aligned}$$

3. FINITE ELEMENT DISCRETIZATION

We introduce a regular family of tetrahedra $(\tau_h)_h$ in the sens that :

- for each $h, \bar{\Omega}$ in the union of all element of τ_h ;
- for each h , the intersection of two different elements of τ_h , if not empty, is a corner, a whole edge or a whole face of both of them;
- the ratio of the diameter h_κ of an element κ in τ_h to the diameter of its inscribed sphere is bounded by a constant independent of κ and h ;

As usual, h denotes the maximum of the diameters of the elements of τ_h .

Next, for each κ in τ_h , we introduce the spaces $\mathbb{P}_0(\kappa)$ of the restrictions to κ of constant functions on $\mathbb{R}^3$, $\mathbb{P}_1(\kappa)$ of the restrictions to κ of affine functions on $\mathbb{R}$ and the space $\mathbb{P}_K(\kappa)$ of the restrictions to κ of polynomials $\mathbf{v}$ of the form :

$$\mathbf{v}(x) = a + b \times x, \quad a \in \mathbb{R}^3, b \in \mathbb{R}^3.$$

The space $\mathbb{P}_K(\kappa)$ and the corresponding finite elements are studied in [20].

Their degrees of freedom are the average flux along the edges $\int_l (\mathbf{v} \cdot \mathbf{t}) dl$, for the six edges l of κ , $\mathbf{t}$ is the direction vector of l .

Hence, we associate the operator r_κ where $r_\kappa(\mathbf{u})$ is the unique polynomial of $\mathbb{P}_K$ that has the same flux along the edges as $\mathbf{u}$. We define also the operator I_κ where $I_\kappa(q)$ is the unique polynomial of $\mathbb{P}_1(\kappa)$ that has the same values on the vertex of κ as q .

Next, let us introduce the discrete spaces :

$$M_h = \{\mathbf{u}_h \in H(\mathbf{curl}, \Omega); \mathbf{u}_h|_\kappa \in \mathbb{P}_K(\kappa), \forall \kappa \in \tau_h\}, \quad (3.1)$$

$$M_{0h} = M_h \cap H_0(\mathbf{curl}, \Omega), \quad (3.2)$$

$$Q_h = \{q_h \in C^0(\bar{\Omega}); q_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \tau_h\}, \quad (3.3)$$

$$Q_{0h} = Q_h \cap H_0^1(\Omega). \quad (3.4)$$

With these spaces, the finite dimensional analogues of V_0 and U are :

$$V_{0h} = \{\mathbf{v}_h \in M_{0h} ; (\nabla q_h, \mathbf{v}_h) = 0, \quad \forall q_h \in Q_h\}$$

and

$$U_h = \{\mathbf{v}_h \in M_h ; (\nabla q_h, \mathbf{v}_h) = 0 \quad \forall q_h \in Q_h\}.$$

We define the interpolation operators r_h from $H^1(\Omega)^3$ onto M_h , I_h from $H^2(\Omega)$ onto Q_h by

$$r_h u = r_\kappa(u) \text{ on } \kappa, \quad \forall \kappa \in \tau_h \quad (\text{similarly for } I_h).$$

Theorem 3.1. *Assume that the triangulation τ_h is regular. For all $k \geq 1$ we have :*

$$\| \mathbf{u} - r_h \mathbf{u} \|_{0,\Omega} + h \| \mathbf{curl}(\mathbf{u} - r_h \mathbf{u}) \|_{0,\Omega} \leq Ch | \mathbf{u} |_{1,t,\Omega}, \quad \forall \mathbf{u} \in W^{1,t}(\Omega)^3, \quad \text{for some } t > 2.$$

Moreover, when $\mathbf{u} \in (H^k(\Omega))^3$ and $\forall k \geq 2$ we have :

$$\| \mathbf{u} - r_h \mathbf{u} \|_{0,\Omega} \leq Ch^k | \mathbf{u} |_{k,\Omega}$$

and, when $\mathbf{u} \in (H^{k+1}(\Omega))^3$ and $\forall k \geq 1$ we have :

$$\| \mathbf{curl}(\mathbf{u} - r_h \mathbf{u}) \|_{0,\Omega} \leq Ch^k | \mathbf{u} |_{k+1,\Omega}$$

We have also an important result given by V. Girault [16] which shows an impact imbedding between the spaces V_{0h} or U_h and $L^4(\Omega)^3$.

Theorem 3.2. *Let τ_h a uniformly regular triangulation of Ω . For each space V_{0h} and U_h , there exists constants C and c' , independent of h , such that*

$$\| \mathbf{u}_h \|_{2,\Omega} \leq C \| \mathbf{u}_h \|_{0,4,\Omega} \leq C' \| \mathbf{curl} \mathbf{u}_h \|_{0,\Omega} \quad \forall \mathbf{u}_h \in V_{0h} \text{ or } U_h \quad (3.5)$$

To discretize in time, we divide the interval $[0, T]$ into N subintervals of equal length $k = \frac{T}{N}$, with grid-points $t^n = nk, 0 \leq n \leq N$.

We discretize (*Problem1*) by : knowing u_h^n , find (u_h^{n+1}, p_h^{n+1}) with values in $V_{0h} \times Q_{0h}$, solution of

$$\frac{1}{k} (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \nu (\mathbf{curl} \mathbf{u}_h^{n+1}, \mathbf{curl} \mathbf{v}_h) + (\nabla p_h^{n+1}, \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in M_{0h} \quad (3.6)$$

Similarly, we discretize (*Problem2*) by : knowing u_h^n , find (u_h^{n+1}, p_h^{n+1}) with values in $U_h \times Q_h / \mathbb{R}$, solution of

$$\frac{1}{k} (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \nu (\mathbf{curl} \mathbf{u}_h^{n+1}, \mathbf{curl} \mathbf{v}_h) + (\nabla p_h^{n+1}, \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in M_h. \quad (3.7)$$

Where we assume that $\mathbf{f}^{n+1} = \mathbf{f}(t_{n+1})$ and $\mathbf{u}_h^0 = \mathbf{0}$.

As in the continuous way, the problem (3.6) can be splitted to

Find $\mathbf{u}_h^{n+1} \in V_{0h}$ such that

$$\frac{1}{k} (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \nu (\mathbf{curl} \mathbf{u}_h^{n+1}, \mathbf{curl} \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in V_{0h}, \quad (3.8)$$

Find $p_h^{n+1} \in Q_{0h}$ such that

$$(\nabla p_h^{n+1}, \nabla q_h) = (\mathbf{f}^{n+1}, \nabla q_h), \quad \forall q_h \in Q_{0h}. \quad (3.9)$$

And the problem (3.7) can be splitted to

Find $\mathbf{u}_h^{n+1} \in U_h$ such that

$$\frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}_h) + \nu(\mathbf{curl} \mathbf{u}_h^{n+1}, \mathbf{curl} \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h), \quad \forall \mathbf{v}_h \in U_h, \quad (3.10)$$

Find $p_h^{n+1} \in Q_h/\mathbb{R}$ such that

$$(\nabla p_h^{n+1}, \nabla q_h) = (\mathbf{f}^{n+1}, \nabla q_h), \quad \forall q_h \in Q_h. \quad (3.11)$$

It is easy to show that these two last discrete problems have a unique solution.

The pressure is entirely dissociated from the velocity, i.e. can be computed without knowing the velocity.

We have also for both discrete problems :

$$\sup_{0 \leq n \leq N-1} \|\mathbf{u}_h^{n+1}\|_{0,\Omega}^2 + \sum_{n=0}^{N-1} \|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|_{0,\Omega}^2 + \sum_{n=0}^{N-1} k \|\mathbf{curl} \mathbf{u}_h^{n+1}\|_{0,\Omega}^2 \leq C \|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)}^2$$

and

$$\sup_{0 \leq n \leq N-1} |p_h^{n+1}|_{1,\Omega} \leq \|\mathbf{f}\|_{L^\infty(0,T;L^2(\Omega)^3)}$$

4. A PRIORI ERROR ANALYSIS

In this section, we will establish the a priori error estimates for the pressure and the velocity.

