Spectral inequalities for non-selfadjoint elliptic operators and application to the null-controllability of parabolic systems

Matthieu Léautaud*†

November, 2009

Abstract

We consider elliptic operators A on a bounded domain, that are compact perturbations of a selfadjoint operator. We first recall some spectral properties of such operators: localization of the spectrum and resolvent estimates. We then derive a spectral inequality that measures the norm of finite sums of root vectors of A through an observation, with an exponential cost. Following the strategy of G. Lebeau and L. Robbiano (1995) [LR95], we deduce the construction of a control for the non-selfadjoint parabolic problem $\partial_t u + Au = Bg$. In particular, the L^2 norm of the control that achieves the extinction of the lower modes of A is estimated. Examples and applications are provided for systems of weakly coupled parabolic equations and for the measurement of the level sets of finite sums of root functions of A.

Keywords

Non-selfadjoint elliptic operators; Spectral theory; Parabolic systems; Controllability.

Contents

*Université Pierre et Marie Curie Paris 6, UMR 7598, Laboratoire Jacques-Louis Lions, F-75005, Paris, France. e-mail: leautaud@ann.jussieu.fr

†Laboratoire POEMS, INRIA Paris-Rocquencourt/ENSTA, CNRS UMR 2706, France.

1 Introduction

1.1 Results in an abstract setting

Various methods have been developed to prove the null-controllability of parabolic equations

$$\begin{cases} \partial_t u + Au = Bg \\ u_{|t=0} = u_0 \end{cases} \tag{1}$$

where A is of elliptic type, and B is a bounded operator (for instance a distributed control). A large number of such results are based on the seminal papers of either G. Lebeau and L. Robbiano [LR95] or A. Fursikov and O.Yu. Imanuvilov [FI96]. The first approach has been used for the treatment of self-adjoint operators A. The second approach also permits to address non-selfadjoint operators and semilinear equations. Each method has advantages. We shall focus on those of the Lebeau-Robbiano method. First, this method only relies on elliptic Carleman estimates that are simpler to produce than their parabolic counterparts as needed for the Fursikov-Imanuvilov approach. Second, it enlights some fundamental properties of the elliptic operator A through a spectral inequality. This type of spectral inequality, further investigated in [LZ98] and [JL99], has a large field of applications, like the measurement of the level sets of sums of root functions (see [JL99] or Section 7). For control issues, it yields the cost of the null-controllability of the low frequencies of the elliptic operator A. Finally, based on this fine spectral knowledge of A, this approach gives an explicit iterative construction of the control function, using both the partial controllability result and the natural parabolic dissipation.

In this paper, we extend the method of [LR95] for the abstract parabolic system (1) to cases in which A is a non-selfadjoint elliptic operator. More precisely, we treat the case where A is a small perturbation of a selfadjoint operator, under certain spectral assumptions. We suppose $A = A_0 + A_1$ where A_1 is a relatively compact perturbation of an elliptic selfadjoint operator A_0. We first obtain spectral inequalities of the following type. Denoting by Π_α the projector on the spectral subspaces of A associated to eigenvalues with real part less than α, we have, for some $\theta \geq 1/2$

$$\|\Pi_\alpha w\|_H \leq C e^{D\alpha^\theta} \|B^* \Pi_\alpha w\|_Y, \quad w \in H. \tag{2}$$

Here, $B^* \in \mathcal{L}(H;Y)$ denotes a bounded observation operator and the state space H and the observation space Y are Hilbert spaces. Typically, $B^* = \mathbb{1}_\omega$ is a distributed observation. This spectral inequality yields the cost of the measurement of some finite sums of root-vectors of A through the partial observation B^*, in terms of the largest eigenvalue of A in the considered spectral subspace. The difficulty here is not in the elliptic estimates since Carleman inequalities remains unchanged under the addition of lower order terms, but in the spectral theory of the non-selfadjoint operator A. This motivates the exposition of the spectral theory of such operators in Section 2, following [Agr94].

Inequalities of the type of (2) were first established in [LR95] for the Laplace operator under a relatively different form. In fact, the inequality proved in [LR95] reads

$$\|\Pi_\alpha w_+\|_H^2 + \|\Pi_\alpha w_-\|_H^2 \leq C e^{D\sqrt{\alpha}} \left\| \varphi B^* \left(e^{t\sqrt{A}} \Pi_\alpha w_+ + e^{-t\sqrt{A}} \Pi_\alpha w_- \right) \right\|_{L^2(0,T;Y)}^2, \quad w_+, w_- \in H, \tag{3}$$

where $\varphi(t)$ is a smooth cut-off function. Here, in the non-selfadjoint case, we prove a spectral inequality of the form

$$\|\Pi_\alpha w\|_H \leq C e^{D\alpha^\theta} \left\| \varphi B^* \left(e^{t\sqrt{A}} \Pi_\alpha + e^{-t\sqrt{A}} \Pi_\alpha \right) w \right\|_{L^2(0,T;Y)}, \quad w \in H. \tag{4}$$

We also prove that such an inequality is sufficient to deduce controllability results (in the case $\theta < 1$). Spectral inequality of the form (2) first appeared in [LZ98] and [JL99] for the Laplace operator with $\theta = 1/2$. Comments can be made about the two different forms of spectral inequalities, (2) on the one hand, and (3), (4) on the other hand, that involves an integration with respect to an additional variable. Both types can be proved with interpolation inequalities (like (12) or (14) below), themselves following from local Carleman inequalities. The interpolation inequality (12) used to prove (3) exhibits a spacialy distributed observation, whereas the interpolation inequality (14), used to prove (2) is characterized by a boundary observation at time $t = 0$. This latter form is more convenient to use for control results (see Section 6). Yet, for systems, the derivation of such an interpolation inequality with boundary observation at time $t = 0$ is open (see Section 5).

In Section 4, we deduce from the spectral inequality (4) the construction of a control function for the parabolic problem (1). Following [LR95] and an idea that goes back to the work of D.L. Russell [Rus73], the spectral inequality yields the controllability of the parabolic system on the related finite dimensional spectral subspace $\Pi_\alpha H$ with a control cost of the form $Ce^{C\alpha^\theta}$. In the case $\theta < 1$, we can then construct a control to the full parabolic equation (1). We improve the method of [LR95] from a spectral point of view. The proof of the controllability in [LR95] relies on the Hilbert basis property of the eigenfunctions of the Laplace operator. Here we only use some resolvent estimate away from the spectrum. No (Hilbert or Riesz) basis property is required in the construction.

The main results of this article can be summarized as follows

- Starting from an interpolation inequality of the Lebeau-Robbiano type (12) (resp. (14)), the spectral inequality (4) (resp. (2)) holds (see Theorem 3.2, resp. 3.3).
- The spectral inequality (4) (resp. (2)) implies that the control cost needed to drive any initial datum to $(I - \Pi_\alpha)H$ is bounded by $Ce^{C\alpha^\theta}$ (see Theorem 4.10, resp. Proposition 6.2).
- In the case $\theta < 1$, the non-selfadjoint parabolic system (1) is null-controllable in any positive time (see Theorem 4.13).

The question of the optimality of the power θ in these spectral inequalities remains open. For an elliptic operator A, in the results we obtain, the power θ increases linearly with the space dimension n: $\theta \approx \max\{1/2; n/2 - cst\}$. In the selfadjoint case however, the spectral inequalities (2) and (4) always hold with $\theta = 1/2$, and the controllability result always follows. The power $1/2$ is known to be optimal in this case (see [JL99] or [LL09]). Moreover, thanks to global parabolic inequalities, the controllability result of Theorem 4.13 is in general known to be true, independently on θ (see [FI96]). We are thus led to consider that the spectral inequalities should be true with $\theta = 1/2$: the power θ we obtain does not seem to be natural but only technical.

Since $\theta \approx \max\{1/2; n/2 - cst\}$, our controllability results, obtained for $\theta < 1$, are limited to small dimensions n. This problem arises in all the applications we give. In fact, in each of them, the power θ is the limiting point of the theory. The origin of the dimension-depending term $n/2 - cst$ in θ cannot be found in the elliptic estimates or in the control theory described above, but only in the resolvent estimates we use (see Section 2). If one wants to improve our results, one has to improve the resolvent estimates of [Agr94] (maybe taking into account that

A is a differential operator). On the contrary, if one wants to prove the optimality of the power θ, one needs to produce lower bounds for the resolvent of A. In any case, it does not seem to be an easy task at all.

Remark 1.1. Note that we can replace A by $A+\lambda$ in (1) without changing the controllability properties of the parabolic problem. More precisely, suppose that the problem

$$\begin{cases} \partial_t v + (A+\lambda)v = Bf \\ v_{|t=0} = u_0 \end{cases}$$

is null-controllable in time $T > 0$ by a control function f. Then, the control function $g = e^{\lambda t} f$ drives the solution of Problem (1) to zero in time T and has the same regularity as f. Hence, in the sequel, we shall consider that the operator A is sufficiently positive without any loss of generality.

Remark 1.2. In the sequel, C will denote a generic constant, whose value may change from line to line.

1.2 Some applications

We give several applications of such a spectral approach in Sections 5, 6 and 7.

This work has first been motivated by the problem of Section 5, i.e. the null-controllability in any time $T > 0$ of parabolic systems of the type

$$\begin{cases} \partial_t u_1 + P_1 u_1 + a u_1 + b u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 + P_2 u_2 + c u_1 + d u_2 = \mathbb{1}_\omega g & \text{in } (0,T)\times\Omega, \\ u_{1|t=0} = u_1^0,\ u_{2|t=0} = u_2^0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{5}$$

where $P_i = -\operatorname{div}(c_i \nabla \cdot), i = 1, 2$, are selfadjoint elliptic operators, ω is a non-empty subset of the bounded open set $\Omega \subset \mathbb{R}^n$, and a, b, c, d are bounded functions of $x \in \Omega$ and the coupling factor b does not vanish in an open subset $\mathcal{O} \subset \Omega$. Such parabolic systems have already been investigated. The first result was stated in [Ter00] in the context of insensitizing control (thus, one of the equations is backward in time). In this work, a positive answer to the null-controllability problem (5) is given in the case $a = c = d = 0$, $b = \mathbb{1}_\mathcal{O}$ and $\mathcal{O}\cap\omega \neq \emptyset$. The control problem (5) was then solved in [ABD06] and [GBPG06] in the case $\mathcal{O} \cap \omega \neq \emptyset$. In all these references, the authors used global parabolic Carleman estimates to obtain an observability estimate. They can thus treat time-dependant coefficients and semilinear reaction-diffusion problems. To the author's knowledge, the null-controllability in the case $\mathcal{O} \cap \omega = \emptyset$ remains an open problem. A recent step in this direction has been achieved in [KT09], proving approximate controllability by a spectral method in the case $a = c = d = 0$, $b = \mathbb{1}_\mathcal{O}$ without any condition on $\mathcal{O} \cap \omega$.

The field of coupled systems of d parabolic equations, $d \geq 2$, has also been investigated in [ABDGB07] where the authors prove a Kalman-type rank condition in the case where the coefficients are constant and in [GBT09] where the authors prove the controllability of a cascade system with nonvanishing variable coupling coefficients by one control force.

Here, we obtain a spectral inequality of the type (4) with $\theta = \max\{1/2; n/2 - 1\}$ for the operator

$$A = \begin{pmatrix} P_1 + a & b \\ c & P_2 + d \end{pmatrix}$$

with the measurement on only one component. As opposed to the results of [ABD06] and [GBPG06], we can provide an estimation on the control cost of low modes for system (5). The null-controllability of (5) follows from the method described above in the case $n \leq 3$ (corresponding to $\theta < 1$ in the spectral inequality).

In Section 6, we address the fractional power controllability problem

$$\begin{cases} \partial_t u + A^\nu u = Bg \\ u_{|t=0} = u_0 \in H, \end{cases}$$

that has been solved in the selfadjoint case in [MZ06] and [Mil06]. We prove the null-controllability in any time $T > 0$ when $\nu > \theta$. The case $\nu \leq \theta$ is open. In particular when $\nu = 1$, it allows us to give an estimation on the control cost of low modes for the following heat equation with a transport term

$$\begin{cases} \partial_t u - \Delta u + b \cdot \nabla u + cu = \mathbb{1}_\omega g & \text{in } (0,T) \times \Omega, \\ u_{|t=0} = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0,T) \times \partial\Omega. \end{cases} \tag{6}$$

In the case $n \leq 2$ (corresponding to $\theta < 1$ in the spectral inequality), this estimate is sufficient to recover the null-controllability of (6).

In Section 7, we are not concerned with controllability issues, but with the measurement of the $(n-1)$-dimensional Hausdorff measure of the level sets of sums of root functions of $A = -\Delta + b \cdot \nabla + c$ on the n-dimensional compact manifold Ω. Yet, the technique and the estimates used here are the same as in the other sections. We obtain the following estimate, for sums of root functions associated with eigenvalues of real part lower than α, i.e., $\varphi \in \Pi_\alpha L^2(\Omega)$, we have

$$\mathcal{H}_{n-1}(\{\varphi = K\}) \leq C_1 \alpha^\theta + C_2 \ , \ \theta = \max\left\{\frac{1}{2}; \frac{n-1}{2}\right\}.$$

This type of inequality has already been proved in the selfadjoint case for the Laplace operator. In [DF88] and [DF90], H. Donnelly and C. Fefferman proved, for φ an eigenfunction associated with the eigenvalue α, the estimate $\mathcal{H}_{n-1}(\{\varphi = K\}) \leq C\alpha^{1/2}$. This was generalized to sums of eigenfunctions associated with eigenvalues lower than α in [JL99]. We generalize this last result for operators that are lower order perturbations of the Laplace operator. Here, however, θ is in general strictly greater than $1/2$.

Acknowledgements. The author wishes to thank O. Glass and J. Le Rousseau for very fruitful discussions, helpful comments and encouragements, and also L. Robbiano for discussions about the article [LR95]. The author was partially supported by l'Agence Nationale de la Recherche under grant ANR-07-JCJC-0139-01.

2 Spectral theory of perturbated selfadjoint operators

To prepare the following sections, we shall first recall a theorem about the spectral theory of some operators close to being selfadjoint. This result can be presented with numerous variations and refinements. This subject has been a well-developed research field since the $60'$s. The first step of the theory is the Keldysh theorem on the completeness of the root vectors of a weakly perturbed compact selfadjoint operator (see [GK69] for more precisions). The main result here, Theorem 2.5, was first proved by A.S. Markus and by D.S. Katsnel'son [Kat67]. A simplified proof was given later by A.S. Markus and V.I. Matstaev with Matsaev's method of "artificial lacuna". This proof is presented in the book [Mar88, Chapter 1] by A.S. Markus in the more general case of weak perturbations of certain normal operators. For later use in the following sections, we sharpen some of the results as given by M.S. Agranovich [Agr94], following the steps of his proof.

Let H be a separable Hilbert space, $(\cdot,\cdot)_H$ be the scalar product on H and $\|.\|_H$ be the associated norm. The set of bounded linear operators on H is denoted by $\mathcal{L}(H)$ and is equiped with the subordinated norm $\|\cdot\|_{\mathcal{L}(H)}$. As usual, for a given operator A on H, we denote by $\mathcal{D}(A)$ its domain, by $\rho(A)$ its resolvent set, i.e., the subset of $\mathbb{C}$ where the resolvent $R_A(z) = (z-A)^{-1}$ is defined and bounded, and by $\mathrm{Sp}(A) = \mathbb{C}\setminus\rho(A)$ its spectrum.

Proposition 2.1. *Let $A = A_0 + A_1$ be an operator on H. Assume that*

(a) $\mathrm{Re}(Au,u)_H \geq 0$ *for all* $u \in \mathcal{D}(A)$*,*

(b) $A_0 : \mathcal{D}(A_0) \subset H \to H$ *is selfadjoint, positive, with a dense domain and compact resolvent,*

(c) $A_1 : \mathcal{D}(A_1) \subset H \to H$ *is q-subordinate to* A_0*, i.e., there exist* $k_0 > 0$ *and* $q \in [0,1)$ *such that for every* $u \in \mathcal{D}(A_0^{1/2})$, $|(A_1u,u)_H| \leq \frac{k_0}{2}\|A_0^{1/2}u\|_H^{2q}\|u\|_H^{2-2q}$.

We then have,

(i) $\mathcal{D}(A) = \mathcal{D}(A_0) \subset \mathcal{D}(A_1) \subset H$ *and* A *has a dense domain and a compact resolvent,*

(ii) $\mathrm{Sp}(A) \subset O_{k_0}^q = \{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0, |\mathrm{Im}(z)| < k_0|z|^q\}$

(iii) $\|R_A(z)\|_{\mathcal{L}(H)} \leq \frac{2}{d(z,Sp(A_0))} \leq \frac{2}{d(z,\mathbb{R}_+)}$ *for* $z \in \mathbb{C}\setminus O_{2k_0}^q$*, where* $d(z,S)$ *denotes the euclidian distance from* z *to the subset* S *of* $\mathbb{C}$*.*

Remark 2.2. Note that the subordination assumption (b) can be found under the different forms:

- $\mathcal{D}(A_0^q) \subset \mathcal{D}(A_1)$, and $\|A_1u\|_H \leq C\|A_0^qu\|_H$ for $u \in \mathcal{D}(A_0^q)$ in [Agr76];
- $\mathcal{D}(A_0) \subset \mathcal{D}(A_1)$, and $\|A_1u\|_H \leq C\|A_0u\|_H^q\|u\|_H^{1-q}$ for $u \in \mathcal{D}(A_0)$ in [Mar88, Chapter 1];
- $\|A_1A_0^{-q}\|_{\mathcal{L}(H)} = C < +\infty$ in [Dzh94].

Replacing assumption (b) by one of these assertions does not change the conclusions of Proposition 2.1 and Theorem 2.5.

Remark 2.3. Point (i) implies that $\mathrm{Sp}(A)$ contains only isolated eingenvalues, with finite multiplicity and without any accumulation point. Furthermore, for every $\lambda \in \mathrm{Sp}(A)$, the sequence of iterated null-spaces $N_k = N((A-\lambda)^k)$ is stationary.

For what follows, we shall need the spectrum of A to be located in a "parabolic neighborhood" of the real positive axis. We note that there exist $\lambda_0 > 0$, $K_0 > 0$ such that

$$\mathrm{Sp}(A+\lambda_0) \subset O_{k_0}^q + \lambda_0 \subset O_{2k_0}^q + \lambda_0 \subset \mathcal{P}_{K_0}^q = \{z \in \mathbb{C}, \mathrm{Re}(z) \geq 0, |\mathrm{Im}(z)| < K_0\,\mathrm{Re}(z)^q\}, \quad (7)$$

see Fig. 1. We now fix λ_0 and K_0 satisfying this assumption. Referring to Remark 1.1, we shall work with the operator $A+\lambda_0$, yet writing A for simplicity.

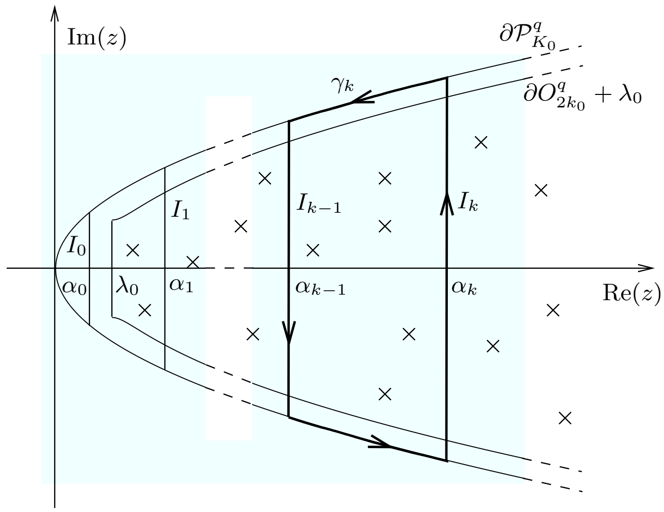

Figure 1: Complex contours around the spectrum of A

Definition 2.4. Let $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ be an increasing sequence, $0 < \alpha_0 < \alpha_1 < \cdots < \alpha_k < \cdots$ tending to $+\infty$, and such that $\alpha_k \notin \mathrm{Re}(\mathrm{Sp}(A))$, for every $k \in \mathbb{N}$. Then, we set $I_k = \{z \in \mathbb{C}, \mathrm{Re}(z) = \alpha_k, |\,\mathrm{Im}(z)| \leq K_0\alpha_k^q\}$ and by γ_k, we denote the positively oriented contour delimited by the vertical line segments I_k on the right and I_{k-1} on the left and by the parabola $\partial\mathcal{P}^q_{K_0}$ on the upper and the lower side (see Fig. 1). We also define the associated spectral projectors

$$P_k = \frac{1}{2i\pi}\int_{\gamma_k} R_A(z)dz.$$

Note that the spectral projector P_k is a projector on the characteristic subspaces of A associated with the eigenvalues that are inside γ_k. The projectors satisfy the identity $P_kP_j = \delta_{jk}P_k$. Moreover, thanks to Remark 2.3, the projectors P_k, $k \in \mathbb{N}$, have finite rank.

We can now state the main spectral result, that can (at least partially) be found under different forms in [Agr76], [Mar88, Chapter 1], [Agr94], [Dzh94].

Theorem 2.5. *Let $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots$ be the spectrum of the selfadjoint operator A_0. We assume the additional condition that there exists $p > 0$ such that $\limsup_{j\to\infty} \lambda_j j^{-p} > 0$. Then, setting $\beta = \max\{0, p^{-1} - (1-q)\}$, there exists a sequence $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ as in Definition 2.4 such that for some $C > 0$*

$$\|R_A(z)\|_{\mathcal{L}(H)} \leq e^{C\alpha_k^\beta}, \quad k \in \mathbb{N}, z \in I_k. \tag{8}$$

Remark 2.6. Note that Proposition 2.1 point (iii) implies that $\|R_A(z)\|_{\mathcal{L}(H)} \leq \frac{2}{K_0\alpha_k^q} \leq \frac{2}{K_0\alpha_0^q}$ for $z \in \gamma_k \cap \partial\mathcal{P}^q_{K_0}$. Thus the resolvent estimate (8), $\|R_A(z)\|_{\mathcal{L}(H)} \leq e^{C\alpha_k^\beta}$ holds for all $z \in \gamma_k$.

Remark 2.7. Comments can be made about this theorem and its proof:

- The idea of the proof of Theorem 2.5 is to find uniform gaps around vertical lines $\mathrm{Re}(z) = \alpha_k$ in the spectrum of A, to be sure that the resolvent R_A is well-defined in these regions. To find such gaps, one proves the existence of sufficiently large gaps in the spectrum of

A_0 and places α_k in these zones. The results presented in [Agr76], [Mar88, Chapter 1], [Agr94], [Dzh94] are in fact a bit stronger than Theorem 2.5 since they contain not only the resolvent estimate (8) but also some basis properties.

- In the case $p(1-q) \geq 1$ one can prove a Riesz basis property for the subspaces $P_k H$, i.e. one can write $H = \bigoplus_{k\in\mathbb{N}} P_k H$ and there exists $c > 0$ such that for all $u \in H$, $c^{-1}\|u\|_H^2 \leq \sum_{k\in\mathbb{N}} \|P_k u\|_H^2 \leq c\|u\|_H^2$.
- In the case $p(1-q) < 1$ one can prove a weaker basis property (so-called Abel basis) for the subspaces $P_k H$.

For $\alpha \in \mathbb{R}_+$ we define the spectral projector on the characteristic subspace of H associated with the eigenvalues of real part less than $\max\{\alpha_k; \alpha_k \leq \alpha\}$:

$$\Pi_\alpha = \sum_{\alpha_k\leq\alpha} P_k = \frac{1}{2i\pi}\int_{\Gamma_\alpha} R_A(z)dz = \frac{1}{2i\pi}\int_{\bigcup_{\alpha_k\leq\alpha}\gamma_k} R_A(z)dz,$$

where Γ_α is the positively oriented contour delimited by the vertical line segments $\mathrm{Re}(z) = \max\{\alpha_k; \alpha_k \leq \alpha\}$ on the right and $\mathrm{Re}(z) = \alpha_0 > 0$ on the left and by the parabola $\mathcal{P}_{K_0}^q$ on the upper and the lower side.

On each finite dimensional subspace $P_k H$ (or equivalently $\Pi_\alpha H$) we have a holomorphic calculus for A (e.g. see [Kat80]); given a holomorphic function f, we have

$$f(A)P_k = f(AP_k) = \frac{1}{2i\pi}\int_{\gamma_k} f(z)R_A(z)dz \in \mathcal{L}(P_k H).$$

In the subsequent sections, we shall consider the adjoint problem of the abstract parabolic system (1), involving $A^* = A_0 + A_1^*$. The spectral theory of A and A^* and their respective functional calculus are connected by the following proposition (see [Kat80]).

Proposition 2.8. *Let f be a holomorphic function, γ a positively oriented contour in $\mathbb{C}\backslash\mathrm{Sp}(A)$. We denote $\check{f} : z \mapsto \overline{f(\overline{z})}$, $f^\gamma(A) = \frac{1}{2i\pi}\int_\gamma f(z)R_A(z)dz$ and $\overline{\gamma}$ the positively oriented complex conjugate contour of γ. Then, $\overline{\gamma}$ is a contour in $\mathbb{C}\setminus\mathrm{Sp}(A^*)$ and $(f^\gamma(A))^* = \check{f}^{\overline{\gamma}}(A^*)$.*

With this new notation, $f^{\gamma_k}(A) = f(A)P_k = f(AP_k)$. Noting that with the choices made above, $\overline{\gamma_k} = \gamma_k$, $\overline{\Gamma_\alpha} = \Gamma_\alpha$, we obtain

$$P_k(A^*) = 1^{\gamma_k}(A^*) = (1^{\gamma_k}(A))^* = P_k(A)^* = P_k^*$$

and $\Pi_\alpha(A^*) = (\Pi_\alpha(A))^* = \Pi_\alpha^*$. More generally, if $\check{f} = f$, we have $f^{\gamma_k}(A^*) = (f^{\gamma_k}(A))^*$ and $f^{\Gamma_\alpha}(A^*) = \left(f^{\Gamma_\alpha}(A)\right)^*$.

Example 2.9. For $t \in \mathbb{R}$, the functions $f(z) = e^{tz}$ or $f(z) = e^{t\sqrt{z}}$ (taking for $\sqrt{z}$ the principal value of the square root of $z \in \mathbb{C}$) or $f(z) = \int_\mathbb{R} \psi(t)e^{-itz}dt$ (ψ being a real function) fulfill the property $\check{f} = f$. For all these functions, we have $f^{\gamma_k}(A^*) = (f^{\gamma_k}(A))^*$, $f^{\Gamma_\alpha}(A^*) = \left(f^{\Gamma_\alpha}(A)\right)^*$. This will be used in the following sections.

Remark 2.10. Note that $\|R_{A^*}(z)\|_{\mathcal{L}(H)} = \|R_A(\overline{z})\|_{\mathcal{L}(H)}$. As a consequence, any sequence satisfying (8) for A also satisfies (8) for A^*.

In the course of the construction of a control function that we present below, we shall need a precise asymptotics of the increasing sequence $(\alpha_k)_{k\in\mathbb{N}}$, that is not given in [Agr94]. We first remark that if $(\alpha_k)_{k\in\mathbb{N}}$ is a sequence satisfying properties $(i)-(iii)$ of Theorem 2.5, then every subsequence of $(\alpha_k)_{k\in\mathbb{N}}$ also satisfies these properties. We shall thus seek a minimal growth for the asymptotics of $(\alpha_k)_{k\in\mathbb{N}}$.

For $\mu\in\mathbb{R}$, we set $\mathcal{N}(\mu)=\#\{k\in\mathbb{N},\lambda_k\in \mathrm{Sp}(A_0),\lambda_k\leq\mu\}$. Here, we prove the following proposition.

Proposition 2.11. *If the eigenvalues of A_0 satisfy the following asymptotics: $\mathcal{N}(\mu)=m_0\mu^{\frac{1}{p}}+o(\mu^{\frac{1}{p}})$, as $\mu\to+\infty$, then, for all $\delta>1$ we can choose the sequence $(\alpha_n)_{n\in\mathbb{N}}$ such that there exists $N\in\mathbb{N}$ and for every $n\geq N$, we have $\delta^{n-1}\leq\alpha_n\leq\delta^n$.*

First note that the assumption we make here for the asymptotics of the eigenvalues is stronger than that made in Theorem 2.5 for it implies $\lambda_k\sim_{k\to\infty}Ck^p$.
To prove Proposition 2.11, we briefly recall how the sequence $(\alpha_k)_{k\in\mathbb{N}}$ is built in [Agr94]. Every α_n is in the interval $[\mu_n-h\mu_n^q,\mu_n+h\mu_n^q]$, where $(\mu_n)_{n\in\mathbb{N}}$ is a sequence increasing to infinity such that $\#\{k\in\mathbb{N},\lambda_k\in\mathrm{Sp}(A_0),\lambda_k\in[\mu_n-h\mu_n^q,\mu_n+h\mu_n^q]\}\leq C\mu_n^\beta$. We thus need to show the existence of such a sequence $(\mu_n)_{n\in\mathbb{N}}$, having a precise asymptotics as n goes to infinity. This is the aim of Lemma 2.12 below, replacing [Agr94, Lemma 7]. With such a result, for all fixed $\delta>1$ we can build $(\alpha_n)_{n\in\mathbb{N}}$ such that there exists $N\in\mathbb{N}$ and for every $n\geq N$, α_n satisfies $\delta^{n-1}\leq\alpha_n\leq\delta^n$. We can then follow the proof of [Agr94] to finish that of Proposition 2.11, estimating the resolvent on vertical lines $\mathrm{Re}(z)=C\in[\mu_n-h\mu_n^q,\mu_n+h\mu_n^q]$.

Lemma 2.12. *We set $r=p^{-1}$ and recall that $\beta=\max\{0,r-1+q\}$. Assume that*

$$\mathcal{N}(\mu)=m_0\mu^r+o(\mu^r)\ \text{as}\ \mu\to+\infty. \tag{9}$$

Then, for every $h>0$, $\delta>1$, there exist $C>0$ and $N\in\mathbb{N}$ such that for every $n\geq N$, there exists μ_n such that $[\mu_n-h\mu_n^q,\mu_n+h\mu_n^q]\subset[\delta^{n-1},\delta^n]$ and $\mathcal{N}(\mu_n+h\mu_n^q)-\mathcal{N}(\mu_n-h\mu_n^q)\leq C\mu_n^\beta$.

Proof. We proceed by contradiction. Let $h>0$, $\delta>0$, and suppose that

$$\begin{gathered}\forall C>0,\forall N\in\mathbb{N},\exists n_0\geq N+1,\forall\mu \text{ satisfying } [\mu-h\mu^q,\mu+h\mu^q]\subset[\delta^{n_0-1},\delta^{n_0}],\\ \text{we have } \mathcal{N}(\mu+h\mu^q)-\mathcal{N}(\mu-h\mu^q)>C\mu^\beta.\end{gathered} \tag{10}$$

We choose $0<\varepsilon<m_0$. From the asymptotics (9) of $\mathcal{N}$, there exists $N_0\in\mathbb{N}$ such that for every $n\geq N_0$,

$$(m_0-\varepsilon)\delta^{nr}\leq\mathcal{N}(\delta^n)\leq(m_0+\varepsilon)\delta^{nr}. \tag{11}$$

Let us fix $C\geq\frac{2h\delta^{\beta+1}}{\delta-1}(m_0+\varepsilon+1)$ and $N\in\mathbb{N}$ such that $N\geq N_0$, and for all $n\geq N+1$, $(m_0-\varepsilon)\delta^{nr}-\frac{2h\delta^{\beta+1}}{\delta-1}(m_0+\varepsilon+1)\delta^{n\beta}\geq 0$. Such a N exists since $\beta\leq r+q-1<r$. Then for these C and N, assumption (10) gives the existence of $n_0\geq N+1\geq N_0+1$ such that for every μ satisfying $[\mu-h\mu^q,\mu+h\mu^q]\subset[\delta^{n_0-1},\delta^{n_0}]$, we have $\mathcal{N}(\mu+h\mu^q)-\mathcal{N}(\mu-h\mu^q)>C\mu^\beta$.
We denote by $\lfloor x\rfloor$ the floor function. In the interval $[\delta^{n_0-1},\delta^{n_0}]$, there are at least $\left\lfloor\frac{\delta^{n_0}-\delta^{n_0-1}}{2h\delta^{n_0q}}\right\rfloor$ disjoint intervals of the type $[\mu-h\mu^q,\mu+h\mu^q]$, each containing more than $C\delta^{(n_0-1)\beta}$ eigenvalues of A_0. Hence,

$$\mathcal{N}(\delta^{n_0})-\mathcal{N}(\delta^{n_0-1})\geq C\delta^{(n_0-1)\beta}\left\lfloor\frac{\delta^{n_0}-\delta^{n_0-1}}{2h\delta^{n_0q}}\right\rfloor\geq C\delta^{(n_0-1)\beta}\left(\frac{\delta-1}{2h\delta}\delta^{n_0(1-q)}-1\right).$$

Then, taking into account the lower bound on C and the asymptotics (11) for $n = n_0 - 1 \geq N \geq N_0$, we obtain

$$\begin{array}{rcl} \mathcal{N}(\delta^{n_0}) & \geq & (m_0+\varepsilon+1)\delta^{n_0\beta+n_0(1-q)} - \frac{2h\delta^{\beta+1}}{\delta-1}(m_0+\varepsilon+1)\delta^{(n_0-1)\beta} + (m_0-\varepsilon)\delta^{r(n_0-1)} \\ & \geq & (m_0+\varepsilon+1)\delta^{n_0(\beta+1-q)}, \end{array}$$

since $n_0 \geq N+1$. The asymptotics (11) for n_0 gives $\mathcal{N}(\delta^{n_0}) \leq (m_0+\varepsilon)\delta^{n_0 r} \leq (m_0+\varepsilon)\delta^{n_0(\beta+1-q)}$ since $\beta \geq r-1+q$. This yields a contradiction and concludes the proof of the lemma. □

3 Spectral inequality for perturbated selfadjoint elliptic operators

In this section, we prove in an abstract setting some spectral inequalities where the norm of a finite sum of root vectors of A is bounded by a partial measurement of these root vectors. For the proof, we assume that some interpolation inequality holds. This inequality will be proved in the case of different elliptic operators in Section 5.

Such interpolation inequality were used in [LR95] and [LZ98] to achieve a spectral inequality of the type we prove here. Note that this type of spectral inequality can however be obtained by other means (e.g. doubling properties [AE08], or directly from global Carleman estimates [BHL08]).

Here, we suppose that $A = A_0 + A_1 : \mathcal{D}(A) \subset H \longrightarrow H$ satisfies some of the spectral properties of Section 2, i.e. $(a)-(c)$ of Proposition 2.1 and the resolvent estimate (8) of Theorem 2.5. We define the following Sobolev spaces based on the selfadjoint operator A_0.

Definition 3.1. For $s \in \mathbb{N}$ and $\tau_1 < \tau_2$, we set

$$\mathcal{H}^s(\tau_1,\tau_2) = \bigcap_{n=0}^{s} H^{s-n}\left(\tau_1,\tau_2;\mathcal{D}(A_0^{n/2})\right),$$

which is a Hilbert space with the natural norm

$$\|v\|_{\mathcal{H}^s(\tau_1,\tau_2)} = \left(\sum_{n+m\leq s}\|\partial_t^m A_0^{n/2}v\|^2_{L^2(\tau_1,\tau_2;H)}\right)^{1/2} \approx \left(\sum_{n=0}^{s}\|v\|^2_{H^{s-n}\left(\tau_1,\tau_2;\mathcal{D}(A_0^{n/2})\right)}\right)^{1/2}.$$

Note that $\mathcal{H}^0(\tau_1,\tau_2) = L^2(\tau_1,\tau_2;H)$.

Let Y be another Hilbert space, $B^* \in \mathcal{L}(H,Y)$ be a bounded operator. Let T_0 be a positive number, $\varphi \in \mathcal{C}_0^\infty(0,T_0;\mathbb{C})$, $\varphi \neq 0$ and $\theta = \max\{1/2, p^{-1}-(1-q)\} = \max\{1/2,\beta\}$.

Theorem 3.2. *Suppose that there exist $C' > 0$, $\zeta \in (0,T_0/2)$ and $\nu \in (0,1]$ such that for every $v \in \mathcal{H}^2(0,T_0)$*

$$\|v\|_{\mathcal{H}^1(\zeta,T_0-\zeta)} \leq C'\|v\|^{1-\nu}_{\mathcal{H}^1(0,T_0)}\left(\|\varphi B^* v\|_{L^2(0,T_0;Y)} + \|(-\partial_t^2+A^*)v\|_{\mathcal{H}^0(0,T_0)}\right)^\nu. \tag{12}$$

Then, there exist positive constants C, D such that for every positive α, for all $w \in \Pi_\alpha^* H$,

$$\|w\|_H \leq Ce^{D\alpha^\theta}\left\|\varphi B^*\left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right)w\right\|_{L^2(0,T_0;Y)}. \tag{13}$$

In other situations, we can prove another interpolation inequality with an observation at the boundary $t=0$. In this case, we obtain a simpler spectral inequality, involving no time integration in the observation term.