Theorem 4.1. *We suppose that the theoretical solution $(\mathbf{u}, p)$ of the problem (2.8)-(2.9) (resp. (2.10)-(2.11)) verify: $\mathbf{u} \in L^\infty(0, T; H^2(\Omega)^3)$, $\mathbf{u}' \in L^2(0, T; H^1(\Omega)^3)$, $\mathbf{u}'' \in L^\infty(0, T; L^2(\Omega)^3)$ and $p \in L^\infty(0, T; H^2(\Omega))$. Then they verify with the numerical solution $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ of the problem (3.8)-(3.9) (resp. (3.10)-(3.11)) the error estimates :*

$$\sup_{0 \leq n \leq N-1} \|\nabla(p(t_{n+1}) - p_h^{n+1})\|_{0,\Omega} \leq C(p, T, \Omega) h \quad (4.1)$$

$$\begin{aligned} \sup_{0 \leq n \leq N-1} \|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|_{0,\Omega}^2 &+ \sum_{n=0}^{N-1} \|(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}) - (\mathbf{u}(t_n) - \mathbf{u}_h^n)\|_{0,\Omega}^2 \\ &+ \sum_{n=0}^{N-1} k \|\mathbf{curl}(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1})\|_{0,\Omega}^2 \leq C(p, \mathbf{u}, \mathbf{f}, T, \Omega) (h^2 + k^2) \end{aligned} \quad (4.2)$$

Proof:

For the pressure, let us choose $q = q_h$ in (2.9) and take, for $t = t_{n+1}$, the difference with (3.9) (resp. (2.11) and (3.11)), we obtain:

$$(\nabla(p(t_{n+1}) - p_h^{n+1}), \nabla q_h) = 0, \quad \forall q_h \in Q_{0h} \quad (\text{resp. } Q_h) \quad (4.3)$$

then $\forall q_h \in Q_{0h}$ (resp. Q_h)

$$(\nabla(I_h p(t_{n+1}) - p_h^{n+1}), \nabla q_h) = -(\nabla(p(t_{n+1}) - I_h p(t_{n+1})), \nabla q_h) \quad (4.4)$$

We take $q_h = I_h p(t_{n+1}) - p_h^{n+1}$ and we obtain, using the triangle inequality, the relation:

$$\|\nabla(p(t_{n+1}) - p_h^{n+1})\|_{0,\Omega} \leq Ch |p(t_{n+1})|_{2,\Omega} \quad (4.5)$$

which lead to the first result (4.1).

For the velocity, we choose $\mathbf{v} = \mathbf{v}_h$ in the equation (2.8) taken for $t = t_{n+1}$ and multiplied by k , and take the difference with (3.8) multiplied by k (resp. (2.10) and (3.10)), we obtain:

$$\begin{aligned} (k \frac{d\mathbf{u}}{dt}(t_{n+1}) - (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n), \mathbf{v}_h) + \nu k (\mathbf{curl}(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}), \mathbf{curl} \mathbf{v}_h) \\ + k (\nabla(p(t_{n+1}) - p_h^{n+1}), \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in M_{0h} \quad (\text{resp. } M_h) \end{aligned}$$

Then a simple manipulation gives:

$$\begin{aligned} & ((r_h \mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}) - (r_h \mathbf{u}(t_n) - \mathbf{u}_h^n), \mathbf{v}_h) + \nu k (\mathbf{curl}(r_h \mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}), \mathbf{curl} \mathbf{v}_h) = \\ & -((\mathbf{u}(t_{n+1}) - r_h \mathbf{u}(t_{n+1})) - (\mathbf{u}(t_n) - r_h \mathbf{u}(t_n)), \mathbf{v}_h) + ((\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)) - k \mathbf{u}'(t_{n+1}), \mathbf{v}_h) \\ & - \nu k (\mathbf{curl}(\mathbf{u}(t_{n+1}) - r_h \mathbf{u}(t_{n+1})), \mathbf{curl} \mathbf{v}_h) - k (\nabla(p(t_{n+1}) - p_h^{n+1}), \mathbf{v}_h) \\ & \forall \mathbf{v}_h \in M_{0h} \quad (\text{resp. } M_h) \end{aligned}$$

We take $\mathbf{v}_h = \mathbf{v}_h^{n+1} = r_h \mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}$ and use the formula $(a - b, a) = \frac{1}{2}a^2 - \frac{1}{2}b^2 + \frac{1}{2}(a - b)^2$, we obtain :

$$\begin{aligned} & \frac{1}{2} \|\mathbf{v}_h^{n+1}\|_{0,\Omega}^2 - \frac{1}{2} \|\mathbf{v}_h^n\|_{0,\Omega}^2 + \frac{1}{2} \|(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\|_{0,\Omega}^2 + \nu k \|\mathbf{curl}(\mathbf{v}_h^{n+1})\|_{0,\Omega}^2 \leq \\ & \left| \left(\int_{t_n}^{t_{n+1}} (\mathbf{u}'(t) - r_h \mathbf{u}'(t)) dt, \mathbf{v}_h^{n+1} \right) \right| + C_2 k^2 \|\mathbf{u}''\|_{L^\infty(0,T;L^2(\Omega)^3)} \|\mathbf{v}_h^{n+1}\|_{0,\Omega} \\ & + \nu C_3 k h \|\mathbf{u}(t_{n+1})\|_{2,\Omega} \|\mathbf{curl} \mathbf{v}_h^{n+1}\|_{0,\Omega} + \left| k (\nabla(p(t_{n+1}) - p_h^{n+1}), \mathbf{v}_h^{n+1}) \right| \end{aligned}$$

The first term of the right member can be treated as :

$$\left| \left(\int_{t_n}^{t_{n+1}} (\mathbf{u}'(t) - r_h \mathbf{u}'(t)) dt, \mathbf{v}_h^{n+1} \right) \right| \leq C_1 \left(\frac{1}{2\varepsilon_1} h^2 \|\mathbf{u}'\|_{L^2(t_n, t_{n+1}; H_0^1(\Omega)^3)}^2 + \frac{\varepsilon_1}{2} k \|\mathbf{v}_h^{n+1}\|_{0,\Omega}^2 \right)$$

The second term :

$$\left| C_2 k^2 \|\mathbf{u}''\|_{L^\infty(0,T;L^2(\Omega)^3)} \|\mathbf{v}_h^{n+1}\|_{0,\Omega} \right| \leq C_2 \left\{ \frac{1}{2\varepsilon_2} k^3 \|\mathbf{u}''\|_{L^\infty(0,T;L^2(\Omega)^3)}^2 + \frac{\varepsilon_2}{2} k \|\mathbf{v}_h^{n+1}\|_{0,\Omega}^2 \right\}$$

The third term :

$$\left| \nu C_3 k h \|\mathbf{u}(t_{n+1})\|_{2,\Omega} \|\mathbf{curl} \mathbf{v}_h^{n+1}\|_{0,\Omega} \right| \leq C_3 \left\{ \frac{\nu^2}{2\varepsilon_3} k h^2 \|\mathbf{u}\|_{L^\infty(0,T;H^2(\Omega)^3)}^2 + \frac{\varepsilon_3}{2} k \|\mathbf{curl} \mathbf{v}_h^{n+1}\|_{0,\Omega}^2 \right\}$$

The last term: for all functions $\mathbf{v}_h^{n+1}$ of M_{0h} (resp M_h), we define q_h^{n+1} in Q_{0h} (resp. Q_h) by

$$(\nabla q_h^{n+1}, \nabla \mu_h) = (\mathbf{v}_h^{n+1}, \nabla \mu_h) \quad \forall \mu_h \in Q_{0h} \quad (\text{resp. } Q_h)$$

and set $\mathbf{w}_h^{n+1} = \mathbf{v}_h^{n+1} - \nabla q_h^{n+1}$. Then $\mathbf{w}_h^{n+1}$ belongs to V_{0h} (resp. U_h) and $\mathbf{curl} \mathbf{w}_h^{n+1} = \mathbf{curl} \mathbf{v}_h^{n+1}$. Then we obtain with the relation (4.3) :

$$\begin{aligned} \left| k (\nabla(p(t_{n+1}) - p_h^{n+1}), \mathbf{v}_h^{n+1}) \right| &= \left| k (\nabla(p(t_{n+1}) - p_h^{n+1}), \mathbf{w}_h^{n+1}) \right| \\ &\leq C_4 \left(\frac{1}{2\varepsilon_4} k \|\nabla(p(t_{n+1}) - p_h^{n+1})\|_{0,\Omega}^2 + \frac{\varepsilon_4}{2} k \|\mathbf{curl} \mathbf{v}_h^{n+1}\|_{0,\Omega}^2 \right) \end{aligned}$$