Theorem 3.3. *Suppose that there exist $C'>0$, $\zeta\in(0,T_0/2)$ and $\nu\in(0,1]$ such that for every $v\in\mathcal{H}^2(0,T_0)$ satisfying $v(0)=0$, we have*

$$\|v\|_{\mathcal{H}^1(\zeta,T_0-\zeta)}\leq C'\|v\|^{1-\nu}_{\mathcal{H}^1(0,T_0)}\left(\|B^*\partial_t v(0)\|_Y+\|(-\partial_t^2+A^*)v\|_{\mathcal{H}^0(0,T_0)}\right)^{\nu}. \tag{14}$$

Then, there exist positive constants C,D such that for every positive α, for all $w\in\Pi^*_\alpha H$,

$$\|w\|_H\leq Ce^{D\alpha^\theta}\left\|B^*w\right\|_Y. \tag{15}$$

The estimation of the constant in the inequality in terms of the maximal eigenvalue in the finite sum is the key point in the control applications below.

Proof of Theorem 3.2. For $w\in\Pi^*_\alpha H$, we set

$$v(t)=\left(e^{t\sqrt{A^*}}+e^{-t\sqrt{A^*}}\right)w=\frac{1}{2i\pi}\int_{\Gamma_\alpha}\left(e^{t\sqrt{z}}+e^{-t\sqrt{z}}\right)R_{A^*}(z)w\,dz.$$

We have $v\in H^2(0,T_0;H)\cap H^1(0,T_0;\mathcal{D}(A))\subset\mathcal{H}^2(0,T_0)$ as $\mathcal{D}(A)=\mathcal{D}(A_0)$. We first notice that $(-\partial_t^2+A^*)v=0$. Second, we have to estimate every single term of

$$\|v\|^2_{\mathcal{H}^1(0,T_0)}=\|v\|^2_{L^2(0,T_0;H)}+\|\partial_t v\|^2_{L^2(0,T_0;H)}+\|A_0^{1/2}v\|^2_{L^2(0,T_0;H)}:$$

$$\begin{aligned}\|v\|_H&\leq\left\|\left(e^{t\sqrt{A^*}}+e^{-t\sqrt{A^*}}\right)\Pi^*_\alpha\right\|_{\mathcal{L}(H)}\|w\|_H\\&\leq\left\|\frac{1}{2i\pi}\int_{\Gamma_\alpha}\left(e^{t\sqrt{z}}+e^{-t\sqrt{z}}\right)R_{A^*}(z)dz\right\|_{\mathcal{L}(H)}\|w\|_H\\&\leq C\,\mathrm{meas}\,(\Gamma_\alpha)\sup_{z\in\Gamma_\alpha}\|R_{A^*}(z)\|_{\mathcal{L}(H)}\sup_{z\in\Gamma_\alpha}\left|e^{t\sqrt{z}}+e^{-t\sqrt{z}}\right|\|w\|_H\\&\leq C\alpha e^{C(\alpha^\beta+t\sqrt{\alpha})}\|w\|_H\end{aligned}$$

since $\mathrm{meas}\,(\Gamma_\alpha)\leq C\alpha$, $\|R_{A^*}(z)\|_{\mathcal{L}(H)}\leq e^{C\alpha^\beta}$ from estimate (8) of Theorem 2.5, Remark 2.10 and,

$$\left|e^{t\sqrt{z}}+e^{-t\sqrt{z}}\right|\leq 2e^{t\,\mathrm{Re}(\sqrt{z})},$$

with $\mathrm{Re}(\sqrt{z})\leq\sqrt{|z|}\leq C\sqrt{\alpha}$ for $z\in\Gamma_\alpha$.

Thus, for some $C>0$, $\|v\|_{L^2(0,T_0;H)}\leq Ce^{C\alpha^\theta}\|w\|_H$. Concerning the second term in $\|v\|_{\mathcal{H}^1(0,T_0)}$, we have

$$\|\partial_t v\|_{L^2(0,T_0;H)}=\left\|\sqrt{A^*}\left(e^{t\sqrt{A^*}}-e^{-t\sqrt{A^*}}\right)w\right\|_{L^2(0,T_0;H)},$$

and similar computations show that $\|\partial_t v\|_{L^2(0,T_0;H)}\leq Ce^{C\alpha^\theta}\|w\|_H$. In order to estimate the third term in $\|v\|_{\mathcal{H}^1(0,T_0)}$, we use assumption (a) of Proposition 2.1, observing that for all $u\in H$,

$$\begin{aligned}\|A_0^{1/2}u\|^2_H&=(A_0u,u)_H=(A^*u,u)_H-(A_1^*u,u)_H\\&\leq(A^*u,u)_H+\frac{k_0}{2}\|A_0^{1/2}u\|^{2q}_H\|u\|^{2-2q}_H\\&\leq(A^*u,u)_H+\frac{k_0}{2}\left(q\varepsilon^{\frac{1}{q}}\|A_0^{1/2}u\|^2_H+(1-q)\varepsilon^{-\frac{1}{1-q}}\|u\|^2_H\right)\end{aligned}$$

for every positive ε and every $q \in (0,1)$, thanks to Young's inequality. We then choose ε such that $\frac{\gamma}{2} q \varepsilon^{\frac{1}{q}} \leq \frac{1}{2}$ and we obtain for every $u \in H$,

$$\|A_0^{1/2} u\|_H^2 \leq C \left((A^* u, u)_H + \|u\|_H^2 \right) \leq C \left(\|A^* u\|_H \|u\|_H + \|u\|_H^2 \right).$$

The same estimate is obvious in the case $q = 0$. Similar computations as those performed above yield $\|A_0^{1/2} v\|_{L^2(0,T_0;H)} \leq C e^{C\alpha^\theta} \|w\|_H$, and we have finally

$$\|v\|_{\mathcal{H}^1(0,T_0)} \leq C e^{C\alpha^\theta} \|w\|_H, \quad w \in \Pi_\alpha^* H.$$

We now produce a lower bound for the left-hand side of (12). We first notice that the operator $\left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \Pi_\alpha^*$ is an isomorphism on $\Pi_\alpha^* H$ and we compute an upper bound for its inverse:

$$\begin{aligned}
\left\| \left[\left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \Pi_\alpha^* \right]^{-1} \right\|_{\mathcal{L}(H)} &= \left\| \frac{1}{2i\pi} \int_{\Gamma_\alpha} \left(e^{t\sqrt{z}} + e^{-t\sqrt{z}}\right)^{-1} R_{A^*}(z) dz \right\|_{\mathcal{L}(H)} \\
&\leq C \,\mathrm{meas}\,(\Gamma_\alpha) e^{C\alpha^\beta} \sup_{z \in \Gamma_\alpha} \left| e^{t\sqrt{z}} + e^{-t\sqrt{z}} \right|^{-1} \\
&\leq C\alpha e^{C(\alpha^\beta + t\sqrt{\alpha})} \sup_{z \in \Gamma_\alpha} \frac{1}{|e^{2t\sqrt{z}} + 1|}.
\end{aligned}$$

Then, we have

$$|e^{2t\sqrt{z}} + 1| \geq e^{2t\,\mathrm{Re}(\sqrt{z})} - 1 \geq 2t\,\mathrm{Re}(\sqrt{z}) \geq 2t\sqrt{\alpha_0}, \tag{16}$$

since $\mathrm{Re}(\sqrt{z}) \geq \sqrt{\alpha_0} > 0$ on Γ_α. Hence, for some constant $C > 0$,

$$\left\| \left[\left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \Pi_\alpha^* \right]^{-1} \right\|_{\mathcal{L}(H)} \leq \frac{C}{t} e^{C(\alpha^\beta + t\sqrt{\alpha})}.$$

Concerning the left-hand-side of (12), we thus have the following lower bound:

$$\begin{aligned}
\|v\|^2_{\mathcal{H}^1(\zeta, T_0-\zeta)} &\geq \|v\|^2_{L^2(\zeta, T_0-\zeta; H)} \geq \int_\zeta^{T_0-\zeta} \left\| \left[\left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \Pi_\alpha^* \right]^{-1} \right\|_{\mathcal{L}(H)}^{-2} dt\, \|w\|_H^2 \\
&\geq C e^{-2C\alpha^\beta} \int_\zeta^{T_0-\zeta} t^2 e^{-2tC\sqrt{\alpha}} dt\, \|w\|_H^2 \\
&\geq C e^{-2C\alpha^\beta} \zeta^2 \int_\zeta^{T_0-\zeta} e^{-2tC\sqrt{\alpha}} dt\, \|w\|_H^2 \\
&\geq C e^{-2C\alpha^\beta} \zeta^2 \frac{e^{-T_0 C\sqrt{\alpha}}}{\sqrt{\alpha}} \sinh\left(C\sqrt{\alpha}(T_0 - 2\zeta) \right) \|w\|_H^2 \\
&\geq C\zeta^2 (T_0 - 2\zeta) e^{-C(\alpha^\beta + \sqrt{\alpha})} \|w\|_H^2 \geq C e^{-C\alpha^\theta} \|w\|_H^2.
\end{aligned}$$

The interpolation inequality (12) then gives

$$C e^{-C\alpha^\theta} \|w\|_H \leq \left(C e^{C\alpha^\theta} \|w\|_H \right)^{1-\nu} \left\| \varphi B^* \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}} \right) w \right\|^\nu_{L^2(0,T_0;Y)}.$$

Dividing both side by $\|w\|_H^{1-\nu}$, we finally obtain the existence of positive constants C, D such that for every positive α, for all $w \in \Pi_\alpha^* H$,

$$\|w\|_H \leq C e^{D\alpha^\theta} \left\| \varphi B^* \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}} \right) w \right\|_{L^2(0,T_0;Y)}.$$

□

Proof of Theorem 3.3. This proof follows the same; for $w \in \Pi_\alpha^* H$, we take

$$v(t) = (A^*)^{-1/2}\sinh(t\sqrt{A^*})w = \frac{1}{2i\pi}\int_{\Gamma_\alpha}\frac{\sinh(t\sqrt{z})}{\sqrt{z}}R_{A^*}(z)w\,dz,$$

instead of $v(t) = \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right)w$, see [LZ98], [JL99]. It satisfies $(-\partial_t^2 + A^*)v = 0$, $v(0) = 0$ and $\partial_t v(0) = w$. We also have $v \in \mathcal{H}^2(0, T_0)$ and $\|v\|_{\mathcal{H}^1(0,T_0)} \leq Ce^{C\alpha^\theta}\|w\|_H$ as above. Only the lower bound for $\|v\|_{\mathcal{H}^1(\zeta,T_0-\zeta)}$ has to be proved. To begin with, the operator $(A^*)^{-1/2}\sinh(t\sqrt{A^*})\Pi_\alpha^*$ is an isomorphism on $\Pi_\alpha^* H$ and we compute an upper bound for its inverse:

$$\begin{aligned}
\left\|\left[(A^*)^{-1/2}\sinh(t\sqrt{A^*})\Pi_\alpha^*\right]^{-1}\right\|_{\mathcal{L}(H)} &= \left\|\frac{1}{2i\pi}\int_{\Gamma_\alpha}\left(\frac{\sinh(t\sqrt{z})}{\sqrt{z}}\right)^{-1}R_{A^*}(z)dz\right\|_{\mathcal{L}(H)}\\
&\leq C\alpha e^{C\alpha^\beta}\sup_{z\in\Gamma_\alpha}\left|\frac{\sqrt{z}}{\sinh(t\sqrt{z})}\right|\\
&\leq C\alpha^{3/2}e^{C\alpha^\beta}\sup_{z\in\Gamma_\alpha}\frac{1}{|e^{t\sqrt{z}} - e^{-t\sqrt{z}}|}\\
&\leq C\alpha^{3/2}e^{C(\alpha^\beta + t\sqrt{\alpha})}\sup_{z\in\Gamma_\alpha}\frac{1}{|e^{2t\sqrt{z}} - 1|}.
\end{aligned}$$

Then, $|e^{2t\sqrt{z}}-1| \geq e^{2t\,\mathrm{Re}(\sqrt{z})}-1$, and (16) gives a lower bound for $\|v\|_{\mathcal{H}^1(\zeta,T_0-\zeta)} \geq \|v\|_{L^2(\zeta,T_0-\zeta;H)}$ as in the preceding proof. The conclusion follows as in the proof of Theorem 3.2. □

4 From the spectral inequality to a parabolic control

In this section we construct a control for the parabolic abstract problem (1). We follow the method introduced by G. Lebeau and L. Robbiano in [LR95]. The non-selfadjoint nature of the problem requires however modifications in their approach.

Let H and Y be two Hilbert spaces, H standing for the state space and Y the control space. We suppose that $B \in \mathcal{L}(Y, H)$ is a bounded control operator and A is an unbounded operator $A : \mathcal{D}(A) \subset H \longrightarrow H$ that satisfies all the spectral properties of Section 2, i.e. $(a) - (c)$ (and thus also $(i) - (iii)$) of Proposition 2.1, the resolvent estimate (8) of Theorem 2.5, and the asymptotics given by Proposition 2.11. In particular, the properties $(a) - (c)$ of Proposition 2.1 imply that $-A$ generates a $\mathcal{C}^0$-semigroup of contraction on H. If we take u_0 in H, problem (1) is then well-posed in H.

Let T_0 be a positive number, $\varphi \in \mathcal{C}_0^\infty(0, T_0; \mathbb{C})$, $\varphi \neq 0$, and $B^* \in \mathcal{L}(H, Y)$ the adjoint operator of B, i.e., such that $(By, h)_H = (B^*h, y)_Y$ for every $y \in Y$, $h \in H$. We assume that the result of Theorem 3.2, (i.e. the spectral inequality (13)) holds. An example will be given in Section 5. We shall first interpret this spectral inequality (13) of Theorem 3.2 as an observability estimate for an elliptic evolution problem.

Remark 4.1. Note that if we suppose the spectral inequality (15) of Theorem 3.3 instead of (13), the construction of the control function follows that of [Mil06] or [LL09] and is much simpler. In fact, the spectral inequality (15) directly yields an observability inequality for the partial problem (27) and implies an analogous of Theorem 4.9. This proof can be found in Section 6, taking $\nu = 1$. Section 4.4 then ends the proof of the null-controllability. Note that in this case, there is no restriction on the subordination number q (in Proposition 4.5, we require $q < 3/4$) since there is no need of the regularisation with a Gevrey function.

4.1 Elliptic controllability on $\Pi_\alpha H$ with initial datum in $P_k H$

From the spectral inequality (13), we deduce a controllability result for a family of (finite dimensional) elliptic evolution problems. Let first $\mathcal{G}_\alpha$ be the following gramian operator

$$\mathcal{G}_\alpha = \int_0^{T_0} \left(e^{t\sqrt{A}} + e^{-t\sqrt{A}}\right) \Pi_\alpha B |\varphi(t)|^2 B^* \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \Pi_\alpha^* dt.$$

Lemma 4.2. *The operator $\mathcal{G}_\alpha$ in an isomorphism from $\Pi_\alpha^* H$ onto $\Pi_\alpha H$. We denote by $\mathcal{G}_\alpha^{-1}$ the inverse of $\mathcal{G}_\alpha$ on $\Pi_\alpha H$.*

Then, there exists $D_0 > 0$ and for every $s \in \mathbb{N}$ there exists C_s such that for every $T_0 > 0$, $\alpha > 0$, $k \in \mathbb{N}^*$ satisfying $\alpha_k \leq \alpha$, for every $v_k \in P_k H$, the function

$$h_k(t) = |\varphi(t)|^2 B^* \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \mathcal{G}_\alpha^{-1} v_k \tag{17}$$

satisfies

(i) $h_k \in \mathcal{C}_0^\infty(0, T_0; Y)$;

(ii) $\|\partial_t^s h_k\|_{L^\infty(0,T_0;Y)} \leq C_s \alpha^{\frac{s}{2}} e^{D_0 \alpha^\theta} \|v_k\|_H$;

(iii) for all $w \in \Pi_\alpha^* H$, $(v_k, w)_H = \left(B h_k, \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) w\right)_{L^2(0,T_0;H)}$.

Remark 4.3. Here, we could actually take the "initial datum" v in $\Pi_\alpha H$ and the result and its proof remain the same. We have chosen to take v in $P_k H$ since it is the precise result we use in Proposition 4.5, in particular to prove the estimate (iii) of Proposition 4.5, which in turn is a key point in the proof of Theorem 4.10 below.

Proof. We first observe that the spectral inequality (13) implies that $\mathcal{G}_\alpha$ in an isomorphism from $\Pi_\alpha^* H$ onto $\Pi_\alpha H$.

Then, we note that point (ii) implies (i), and is itself a direct consequence of expression (17) where B^* is bounded and $\|\mathcal{G}_\alpha^{-1}\|_{\mathcal{L}(H)} \leq C e^{D_0 \alpha^\theta}$ from the spectral inequality (13).

Finally, we check that (iii) holds. For $w \in \Pi_\alpha^* H$, we compute

$$\begin{aligned}
&\left(B h_k, \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) w\right)_{L^2(0,T_0;H)} \\
&= \left(B |\varphi(t)|^2 B^* \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \mathcal{G}_\alpha^{-1} v_k, \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) w\right)_{L^2(0,T_0;H)} \\
&= \Bigg(\underbrace{\left(\int_0^{T_0} \left(e^{t\sqrt{A}} + e^{-t\sqrt{A}}\right) \Pi_\alpha B |\varphi(t)|^2 B^* \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) \Pi_\alpha^* dt\right)}_{\mathcal{G}_\alpha} \mathcal{G}_\alpha^{-1} v_k, w\Bigg)_H \\
&= (v_k, w)_H.
\end{aligned}$$

□

Remark 4.4. Lemma 4.2 corresponds to a null-controllability property on $[0, T_0]$ in $\Pi_\alpha H$ for the elliptic control problem with initial condition in $P_k H$

$$\begin{cases} -\partial_t^2 u + A u = \Pi_\alpha B h_k, \\ u_{|t=0} = 0, \\ \partial_t u_{|t=0} = v \in P_k H, \\ u_{|t=T_0} = \partial_t u_{|t=T_0} = 0, \end{cases}$$

whose dynamics remains in $\Pi_\alpha H$ for every $t \in [0, T_0]$. The adjoint problem is the following, well-posed in $\Pi^*_\alpha H$:

$$\begin{cases} -\partial_t^2 z + A^* z = 0, \\ z_{|t=T_0} = w \in \Pi^*_\alpha H, \\ \partial_t z_{|t=T_0} = 0. \end{cases}$$

The estimation of the "cost" of the control h_k is the key point for the following parabolic partial controllability properties.

4.2 Parabolic controllability on $\Pi_\alpha H$ with initial datum in $P_k H$

From the elliptic controllability result of Lemma 4.2, we now deduce a corresponding parabolic controllability result. The main tools here are the transformation introduced in [Rus73] and a Paley-Wiener-type theorem.

Proposition 4.5. *We suppose that the subordination number satisfies $q < \frac{3}{4}$ and we fix $\gamma > \max\{\frac{2(1-q)}{3-4q}, 1\}$. Then, for all $\mathcal{T} > 0$ there exists $D_1 > 0$ and for every $s \in \mathbb{N}$ there exists C_s such that for every $0 < T < \mathcal{T}$, $\alpha > 0$, $k \in \mathbb{N}^*$ satisfying $\alpha_k \leq \alpha$, for every $u_{k,0} \in P_k H$, there exists a control function G_k such that:*

(i) $G_k \in \mathcal{C}_0^\infty(0, T; Y)$;

(ii) $\|\partial_t^s G_k\|_{L^\infty(0,T;Y)} \leq C_s T^{-2\gamma s} \exp\left(D_1(\alpha^\theta + \frac{1}{T^\gamma} + (T\alpha_k)^{\frac{\gamma}{2\gamma-1}}) - T\alpha_{k-1}\right) \|u_{k,0}\|_H$;

(iii) for all $w \in \Pi^*_\alpha H$, $-(u_{k,0}, e^{-TA^*}w)_H = \left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)}$.

Remark 4.6. For technical requirements, that can be found in the proof of Lemma 8.1 in Appendix 8.1, we have assumed that the subordination q of A_1 to A_0 is less than $\frac{3}{4}$ here. This will be the case in all the applications we present in Section 5.

Remark 4.7. The new variable $\mathcal{T}$ here is the time in which we want to control the full equation (29), appearing in Section 4.4.

Proof. From Lemma 4.2, we have for any positive T_0 (that will be fixed equal to 1 below): for all $v_k \in P_k H$, there exists h_k (that we know with precision) such that for all $w \in \Pi^*_\alpha H$,

$$\begin{aligned} (v_k, w)_H &= (v_k, P_k^* w)_H = \left(Bh_k, \left(e^{t\sqrt{A^*}} + e^{-t\sqrt{A^*}}\right) P_k^* w\right)_{L^2(0,T_0;H)} \\ &= \left(\int_0^{T_0} \left(e^{t\sqrt{A}} + e^{-t\sqrt{A}}\right) P_k B h_k(t) dt, w\right)_H \\ &= \left(\int_0^{T_0} \left(\frac{1}{2i\pi} \int_{\gamma_k} \left(e^{t\sqrt{z}} + e^{-t\sqrt{z}}\right) R_A(z) dz\right) B h_k(t) dt, w\right)_H \\ &= \left(\frac{1}{2i\pi} \int_{\gamma_k} R_A(z) B \int_0^{T_0} \left(e^{t\sqrt{z}} + e^{-t\sqrt{z}}\right) h_k(t) dt\, dz, w\right)_H . \end{aligned}$$

We introduce the Fourier-Laplace transform of a function:

$$\hat{f}(z) = \int_{\mathbb{R}} f(t) e^{-itz} dt,\ z \in \mathbb{C}.$$

If $f \in \mathcal{C}_0^\infty(0,T_0;Y)$, then $\hat{f}$ is an entire function with values in Y. We write $\hat{f} \in \mathcal{H}(\mathbb{C};Y)$. Recalling that $h_k \in \mathcal{C}_0^\infty(0,T_0;Y)$, we have $\hat{h}_k \in \mathcal{H}(\mathbb{C};Y)$ and

$$(v_k, w)_H = \left(\frac{1}{2i\pi} \int_{\gamma_k} R_A(z) B \left(\hat{h}_k(i\sqrt{z}) + \hat{h}_k(-i\sqrt{z}) \right) dz, w \right)_H . \tag{18}$$

From Lemma 4.2 (ii), taken with $s = 0$, we obtain $\|\hat{h}_k(z)\|_Y \leq C_0 e^{D_0 \alpha^\theta} e^{T_0 |\operatorname{Im}(z)|} \|v_k\|_H$. Following [Rus73] and [LR95], we set

$$Q_k(-iz^2) = \hat{h}_k(iz) + \hat{h}_k(-iz),$$

and note that $Q_k(z)$ is an entire function with respect to z. We now have for every $v_k \in P_k H$, the existence of $Q_k \in \mathcal{H}(\mathbb{C};Y)$ such that for all $w \in \Pi_\alpha^* H$,

$$(v_k, w)_H = \left(\frac{1}{2i\pi} \int_{\gamma_k} R_A(z) B Q_k(-iz) dz, w \right)_H , \tag{19}$$

with Q_k satisfying

$$\|Q_k(z)\|_Y \leq C_0 e^{D_0 \alpha^\theta} e^{T_0 \sqrt{|z|}} \|v_k\|_H. \tag{20}$$

The goal is now to see Q_k as the Fourier-Laplace transform of a regular function with compact support in $(0,T)$. However, (a Hilbert-valued version of) the Paley-Wiener theorem [Hör83, Theorem 15.1.5] indicates that the inverse Fourier transform of Q_k is only in the dual space $(G^2(\mathbb{R};Y))'$ of the space of Gevrey functions of order 2. With the convolution by a function $e \in G^\sigma$, $\sigma \in (1,2)$, i.e. by multiplying Q_k by $\hat{e}$, we can now regularize the inverse Fourier transform of Q_k.

We fix $T_0 = 1$ and set $\sigma = 2 - \frac{1}{\gamma} \in (1,2)$. Since we have required that $q < \frac{3}{4}$ and $\gamma > \max\{\frac{2(1-q)}{3-4q}, 1\}$, the Gevrey index $\sigma = 2 - \frac{1}{\gamma}$ satisfies $\sigma > \frac{1}{2(1-q)}$ what is equivalent to $q < 1 - \frac{1}{2\sigma}$. Under these conditions, Lemma 8.1 of Appendix 8.1 gives the existence of a Gevrey function $e \in G^\sigma$ satisfying (the constants c_i are positive)

$$\begin{cases} \operatorname{supp}(e) = [0,1] \text{ and } 0 < e(t) \leq 1 \text{ for all } t \in (0,1), \\ |\hat{e}(z)| \leq c_1 e^{-c_2 |z|^{\frac{1}{\sigma}}} \text{ if } \operatorname{Im}(z) \leq 0, \\ |\hat{e}(z)| \geq c_3 e^{-c_4 |z|^{\frac{1}{\sigma}}} \text{ in } -i\mathcal{P}^q_{K_0 \mathcal{T}^{1-q}} = -i\{z \in \mathbb{C}, \operatorname{Re}(z) \geq 0, |\operatorname{Im}(z)| < K_0 \mathcal{T}^{1-q} \operatorname{Re}(z)^q\}. \end{cases} \tag{21}$$

The parabola $\mathcal{P}^q_{K_0 \mathcal{T}^{1-q}}$ is chosen here so that for every $k \in \mathbb{N}$, for every $z \in \gamma_k \subset \overline{\mathcal{P}^q_{K_0}}$ (defined in (7)), we have $\mathcal{T} z \in \mathcal{P}^q_{K_0 \mathcal{T}^{1-q}}$ and the lower bound of (21) holds.

We set, for $T < \mathcal{T}$,

$$\hat{g}_k(z) = \hat{e}(Tz) Q_k(z) \in \mathcal{H}(\mathbb{C};Y),$$

and because of (20) and the first two points of (21), $\hat{g}_k(z)$ satisfies the following estimates

$$\|\hat{g}_k(z)\|_Y \leq C_0 e^{T \operatorname{Im}(z)} e^{D_0 \alpha^\theta} e^{\sqrt{|z|}} \|v_k\|_H \text{ for every } z \in \mathbb{C}, \tag{22}$$

$$\|\hat{g}_k(z)\|_Y \leq C_0 c_1 e^{-c_2 |Tz|^{\frac{1}{\sigma}}} e^{D_0 \alpha^\theta} e^{\sqrt{|z|}} \|v_k\|_H \text{ if } \operatorname{Im}(z) \leq 0. \tag{23}$$

From (22), (23) and the Paley-Wiener-type theorem given in Proposition 8.3 in Appendix 8.2, there exists $g_k \in \mathcal{C}_0^\infty(0,T;Y)$ such that $\hat{g}_k(z) = \int_{\mathbb{R}} g_k(t)e^{-izt}dt$ for every $z \in \mathbb{C}$. The function g_k is then given by $g_k(t) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{g}_k(\tau)e^{i\tau t}d\tau$. From (23), g_k satisfies the estimates

$$\|\partial_t^s g_k\|_{L^\infty(0,T;Y)} \leq \frac{C_0c_1}{2\pi}e^{D_0\alpha^\theta}\int_{\mathbb{R}}|\tau|^s e^{-c_2|T\tau|^{\frac{1}{\sigma}}}e^{\sqrt{|\tau|}}d\tau\|v_k\|_H, \quad s \in \mathbb{N}.$$

The Laplace method [Erd56] applied to this integral (only dependent on s and γ) finally implies that there exists $C > 0$ and for every $s \in \mathbb{N}$ there exists $C_s > 0$ such that for every $T > 0$, $\alpha > 0$, $k \in \mathbb{N}^*$ such that $\alpha_k \leq \alpha$, for every $v_k \in P_kH$,

$$\|\partial_t^s g_k\|_{L^\infty(0,T;Y)} \leq C_s e^{D_0\alpha^\theta}T^{-2\gamma s}e^{\frac{C}{T^\gamma}}\|v_k\|_H. \tag{24}$$

Let us now properly construct the control function that satisfies the three assertions of the proposition. We first note that for $T < \mathcal{T}$, the operators $\hat{e}(-iTA)$ and e^{-TA} are two isomorphisms of P_kH since the holomorphic functions $\hat{e}(-iTz)$ and e^{-Tz} do not vanish in $\mathcal{P}^q_{K_0}$. Given $u_{k,0} \in P_kH$, we set $v_k = -[\hat{e}(-iTA)]^{-1}e^{-TA}u_{k,0} \in P_kH$ and g_k the associated control function given as preceding. We set $G_k(t) = g_k(T-t)$, which satisfies point (i) of the proposition. From (24), we obtain

$$\|\partial_t^s G_k\|_{L^\infty(0,T;Y)} \leq C_s e^{D_0\alpha^\theta}T^{-2\gamma s}e^{\frac{C}{T^\gamma}}\|[\hat{e}(-iTA)]^{-1}P_k\|_{\mathcal{L}(H)}\|e^{-TA}P_k\|_{\mathcal{L}(H)}\|u_{k,0}\|_H. \tag{25}$$

We can estimate

$$\begin{aligned}
\|[\hat{e}(-iTA)]^{-1}P_k\|_{\mathcal{L}(H)} &= \left\|\frac{1}{2i\pi}\int_{\gamma_k}\frac{1}{\hat{e}(-iTz)}R_A(z)dz\right\|_{\mathcal{L}(H)} \\
&\leq \frac{1}{2\pi}\,\mathrm{meas}(\gamma_k)\sup_{z\in\gamma_k}\frac{1}{|\hat{e}(-iTz)|}\sup_{z\in\gamma_k}\|R_A(z)\|_{\mathcal{L}(H)} \\
&\leq C\alpha_k\sup_{z\in\gamma_k}\frac{1}{|\hat{e}(-iTz)|}\sup_{z\in\gamma_k}\|R_A(z)\|_{\mathcal{L}(H)} \\
&\leq C'\alpha_k\sup_{z\in\gamma_k}c_3^{-1}e^{c_4|Tz|^{\frac{1}{\sigma}}}e^{C\alpha_k^\beta} \\
&\leq C''e^{\tilde{c_4}(|T\alpha_k|^{\frac{1}{\sigma}}+\alpha^\theta)},
\end{aligned}$$

where we have used the third property of the Gevrey function e given in (21) and the resolvent estimate (8) of Theorem 2.5 on γ_k. A similar estimate for $\|e^{-TA}P_k\|_{\mathcal{L}(H)}$ gives

$$\|e^{-TA}P_k\|_{\mathcal{L}(H)} \leq C\alpha_k e^{-T\alpha_{k-1}+C\alpha^\theta}.$$

We finally obtain from (25) that for all $\mathcal{T} > 0$ there exists $D_1 > 0$ and for every $s \in \mathbb{N}$ there exists $C_s > 0$ such that for every $0 < T < \mathcal{T}$, $\alpha > 0$, $k \in \mathbb{N}^*$ such that $\alpha_k \leq \alpha$, for every $u_{k,0} \in P_kH$,

$$\|\partial_t^s G_k\|_{L^\infty(0,T;Y)} \leq C_s e^{D_1\alpha^\theta}T^{-2\gamma s}e^{\frac{D_1}{T^\gamma}}e^{D_1|T\alpha_k|^{\frac{1}{\sigma}}}e^{-T\alpha_{k-1}}\|u_{k,0}\|_H. \tag{26}$$

Point (ii) of the proposition is thus proved recalling that $\frac{1}{\sigma} = \frac{\gamma}{2\gamma-1}$. To prove (iii), we compute $\left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)}$, with $w \in \Pi_\alpha^* H$. We have

$$\begin{aligned}
\left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)} &= \left(Bg_k, e^{-tA^*}w\right)_{L^2(0,T;H)} = \left(e^{-tA}Bg_k, w\right)_{L^2(0,T;H)} \\
&= \left(\int_0^T\left(\frac{1}{2i\pi}\int_{\gamma_k}e^{-tz}R_A(z)dz\right)Bg_k(t)dt, w\right)_H \\
&= \left(\frac{1}{2i\pi}\int_{\gamma_k}R_A(z)B\int_0^T e^{-tz}g_k(t)dtdz, w\right)_H
\end{aligned}$$

As $\operatorname{supp}(g_k) \subset (0,T)$, we obtain

$$\begin{aligned}
\left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)} &= \left(\frac{1}{2i\pi}\int_{\gamma_k} R_A(z)B\hat{g}_k(-iz)dz, w\right)_H \\
&= \left(\frac{1}{2i\pi}\int_{\gamma_k} R_A(z)B\hat{e}(-iTz)Q_k(-iz)dz, w\right)_H \\
&= \left(\hat{e}(-iTA)\frac{1}{2i\pi}\int_{\gamma_k} R_A(z)BQ_k(-iz)dz, w\right)_H
\end{aligned}$$

because the holomorphic calculus gives for $\phi \in \mathcal{H}(\mathbb{C};\mathbb{C})$ and $\psi \in \mathcal{H}(\mathbb{C};H)$

$$\left(\frac{1}{2i\pi}\int_{\gamma_k} \phi(z)R_A(z)dz\right)\left(\frac{1}{2i\pi}\int_{\gamma_k} R_A(z)\psi(z)dz\right) = \frac{1}{2i\pi}\int_{\gamma_k} \phi(z)R_A(z)\psi(z)dz.$$

From (19), we then have

$$\begin{aligned}
\left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)} &= \left(\frac{1}{2i\pi}\int_{\gamma_k} R_A(z)BQ_k(-iz)dz, \hat{e}(-iTA^*)w\right)_H \\
&= (v_k, \hat{e}(-iTA^*)w)_H
\end{aligned}$$

as $\hat{e}(-iTA^*)w \in \Pi_\alpha^* H$. Recalling that $v_k = -[\hat{e}(-iTA)]^{-1}e^{-TA}u_{k,0}$, we have obtained that the function G_k constructed here satisfies for all $w \in \Pi_\alpha^* H$

$$\begin{aligned}
\left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)} &= \left(-[\hat{e}(-iTA)]^{-1}e^{-TA}u_{k,0}, \hat{e}(-iTA^*)w\right)_H \\
&= -\left(e^{-TA}u_{k,0}, w\right)_H,
\end{aligned}$$

for every $u_{k,0} \in P_kH$. Point (iii) is thus proved. This concludes the proof of Proposition 4.5. $\square$

Remark 4.8. Proposition 4.5 is a null-controllability property on $[0,T]$ in the finite dimensional space $\Pi_\alpha H$ for the parabolic control problem :

$$\begin{cases} \partial_t u + Au = \Pi_\alpha BG_k, \\ u_{|t=0} = u_{k,0} \in P_kH, \\ u_{|t=T} = 0. \end{cases}$$

This also means that for every initial datum $u_{0,k} \in P_kH$ and $T > 0$, there exists $G_k \in \mathcal{C}_0^\infty(0,T;Y)$ such that the solution of

$$\begin{cases} \partial_t u + Au = BG_k, \\ u_{|t=0} = u_{k,0} \in P_kH, \end{cases}$$

satisfies $\Pi_\alpha u(T) = 0$.

4.3 Parabolic controllability on $\Pi_\alpha H$

We shall now combine the controllability results for initial datum in P_kH, $\alpha_k \leq \alpha$ to obtain a null-controllability result for an initial datum in $\Pi_\alpha H$, i.e., for the problem

$$\begin{cases} \partial_t u + Au = \Pi_\alpha BG_\alpha \\ u_{|t=0} = u_0 \in \Pi_\alpha H \\ u_{|t=T} = 0. \end{cases} \tag{27}$$

The norm of the control function G_α will be estimated to prepare for the next section, where a control for an arbitrary initial condition in H is constructed.

Theorem 4.9. *For the control problem* (27)*, for every positive T, for every γ sufficiently large, there exists a control function $G_\alpha \in \mathcal{C}_0^\infty(0,T;Y)$ driving u_0 to 0 in time T with a cost given by* $\|G_\alpha\|_{L^\infty(0,T;Y)} \le C \exp\left(D(\alpha^\theta + \frac{1}{T^\gamma})\right) \|u_0\|_H$.

Actually, we prove the following more precise result.

Theorem 4.10. *Let $q < \frac{3}{4}$ and $\gamma > \max\{\frac{2(1-q)}{3-4q}, 1\}$. For all $\mathcal{T} > 0$, there exists $D > 0$ and for every $s \in \mathbb{N}$ there exists C_s such that for every $0 < T < \mathcal{T}$, $\alpha > 0$, for every $u_0 \in \Pi_\alpha H$, there exists a control function G_α such that :*

(i) $G_\alpha \in \mathcal{C}_0^\infty(0,T;Y)$*;*

(ii) $\|\partial_t^s G_\alpha\|_{L^\infty(0,T;Y)} \le C_s T^{-2\gamma s} \exp\left(D(\alpha^\theta + \frac{1}{T^\gamma})\right) \|u_0\|_H$*;*

(iii) for all $w \in \Pi_\alpha^* H$, $-(u_0, e^{-TA^*}w)_H = \left(BG_\alpha, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)}$.