Taking all the last inequality, summing over n from 0 to $m-1$, we obtain by choosing $\varepsilon_1 = \frac{1}{8C_1 T}$; $\varepsilon_2 = \frac{1}{8C_2 T}$, $\varepsilon_3 = \frac{\nu}{2C_3}$, $\varepsilon_4 = \frac{\nu}{2C_4}$:

$$\frac{1}{2} \|\mathbf{v}_h^m\|_{0,\Omega}^2 + \frac{1}{2} \sum_{n=0}^{m-1} \|(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\|_{0,\Omega}^2 + \frac{\nu}{2} \sum_{n=0}^{m-1} k \|\mathbf{curl}(\mathbf{v}_h^{n+1})\|_{0,\Omega}^2 \leq C(h^2 + k^2) + \frac{1}{8} \sup_{1 \leq n \leq N-1} \|\mathbf{v}_h^{n+1}\|_{0,\Omega}^2$$

which leads to :

$$\frac{3}{8} \sup_{1 \leq n \leq N-1} \|\mathbf{v}_h^{n+1}\|_{0,\Omega}^2 + \frac{1}{2} \sum_{n=0}^{N-1} \|(\mathbf{v}_h^{n+1} - \mathbf{v}_h^n)\|_{0,\Omega}^2 + \frac{\nu}{2} \sum_{n=0}^{N-1} k \|\mathbf{curl}(\mathbf{v}_h^{n+1})\|_{0,\Omega}^2 \leq C'(h^2 + k^2)$$

We obtain the result by using the triangular inequality:

$$\|\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|_{0,\Omega} \leq \|\mathbf{u}(t_{n+1}) - r_h \mathbf{u}(t_{n+1})\|_{0,\Omega} + \|r_h \mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}\|_{0,\Omega}$$

and remarking that:

$$\begin{aligned} \|(\mathbf{u}(t_{n+1}) - \mathbf{u}_h^{n+1}) - (\mathbf{u}(t_n) - \mathbf{u}_h^n)\|_{0,\Omega}^2 &\leq 2\left\|\int_{t_n}^{t_{n+1}} (\mathbf{u}'(\tau) - r_h \mathbf{u}'(\tau)) d\tau\right\|_{0,\Omega}^2 + 2\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,\Omega}^2 \\ &\leq Ckh^2\|\nabla \mathbf{u}'\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + 2\|\mathbf{v}_h^{n+1} - \mathbf{v}_h^n\|_{0,\Omega}^2 \end{aligned}$$

□

Remark: We denote by $\mathbf{u}_h$ and p_h the time dependent functions which take in the interval $[t_n, t_{n+1}]$ the values:

$$\begin{aligned} \mathbf{u}_h(t) &= \frac{t - t_n}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) + \mathbf{u}_h^n \\ p_h(t) &= \frac{t - t_n}{k}(p_h^{n+1} - p_h^n) + p_h^n \end{aligned}$$

We have, with some regularities over $(\mathbf{u}, p)$, the same a priori estimates as the last theorem:

$$\|\nabla(p - p_h)\|_{L^\infty(0,T;L^2(\Omega))}^2 \leq C(h^2 + k^2) \quad (4.6)$$

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(0,T;L^2(\Omega)^3)}^2 + \nu\|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(0,T;L^2(\Omega)^3)}^2 \leq C(h^2 + k^2) \quad (4.7)$$

If fact, it suffices to use the inequality:

$$|\nabla(p(t) - p_h(t))| \leq |\nabla(p(t) - p(t_{n+1}))| + |\nabla(p(t_{n+1}) - p_h^{n+1})| + |\nabla(p_h^{n+1} - p_h(t))|$$

and remark that:

$$|\nabla(p_h^{n+1} - p_h(t))| \leq |\nabla(p_h^{n+1} - p_h^n)| \leq |\nabla(p_h^{n+1} - p(t_{n+1}))| + |\nabla(p(t_{n+1}) - p(t_n))| + |\nabla(p(t_n) - p_h^n)|$$

5. A POSTERIORI ERROR ANALYSIS

We now intend to prove a posteriori error estimates between the exact solution $(\mathbf{u}, p)$ of the problem (2.8)-(2.9) and the numerical solution $(\mathbf{u}_h, p_h)$ of the problem (3.8)-(3.9). By the same way, we can prove a posteriori error estimates between the solution $(\mathbf{u}, p)$ of the exact problem (2.10)-(2.11) and $(\mathbf{u}_h, p_h)$ of the numerical problem (3.10)-(3.11). In all the rest of the paper, we suppose that $\mathbf{f} \in L^\infty(0, T; H(\text{div}, \Omega))$.

We first introduce the space

$$Z_h = \{\mathbf{g}_h \in L^2(\Omega)^3; \forall \kappa \in \tau_h, \mathbf{g}_h|_\kappa \in \mathbb{P}_0(\kappa)\}$$

and we fix an approximation $\rho_h(\mathbf{f})$ of the data $\mathbf{f}$ in Z_h .

Next, we denote by ε_h the set of all faces of the elements of τ_h that are not contained in Γ . For every element κ in τ_h , we denote by ε_κ the set of faces of κ that are not contained in Γ , Δ_κ the set of union of elements of τ_h that intersect κ , Δ_e the union of elements of τ_h that intersect the face e , h_κ the diameter of κ and h_e the diameter of the face e . Also, $\mathbf{n}_\kappa$ stands for the unit outward normal vector to κ on $\partial\kappa$ and $[\cdot]_e$ the jump through the face e of κ .

For the demonstration of the next theorems, we introduce for an element κ of τ_h , the bull function ψ_κ (resp. ψ_e of the face e) which is equal to the product of the $d+1$ barycentric coordinates associated with the vertices of κ (resp. of e) and $\mathcal{L}_e$ the lifting operator from polynomials defined on e into polynomials defined on the elements κ and κ' contained e , which is constructed by affine transformations from a fixed operator on the reference element.

Property 5.1. Denoting by $P_r(\kappa)$ the polynomial of degrees r on κ , we have

$$\forall v \text{ polynom of } P_r(\kappa) \quad \begin{cases} c\|v\|_{0,\kappa} \leq \|v\psi_\kappa^{1/2}\|_{0,\kappa} \leq c'\|v\|_{0,\kappa} \\ \|v\|_{1,\kappa} \leq ch_\kappa^{-1}\|v\|_{0,\kappa} \end{cases} \quad (5.1)$$

Property 5.2. Denoting by $P_r(e)$ the polynomial of degrees r on e , we have

$$\forall v \text{ polynom of } P_r(e) \quad c \|v\|_{0,e} \leq \|v\psi_e^{1/2}\|_{0,e} \leq c' \|v\|_{0,e}$$

and $\forall v$ polynom of $P_r(e)$ which vanishes on ∂e , κ is a element which contains e , we have

$$\|\mathcal{L}_e v\|_{0,\kappa} + h_e \|\mathcal{L}_e v\|_{1,\kappa} \leq ch_e^{1/2} \|v\|_{0,e}.$$

We denote by R_h the Clment operator [10]. We have for all function $q \in H_0^1(\Omega)$, $R_h q \in Q_{0h}$ verifies

$$\begin{aligned} \|q - R_h q\|_{0,\kappa} &\leq ch_\kappa \|q\|_{1,\Delta\kappa} \\ \|q - R_h q\|_{0,e} &\leq ch_e^{1/2} \|q\|_{1,\Delta e} \end{aligned} \quad (5.2)$$

Furthermore, we denote by $\mathcal{R}_h$ the Raviart-Thomas operator: for any smooth enough vectorial function $\mathbf{v}$, $\mathcal{R}_h \mathbf{v}$ belongs to M_{0h} and satisfies

$$\forall e \in \varepsilon_h, \quad \int_e (\mathbf{v} - \mathcal{R}_h \mathbf{v}) \cdot \mathbf{n} \, d\tau = 0.$$

Moreover, this operator satisfies, see [24]: $\forall \mathbf{v}$ in $H^1(\Omega)^3$ and $\forall \kappa$ in τ_h ,

$$\begin{aligned} \|\mathbf{v} - \mathcal{R}_h \mathbf{v}\|_{0,\kappa} &\leq ch_\kappa \|\mathbf{v}\|_{1,\kappa} \\ \|\mathbf{v} - \mathcal{R}_h \mathbf{v}\|_{0,e} &\leq ch_e^{1/2} \|\mathbf{v}\|_{1,\Delta e} \end{aligned} \quad (5.3)$$

Let us begin with a posteriori error for the pressure. We introduce the function:

$$\mathbf{f}^k(t) = \frac{t - t_n}{k} (\mathbf{f}(t_{n+1}) - \mathbf{f}(t_n)) + \mathbf{f}(t_n) \quad \forall t \in [t_n, t_{n+1}]$$

and we deduce that:

$$(\nabla p_h(t) - \mathbf{f}^k(t), \nabla q_h(t)) = 0$$

So, we have:

$$\forall q(t) \in H_0^1(\Omega) \quad (\text{resp. } H^1(\Omega)/\mathbb{R})$$

$$(\nabla(p(t) - p_h(t)), \nabla q(t)) = (\mathbf{f}(t) - \nabla p_h(t), \nabla q(t)) = (\mathbf{f}(t) - \mathbf{f}^k(t), \nabla q(t)) + (\mathbf{f}^k(t) - \nabla p_h(t), \nabla(q(t) - q_h(t))) \quad (5.4)$$

We define the error indicator by

$$\eta_{n,\kappa}^2 = \sum_{e \in \varepsilon_\kappa} h_e \left\| [(\rho_h \mathbf{f}^k - \nabla p_h) \cdot \mathbf{n}] \right\|_{L^2(t_n, t_{n+1}; L^2(e))}^2$$

Lemma 5.3. The following estimate hold

$$\begin{aligned} \|\nabla(p - p_h)\|_{L^2(0,T;L^2(\Omega))}^2 &\leq C \left\{ \sum_{n=0}^{N-1} \sum_{\kappa \in \tau_h} \left(\eta_{n,\kappa}^2 + \|\mathbf{f} - \mathbf{f}^k\|_{L^2(t_n, t_{n+1}; L^2(\kappa)^3)}^2 \right. \right. \\ &\quad \left. \left. + h_\kappa^2 \|\operatorname{div} \mathbf{f}^k\|_{L^2(t_n, t_{n+1}; L^2(\kappa)^3)}^2 + \sum_{e \in \varepsilon_\kappa} h_e \left\| [(\mathbf{f}^k - \rho_h \mathbf{f}^k) \cdot \mathbf{n}] \right\|_{L^2(t_n, t_{n+1}; L^2(e))}^2 \right) \right\} \end{aligned}$$

Proof: For any $q_h(t) \in M_{0h}$, we have

$$\begin{aligned} &(\nabla(p(t) - p_h(t)), \nabla q(t)) \\ &= (\mathbf{f}(t) - \mathbf{f}^k(t), \nabla q(t)) + (\mathbf{f}^k(t) - \rho_h \mathbf{f}^k(t), \nabla(q(t) - q_h(t))) + (\rho_h \mathbf{f}^k(t) - \nabla p_h(t), \nabla(q(t) - q_h(t))) \\ &= \sum_{\kappa \in \tau_h} \left(\int_\kappa (\mathbf{f}(t) - \mathbf{f}^k(t)) \nabla q(t) + \int_\kappa (\mathbf{f}^k(t) - \rho_h \mathbf{f}^k(t)) \nabla(q(t) - q_h(t)) + \int_\kappa (\rho_h \mathbf{f}^k(t) - \nabla p_h(t)) \nabla(q(t) - q_h(t)) \right) \\ &= \sum_{\kappa \in \tau_h} \left\{ \int_\kappa (\mathbf{f}(t) - \mathbf{f}^k(t)) \nabla q(t) - \int_\kappa \operatorname{div} \mathbf{f}^k(t) (q(t) - q_h(t)) \right. \\ &\quad \left. + \frac{1}{2} \sum_{e \in \varepsilon_\kappa} \int_e \left([(\mathbf{f}^k(t) - \rho_h \mathbf{f}^k(t)) \cdot \mathbf{n}] (q(t) - q_h(t)) + [(\rho_h \mathbf{f}^k(t) - \nabla p_h(t)) \cdot \mathbf{n}] (q(t) - q_h(t)) \right) \right\} \end{aligned} \quad (5.5)$$

We take $q(t) = (p(t) - p_h(t))$ and $q_h(t) = R_h q(t)$, the image of $q(t)$ by the Clément type regularisation operator, and we obtain:

$$\begin{aligned} \|\nabla q(t)\|_{L^2(\Omega)}^2 &\leq C \sum_{\kappa \in \mathcal{T}_h} \left\{ \frac{1}{2\varepsilon_1} (\|\mathbf{f}(t) - \mathbf{f}^k(t)\|_{0,\kappa}^2 + \frac{1}{2}\varepsilon_1 \|\nabla q(t)\|_{0,\kappa}^2) \right. \\ &\quad + \left(\frac{1}{2\varepsilon_2} h_k^2 \|\operatorname{div} \mathbf{f}^k(t)\|_{0,\kappa}^2 + \frac{1}{2}\varepsilon_2 \|\nabla q(t)\|_{0,\Delta_\kappa}^2 \right) \\ &\quad + \sum_{e \in \varepsilon_\kappa} \left(\left(\frac{1}{2\varepsilon_3} h_e \left\| \left[(\mathbf{f}^k(t) - \rho_h \mathbf{f}^k(t)) \cdot \mathbf{n} \right] \right\|_{0,e}^2 + \frac{1}{2}\varepsilon_3 \|\nabla q(t)\|_{0,\Delta_e}^2 \right) \right. \\ &\quad \left. \left. + \left(\frac{1}{2\varepsilon_4} h_e \left\| \left[(\rho_h \mathbf{f}^k(t) - \nabla p_h(t)) \cdot \mathbf{n} \right] \right\|_{0,e}^2 + \frac{1}{2}\varepsilon_4 \|\nabla q(t)\|_{0,\Delta_e}^2 \right) \right) \right\} \end{aligned} \quad (5.6)$$

As $\Delta_e \subset \Delta_\kappa$ and after a suitable choice of $\varepsilon_i, i = 1, 2, 3, 4$, we integrate between t_n and t_{n+1} and we sum over n from 0 to $N - 1$ to obtain the result. $\square$

Proposition 5.4. *The error indicators verify the following optimality conditions*

$$\begin{aligned} \eta_{n,\kappa}^2 &\leq C' \left(\|p - p_h\|_{L^2(t_n, t_{n+1}; H^1(\Delta_\kappa))}^2 + \|\mathbf{f} - \mathbf{f}^k\|_{L^2(t_n, t_{n+1}; L^2(\Delta_\kappa)^3)}^2 \right. \\ &\quad \left. + h_e^2 \|\operatorname{div} \mathbf{f}^k\|_{L^2(t_n, t_{n+1}; L^2(\Delta_\kappa))}^2 + \sum_{e \in \varepsilon_\kappa} h_e \left\| \left[(\mathbf{f}^k - \rho_h \mathbf{f}^k) \cdot \mathbf{n} \right] \right\|_{L^2(t_n, t_{n+1}; (L^2(e))^3)}^2 \right) \end{aligned} \quad (5.7)$$

Proof: We consider the equation (5.5) with $q_h = 0$ and we take $q(t) = q_e(t) = \mathcal{L}_e([\rho_h \mathbf{f}^k(t) - \nabla p_h(t)] \cdot \mathbf{n}) \psi_e$:

$$\begin{aligned} \int_{\kappa \cup \kappa'} \nabla(p(t) - p_h(t)) \nabla q_e(t) &= \int_{\kappa \cup \kappa'} (\mathbf{f}(t) - \mathbf{f}^k(t)) \cdot \nabla q_e(t) - \int_{\kappa \cup \kappa'} \operatorname{div} \mathbf{f}^k(t) q_e(t) \\ &\quad + \frac{1}{2} \int_e \left(\left[(\mathbf{f}^k(t) - \rho_h \mathbf{f}^k(t)) \cdot \mathbf{n} \right] q_e(t) + \left[(\rho_h \mathbf{f}^k(t) - \nabla p_h(t)) \cdot \mathbf{n} \right]^2 \psi_e \right) \end{aligned} \quad (5.8)$$

then by using the properties 5.1 and 5.2

$$\begin{aligned} \left\| \left[(\rho_h \mathbf{f}^k(t) - \nabla p_h(t)) \cdot \mathbf{n} \right] \right\|_{0,e}^2 &\leq C' \left(h_e^{-1} \|p(t) - p_h(t)\|_{1,\kappa \cup \kappa'}^2 + h_e^{-1} \|\mathbf{f}(t) - \mathbf{f}^k(t)\|_{0,\kappa \cup \kappa'}^2 \right. \\ &\quad \left. + h_e \|\operatorname{div} \mathbf{f}^k(t)\|_{0,\kappa \cup \kappa'}^2 + \left\| \left[(\mathbf{f}^k(t) - \rho_h \mathbf{f}^k(t)) \cdot \mathbf{n} \right] \right\|_{0,e}^2 \right), \end{aligned}$$

multiplying by h_e , integrating over t from t_n to t_{n+1} and summing over ε_κ , we obtain the result. $\square$