Proof. We write $u_0 = \sum_{\alpha_k \le \alpha} P_k u_0 \in \Pi_\alpha H$, with $P_k u_0 \in P_k H$ and Proposition 4.5 gives for every k the existence of a control function G_k satisfying:

$$\left\{\begin{array}{l} \text{for all } w \in \Pi_\alpha^* H, \quad -(P_k u_0, e^{-TA^*}w)_H = \left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)}; \\ \|\partial_t^s G_k\|_{L^\infty(0,T;Y)} \le C_s T^{-2\gamma s} \exp\left(D_1(\alpha^\theta + \frac{1}{T^\gamma} + (T\alpha_k)^{\frac{\gamma}{2\gamma-1}}) - T\alpha_{k-1}\right) \|P_k u_0\|_H \end{array}\right. \tag{28}$$

We set

$$G_\alpha = \sum_{\alpha_k \le \alpha} G_k \in \mathcal{C}_0^\infty(0,T;Y).$$

and (i) is clear as the sum is finite. To prove (iii), given $w \in \Pi_\alpha^* H$, we simply compute

$$\begin{aligned} -(u_0, e^{-TA^*}w)_H &= \sum_{\alpha_k \le \alpha} -\left(P_k u_0, e^{-TA^*}w\right)_H = \sum_{\alpha_k \le \alpha} \left(BG_k, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)} \\ &= \left(BG_\alpha, e^{-(T-t)A^*}w\right)_{L^2(0,T;H)}. \end{aligned}$$

We now prove point (ii). Here, we use the asymptotic estimation for the sequence $(\alpha_k)_{k\in\mathbb{N}}$ given in Proposition 2.11. Let $\delta > 1$, there exists $N \in \mathbb{N}$ such that $\delta^{k-1} \le \alpha_k \le \delta^k$ if $k \ge N$. We then have

$$\begin{aligned} \|\partial_t^s G_\alpha\|_{L^\infty(0,T;Y)} &\le \sum_{\alpha_k \le \alpha} \|\partial_t^s G_k\|_{L^\infty(0,T;Y)} \\ &\le C_s T^{-2\gamma s} e^{D_1(\alpha^\theta + \frac{1}{T^\gamma})} \sum_{\alpha_k \le \alpha} e^{D_1 (T\alpha_k)^{\frac{\gamma}{2\gamma-1}} - T\alpha_{k-1}} \|P_k u_0\|_H \\ &\le C_s T^{-2\gamma s} e^{D_1(\alpha^\theta + \frac{1}{T^\gamma})} \|u_0\|_H \sum_{\alpha_k \le \alpha} e^{D_1 (T\delta^k)^{\frac{\gamma}{2\gamma-1}} - T\delta^{k-2}}. \end{aligned}$$

It remains to estimate the sum. Recalling that $\frac{\gamma}{2\gamma-1} < 1$, the function $x \mapsto e^{D_1 x^{\frac{\gamma}{2\gamma-1}} - \delta^{-2}x}$ is bounded on $\mathbb{R}_+$ by a constant $\kappa = \kappa(\gamma, \delta)$. We thus have

$$\sum_{\alpha_k \le \alpha} e^{D_1 (T\delta^k)^{\frac{\gamma}{2\gamma-1}} - T\delta^{k-2}} \;\le\; N\kappa + \sum_{\substack{k \ge N \\ \alpha_k \le \alpha}} \kappa \;\le\; \kappa \left(N + \frac{\ln(\alpha)}{\ln(\delta)}\right).$$

Finally, changing the constants C_s, we conclude that

$$\|\partial_t^s G_\alpha\|_{L^\infty(0,T;Y)} \le C_s T^{-2\gamma s} \exp\left(D(\alpha^\theta + \frac{1}{T^\gamma})\right) \|u_0\|_H.$$

□

Remark 4.11. Here, we have constructed a control in $\mathcal{C}_0^\infty(0,T;Y)$. In the case $Y = H$ and for every $p \in \mathbb{N}$, $B^* \in \mathcal{L}(\mathcal{D}(A^p))$, we are able to construct with the same techniques a control function in $\mathcal{C}_0^\infty(0,T;\bigcap_{p\in\mathbb{N}}\mathcal{D}(A^p))$, following [LR95].

4.4 Decay property for the semigroup and construction of the final control

We shall now conclude the proof of the main controllability theorem. We consider the full controllability problem: given $\mathcal{T} > 0$, we construct a control function g such that the solution of the problem

$$\begin{cases} \partial_t u + Au = Bg, \\ u_{|t=0} = u_0 \in H, \end{cases} \tag{29}$$

satisfies $u(\mathcal{T}) = 0$ in H. The proof uses both the partial control result of Theorem 4.9 and the decay rate of the semigroup generated by $-A$, once restricted to $(I - \Pi_\alpha)H$. We first prove an estimate of this decay rate. We denote by $(S_A(t))_{t\in\mathbb{R}_{+*}}$ the $\mathcal{C}^0$-semigroup of contraction generated by $-A$.

Proposition 4.12. *There exist $C > 0$ and $\widetilde{N} \in \mathbb{N}$ such that for every $k \geq \widetilde{N}$, $t \geq \frac{1}{\alpha_k}$,*

$$\|S_A(t)(I - \Pi_{\alpha_k})\|_{\mathcal{L}(H)} \leq Ce^{C\alpha_k^\theta - t\alpha_k}.$$

Proof. From [Paz83, Theorem 1.7.7] we first write the semigroup generated by $-A(I - \Pi_{\alpha_k})$ as an integral over the infinite positively oriented contour $\partial\mathcal{P}_{K_0}^q$

$$S_A(t)(I - \Pi_{\alpha_k}) = S_{A(I-\Pi_{\alpha_k})}(t) = \frac{1}{2i\pi}\int_{\partial\mathcal{P}_{K_0}^q} e^{-tz} R_{A(I-\Pi_{\alpha_k})}(z)\,dz.$$

We set $\Lambda_k = \{z \in \mathcal{P}_{K_0}^q, \mathrm{Re}(z) \geq \alpha_k\}$, $\partial\Lambda_k^+ = \{z \in \partial\Lambda_k, \mathrm{Re}(z) > \alpha_k, \mathrm{Im}(z) \geq 0\}$ and $\partial\Lambda_k^- = \{z \in \partial\Lambda_k, \mathrm{Re}(z) > \alpha_k, \mathrm{Im}(z) \leq 0\}$ so that $\partial\Lambda_k = \partial\Lambda_k^+ \cup I_k \cup \partial\Lambda_k^-$ and is a positively oriented contour.

Since $R_{A(I-\Pi_{\alpha_k})}$ is holomorphic in $\mathbb{C}\setminus\Lambda_k$, we may shift the path of integration from $\partial\mathcal{P}_{K_0}^q$ to $\partial\Lambda_k$ without changing the value of the integral. Hence,

$$\begin{aligned} S_A(t)(I - \Pi_{\alpha_k}) &= \frac{1}{2i\pi}\int_{\partial\Lambda_k} e^{-tz} R_A(z)\,dz \\ &= \frac{1}{2i\pi}\int_{\partial\Lambda_k^+} e^{-tz} R_A(z)\,dz + \frac{1}{2i\pi}\int_{I_k} e^{-tz} R_A(z)\,dz + \frac{1}{2i\pi}\int_{\partial\Lambda_k^-} e^{-tz} R_A(z)\,dz, \end{aligned}$$

where

$$\left\|\int_{\partial\Lambda_k^+} e^{-tz} R_A(z)\,dz\right\|_{\mathcal{L}(H)} \leq \int_{\alpha_k}^\infty Ce^{-tx}\,dx \leq C\frac{e^{-t\alpha_k}}{t},$$

since $R_A(z)$ is uniformly bounded on $\partial\Lambda_0$ from Proposition 2.1 point (iii). The same estimate holds for the integral over $\partial\Lambda_k^-$. Finally estimate (8) of Theorem 2.5 gives

$$\left\|\int_{I_k} e^{-tz} R_A(z)\,dz\right\|_{\mathcal{L}(H)} \leq C\alpha_k^q e^{C\alpha_k^\theta - t\alpha_k}.$$

Then, for some $C > 0$, we obtain the following estimation of the semigroup

$$\|S_A(t)(I - \Pi_{\alpha_k})\|_{\mathcal{L}(H)} \leq C \left(e^{C\alpha_k^\theta - t\alpha_k} + \frac{e^{-t\alpha_k}}{t} \right), \quad t > 0.$$

Thus, taking $\widetilde{N} \in \mathbb{N}$ such that for $k \geq \widetilde{N}$, $\frac{1}{\alpha_k} \geq e^{-C\alpha_k^\theta}$, we finally obtain

$$\|S_A(t)(I - \Pi_{\alpha_k})\|_{\mathcal{L}(H)} \leq Ce^{C\alpha_k^\theta - t\alpha_k}, \quad k \geq \widetilde{N}, t \geq \frac{1}{\alpha_k},$$

and the proposition is proved. □

For the sake of completeness, we now construct the control function for the parabolic problem (29), following [LR95]. The decay rate proved in Proposition 4.12 shows that we have now to restrict ourselves to the case $\theta < 1$, i.e. $p^{-1} + q < 2$. We recall that N is an integer such that for all $n \geq N$, $\delta^{n-1} \leq \alpha_n \leq \delta^n$ (see Proposition 2.11).

Theorem 4.13. *Suppose that $q < \frac{3}{4}$ and $\theta < 1$. Then, for every $\mathcal{T} > 0$, $u_0 \in H$, there exists a control function $g \in \mathcal{C}_0^\infty(0, \mathcal{T}; Y)$ such that the solution u of the problem (29) satisfies $u(\mathcal{T}) = 0$.*

Proof. We first fix ρ and $\gamma > \max\{\frac{2(1-q)}{3-4q}, 1\}$ such that $0 < \rho < \min\{\frac{1}{\gamma}, \frac{1-\theta}{\theta}\}$. We set $T_j = K\delta^{-j\rho\theta}$, K being such that $\sum_{j\in\mathbb{N}} 2T_j = \mathcal{T}$. We divide the time interval $[0, \mathcal{T}] = \bigcup_{j\in\mathbb{N}}[a_j, a_{j+1}]$, with $(a_j)_{j\in\mathbb{N}}$ such that $a_0 = 0$ and $a_{j+1} = a_j + 2T_j$. Hence, $\lim_{j\to\infty} a_j = \mathcal{T}$.
For $u_j \in \Pi_{\alpha_j} H$, we define by $G_{\alpha_j}(u_j, T_j)$ the control function given by Theorem 4.10 that drives u_j to zero in time T_j, which in particular satisfies the estimate (ii) of this theorem.
We set J_0 an integer such that $J_0 \geq \max\{N, \widetilde{N}\}$ (so that $\delta^{j-1} \leq \alpha_j \leq \delta^j$ for $j \geq J_0$ and the decay rate of Proposition 4.12 holds) and $T_j \geq \delta^{-(j-1)} \geq \alpha_j^{-1}$ for every $j \geq J_0$. We now define the control function g:

- if $t \leq a_{J_0}$, we set $g = 0$, and $u(t) = S_A(t)u_0$;
- if $j \geq J_0$, $t \in (a_j, a_j + T_j]$, we set $g = G_{\alpha_j}(\Pi_{\alpha_j} u(a_j), T_j)$, and
$$u(t) = S_A(t - a_j)u(a_j) + \int_{a_j}^t S_A(t - s)Bg(s)ds;$$
- if $j \geq J_0$, $t \in (a_j + T_j, a_{j+1}]$, we set $g = 0$, and
$$u(t) = S_A(t - a_j - T_j)u(a_j + T_j).$$

We recall that $\|S_A(t)\|_{\mathcal{L}(H)} \leq 1$ because we have required A to be positive. During the first phase $0 \leq t \leq a_{J_0}$, we simply have $\|u(a_{J_0})\|_H \leq \|u_0\|_H$. The choice of the control function during the second phase implies for $j \geq J_0$, $\Pi_{\alpha_j} u(a_j + T_j) = 0$ and

$$\|u(a_j + T_j)\|_H \leq C \exp\left(C\delta^{j\theta} + C\delta^{j\gamma\rho\theta}\right) \|u(a_j)\|_H \leq \exp\left(C'\delta^{j\theta}\right) \|u(a_j)\|_H,$$

as $\gamma\rho < 1$.

Finally, during the third phase, the decay rate of the semigroup is given for $j \geq J_0$ by Proposition 4.12 and we then have

$$\|u(a_{j+1})\|_H \leq C \exp\left(C\delta^{j\theta} - T_j\delta^{j-1}\right) \|u(a_j + T_j)\|_H \leq \exp\left(C''\delta^{j\theta} - K\delta^{j(1-\rho\theta)}\right) \|u(a_j + T_j)\|_H$$

Combining the estimations given on the three phases, we obtain for $j \geq J_0$

$$\|u(a_{j+1})\|_H \leq \exp\left(\sum_{k=J_0}^{j} C''\delta^{k\theta} - K\delta^{k(1-\rho\theta)}\right)\|u_0\|_H.$$

Because of our choice $\rho < \frac{1-\theta}{\theta}$, we have $1-\rho\theta > \theta$, and for some $c > 0$,

$$\left(\sum_{k=J_0}^{j} C''\delta^{k\theta} - K\delta^{k(1-\rho\theta)}\right) \leq c\exp\left(-c\delta^{j(1-\rho\theta)}\right),$$

and thus for every $j \geq J_0$,

$$\|u(a_{j+1})\|_H \leq c\exp\left(-c\delta^{j(1-\rho\theta)}\right)\|u_0\|_H. \tag{30}$$

From Theorem 4.10 point (ii) and Estimate (30), we have

$$\begin{aligned}\|\partial_t^s g\|_{L^\infty(0,\mathcal{T};Y)} &\leq \sup_{j\geq J_0}\left\{C_s T_j^{-2\gamma s}\exp\left(D(\alpha_j^\theta + \frac{1}{T_j^\gamma})\right)\|u(a_j)\|_H\right\} \\ &\leq \sup_{j\geq J_0}\left\{C_s\delta^{2j\rho\theta\gamma s}\exp\left(D(\delta^{j\theta} + K\delta^{j\rho\gamma\theta}) - c\delta^{(j-1)(1-\rho\theta)}\right)\|u_0\|_H\right\} < +\infty,\end{aligned}$$

following the same estimations as above. Thus, $g \in \mathcal{C}_0^\infty(0,\mathcal{T};Y)$. This implies in particular that the solution u of (29) is continuous with values in H. Hence, from (30) we directly obtain

$$\|u(\mathcal{T})\|_H = \lim_{j\to\infty}\|u(a_{j+1})\|_H = 0,$$

and $u(\mathcal{T}) = 0$ in H. □

5 Application to the controllability of parabolic coupled systems

In this section, we apply the abstract results proved in the previous sections to second order elliptic operators and to the controllability of parabolic systems. In the following, we first check that the assumptions of Proposition 2.1 and Theorem 2.5 for these elliptic operators are fulfilled and we prove the interpolation inequality (12). Sections 3 and 4 then directly yield the spectral inequality and the controllability results.

We are concerned with the system

$$\begin{cases} \partial_t u_1 + P_1 u_1 + a u_1 + b u_2 = 0 & \text{in } (0,T)\times\Omega, \\ \partial_t u_2 + P_2 u_2 + c u_1 + d u_2 = \mathbb{1}_\omega g & \text{in } (0,T)\times\Omega, \\ u_{1|t=0} = u_1^0,\ u_{2|t=0} = u_2^0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } (0,T)\times\partial\Omega, \end{cases} \tag{31}$$

where Ω is an open connected subset of $\mathbb{R}^n$ with $n \leq 3$ (a compact connected Riemannian manifold with or without boundary of dimension $n \leq 3$). We suppose that $\partial\Omega$ is at least of class $\mathcal{C}^2$. The function $\mathbb{1}_\omega(x)$ stands for the characteristic function of the open subset $\omega \subset \Omega$ and $a, b, c, d \in L^\infty(\Omega)$. Here, P_i, $i = 1,2$, denotes a positive elliptic selfadjoint operator on Ω:

$P_i u = -\operatorname{div}_x(c_i(x)\nabla_x u)$ where $c_i(x)$ is a symmetric uniformly elliptic matrix, i.e. $c_i \in W^{1,\infty}(\Omega)$ and there exists $C > 0$ such that for every $x \in \Omega$, $\xi \in \mathbb{R}^n$, $\xi \cdot c_i(x)\xi \geq C|\xi|^2$. We set $H = (L^2(\Omega))^2$, $\mathcal{D}(A_0) = (H^2(\Omega) \cap H_0^1(\Omega))^2$, and

$$A_0 = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Referring to Remark 1.1, we shift $A = A_0 + A_1$ by a $\lambda_0 > 0$ sufficiently large so that Eq. (7) is satisfied. We carry on the analysis with the operator $A + \lambda_0$, which we write A by abuse of notation. Then, the operator A satisfies the assumptions $(a) - (c)$ of Proposition 2.1 with $q = 0$ since A_1 is bounded in H. Moreover, for $\mu \in \mathbb{R}_+$ the number of eigenvalues of the operator P_i lower than μ is given by the Weyl asymptotics $\mathcal{N}_i(\mu) = m_i \mu^{n/2} + o(\mu^{n/2})$ as $\mu \to +\infty$. Thus, for A, $\mathcal{N}$ is given by $\mathcal{N}(\mu) = \mathcal{N}_1(\mu) + \mathcal{N}_2(\mu) = (m_1 + m_2)\mu^{n/2} + o(\mu^{n/2})$ and A satisfies the assumption of Proposition 2.11 and that of Theorem 2.5 with $p = 2/n$. The assumption $\theta < 1$ of Theorem 4.13 is satisfied if and only if $n/2 - 1 < 1$, i.e., $n \leq 3$.

We set $Y = L^2(\Omega)$ as the control space and the operator B is given by $B : g \longrightarrow (0, \mathbb{1}_\omega g)^T$, and is bounded from $L^2(\Omega)$ to $(L^2(\Omega))^2$ and its adjoint is $B^* : (u_1, u_2)^T \longrightarrow \mathbb{1}_\omega u_2 \in \mathcal{L}\left((L^2(\Omega))^2; L^2(\Omega)\right)$.

Now, it remains to prove the interpolation inequality (12) to apply Theorem 3.2.

Proposition 5.1. *Let $T_0 > 0$, $\zeta \in (0, T_0/2)$. Suppose that there exists an open subset $\mathcal{O} \subset \Omega$, $\mathcal{O} \cap \omega \neq \emptyset$ such that the coupling coefficient $b \in L^\infty(\Omega)$ satisfies $|b(x)| \geq b_0 > 0$ for almost every $x \in \mathcal{O}$. Then, there exist $C > 0$, $\varphi \in \mathcal{C}_0^\infty(0, T_0)$ and $\nu \in (0, 1)$ such that for every $v \in \left(H^2((0, T_0) \times \Omega)\right)^2$, $v_{|(0,T_0)\times\partial\Omega} = 0$, we have*

$$\|v\|_{(H^1((\zeta, T_0-\zeta)\times\Omega))^2} \leq C\|v\|^{1-\nu}_{(H^1((0,T_0)\times\Omega))^2} \left(\|\varphi B^* v\|_{L^2((0,T_0)\times\Omega)} + \|(-\partial_t^2 + A^*)v\|_{(L^2((0,T_0)\times\Omega))^2}\right)^\nu. \quad (32)$$

Proof. We first prove (32) with $\|\varphi B^* v\|_{L^2((0,T_0)\times\Omega)}$ replaced by $\|v\|_{L^2(V)}$ with $\overline{V} \subset (0, T_0) \times \mathcal{O} \cap \omega$. In a second step, thanks to local elliptic energy estimates, we eliminate the first component and obtain (32).