Now, let us establish a posteriori error for the velocity. The error function $\mathbf{u}(t) - \mathbf{u}_h(t)$ belongs to $H_0(\operatorname{curl}, \Omega)$, there exists a function $\lambda(t) \in H_0^1(\Omega)$ solution of the problem:

$$\forall \mu \in H_0^1(\Omega), \quad \int_{\Omega} \nabla \lambda(t) \nabla \mu = \int_{\Omega} (\mathbf{u}(t) - \mathbf{u}_h(t)) \nabla \mu$$

Then the function $\mathbf{w}(t) = (\mathbf{u}(t) - \mathbf{u}_h(t)) - \nabla \lambda(t)$ belongs to V_0 and we have $\operatorname{curl} \mathbf{w}(t) = \operatorname{curl} (\mathbf{u}(t) - \mathbf{u}_h(t))$. We obtain

$$\begin{aligned} \|\mathbf{u}(t) - \mathbf{u}_h(t)\|_{0,\Omega}^2 &= \|\nabla \lambda(t)\|_{0,\Omega}^2 + \|\mathbf{w}(t)\|_{0,\Omega}^2 \\ \|\operatorname{curl} (\mathbf{u}(t) - \mathbf{u}_h(t))\|_{0,\Omega}^2 &= \|\operatorname{curl} \mathbf{w}(t)\|_{0,\Omega}^2 \end{aligned} \quad (5.9)$$

In order to find the upper and lower bounds of the velocity, we start by finding the upper and lower bounds of the right hand sides of the last equations.

For the first term of the right hand side of (5.9), we have for a $t \in [t_n, t_{n+1}]$ and $\forall \mu(t) \in H_0^1(\Omega)$,

$$\begin{aligned} \int_{\Omega} \nabla \lambda(t) \nabla \mu(t) &= \int_{\Omega} (\mathbf{w}(t) + \nabla \lambda(t)) \nabla \mu(t) = \int_{\Omega} (\mathbf{u}(t) - \mathbf{u}_h(t)) \nabla \mu(t) \\ &= - \int_{\Omega} \mathbf{u}_h(t) \nabla \mu(t) = - \int_{\Omega} \mathbf{u}_h(t) \nabla (\mu(t) - \mu_h(t)) \quad \forall \mu_h(t) \in Q_{0h} \end{aligned}$$

then, using the fact that $\operatorname{div} \mathbf{u}_h(t) = 0$ on every element $\kappa \in \tau_h$, we have

$$\int_{\Omega} \nabla \lambda(t) \nabla \mu(t) = -\frac{1}{2} \sum_{\kappa \in \tau_h} \left(\sum_{e \in \varepsilon_{\kappa}} \int_e [\mathbf{u}_h(t) \cdot \mathbf{n}] (\mu(t) - \mu_h(t)) \right) \quad (5.10)$$

The same steps with $\lambda'(t)$ gives:

$$\int_{\Omega} \nabla \lambda'(t) \nabla \mu(t) = -\frac{1}{2} \sum_{\kappa \in \tau_h} \left(\sum_{e \in \varepsilon_{\kappa}} \int_e [\mathbf{u}'_h(t) \cdot \mathbf{n}] (\mu(t) - \mu_h(t)) \right) \quad (5.11)$$

with

$$\mathbf{u}'_h(t) = \frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{k} \quad \text{for } t \in [t_n, t_{n+1}[\text{ for all } n = 0, \dots, N-1$$

We introduce the indicators

$$\xi_{n,\kappa}^2 = \sum_{e \in \varepsilon_{\kappa}} h_e \left(\|\mathbf{u}_h \cdot \mathbf{n}\|_{L^\infty(t_n, t_{n+1}; L^2(e))}^2 + \|\mathbf{u}'_h \cdot \mathbf{n}\|_{L^2(t_n, t_{n+1}; L^2(e))}^2 \right) \quad (5.12)$$

Theorem 5.5. *The following bounds hold*

$$\sum_{n=0}^{N-1} \left(\|\nabla \lambda\|_{L^\infty(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + \|\nabla \lambda'\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 \right) \leq C_1 \left(\sum_{n=0}^{N-1} \sum_{\kappa \in \tau_h} \xi_{n,\kappa}^2 \right)$$

and

$$\xi_{n,\kappa}^2 \leq C_2 \left(\|\nabla \lambda\|_{L^\infty(t_n, t_{n+1}; L^2(\Delta_\kappa)^3)}^2 + \|\nabla \lambda'\|_{L^2(t_n, t_{n+1}; L^2(\Delta_\kappa)^3)}^2 \right) \quad (5.13)$$

Proof: For the upper bound, first we take $\mu = \lambda(t)$ and $\mu_h = R_h \mu$ in (5.10) and integrate over t from t_n to t_{n+1} , and second, we take $\mu = \lambda'(t)$ and $\mu_h = R_h \mu$ in (5.11), and we obtain the result after summing over n .

In order to find the lower bound, we take in the equation (5.10) (resp. (5.11)), $\mu_h = 0$ and $\mu = \mathcal{L}_e([\mathbf{u}_h(t) \cdot \mathbf{n}] \psi_e)$ (resp. $\mu = \mathcal{L}_e([\mathbf{u}'_h(t) \cdot \mathbf{n}] \psi_e)$) and we obtain by using the property 5.2

$$\|\mathbf{u}_h(t) \cdot \mathbf{n}\|_{0,e}^2 \leq C_2 \left(h_e^{-1} \|\nabla \lambda(t)\|_{0,\kappa \cup \kappa'}^2 \right); \quad \|\mathbf{u}'_h(t) \cdot \mathbf{n}\|_{0,e}^2 \leq C_2 \left(h_e^{-1} \|\nabla \lambda'(t)\|_{0,\kappa \cup \kappa'}^2 \right);$$

which leads, after summing over $e \in \varepsilon_{\kappa}$, to

$$\xi_{n,\kappa}^2 \leq C_2 \left(\|\nabla \lambda\|_{L^\infty(t_n, t_{n+1}; L^2(\Delta_\kappa)^3)}^2 + \|\nabla \lambda'\|_{L^2(t_n, t_{n+1}; L^2(\Delta_\kappa)^3)}^2 \right) \quad (5.14)$$

□

Now, we will bound the second and third terms of the right hand side of (5.9):

For $t \in [t_n, t_{n+1}[$, $\forall \mathbf{v}(t) \in H_0(\mathbf{curl}, \Omega)$:

$$\begin{aligned} & \left(\frac{d}{dt} (\mathbf{u} - \mathbf{u}_h)(t), \mathbf{v}(t) \right) + \nu (\mathbf{curl} (\mathbf{u}(t) - \mathbf{u}_h(t)), \mathbf{curl} \mathbf{v}(t)) + (\nabla(p(t) - p_h(t)), \mathbf{v}(t)) \\ &= (\mathbf{f}(t), \mathbf{v}(t)) - \frac{1}{k} (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n, \mathbf{v}(t)) - \nu (\mathbf{curl} \mathbf{u}_h(t), \mathbf{curl} \mathbf{v}(t)) - (\nabla p_h(t), \mathbf{v}(t)) \end{aligned} \quad (5.15)$$

We introduce the residual $R(\mathbf{u}_h) \in L^2(0, T; (H_0(\mathbf{curl}, \Omega))')$: for $t \in [t_n, t_{n+1}[$, $\forall \mathbf{v}(t) \in H_0(\mathbf{curl}, \Omega)$

$$\langle R(\mathbf{u}_h)(t), \mathbf{v}(t) \rangle = (\mathbf{f}(t), \mathbf{v}(t)) - \left(\frac{d\mathbf{u}_h}{dt}(t), \mathbf{v}(t) \right) - \nu (\mathbf{curl} \mathbf{u}_h(t), \mathbf{curl} \mathbf{v}(t)) - (\nabla p_h(t), \mathbf{v}(t)) \quad (5.16)$$