Here, the time variable does not play a particular role. Thus, for the sake of clarity, we simplify the notation, denoting by ∇ the time-space gradient $(\partial_t, \nabla_x)^T$, by div the time-space divergence $\partial_t + \operatorname{div}_x$, by C_i the time-space diffusion matrices $C_i = \begin{pmatrix} 1 & 0 \\ 0 & c_i \end{pmatrix}$, and by $-\Delta_i$ the time-space elliptic operators

$$-\Delta_i = -\partial_t^2 + P_i = -\partial_t^2 - \operatorname{div}_x(c_i \nabla_x \cdot) = -\operatorname{div}(C_i \nabla \cdot).$$

We also set

$$\widetilde{A^*} = -\partial_t^2 + A^* = \begin{pmatrix} -\Delta_1 + a & c \\ b & -\Delta_2 + d \end{pmatrix}.$$

We also denote by $(\cdot, \cdot)$ the L^2 scalar product on $L^2((0, T_0) \times \Omega)$ or $(L^2((0, T_0) \times \Omega))^{n+1}$, $\|\cdot\|$ the associated norms, and by $(\cdot, \cdot)_i$, $i = 1, 2$, the L^2 scalar product defined by

$$(\xi, \xi')_i = (c_i \xi, \xi'), \quad \xi, \xi' \in (L^2((0, T_0) \times \Omega))^{n+1}, \; i = 1, 2,$$

and $\|\cdot\|_i$ the associated norm. With the assumptions made on c_i, the norms $\|\cdot\|_i$ and $\|\cdot\|$ are equivalent on $(L^2((0,T_0)\times\Omega))^{n+1}$.

We first state local Carleman estimates for $\widetilde{A^*}$. These are direct consequences of the classical local Carleman estimates for the elliptic operators Δ_i. We first choose a local weight function ϕ satisfying a subellipticity condition with respect to both Δ_1 and Δ_2 (which can be done taking $\phi = e^{\lambda\psi}$ for λ sufficiently large and ψ satisfying $|\nabla\psi| \geq C > 0$, see [Hör63, Chapter 8], [LR95] or [LL09]). Then there exists $h_1 > 0$ and $C > 0$ such that for every $w \in (\mathcal{C}_0^\infty((0,T_0)\times\Omega))^2$, $w = (w_1, w_2)^T$, and $0 < h < h_1$,

$$h\|e^{\phi/h}w_i\|^2 + h^3\|e^{\phi/h}\nabla w_i\|^2 \leq Ch^4\|e^{\phi/h}\Delta_i w_i\|^2\,, \quad i = 1,2$$

(see [LR95], or [FI96] in the case $c_i \in W^{1,\infty}$). Adding these two estimates and absorbing the zero-order terms for h sufficiently small, we directly obtain the same estimate for $\widetilde{A^*}$:

$$h\|e^{\phi/h}w\|^2 + h^3\|e^{\phi/h}\nabla w\|^2 \leq Ch^4\|e^{\phi/h}\widetilde{A^*}w\|^2\,, \quad \nabla w = \begin{pmatrix} \nabla w_1 \\ \nabla w_2 \end{pmatrix}.$$

By optimizing in h (see [LR95]), these local Carleman estimates yield local interpolation estimates of the form

$$\|v\|_{(H^1(B(3r)))^2} \leq C\|v\|^{1-\nu}_{(H^1((0,T_0)\times\Omega))^2}\left(\|v\|_{(H^1(B(r)))^2} + \|\widetilde{A^*}v\|_{(L^2((0,T_0)\times\Omega))^2}\right)^\nu,$$

where $B(r)$ denote concentric balls of radius r.

Similar estimates at the boundary $(0,T_0)\times\partial\Omega$ are also direct consequences of the Carleman estimates at the boundary for a scalar elliptic operator.

Then, following [LR95], these local interpolation inequalities can be "propagated", so that we obtain the following global interpolation inequality, with two observations in H^1 norm, localized in any nonempty open subset W of $(\zeta, T_0-\zeta)\times\Omega$:

$$\|v\|_{(H^1((\zeta,T_0-\zeta)\times\Omega))^2} \leq C\|v\|^{1-\nu}_{(H^1((0,T_0)\times\Omega))^2}\left(\|v\|_{(H^1(W))^2} + \|\widetilde{A^*}v\|_{(L^2((0,T_0)\times\Omega))^2}\right)^\nu. \tag{33}$$

Let us take the open subsets W, V, and U such that $\overline{W}\subset V$, $\overline{V}\subset U$, and $\overline{U}\subset(0,T_0)\times\mathcal{O}\cap\omega$. Elliptic regularity for the operators Δ_i, $i=1,2$, shows that there exists $C>0$ such that

$$\|v\|_{H^1(W)} \leq C(\|\widetilde{A^*}v\| + \|v\|_{L^2(V)}).$$

It finally remains to eliminate one of the two observations with energy estimates. In fact, we prove that

$$\|v_1\|_{L^2(V)} \leq C(\|\widetilde{A^*}v\| + \|v_2\|_{L^2(U)}).$$

We write $f = (f_1, f_2)^T = \widetilde{A^*}v$, i.e.

$$\begin{cases} f_1 = -\Delta_1 v_1 + av_1 + cv_2, \\ f_2 = -\Delta_2 v_2 + bv_1 + dv_2. \end{cases} \tag{34}$$

Let χ be a cut-off function such that $\chi \in \mathcal{C}_0^\infty(U)$, $0\leq\chi\leq 1$, and $\chi = 1$ on $V\subset U$. We set

$$\eta = \chi^\tau, \quad \eta_1 = \chi^{\tau+1}, \quad \eta_2 = \chi^{\tau-1},$$

for a real number $\tau > 2$, so that η, η_1, η_2 and $\chi^{\tau-2}$ are also cut-off functions of the same type. We notice that $\nabla\eta_1 = \eta(\tau+1)\nabla\chi$ and $\nabla\eta = \eta_2\tau\nabla\chi$, where $\nabla\chi$ is a bounded function.

We form the scalar product of the second equation of (34) by $\eta^2 v_1$

$$(\eta^2 v_1, bv_1) = (\eta^2 v_1, f_2) - (\eta^2 v_1, dv_2) + (\eta^2 v_1, \Delta_2 v_2).$$

The third term can be estimated as follows, using the equivalence of the norms $\|\cdot\|_2$ and $\|\cdot\|$:

$$\begin{array}{rcl} (\eta^2 v_1, \Delta_2 v_2) & = & -(\eta\nabla v_1, \eta\nabla v_2)_2 - (v_1(2\eta\nabla\eta), \nabla v_2)_2 \\ & \leq & C\left(\varepsilon_1\|\eta_1\nabla v_1\|^2 + \dfrac{1}{\varepsilon_1}\|\eta_2\nabla v_2\|^2 + \varepsilon_2\|\eta v_1\|^2 + \dfrac{1}{\varepsilon_2}\|\eta_2\nabla v_2\|^2\right), \end{array}$$

for every positive ε_1 and ε_2, thanks to Young's inequality. Hence,

$$\begin{array}{rcl} (\eta^2 v_1, bv_1) & \leq & C\left(\varepsilon_1\|\eta_1\nabla v_1\|^2 + \frac{1}{\varepsilon_1}\|\eta_2\nabla v_2\|^2 + \varepsilon_2\|\eta v_1\|^2 + \frac{1}{\varepsilon_2}\|\eta_2\nabla v_2\|^2\right) \\ & & +|(\eta^2 v_1, f_2)| + |(\eta^2 v_1, dv_2)|. \end{array} \tag{35}$$

Moreover, forming the scalar product of the first equation of (34) by $\delta_1\eta_1^2 v_1$ and the second one by $\delta_2\eta_2^2 v_2$ for $\delta_1, \delta_2 > 0$, we obtain

$$\left\{\begin{array}{l} 0 = \delta_1(\eta_1^2 v_1, \Delta_1 v_1) + \delta_1(\eta_1^2 v_1, f_1) - \delta_1(\eta_1^2 v_1, av_1) - \delta_1(\eta_1^2 v_1, cv_2), \\ 0 = \delta_2(\eta_2^2 v_2, \Delta_2 v_2) + \delta_2(\eta_2^2 v_2, f_2) - \delta_2(\eta_2^2 v_2, dv_2) - \delta_2(\eta_2^2 v_2, bv_1), \end{array}\right. \tag{36}$$

with

$$\begin{array}{rcl} \delta_1(\eta_1^2 v_1, \Delta_1 v_1) & = & -\delta_1\|\eta_1^2\nabla v_1\|_1^2 - \delta_1(v_1(2\eta_1\nabla\eta_1), \nabla v_1)_1 \\ & \leq & -\delta_1\|\eta_1\nabla v_1\|_1^2 + C\left(\delta_1\varepsilon_3\|\eta_1\nabla v_1\|_1^2 + \frac{\delta_1}{\varepsilon_3}\|\eta v_1\|_1^2\right), \end{array} \tag{37}$$

and similarly

$$\delta_2(\eta_2^2 v_2, \Delta_2 v_2) \leq -\delta_2\|\eta_2\nabla v_2\|_2^2 + C\left(\delta_2\varepsilon_3\|\eta_2\nabla v_2\|_2^2 + \frac{\delta_2}{\varepsilon_3}\|v_2\nabla\eta_2\|_2^2\right) \tag{38}$$

for all positive ε_3.

Replacing (37) and (38) in (36), and adding (35) and (36), we obtain, for positive constants C_0, K_0

$$\begin{array}{rl} (\eta^2 v_1, bv_1) \leq & C_0\Big\{|(\eta^2 v_1, f_2)| + |(\eta^2 v_1, dv_2)| + \varepsilon_1\|\eta_1\nabla v_1\|^2 + \frac{1}{\varepsilon_1}\|\eta_2\nabla v_2\|^2 + \varepsilon_2\|\eta v_1\|^2 \\ & +\frac{1}{\varepsilon_2}\|\eta_2\nabla v_2\|^2 + \delta_1|(\eta_1^2 v_1, f_1)| + \delta_1|(\eta_1^2 v_1, av_1)| + \delta_1|(\eta_1^2 v_1, cv_2)| \\ & +\delta_2|(\eta_2^2 v_2, f_2)| + \delta_2|(\eta_2^2 v_2, dv_2)| + \delta_2|(\eta_2^2 v_2, bv_1)| + \delta_1\varepsilon_3\|\eta_1\nabla v_1\|^2 + \frac{\delta_1}{\varepsilon_3}\|\eta v_1\|^2 \\ & +\delta_2\varepsilon_3\|\eta_2\nabla v_2\|^2 + \frac{\delta_2}{\varepsilon_3}\|v_2\nabla\eta_2\|^2\Big\} - K_0\left\{\delta_1\|\eta_1\nabla v_1\|^2 + \delta_2\|\eta_2\nabla v_2\|^2\right\}, \end{array}$$

where we used the equivalence between the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|$ to write everything in terms of $\|\cdot\|$. Note that all the positive parameters δ_i, ε_j have not been fixed yet. Now we suppose that $b \geq b_0 > 0$ on U. The case $b \leq -b_0 < 0$ follows the same. We thus have,

$$b_0\|\eta v_1\|^2 \leq (\eta^2 v_1, bv_1) \leq C_0\{J_1 + J_2\} + K_0 J_3 \tag{39}$$

where J_1 contains only the terms without gradient, and

$$\left\{\begin{array}{rcl} J_3 & = & -\delta_1\|\eta_1\nabla v_1\|^2 - \delta_2\|\eta_2\nabla v_2\|^2, \\ J_2 & = & \varepsilon_1\|\eta_1\nabla v_1\|^2 + \frac{1}{\varepsilon_1}\|\eta_2\nabla v_2\|^2 + \frac{1}{\varepsilon_2}\|\eta_2\nabla v_2\|^2 + \delta_1\varepsilon_3\|\eta_1\nabla v_1\|^2 \\ & & +\delta_2\varepsilon_3\|\eta_2\nabla v_2\|^2 + \frac{\delta_2}{\varepsilon_3}\|v_2\nabla\eta_2\|^2. \end{array}\right.$$

The term $C_0J_2 + K_0J_3$ in (39) is thus non-positive as soon as the conditions

$$\left\{ \begin{array}{rcl} C_0(\varepsilon_1 + \delta_1\varepsilon_3) - K_0\delta_1 & \leq & 0, \\ C_0(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \delta_2\varepsilon_3) - K_0\delta_2 & \leq & 0 \end{array} \right.$$

are satisfied. In this case, we obtain from (39) $b_0\|\eta v_1\|^2 \leq C_0J_1$. Let us now estimate C_0J_1, using that $a, b, c, d \in L^\infty$ and Young's inequality with the parameters 1, ε_2 or $\varepsilon_4 > 0$:

$$\begin{array}{rcl} C_0J_1 & \leq & C_1 \Big\{ \varepsilon_2\|\eta v_1\|^2 + \frac{1}{\varepsilon_2}\|f_2\|^2 + \frac{1}{\varepsilon_2}\|\eta v_2\|^2 + \delta_1\|\eta v_1\|^2 + \delta_1\|f_1\|^2 + \delta_1\|\eta v_2\|^2 \\ & & +\delta_2\|\eta_2 v_2\|^2 + \delta_2\|f_2\|^2 + \varepsilon_4\delta_2\|\eta v_1\|^2 + \frac{\delta_2}{\varepsilon_4}\|\chi^{\tau-2}v_2\|^2 + \frac{\delta_1}{\varepsilon_3}\|\eta v_1\|^2 \Big\} \\ & \leq & C_1 \left(\varepsilon_2 + \delta_1 + \varepsilon_4\delta_2 + \frac{\delta_1}{\varepsilon_3} \right) \|\eta v_1\|^2 + C(\delta_i, \varepsilon_j) \left(\|\chi^{\tau-2}v_2\|^2 + \|f_1\|^2 + \|f_2\|^2 \right). \end{array}$$

If we choose the parameters such that $C_1 \left(\varepsilon_2 + \delta_1 + \varepsilon_4\delta_2 + \frac{\delta_1}{\varepsilon_3} \right) \leq \frac{b_0}{2}$, since we now have $b_0\|\eta v_1\|^2 \leq C_0J_1$, we then obtain

$$\begin{array}{l} \|\eta v_1\|^2_{L^2(V)} \leq C(\delta_i, \varepsilon_j) \left(\|v_2\|^2_{L^2(U)} + \|f_1\|^2 + \|f_2\|^2 \right), \\ \|v_1\|_{L^2(V)} \leq C \left(\|v_2\|_{L^2(U)} + \|\widetilde{A^*}v\| \right). \end{array}$$

Recalling that the open subset U is chosen such that $\overline{U} \subset (0, T_0) \times \mathcal{O} \cap \omega$, we take $\varphi \in \mathcal{C}_0^\infty(0, T_0; \mathbb{C})$, with $\varphi = 1$ on the time support of U and we have

$$\|v_2\|_{L^2(U)} \leq \|\varphi B^*v\|_{L^2((0,T_0)\times\Omega)}.$$

The proof of the proposition is complete.

It only remains to note that it is possible to choose the parameters δ_i, ε_j satisfying

$$\left\{ \begin{array}{rcl} \delta_1(C_0\varepsilon_3 - K_0) + C_0\varepsilon_1 & \leq & 0, \\ \delta_2(C_0\varepsilon_3 - K_0) + C_0(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}) & \leq & 0, \\ \varepsilon_2 + \delta_1 + \varepsilon_4\delta_2 + \frac{\delta_1}{\varepsilon_3} & \leq & \frac{b_0}{2C_1}. \end{array} \right.$$

This can be done, fixing first $\varepsilon_2 + \delta_1 \leq \frac{b_0}{6C_1}$ and $\varepsilon_3 < \min\{1, \frac{K_0}{C_0}\}$. Second, choosing $\varepsilon_1 \leq \delta_1(\frac{K_0}{C_0} - \varepsilon_3)$, the first condition is satisfied. Third, we can fix δ_2 sufficiently large so that the second condition is satisfied and finally ε_4 such that $\varepsilon_4\delta_2 \leq \frac{b_0}{6C_1}$ and the last condition is fulfilled. □

As a consequence of Proposition 5.1, the spectral inequality (13) of Theorem 3.2, the partial controllability result of Theorem 4.10 and the null-controllability result of Theorem 4.13 hold in this case. Under the assumptions made above, in particular those of Proposition 5.1, the coupled parabolic system (31) is null-controllable in any positive time by a control function $g \in \mathcal{C}_0^\infty(0, T; L^2(\Omega))$. Note that in this context, the spectral inequality (13) corresponds to the estimation of a finite sum of root vectors of A by a localized measurement of only one component of this finite sum of root vectors.

Remark 5.2. In the local energy estimates made in the proof, we see that the assumption $\mathcal{O} \cap \omega \neq \emptyset$ is crucial. In the case $\mathcal{O} \cap \omega = \emptyset$, the spectral inequality and the null-controllability remain open problems to the author's knowledge .

Remark 5.3. In the case where the operator A is selfadjoint (i.e. $b = c$ in (31)), the spectral inequality (13) is much easier to prove, once the interpolation inequality (32) holds. This spectral inequality can take the following form. We denote by $(\mu_j)_{j\in\mathbb{N}}$ the eigenvalues of $A = A^*$ and $\{(\phi_j, \psi_j)^T\}_{j\in\mathbb{N}}$ the associated eigenfunctions, that form a Hilbert basis of $(L^2(\Omega))^2$. Then, for every open subset $U \subset (0,T) \times \mathcal{O}$, there exist $C > 0$ such that for every sequence $(a_j, b_j)_{j\in\mathbb{N}} \subset \mathbb{C}$ and $\alpha > 0$, we have

$$\sum_{\mu_j\leq\alpha} \left(|a_j|^2 + |b_j|^2\right) \leq Ce^{C\sqrt{\alpha}} \left\| \sum_{\mu_j\leq\alpha} \left(a_j e^{\sqrt{\mu_j}t} + b_j e^{-\sqrt{\mu_j}t}\right)\psi_j \right\|^2_{L^2(U)} .$$

Following the proof of [LR95] or Section 4, it yields the controllability of the coupled problem (31), without restriction on the dimension of Ω.

Remark 5.4. The same proof also yields a spectral inequality, a partial controllability and a null-controllability result for the following cascade system of d equations with one control force

$$\begin{cases} \partial_t u_1 + P_1 u_1 + \mathbb{1}_{\omega_1} u_2 &= 0, \\ \partial_t u_2 + P_2 u_2 + \mathbb{1}_{\omega_2} u_3 &= 0, \\ \cdots & \\ \partial_t u_{d-1} + P_{d-1} u_{d-1} + \mathbb{1}_{\omega_{d-1}} u_d &= 0, \\ \partial_t u_d + P_d u_d &= \mathbb{1}_{\omega_d} g, \\ u_{j|t=0} = u_j^0\,, & j \in \{1\cdots d\}, \\ u_j = 0 \text{ on } (0,T)\times\partial\Omega\,, & j \in \{1\cdots d\}, \end{cases}$$

where $P_j = -\operatorname{div}(c_j(x)\nabla\cdot)$ for some symmetric uniformly elliptic matrices c_j. Note that the null-controllability result is a particular case of the article [GBT09]. We have to suppose that $\bigcap_{j=1}^{d} \omega_j \neq \emptyset$. The spaces here are the same as above, the operator A_0 is $\operatorname{diag}(P_1 \cdots P_d)$ and A_1 is

$$\begin{pmatrix} 0 & \mathbb{1}_{\omega_1} & & \\ & \ddots & \ddots & \\ & & \ddots & \mathbb{1}_{\omega_{d-1}} \\ & & & 0 \end{pmatrix}.$$

The above analysis directly yields the spectral inequality (13) of Theorem 3.2 and the partial control result of Theorem 4.10. The null-controllability result of Theorem 4.13 in any positive time, by only one control function $g \in \mathcal{C}_0^\infty(0,T; L^2(\Omega))$ holds, supposing that $\Omega \subset \mathbb{R}^n, n \leq 3$.

6 Application to the controllability of a fractional order parabolic equation

Following [MZ06] and [Mil06], we give here an application of the spectral inequality (15) of Theorem 3.3 to the null-controllability of the following parabolic-type problem in which the dynamics is given by a fractional power of the non-selfadjoint operator A. We only treat the "good" case, i.e., when the power $\nu > 0$ is sufficiently large. In this case, the selfadjoint problem is null-controllable. We consider

$$\begin{cases} \partial_t u + A^\nu u = Bg \\ u_{|t=0} = u_0 \in H. \end{cases} \tag{40}$$

Here, we define $(\mathcal{D}(A^\nu), -A^\nu)$ as the infinitesimal generator of the strongly continuous semigroup

$$\frac{1}{2i\pi}\int_{\partial\mathcal{P}^q_{K_0}} e^{-tz^\nu} R_A(z)\,dz = S_{A^\nu}(t),$$

or equivalently one of the following expressions (see [Haa06]),

$$\begin{aligned} A^\nu &= \frac{1}{2i\pi}\lim_{t\to 0^+}\int_{\partial\mathcal{P}^q_{K_0}} z^\nu e^{-tz^\nu} R_A(z)\,dz = (A+I)^m \frac{1}{2i\pi}\int_{\partial\mathcal{P}^q_{K_0}} z^\nu (z+1)^{-m} R_A(z)\,dz \\ &= \left[\frac{1}{2i\pi}\int_{\partial\Sigma} z^{-\nu} R_A(z)\,dz\right]^{-1}, \end{aligned}$$

where $m\in\mathbb{N}$, $m\geq \nu+1$, and $\Sigma=\{z\in\mathbb{C}, \arg(z)\leq \arctan(K_0\alpha_0^{q-1}), \mathrm{Re}(z)\geq \alpha_0\}$ denotes a sector containing the spectrum of A. Here, we have to suppose that the operator A is positive, since Remark 1.1 does not hold in the case $\nu\neq 1$. In the case $\nu\notin\mathbb{N}$, we choose the principal value of the fractional root. Hence, on each finite-dimensional subspace P_kH, we can write A^ν in terms of the functional calculus $A^\nu P_k = \dfrac{1}{2i\pi}\displaystyle\int_{\gamma_k} z^\nu R_A(z)\,dz$. Moreover, from Proposition 2.8 we have $(A^\nu P_k)^* = (A^*)^\nu P_k^*$. The same holds with Π_α instead of P_k.

We now assume that the spectral inequality (15) of Theorem 3.3 holds for A and we obtain the following partial controllability result for $\partial_t + A^\nu$. It is the analogous of Theorem 4.9, supposing the spectral inequality (15) instead of (13). Note that in this case we have no additional restriction on the subordination number q (as opposed to the statement of Theorem 4.13). The proof follows that of [Mil06] or [LL09]. However, when A is not selfadjoint, the operator A^ν is not necessarily positive. As a consequence, we also have to treat the possibly non-positive low frequencies of A^ν. This problem does not arise when A is positive selfadjoint since A^ν is always positive.

For $\nu>0$, we define $N_\nu = \min\{k\in\mathbb{N}, \mathrm{Re}(z^\nu)>0, \forall z\in I_k\}$, such that $A^\nu(I-\Pi_{\alpha_k})$ (and also $A^{*\nu}(I-\Pi^*_{\alpha_k})$) is a positive operator if $k\geq N_\nu$.

Proposition 6.1. *Let $\alpha\geq\alpha_{N_\nu}$. The partial control problem*

$$\begin{cases} \partial_t u + A^\nu u = \Pi_\alpha B g \\ u_{|t=0} = u_0 \in \Pi_\alpha H, \end{cases} \tag{41}$$

is null-controllable in any positive time T by a control function satisfying

$$\|g\|_{L^2(0,T;H)} \leq C T^{-1/2} e^{CT\alpha^\nu_{N_\nu} + C\alpha^\theta}\|u_0\|_H.$$

Note that the additional cost $e^{CT\alpha^\nu_{N_\nu}}$ of the control function is needed to handle the exponentially increasing low frequencies.

Proof. The adjoint system of (41) is

$$\begin{cases} -\partial_t w + A^{*\nu} w = 0 \\ w_{|t=T} = w_T \in \Pi^*_\alpha H. \end{cases}$$

Thus $w(0)\in\Pi^*_\alpha H$ and $w(t) = e^{tA^{*\nu}}w(0)$. We first estimate

$$\begin{aligned} \|e^{-tA^{*\nu}}\Pi^*_{\alpha_k}\|_{\mathcal{L}(H)} &= \left\|\frac{1}{2i\pi}\int_{\Gamma_k} e^{-tz^\nu} R_{A^*}(z)\,dz\right\|_{\mathcal{L}(H)} \\ &\leq \frac{1}{2\pi}\left(\int_{\Gamma_{\alpha_{N_\nu}}} \left\|e^{-tz^\nu}R_{A^*}(z)\right\|_{\mathcal{L}(H)} dz + \int_{\bigcup_{\alpha_{N_\nu}<\alpha_k\leq\alpha}\gamma_k} \left\|e^{-tz^\nu}R_{A^*}(z)\right\|_{\mathcal{L}(H)} dz\right). \end{aligned}$$

The second term is bounded by $Ce^{C\alpha^\theta} \sup e^{-t\,\mathrm{Re}(z^\nu)} \leq Ce^{C\alpha^\theta}$ since $\mathrm{Re}(z^\nu) \geq 0$ on each γ_k, $\alpha_{N_\nu} < \alpha_k \leq \alpha$. Concerning the first term, we have $\mathrm{Re}(z^\nu) \geq -C\alpha_{N_\nu}^\nu$ for $z \in \Gamma_{N_\nu}$ and thus

$$\int_{\Gamma_{\alpha_{N_\nu}}} \left\| e^{-tz^\nu} R_{A^*}(z) \right\|_{\mathcal{L}(H)} dz \leq Ce^{CT\alpha_{N_\nu}^\nu + C\alpha^\theta}.$$

We thus obtain the estimate $\|e^{-tA^{*\nu}} \Pi_{\alpha_k}^* \|_{\mathcal{L}(H)} \leq Ce^{CT\alpha_{N_\nu}^\nu + C\alpha^\theta}$. We then have the observability inequality

$$T\|w(0)\|_H^2 \leq \int_0^T \left(Ce^{CT\alpha_{N_\nu}^\nu + C\alpha^\theta} \right)^2 \|w(t)\|_H^2 dt \leq Ce^{CT\alpha_{N_\nu}^\nu + C\alpha^\theta} \|B^* w\|_{L^2(0,T;Y)}^2$$

from (15) applied to $w(t) \in \Pi_\alpha^* H$. By duality, the proposition is proved. □

The same type of estimates as those performed in the proof of Proposition 4.12 gives the following decay property, for k sufficiently large ($k \geq N_\nu$), for some constant $0 < c < 1$,

$$\|S_{A^\nu}(t)(I - \Pi_{\alpha_k})\|_{\mathcal{L}(H)} \leq Ce^{C\alpha_k^\theta - ct\alpha_k^\nu}, \quad t \geq \frac{1}{\alpha_k}.$$

We finally have the analogous of Theorem 4.13, which proof follows the same (choosing $J_0 \geq N_\nu$).

Proposition 6.2. *Suppose that $\nu > \theta$. For every $\mathcal{T} > 0$, for every $u_0 \in H$, there exists a control function $g \in L^2(0, \mathcal{T}; Y)$ such that the solution u of the problem* (40) *satisfies $u(\mathcal{T}) = 0$.*

In the case where A is a second order selfadjoint elliptic operator, the spectral inequality (15) always holds for $\theta = 1/2$, and $\nu > 1/2$ is necessary and sufficient for the null-controllability (see [MZ06] and [Mil06]). Here, with the estimations we have proved, the case $1/2 < \nu \leq \theta$ is open.

Remark 6.3. Using arguments of measure theory given in [Wan08], Proposition 6.2 still holds if we replace the control operator B in (40) by $\mathbb{1}_E B$, for any subset of positive measure $E \subset (0, \mathcal{T})$. This means that the control function g given by Proposition 6.2 can be chosen so that $g(t) = 0$ for $t \notin E$.

Example 6.4. For $\Omega \subset \mathbb{R}^n$ and ω a non-empty subset of Ω, we take $H = Y = L^2(\Omega)$, $\mathcal{D}(A_0) = H^2 \cap H_0^1(\Omega)$, $A_0 = -\Delta$ and $A_1 = b \cdot \nabla + c$ with $b \in W^{1,\infty}(\Omega; \mathbb{C}^n)$, $c \in L^\infty(\Omega; \mathbb{C})$ chosen so that A is positive. Here, B^* a localized observation, i.e. $B^* = B = \mathbb{1}_\omega \in \mathcal{L}(L^2(\Omega))$. Under the conditions above, Proposition 2.1 is valid with $q = 1/2$ and the assumption of Theorem 2.5 is satisfied for $p = 2/n$. Moreover, the interpolation inequality (14) is well known in this case. In fact, it originates from Carleman inequalities [LR95], which form is invariant under changes in the operator by lower order terms. Hence, the spectral inequality (15) of Theorem 3.3 holds for $\theta = \max\{1/2, \frac{n-1}{2}\}$. For any time $\mathcal{T} > 0$ and $E \subset (0, \mathcal{T})$ satisfying $\mathrm{meas}(E) > 0$, Proposition 6.2 and Remark 6.3 give the null-controllability of the problem

$$\begin{cases} \partial_t u + (-\Delta + b \cdot \nabla + c)^\nu u = \mathbb{1}_{E \times \omega}\, g & \text{in } (0, \mathcal{T}) \times \Omega, \\ u_{|t=0} = u_0 & \text{in } \Omega, \\ u = 0 & \text{on } (0, \mathcal{T}) \times \partial\Omega, \end{cases}$$

for any $n \in \mathbb{N}$, $\nu > \max\{1/2, \frac{n-1}{2}\}$.

7 Application to level sets of sums of root functions

Following Jerison and Lebeau [JL99], we give here an application of the spectral inequality (15) of Theorem 3.3 to the measurement of the level sets of finite sums of root functions in term of the largest eigenvalue. The operator involved here is a non-selfadjoint perturbation of the Laplace operator.

Let Ω be a bounded open set in $\mathbb{R}^n$ (or a n-dimensional Riemannian compact manifold with or without boundary). We set $H = Y = L^2(\Omega)$. Let $-\Delta$ be the Laplace operator on Ω and $P(x, D)$ a differential operator of order $d \in \{0, 1\}$, such that $A_1 = P(x, D)$ is a relatively compact perturbation of $A_0 = -\Delta$, with dirichlet boundary conditions. We set $A = A_0 + A_1 = -\Delta + P(x, D)$ and take for B^* a localized observation, i.e. $B^* = B = \mathbb{1}_\omega \in \mathcal{L}(L^2(\Omega))$ for some nonempty open subset $\omega \subset \Omega$.

First note that under the conditions above, Proposition 2.1 is valid with $q = d/2 < 1$ and assumption (a) of Theorem 2.5 is satisfied for $p = 2/n$. Moreover, the interpolation inequality (14) holds in this case (see Example 6.4 above). Note that a function φ is a sum of root functions of the operator A associated with eigenvalues of real part lower than $\max\{\alpha_k; \alpha_k \leq \alpha\}$ if $\varphi \in \Pi_\alpha L^2(\Omega)$. From Theorem 3.3, we have the following spectral inequality: there exists positive constants C, D such that for every positive α, for all $\varphi \in \Pi_\alpha L^2(\Omega)$ (the dual space $\Pi_\alpha^* L^2(\Omega)$ does not play any role here),

$$\|\varphi\|_{L^2(\Omega)} \leq Ce^{D\alpha^\theta} \|\varphi\|_{L^2(\omega)}, \quad \theta = \max\left\{\frac{1}{2}; \frac{n+d}{2} - 1\right\}. \tag{42}$$

Assume now that Ω is real-analytic, and, moreover the differential operator $P(x, D)$ has real-analytic coefficients. Under these conditions, the operator $-\partial_t^2 + A$ is real-analytic hypoelliptic on $\mathbb{R} \times \Omega$ [Tre80, Theorem 5.4], and $\varphi \in \Pi_\alpha L^2(\Omega)$ implies that φ is real-analytic. We denote by $\mathcal{H}_{n-1}$ the $(n-1)$-dimensional Hausdorff measure on Ω. We can now state the analogous of the result of Jerison and Lebeau [JL99] for the class of non-selfadjoint elliptic operators we consider.

Theorem 7.1. *For every level set $K \in \mathbb{R}$, there exist positive constants C_1, C_2 such that for all $\alpha > 0$ and $\varphi \in \Pi_\alpha L^2(\Omega)$,*

$$\mathcal{H}_{n-1}(\{\varphi = K\}) \leq C_1 \alpha^\theta + C_2 \ , \ \theta = \max\left\{\frac{1}{2}; \frac{n+d}{2} - 1\right\}. \tag{43}$$

The proof follows exactly the same of [JL99] and uses arguments from [DF88] and [DF90].

This estimations (42) and (43) are known to be optimal for the Laplace operator, with $\theta = 1/2$, see [JL99]. As a consequence, one cannot hope to have better estimates in the cases where $\frac{n+d}{2} - 1 \leq \frac{1}{2}$, i.e. $n \leq 3$ if $A = -\Delta + c(x)$ and $n \leq 2$ if $A = -\Delta + b(x) \cdot \nabla + c(x)$.

However in the case $\frac{n+d}{2} - 1 > \frac{1}{2}$, the results (42), (43) do not seem to be optimal.

8 Appendix

8.1 Properties of the Gevrey function $e \in G^\sigma$, $1 < \sigma < 2$

Here, we prove the existence of the Gevrey function e that is needed in the proof of Proposition 4.5.

Lemma 8.1. *For every $\sigma > 1$, $B_0 > 0$ and $\kappa \leq 1 - \frac{1}{2\sigma}$, there exists a Gevrey function $e \in G^\sigma$ such that*

(i) $\operatorname{supp}(e) = [0,1]$ *and* $0 < e(t) \leq 1$ *for all* $t \in (0,1)$ *;*

(ii) $|\hat{e}(z)| \leq c_1 e^{-c_2|z|^{\frac{1}{\sigma}}}$ *if* $\operatorname{Im}(z) \leq 0$

(iii) $|\hat{e}(z)| \geq c_3 e^{-c_4|z|^{\frac{1}{\sigma}}}$ *in* $-i\mathcal{P}^\kappa_{B_0}$

where the constants c_i are positive and $-i\mathcal{P}^\kappa_{B_0} = -i\left\{z \in \mathbb{C}, \operatorname{Re}(z) \geq 0, |\operatorname{Im}(z)| < B_0 \operatorname{Re}(z)^\kappa\right\}$.

Proof. The function

$$e_0(t) = \exp\left(-t^{\frac{-1}{\sigma-1}} - (1-t)^{\frac{-1}{\sigma-1}}\right)$$

is in G^σ and satisfies the properties (i) and (ii).
We aim to prove a lower bound for $|\hat{e}_0(z)|$ as $|z| \to \infty$ in the parabola $-i\mathcal{P}^\kappa_{B_0}$. To have a precise estimation, we develop in detail the Laplace method, following [Erd56]. Let κ and B_0 two positive integers. For $\beta < B_0$, we estimate

$$\begin{aligned}\hat{e}_0(-i(s+i\beta s^\kappa)) &= \int_0^1 \exp\left(-t^{\frac{-1}{\sigma-1}} - (1-t)^{\frac{-1}{\sigma-1}}\right)\exp(-(s+i\beta s^\kappa)t)dt \\ &= \int_0^{s^{\frac{\sigma-1}{\sigma}}} \exp\left(s^{\frac{1}{\sigma}}(-u^{\frac{-1}{\sigma-1}} - u)\right)\exp\left(-(1-s^{-\frac{\sigma-1}{\sigma}}u)^{-\frac{1}{\sigma-1}} - i\beta u s^{\kappa+\frac{1}{\sigma}-1}\right)s^{-\frac{\sigma-1}{\sigma}}du\end{aligned}$$

after the rescaling change of variable $t = s^{-\frac{\sigma-1}{\sigma}}u$. We then set $\omega = s^{\frac{1}{\sigma}}$ the increasing parameter, $h(u) = -u^{\frac{-1}{\sigma-1}} - u$ and

$$g_\beta(\omega,u) = \exp\left(-(1-\omega^{-(\sigma-1)}u)^{-\frac{1}{\sigma-1}} - i\beta u\omega^{1+\sigma(\kappa-1)}\right),$$

such that we write

$$\hat{e}_0(-i(s+i\beta s^\kappa)) = I(\omega,\beta) = \int_0^{\omega^{\sigma-1}} \omega^{-(\sigma-1)}e^{\omega h(u)}g_\beta(\omega,u)du.$$

The function $h(u)$ is negative on $\mathbb{R}_{+*}$, concave and $h(u) < h(a) < 0$ for $u \neq a$ with $a = (\sigma-1)^{-\frac{\sigma-1}{\sigma}} > 0$.
Following the Laplace method, we then split the integral $I(\omega,\beta)$ in three parts. The most important contribution comes from the region where h reaches its maximum. We write

$$I(\omega,\beta) = \omega^{-(\sigma-1)}(I_1(\omega,\beta) + I_2(\omega,\beta) + I_3(\omega,\beta)), \tag{44}$$

with

$$\begin{gathered}I_1(\omega,\beta) = \int_0^{a-\eta} e^{\omega h(u)}g_\beta(\omega,u)du \quad , \quad I_2(\omega,\beta) = \int_{a-\eta}^{a+\eta} e^{\omega h(u)}g_\beta(\omega,u)du \\ I_3(\omega,\beta) = \int_{a+\eta}^{\omega^{\sigma-1}} e^{\omega h(u)}g_\beta(\omega,u)du\end{gathered} \tag{45}$$

for $\eta > 0$, sufficiently small, that will be fixed below.