Using (3.6), we introduce the space residual R^h and the time residual R^τ :

$$R(\mathbf{u}_h) = \mathbf{f} - \mathbf{f}(t_{n+1}) + R^h(\mathbf{u}_h) + R^\tau(\mathbf{u}_h) \quad (5.17)$$

with, for $t \in [t_n, t_{n+1}[$, $\forall \mathbf{v}_h(t) \in H_0(\mathbf{curl}, \Omega)$:

$$\begin{aligned}
\langle R^h(\mathbf{u}_h)(t), \mathbf{v}(t) \rangle &= (\mathbf{f}(t_{n+1}) - \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - \nabla p_h^{n+1}, \mathbf{v}(t)) - \nu(\mathbf{curl} \mathbf{u}_h^{n+1}, \mathbf{curl} \mathbf{v}(t)) \\
&= (\mathbf{f}(t_{n+1}) - \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - \nabla p_h^{n+1}, \mathbf{v}(t) - \mathbf{v}_h(t)) - \nu(\mathbf{curl} \mathbf{u}_h^{n+1}, \mathbf{curl}(\mathbf{v}(t) - \mathbf{v}_h(t))) \\
&= \sum_{\kappa \in \tau_h} \left\{ \int_{\kappa} (\mathbf{f}(t_{n+1}) - \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - \nabla p_h^{n+1})(\mathbf{v}(t) - \mathbf{v}_h(t)) \right. \\
&\quad \left. + \frac{\nu}{2} \sum_{e \in \varepsilon_{\kappa}} \int_e ([\mathbf{curl} \mathbf{u}_h^{n+1} \times \mathbf{n}])(\mathbf{v}(t) - \mathbf{v}_h(t)) \right\}
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\langle R^{\tau}(\mathbf{u}_h)(t), \mathbf{v}(t) \rangle &= \nu(\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h(t)), \mathbf{curl} \mathbf{v}(t)) + (\nabla(p_h^{n+1} - p_h(t)), \mathbf{v}(t)) \\
&= \sum_{\kappa \in \tau_h} \left\{ \int_{\kappa} \nabla(p_h^{n+1} - p_h(t)) \mathbf{v}(t) + \nu \int_{\kappa} \mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h(t)) \mathbf{curl} \mathbf{v}(t) \right\}
\end{aligned} \tag{5.19}$$

We introduce, for each $\kappa \in \tau_h$, the indicators:

$$\begin{aligned}
\gamma_{n,\kappa}^2 &= h_{\kappa}^2 \|\rho_h \mathbf{f}(t_{n+1}) - \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - \nabla p_h^{n+1}\|_{0,\kappa}^2 + \sum_{e \in \varepsilon_{\kappa}} h_e \|[\mathbf{curl} \mathbf{u}_h^{n+1} \times \mathbf{n}]\|_{0,e}^2 \\
\delta_{n,\kappa}^2 &= \|\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)\|_{0,\kappa}^2
\end{aligned} \tag{5.20}$$

Theorem 5.6. *We have the upper bound*

$$\begin{aligned}
&\|\mathbf{w}\|_{L^{\infty}(0,t_m;L^2(\Omega)^3)}^2 + \|\mathbf{curl} \mathbf{w}\|_{L^2(0,t_m;L^2(\Omega)^3)}^2 + \|\mathbf{w}'\|_{L^2(0,t_m;L^2(\Omega)^3)}^2 \\
&\leq C \sum_{n=0}^{m-1} \left(\|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n,t_{n+1};L^2(\Omega)^3)}^2 + \sum_{\kappa \in \tau_h} (kh_{\kappa}^2 \|\rho_h \mathbf{f}(t_{n+1}) - \mathbf{f}(t_{n+1})\|_{L^2(\kappa)^3}^2 + k\gamma_{n,\kappa}^2 + k\delta_{n,\kappa}^2) \right)
\end{aligned}$$

Proof: First, we consider the equations (5.15) and (5.17) and the definition of $R(\mathbf{u}_h)$, we replace $\mathbf{u}(t) - \mathbf{u}_h(t) = \mathbf{w}(t) + \nabla \lambda(t)$ and $\mathbf{v}(t) = \mathbf{w}(t)$, integrate over t from t_n to t_{n+1} , sum over n from 0 to $m-1$ and using the theorem 2.2:

$$\begin{aligned}
\frac{1}{2} \|\mathbf{w}(t_m)\|_{0,\Omega}^2 + \nu \|\mathbf{curl} \mathbf{w}\|_{L^2(0,t_m;L^2(\Omega)^3)}^2 &\leq C \left(\sum_{n=0}^{m-1} \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n,t_{n+1};L^2(\Omega)^3)}^2 \right. \\
&\quad \left. + \|R^h(\mathbf{u}_h)\|_{L^2(0,t_m;V_0')} + \|R^{\tau}(\mathbf{u}_h)\|_{L^2(0,t_m;V_0')} \right) \|\mathbf{curl} \mathbf{w}\|_{L^2(0,t_m;L^2(\Omega)^3)}
\end{aligned} \tag{5.21}$$

which leads to

$$\begin{aligned}
&\|\mathbf{w}\|_{L^{\infty}(0,T;L^2(\Omega)^3)}^2 + \|\mathbf{curl} \mathbf{w}\|_{L^2(0,T;L^2(\Omega)^3)}^2 \\
&\leq C' \left(\sum_{n=0}^{N-1} \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n,t_{n+1};L^2(\Omega)^3)}^2 + \|R^h(\mathbf{u}_h)\|_{L^2(0,T;V_0')}^2 + \|R^{\tau}(\mathbf{u}_h)\|_{L^2(0,T;V_0')}^2 \right)
\end{aligned} \tag{5.22}$$

We repeat the same procedure with $\mathbf{v}(t) = \mathbf{w}'(t)$ and combining the tow steps:

$$\begin{aligned}
&\|\mathbf{w}\|_{L^{\infty}(0,T;L^2(\Omega)^3)}^2 + \|\mathbf{curl} \mathbf{w}\|_{L^2(0,T;L^2(\Omega)^3)}^2 + \|\mathbf{w}'\|_{L^2(0,T;L^2(\Omega)^3)}^2 \\
&\leq C'' \left(\sum_{n=0}^{N-1} \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n,t_{n+1};L^2(\Omega)^3)}^2 + \|R^h(\mathbf{u}_h)\|_{L^2(0,T;V_0')}^2 + \|R^{\tau}(\mathbf{u}_h)\|_{L^2(0,T;V_0')}^2 \right)
\end{aligned} \tag{5.23}$$

To find the upper bound of $\mathbf{w}$, it suffices to find the upper bound of the operator R^h and R^{τ} .

We take in the equation (5.18) $\mathbf{v}(t) \in V_0$ and $\mathbf{v}_h(t) = \mathcal{R}_h \mathbf{v}(t)$, we integrate over t from t_n to t_{n+1} and we sum over n from 0 to $N-1$. We obtain, using the properties of $\mathcal{R}_h$, the upper following bound:

$$\|R^h(\mathbf{u}_h)\|_{L^2(0,T;V_0')}^2 \leq C_1 \sum_{n=0}^{N-1} \sum_{\kappa \in \tau_h} \left(kh_{\kappa}^2 \|\rho_h \mathbf{f}(t_{n+1}) - \mathbf{f}(t_{n+1})\|_{L^2(\kappa)^3}^2 + k\gamma_{n,\kappa}^2 \right) \tag{5.24}$$

Similarly, we take the equation (5.19) with $\mathbf{v}(t) \in V_0$; we integrate over t from t_n to t_{n+1} and we sum over n from 0 to $N-1$. We obtain:

$$\begin{aligned} \|R^\tau(\mathbf{u}_h)\|_{L^2(0,T;V'_0)}^2 &\leq C_2 \sum_{n=0}^{N-1} \sum_{\kappa \in \tau_h} \int_{t_n}^{t_{n+1}} \frac{(t_{n+1}-t)^2}{k^2} \|\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)\|_{L^2(\kappa)}^2 \\ &\leq \frac{C_2}{3} \sum_{n=0}^{N-1} \sum_{\kappa \in \tau_h} k \delta_{n,\kappa}^2 \end{aligned} \quad (5.25)$$

The formulas (5.23), (5.24) and (5.25) give the result. $\square$

Next, we will show some properties of the operator R^h . To do that, we introduce, for any element κ , the operators:

$$\begin{aligned} R_{n,\kappa}^h &= \rho_h \mathbf{f}(t_{n+1}) - \frac{1}{k}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) - \nabla p_h^{n+1} \\ R_{n,e}^h &= [\mathbf{curl} \mathbf{u}_h^{n+1} \times \mathbf{n}]_e \end{aligned} \quad (5.26)$$

and, for some $\alpha_1 > 0$ and $\alpha_2 > 0$:

$$\theta_n = \sum_{\kappa \in \tau_h} \theta_{n,\kappa} \quad \text{where} \quad \theta_{n,\kappa} = \alpha_1 h_\kappa^2 R_{n,\kappa}^h \psi_\kappa + \alpha_2 \sum_{e \in \varepsilon_\kappa} h_e \mathcal{L}_e(R_{n,e}^h \psi_e)$$

Theorem 5.7. *There exist constants $C_{1\dagger}$ and $C_{2\dagger}$ such that, on each interval $[t_n, t_{n+1}]$, the following estimates, for some θ_n , hold:*

$$\|\theta_n\|_{0,\Omega}^2 \leq C_{1\dagger} \sum_{\kappa \in \tau_h} h_\kappa^2 \gamma_{n,\kappa}^2 \quad (5.27)$$

$$\|\mathbf{curl} \theta_n\|_{0,\Omega}^2 \leq C_{2\dagger} \sum_{\kappa \in \tau_h} \gamma_{n,\kappa}^2 \quad (5.28)$$

$$\langle R^h(\mathbf{u}_h), \theta_n \rangle \geq (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) + \sum_{\kappa \in \tau_h} \gamma_{n,\kappa}^2 \quad (5.29)$$

Proof: Using the properties 5.2 and 5.1 and the fact that $h_e \leq h_\kappa$, we have:

$$\begin{aligned} \|\theta_n\|_{0,\Omega}^2 &= \sum_{\kappa \in \tau_h} \|\theta_{n,\kappa}\|_{0,\kappa}^2 \\ &= \sum_{\kappa \in \tau_h} \left(\alpha_1^2 h_\kappa^4 \|(R_{n,\kappa}^h \psi_\kappa)\|_{0,\kappa}^2 + \alpha_2^2 \sum_{e \in \varepsilon_\kappa} \sum_{e' \in \varepsilon_\kappa} h_e h_{e'} ((\mathcal{L}(R_{n,e}^h \psi_e), (\mathcal{L}(R_{n,e'}^h \psi_{e'})))_\kappa \right. \\ &\quad \left. + 2\alpha_1 \alpha_2 \sum_{e \in \varepsilon_\kappa} h_\kappa^2 h_e ((R_{n,\kappa}^h \psi_\kappa), (\mathcal{L}(R_{n,e}^h \psi_e)))_\kappa \right) \\ &\leq C_{1\dagger} \sum_{\kappa \in \tau_h} h_\kappa^2 \gamma_{n,\kappa}^2 \end{aligned}$$

and

$$\begin{aligned} \|\mathbf{curl} \theta_n\|_{0,\Omega}^2 &= \sum_{\kappa \in \tau_h} \|\mathbf{curl} \theta_{n,\kappa}\|_{0,\kappa}^2 \\ &= \sum_{\kappa \in \tau_h} \left(\alpha_1^2 h_\kappa^4 \|\mathbf{curl}(R_{n,\kappa}^h \psi_\kappa)\|_{0,\kappa}^2 + \alpha_2^2 \sum_{e \in \varepsilon_\kappa} \sum_{e' \in \varepsilon_\kappa} h_e h_{e'} (\mathbf{curl}(\mathcal{L}(R_{n,e}^h \psi_e)), \mathbf{curl}(\mathcal{L}(R_{n,e'}^h \psi_{e'})))_\kappa \right. \\ &\quad \left. + 2\alpha_1 \alpha_2 \sum_{e \in \varepsilon_\kappa} h_\kappa^2 h_e (\mathbf{curl}(R_{n,\kappa}^h \psi_\kappa), \mathbf{curl}(\mathcal{L}(R_{n,e}^h \psi_e)))_\kappa \right) \\ &\leq C_{2\dagger} \sum_{\kappa \in \tau_h} \gamma_{n,\kappa}^2 \end{aligned}$$

Furthermore, we consider the equation (5.18) with $\mathbf{v}(t) = \theta_n$ and $\mathbf{v}_h(t) = 0$:

$$\begin{aligned}
\langle R^h(\mathbf{u}_h)(t), \theta_n \rangle &= (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) + \sum_{\kappa \in \mathcal{T}_h} (R_{n,\kappa}^h, \theta_n)_\kappa + \frac{\nu}{2} \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} (R_{n,e}^h, \theta_n)_e \\
&= (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) + \alpha_1 \sum_{\kappa \in \mathcal{T}_h} h_\kappa^2 (R_{n,\kappa}^h, R_{n,\kappa}^h \psi_\kappa)_\kappa \\
&\quad + 2\alpha_2 \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} h_e (R_{n,\kappa}^h, \mathcal{L}_e(R_{n,e}^h \psi_e))_\kappa + \nu \alpha_2 \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} h_e (R_{n,e}^h, R_{n,e}^h \psi_e)_e
\end{aligned} \tag{5.30}$$

We have by using the properties 5.2 and 5.1, and remarking that $h_e \leq h_\kappa$:

$$\begin{aligned}
\langle R^h(\mathbf{u}_h)(t), \theta_n \rangle &\geq (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) + C_1 \alpha_1 \sum_{\kappa \in \mathcal{T}_h} h_\kappa^2 \|R_{n,\kappa}^h\|_{0,\kappa}^2 \\
&\quad - 2\alpha_2 C_2 \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} h_\kappa h_e^{1/2} \|R_{n,\kappa}^h\|_{0,\kappa} \|R_{n,e}^h\|_{0,e} + \nu \alpha_2 C_3 \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} h_e \|R_{n,e}^h\|_{0,e}^2 \\
&\geq (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) + C_1 \alpha_1 \sum_{\kappa \in \mathcal{T}_h} h_\kappa^2 \|R_{n,\kappa}^h\|_{0,\kappa}^2 + \nu \alpha_2 C_3 \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} h_e \|R_{n,e}^h\|_{0,e}^2 \\
&\quad - 2\alpha_2 C_2 \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} \left\{ \frac{1}{2\varepsilon} h_\kappa^2 \|R_{n,\kappa}^h\|_{0,\kappa}^2 + \frac{1}{2} \varepsilon h_e \|R_{n,e}^h\|_{0,e}^2 \right\} \\
&\geq (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) + (C_1 \alpha_1 - \frac{4\alpha_2 C_2}{\varepsilon}) \sum_{\kappa \in \mathcal{T}_h} h_\kappa^2 \|R_{n,\kappa}^h\|_{0,\kappa}^2 \\
&\quad + \alpha_2 (\nu C_3 - C_2 \varepsilon) \sum_{\kappa \in \mathcal{T}_h} \sum_{e \in \varepsilon_\kappa} h_e \|R_{n,e}^h\|_{0,e}^2
\end{aligned} \tag{5.31}$$

We choice $\varepsilon = \frac{\nu C_3}{2C_2}$, $\alpha_2 = \frac{2}{\nu C_3}$ and $\alpha_1 = \frac{1}{C_1} (1 + \frac{16C_2^2}{\nu C_3^2})$ to obtain the third estimate. $\square$

Theorem 5.8. *The following upper bound estimate holds for a positive constant c_1*

$$\begin{aligned}
\sum_{\kappa \in \mathcal{T}_h} k \delta_{n,\kappa}^2 &\leq c_1 \left\{ \|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + \|\nabla(p - p_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 \right. \\
&\quad \left. + \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 \right\}
\end{aligned} \tag{5.32}$$

Proof: We take the definition of R^τ with $\mathbf{v}(t) = \mathbf{v}_h = (\mathbf{u}_h^{n+1} - \mathbf{u}_h^n) \in V_{0h}$, we integrate from t_n to t_{n+1} and we obtain:

$$\begin{aligned}
\nu \frac{k}{2} \|\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)\|_{0,\Omega}^2 &= \int_{t_n}^{t_{n+1}} \langle R^\tau(\mathbf{u}_h)(t), \mathbf{v}_h \rangle - \int_{t_n}^{t_{n+1}} (\nabla(p_h^{n+1} - p_h(t)), \mathbf{v}_h) \\
&\leq \int_{t_n}^{t_{n+1}} |\langle R(\mathbf{u}_h)(t), \mathbf{v}_h \rangle| + \int_{t_n}^{t_{n+1}} |\langle R^h(\mathbf{u}_h)(t), \mathbf{v}_h \rangle| \\
&\quad + \int_{t_n}^{t_{n+1}} |(\mathbf{f}(t) - \mathbf{f}(t_{n+1}), \mathbf{v}_h)| + \int_{t_n}^{t_{n+1}} |(\nabla(p_h^{n+1} - p_h(t)), \mathbf{v}_h)|
\end{aligned}$$