We first treat the main contribution I_2: Morse Lemma [GS94] implies that for η sufficiently small, there exists two positive constants ν_1 and ν_2 and a diffeomorphism $\mathcal{H} : (a-\eta, a+\eta) \longrightarrow (-\nu_1, \nu_2)$ such that $h \circ \mathcal{H}^{-1}(x) = h(a) - \frac{x^2}{2}$ for $x \in (-\nu_1, \nu_2)$. Moreover, the jacobian $J(x) = |\det(d\mathcal{H}^{-1})|(x)$ satisfies $J(0)^2 = |h''(a)|^{-1}$.
With this change of variable, we obtain

$$I_2(\omega, \beta) = \int_{-\nu_1}^{\nu_2} g_\beta(\omega, \mathcal{H}^{-1}(x)) e^{\omega h(a) - \frac{\omega x^2}{2}} J(x) dx.$$

Setting $y = \sqrt{\frac{\omega}{2}} x$, we obtain

$$I_2(\omega, \beta) = \sqrt{\frac{2}{\omega}} e^{\omega h(a)} \int_{\mathbb{R}} \mathbb{1}_{(-\nu_1, \nu_2)} \left(\sqrt{\frac{2}{\omega}} y\right) J\left(\sqrt{\frac{2}{\omega}} y\right) g_\beta\left(\omega, \mathcal{H}^{-1}\left(\sqrt{\frac{2}{\omega}} y\right)\right) e^{-y^2} dy.$$

The modulus of the integrand is clearly bounded on $\mathbb{R}$ by Ce^{-y^2}, independent of ω and integrable. Let us study the asymptotics of the integrand as $\omega \to +\infty$.

$$\begin{aligned} g_\beta\left(\omega, \mathcal{H}^{-1}\left(\sqrt{\tfrac{2}{\omega}} y\right)\right) &= \exp\left(-\left(1 - \omega^{-(\sigma-1)} \mathcal{H}^{-1}\left(\sqrt{\tfrac{2}{\omega}} y\right)\right)^{-\frac{1}{\sigma-1}}\right) \\ &\quad \times \exp\left(-i\beta \mathcal{H}^{-1}\left(\sqrt{\tfrac{2}{\omega}} y\right) \omega^{1+\sigma(\kappa-1)}\right). \end{aligned}$$

The first exponential converges when $\omega \to +\infty$. In fact, setting

$$\psi(\omega, y) = \mathbb{1}_{(-\nu_1, \nu_2)} \left(\sqrt{\frac{2}{\omega}} y\right) J\left(\sqrt{\frac{2}{\omega}} y\right) \exp\left(-\left(1 - \omega^{-(\sigma-1)} \mathcal{H}^{-1}\left(\sqrt{\frac{2}{\omega}} y\right)\right)^{-\frac{1}{\sigma-1}}\right),$$

we have

$$\psi(\omega, y) \longrightarrow e^{-1} J(0) = e^{-1} |h''(a)|^{-\frac{1}{2}}, \quad \text{as } \omega \to +\infty.$$

For the second exponential in g_β (the oscillating part), as $\mathcal{H}^{-1}(0) = a$, we write $\mathcal{H}^{-1}(x) = a + xK(x)$ where $K \in \mathcal{C}^\infty(\mathbb{R})$ and we have

$$\begin{aligned} \exp\left(-i\beta \mathcal{H}^{-1}\left(\sqrt{\frac{2}{\omega}} y\right) \omega^{1+\sigma(\kappa-1)}\right) &= \exp\left(-i\beta \omega^{1+\sigma(\kappa-1)} \left(a + \sqrt{\frac{2}{\omega}} y K\left(\sqrt{\frac{2}{\omega}} y\right)\right)\right) \\ &= \exp\left(-i\beta \omega^{1+\sigma(\kappa-1)} a\right) \exp\left(-i\sqrt{2} \beta \omega^{\frac{1}{2}+\sigma(\kappa-1)} y K\left(\sqrt{\frac{2}{\omega}} y\right)\right). \end{aligned}$$

We may thus write

$$I_2(\omega, \beta) = \sqrt{\frac{2}{\omega}} e^{\omega h(a)} \exp\left(-i\beta \omega^{1+\sigma(\kappa-1)} a\right) \widetilde{I_2}(\omega, \beta),$$

where

$$\widetilde{I_2}(\omega, \beta) = \int_{\mathbb{R}} \psi(\omega, y) \exp\left(-i\sqrt{2} \beta \omega^{\frac{1}{2}+\sigma(\kappa-1)} y K\left(\sqrt{\frac{2}{\omega}} y\right)\right) e^{-y^2} dy.$$

The integrand in $\widetilde{I}_2(\omega,\beta)$ converges as $\omega \to +\infty$ under the condition $\frac{1}{2}+\sigma(\kappa-1)\leq 0$. By summated convergence, we have

$$\widetilde{I}_2(\omega,\beta) \longrightarrow L(\beta) = \frac{\sqrt{\pi}}{e}|h''(a)|^{-\frac{1}{2}}, \quad \text{if } \frac{1}{2}+\sigma(\kappa-1)<0,$$

and

$$\widetilde{I}_2(\omega,\beta) \longrightarrow L(\beta) = \frac{\sqrt{\pi}}{e}|h''(a)|^{-\frac{1}{2}} \exp\left(-\frac{\beta^2 K(0)^2}{2}\right), \quad \text{if } \frac{1}{2}+\sigma(\kappa-1)=0.$$

Moreover, similar arguments show that $\widetilde{I}_2(\omega,\beta)-L(\beta)$ is $\mathcal{C}^1([-B_0,B_0])$ with respect to the variable β. Then there exists k_0 such that for every ω sufficiently large

$$\left|\frac{\partial(\widetilde{I}_2-L)}{\partial\beta}(\omega,\beta)\right| \leq k_0, \quad \beta\in[-B_0,B_0].$$

Thus, $\widetilde{I}_2-L$ is uniformly Lipschitz with respect to the variable β and tends to zero for every fixed β as $\omega \longrightarrow +\infty$. Lemma 8.2 below implies that for all $\varepsilon>0$, there exists $\omega_0>0$ such that $|\widetilde{I}_2(\omega,\beta)-L(\beta)|<\varepsilon$ for $\omega>\omega_0$ and $\beta\in[-B_0,B_0]$.

We now address the terms I_1 and I_3 in (45). As $h(u)<h(a)<0$ for $u\neq a$, we can write $h(a-\eta)=h(a)-C_-$ and $h(a+\eta)=h(a)-C_+$ with $C_+,C_->0$, depending only on η. Because h increases on $(0,a]$, decreases on $[a,+\infty)$, and $|g_\beta|\leq 1$, we have

$$|I_1(\omega,\beta)| \leq a e^{\omega h(a)} e^{-\omega C_-}, \quad |I_3(\omega,\beta)| \leq \omega^{\sigma-1} e^{\omega h(a)} e^{-\omega C_+}.$$

Finally, for every fixed $\kappa \leq 1-\frac{1}{2\sigma}$, we can write

$$I(\omega,\beta) = \sqrt{2}\omega^{\frac{1}{2}-\sigma} e^{\omega h(a)} \left(\exp\left(-i\beta\omega^{1+\sigma(\kappa-1)}a\right)\widetilde{I}_2(\omega,\beta)+D(\omega,\beta)\right)$$

with $|D(\omega,\beta)|\leq C\omega^{\sigma-\frac{1}{2}}e^{-C_\pm\omega}$ and $\widetilde{I}_2$ converging to a non-zero limit uniformly in β.

As a consequence, there exist $C_1,C_2>0$ and $\omega_0>0$ such that

$$|I(\omega,\beta)| \geq C_1 e^{-C_2\omega}, \quad \omega>\omega_0, \quad \beta\in[-B_0,B_0].$$

Switching back to the variable $s=\omega^\sigma$, we then have for some $s_0>0$

$$|\hat{e}_0(-i(s+i\beta s^\kappa))| \geq C_1 e^{-C_2 s^{\frac{1}{\sigma}}}, \quad s>s_0, \quad \beta\in[-B_0,B_0].$$

To conclude the lemma, we now set $e(t)=e^{-s_0 t}e_0(t)$ that is also in G^σ and satisfies property (i). For $z\in\mathbb{C}$, we have $\hat{e}(z)=\hat{e}_0(z-is_0)$ and (ii) holds. Property (iii) follows from what precedes. □

Lemma 8.2. *Let K be a compact set and $I(\omega,x)$ a function defined on $\mathbb{R}_+\times K$, that is uniformly Lipschitz on $\mathbb{R}_+\times K$ with respect to the variable $x\in K$, i.e.,*

$$\exists k_0>0, \quad |I(\omega,x_2)-I(\omega,x_1)|\leq k_0|x_2-x_1|, \quad \omega\in\mathbb{R}_+, \quad x_1,x_2\in K.$$

If for every $x\in K$, $\lim_{\omega\to+\infty} I(\omega,x)=0$, then $\lim_{\omega\to+\infty}\max_{x\in K} I(\omega,x)=0$.

8.2 A Paley-Wiener-type theorem

Here we prove a Paley-Wiener-type theorem adapted to the situation of Proposition 4.5

Proposition 8.3. *Let Y be a separable Hilbert space and $f \in \mathcal{H}(\mathbb{C}; Y)$ satisfying for positive constants C_N, C_ε:*

$$\|f(z)\|_Y \le C_\varepsilon e^{\varepsilon|z|} e^{T\,\mathrm{Im}(z)}, \forall \varepsilon > 0, z \in \mathbb{C}; \tag{46}$$

$$\|f(\tau)\|_Y \le C_N (1+|\tau|)^{-N}, \forall N \in \mathbb{N}, \tau \in \mathbb{R}. \tag{47}$$

Then, there exists $u \in \mathcal{C}_0^\infty(0,T;Y)$ such that $\hat{u}(z) = f(z)$, $z \in \mathbb{C}$.

Proof. Let $(e_j)_{j\in\mathbb{N}}$ be a Hilbert basis of Y. For every j, $z \mapsto (f(z), e_j)_Y \in \mathcal{H}(\mathbb{C};\mathbb{C})$. Equation (46) gives $|(f(z), e_j)_Y| \le C_\varepsilon e^{\varepsilon|z|} e^{T\,\mathrm{Im}(z)}, \forall \varepsilon > 0, z \in \mathbb{C}$, and the Paley-Wiener theorem [Hör83, Theorem 15.1.5] then implies that there exists an analytic functional u_j carried by $(0,T)$ (see [Hör90, Chapter 9] for a precise definition) such that $\hat{u}_j(z) = (f(z), e_j)_Y$, $z \in \mathbb{C}$. Moreover, (47) yields $|\hat{u}_j(\tau)| \le C_N(1+|\tau|)^{-N}, \forall N \in \mathbb{N}, \tau \in \mathbb{R}$ and thus, $u_j \in \mathcal{C}_0^\infty(0,T;\mathbb{C})$.

We now set $u = \sum_{j\in\mathbb{N}} u_j e_j$ and observe that $u \in L^2(\mathbb{R};Y)$

$$\|u\|^2_{L^2(\mathbb{R};Y)} = \sum_{j\in\mathbb{N}} \|u_j\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\sum_{j\in\mathbb{N}} \|\hat{u}_j\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\sum_{j\in\mathbb{N}} \|(f(\cdot), e_j)_Y\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{R};Y)}.$$

We note that $\mathrm{supp}(u) \subset (0,T)$ since $\mathrm{supp}(u_j) \subset (0,T)$ for all $j \in \mathbb{N}$. Hence the Fourier-Laplace transform of u is an entire function, satisfying, for $z \in \mathbb{C}$

$$(\hat{u}(z), e_k)_Y = \left(\int_0^T \sum_{j\in\mathbb{N}} u_j(t) e^{-itz} e_j \, dt, e_k\right)_Y = \int_0^T \sum_{j\in\mathbb{N}} u_j(t) e^{-itz} \left(e_j, e_k\right)_Y dt = \hat{u}_k(z).$$

Thus,

$$\hat{u}(z) = \sum_{j\in\mathbb{N}} \hat{u}_j(z) e_j = \sum_{j\in\mathbb{N}} (f(z), e_j)_Y e_j = f(z)$$

and f is the Fourier-Laplace of u. Finally, (47) yields $\|\hat{u}(\tau)\|_Y \le C_N(1+|\tau|)^{-N}, \forall N \in \mathbb{N}, \tau \in \mathbb{R}$ and thus $u \in \mathcal{C}^\infty$. □

References

[ABD06] F. Ammar Khodja, A. Benabdallah, and C. Dupaix. Null controllability of some reaction-diffusion systems with one control force. *J. Math. Anal. Appl.*, 320:928–943, 2006.

[ABDGB07] F. Ammar Khodja, A. Benabdallah, C. Dupaix, and M. González-Burgos. Controllability for a class of reaction-diffusion systems: the generalized kalman's condition. *C. R. Acad. Sci. Paris, Ser I.*, 345:543–548, 2007.

[AE08] G. Alessandrini and L. Escauriaza. Null-controllability of one-dimensional parabolic equations. *ESAIM Control Optim. Calc. Var.*, 14:284–293, 2008.

[Agr76] M.S. Agranovich. Summability of series in root vectors of non-selfadjoint elliptic operators. *Functional analysis and its applications*, 10:165–174, 1976.

[Agr94] M.S. Agranovich. On series with respect to root vectors of operators associated with forms having symmetric principal part. *Functional analysis and its applications*, 28:151–167, 1994.

[BHL08] F. Boyer, F. Hubert, and J. Le Rousseau. Discrete carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations. *Preprint*, `http://hal.archives-ouvertes.fr/hal-00366496/fr/`, 2008.

[DF88] H. Donnelly and C. Fefferman. Nodal sets of eigenfunctions on riemannian manifolds. *Invent. Math.*, 93:161–183, 1988.

[DF90] H. Donnelly and C. Fefferman. Nodal sets of eigenfunctions: Riemannian manifolds with boundary. In *Analysis, et cetera (Moser volume)*, volume 93, pages 251–262. Academic Press, Boston, MA, 1990.

[Dzh94] L.S. Dzhanlatyan. Basis properties of the system of root vectors for weak perturbations of a normal operator. *Functional analysis and its applications*, 28:204–207, 1994.

[Erd56] A. Erdélyi. *Asymptotic expansions.* Dover publications, Inc, New York, 1956.

[FI96] A. Fursikov and O.Yu. Imanuvilov. *Controllability of evolution equations.* Lecture Notes Series. Seoul National University, Research Institute of Mathematics, Global Analysis Research Center, 1996.

[GBPG06] M. González-Burgos and R. Pérez-García. Controllability results for some nonlinear coupled parabolic systems by one control force. *Asymptotic Analysis*, 46:123–162, 2006.

[GBT09] M. González-Burgos and L. de Teresa. Controllability results for cascade systems of m coupled parabolic pdes by one control force. *To appear in Portugaliae Mathematica*, 2009.

[GK69] I.C. Gohberg and M.G. Krein. *Introduction to the theory of linear non-selfadjoint operators*, volume 18 of *Transl. Math. Monogr.* Amer. Math. Soc., Providence, R.I., 1969.

[GS94] A. Grigis and J. Sjöstrand. *Microlocal Analysis for Differential Operators.* Cambridge University Press, Cambridge, 1994.

[Haa06] M. Haase. *The Functional Calculus for Sectorial Operators.* Birkhäuser Verlag, Basel, 2006.

[Hör63] L. Hörmander. *Linear Partial Differential Operators.* Springer-Verlag, Berlin, 1963.

[Hör83] L. Hörmander. *The Analysis of Linear Partial Differential Operators*, volume II. Springer-Verlag, Berlin, 1983.

[Hör90] L. Hörmander. *The Analysis of Linear Partial Differential Operators*, volume I. Springer-Verlag, Berlin, second edition, 1990.

[JL99] D. Jerison and G. Lebeau. Nodal sets of sums of eigenfunctions. In *Harmonic analysis and partial differential equations (Chicago, IL, 1996)*, Chicago Lectures in Math., pages 223–239. Univ. Chicago Press, Chicago, IL, 1999.

[Kat67] V.E. Katsnel'son. Conditions under which systems of eigenvectors of some classes of operators form a basis. *Functional analysis and its applications*, 1:122–132, 1967.

[Kat80] T. Kato. *Perturbation Theory for Linear Operators.* Springer-Verlag, Berlin, 1980.

[KT09] O. Kavian and L. de Teresa. Unique continuation principle for systems of parabolic equations. *ESAIM Control Optim. Calc. Var., doi:10.1051/cocv/2008077*, 2009.

[LL09] J. Le Rousseau and G. Lebeau. On carleman estimates for elliptic and parabolic operators. applications to unique continuation and control of parabolic equations. *Preprint,* `http://hal.archives-ouvertes.fr/hal-00351736/fr/`, 2009.

[LR95] G. Lebeau and L. Robbiano. Contrôle exact de l'équation de la chaleur. *Comm. Partial Differential Equations*, 20:335–356, 1995.

[LZ98] G. Lebeau and E. Zuazua. Null-controllability of a system of linear thermoelasticity. *Arch. Rational Mech. Anal.*, 141:297–329, 1998.

[Mar88] A.S. Markus. *Introduction to the Spectral Theory of Polynomial Operator Pencils*, volume 71 of *Transl. Math. Monogr.* Amer. Math. Soc., Providence, R.I., 1988.

[Mil06] L. Miller. On the controllability of anomalous diffusions generated by the fractional laplacian. *Mathematics of Control, Signals, and Systems*, 3:260–271, 2006.

[MZ06] S. Micu and E. Zuazua. On the controllability of a fractional order parabolic equation. *SIAM J. Cont. Optim.*, 44:1950–1979, 2006.

[Paz83] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, New York, 1983.

[Rus73] D.L. Russell. A unified boundary controllability theory for hyperbolic and parabolic partial differential equations. *Studies in Appl. Math.*, 52:189–221, 1973.

[Ter00] L. de Teresa. Insensitizing controls for a semilinear heat equation. *Comm. Partial Differential Equations*, 25:39–72, 2000.

[Tre80] F. Treves. *Introduction to Pseudodifferential Operators and Fourier Integral Operators*, volume I. Plenum Press, New York, 1980.

[Wan08] G. Wang. L^∞-null controllability for the heat equation and its consequences for the time optimal control problem. *SIAM J. Cont. Optim.*, 47:1701–1720, 2008.