We will bound terms of the right side. We remark that $(\nabla(p_h^{n+1} - p_h(t)), \mathbf{v}_h) = 0$. The definition of $R(\mathbf{u}_h)$ and $R^h(\mathbf{u}_h)$, and the fact that $\mathbf{v}_h \in V_{0h}$ give

$$\langle R^h(\mathbf{u}_h)(t), \mathbf{v}_h \rangle = 0$$

and

$$\begin{aligned} \int_{t_n}^{t_{n+1}} |\langle R(\mathbf{u}_h)(t), \mathbf{v}_h \rangle| &\leq \left(C \|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + C \|\nabla(p - p_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \right. \\ &\quad \left. + \nu \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \right) \|\mathbf{curl} \mathbf{v}_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \end{aligned}$$

and

$$\begin{aligned} \int_{t_n}^{t_{n+1}} |\mathbf{f}(t) - \mathbf{f}(t_{n+1}), \mathbf{v}_h| &\leq \|(\mathbf{f} - \mathbf{f}(t_{n+1}))\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \|\mathbf{v}_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \\ &\leq C \|(\mathbf{f} - \mathbf{f}(t_{n+1}))\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \|\mathbf{curl} \mathbf{v}_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \end{aligned}$$

We deduce:

$$\begin{aligned} \frac{\nu}{\sqrt{2}} k^{1/2} \|\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)\|_{0,\Omega} &\leq C' \left\{ \|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + \|\nabla(p - p_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \right. \\ &\quad \left. + \nu \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \right\} \end{aligned} \quad (5.33)$$

which leads to the result. $\square$

Theorem 5.9. *The following upper bound estimate holds*

$$\begin{aligned} \sum_{\kappa \in \tau_h} k \gamma_{n,\kappa}^2 &\leq c_2 \left\{ \|\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 \right. \\ &\quad + \|\mathbf{f}(t_{n+1}) - \mathbf{f}(t_n)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + \|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 + \|\nabla(p - p_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 \\ &\quad \left. + \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}^2 \right\} \end{aligned} \quad (5.34)$$

Proof: We consider the inequality (5.29):

$$\begin{aligned} \sum_{\kappa \in \tau_h} \gamma_{n,\kappa}^2 &\leq \langle R^h(\mathbf{u}_h), \theta_n \rangle - (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) \\ &\leq \langle R(\mathbf{u}_h), \theta_n \rangle - \langle R^\tau(\mathbf{u}_h), \theta_n \rangle - (\mathbf{f}(t) - \mathbf{f}(t_{n+1}), \theta_n) - (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) \end{aligned} \quad (5.35)$$

We integrate the resulting inequality from t_n to t_{n+1} . We thus arrive at:

$$\begin{aligned} \sum_{\kappa \in \tau_h} k \gamma_{n,\kappa}^2 &\leq \int_{t_n}^{t_{n+1}} \langle R(\mathbf{u}_h), \theta_n \rangle - \int_{t_n}^{t_{n+1}} \langle R^\tau(\mathbf{u}_h), \theta_n \rangle - \int_{t_n}^{t_{n+1}} (\mathbf{f}(t) - \mathbf{f}(t_{n+1}), \theta_n) \\ &\quad - \int_{t_n}^{t_{n+1}} (\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1}), \theta_n) \end{aligned} \quad (5.36)$$

Since θ_n is constant with respect to time, we obtain by using theorem 5.7 and remarking that $\|\nabla(p_h^{n+1} - p_h^n)\|_{0,\Omega} \leq \|f(t_{n+1}) - f(t_n)\|_{0,\Omega}$:

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \langle R^\tau(\mathbf{u}_h), \theta_n \rangle &= \nu (\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n), \mathbf{curl} \theta_n) \int_{t_n}^{t_{n+1}} \left(\frac{t_{n+1} - t}{k} \right) dt \\ &\quad + (\nabla(p_h^{n+1} - p_h^n), \theta_n) \int_{t_n}^{t_{n+1}} \left(\frac{t_{n+1} - t}{k} \right) dt \\ &\leq \frac{1}{2} \nu \sqrt{C_{2\dagger}} k^{1/2} \|\mathbf{curl}(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)\|_{0,\Omega} \left(\sum_{\kappa \in \tau_h} k \gamma_{n,\kappa}^2 \right)^{1/2} \\ &\quad + \frac{1}{2} \sqrt{C_{1\dagger}} k^{1/2} \|\mathbf{f}(t_{n+1}) - \mathbf{f}(t_n)\|_{0,\Omega} \left(\sum_{\kappa \in \tau_h} k h_\kappa^2 \gamma_{n,\kappa}^2 \right)^{1/2} \end{aligned} \quad (5.37)$$

and

$$\begin{aligned} \int_{t_n}^{t_{n+1}} \langle R(\mathbf{u}_h), \theta_n \rangle &\leq (\|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + \|\nabla(p - p_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)}) \sqrt{C_{1\dagger}} \left(\sum_{\kappa \in \mathcal{T}_h} k h_\kappa^2 \gamma_{n,\kappa}^2 \right)^{1/2} \\ &\quad + \nu \sqrt{C_{2\dagger}} \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \left(\sum_{\kappa \in \mathcal{T}_h} k \gamma_{n,\kappa}^2 \right)^{1/2} \end{aligned} \quad (5.38)$$

Combining the last estimates and using the previous theorem give:

$$\begin{aligned} \sum_{\kappa \in \mathcal{T}_h} k \gamma_{n,\kappa}^2 &\leq \hat{C} \left(\sum_{\kappa \in \mathcal{T}_h} k \gamma_{n,\kappa}^2 \right)^{1/2} \left\{ \|\mathbf{f}(t_{n+1}) - \rho_h \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \right. \\ &\quad + \|\mathbf{f}(t_{n+1}) - \mathbf{f}(t_n)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + \|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} + \|\nabla(p - p_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \\ &\quad \left. + \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(t_n, t_{n+1}; L^2(\Omega)^3)} \right\} \end{aligned} \quad (5.39)$$

which leads to the result. $\square$

Theorem 5.10. *Finally, the pressure and the velocity verify the upper bound:*

$$\begin{aligned} &\|\nabla(p - p_h)\|_{L^2(0, T; L^2(\Omega)^3)}^2 + \|\mathbf{u} - \mathbf{u}_h\|_{L^\infty(0, T; L^2(\Omega)^3)}^2 + \|\mathbf{curl}(\mathbf{u} - \mathbf{u}_h)\|_{L^2(0, T; L^2(\Omega)^3)}^2 + \|\mathbf{u}' - \mathbf{u}'_h\|_{L^2(0, T; L^2(\Omega)^3)}^2 \\ &\leq c \left\{ \sum_{n=0}^{N-1} \sum_{\kappa \in \mathcal{T}_h} \left(\eta_{n,\kappa}^2 + \xi_{n,\kappa}^2 + k \gamma_{n,\kappa}^2 + k \delta_{n,\kappa}^2 \right) + \|\mathbf{f} - \mathbf{f}^k\|_{L^2(t_n, t_{n+1}; L^2(\kappa)^3)}^2 \right. \\ &\quad + h_\kappa^2 \|\operatorname{div} \mathbf{f}^k\|_{L^2(t_n, t_{n+1}; L^2(\kappa)^3)}^2 + \sum_{e \in \mathcal{E}_\kappa} h_e \left\| [(\mathbf{f}^k - \rho_h \mathbf{f}^k) \cdot \mathbf{n}] \right\|_{L^2(t_n, t_{n+1}; L^2(e))}^2 \\ &\quad \left. + \|\mathbf{f} - \mathbf{f}(t_{n+1})\|_{L^2(t_n, t_{n+1}; L^2(\kappa)^3)}^2 + k h_\kappa^2 \|\rho_h \mathbf{f}(t_{n+1}) - \mathbf{f}(t_{n+1})\|_{0,\kappa}^2 \right\} \end{aligned} \quad (5.40)$$

and the lower bounds defined by (5.7), (5.13), (5.32) and (5.34).

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