CANONICAL BASES AND AFFINE HECKE ALGEBRAS OF TYPE B

M. Varagnolo, E. Vasserot

Abstract. We prove a series of conjectures of Enomoto and Kashiwara on canonical bases and branching rules of affine Hecke algebras of type B. The main ingredient of the proof is a new graded Ext-algebra associated with quiver with involutions that we compute explicitely.

Introduction

A new family of graded algebras, called KLR algebras, has been recently introduced in [KL1], [R]. These algebras yield a categorification of **f**, the negative part of the quantized enveloping algebra of any type. In particular,one can obtain a new interpretation of the canonical bases, see [VV]. In type A or $A^{(1)}$ the KLR algebras are Morita equivalent to the affine Hecke algebras and their cyclotomic quotients. Hence they give a new way to understand the categorification of the simple highest weight modules and the categorification of the negative part of the quantized enveloping algebra, via some Hecke algebras of type A or $A^{(1)}$. See [BK] for instance. One of the advantages of KLR algebras is that they are graded, while the affine Hecke algebras are not. This explain why KLR algebras are better adapted than affine Hecke algebras to describe canonical bases. Indeed one could view KLR algebras as an intermediate object between the representation theory of affine Hecke algebras and its Kazhdan-Lusztig geometric counterpart in term of perverse sheaves. This is central in [VV], where KLR algebras are proved to be isomorphic to the Ext-algebras of some complex of constructible sheaves.

In the other hand, the branching rules for affine Hecke algebras of type B have been investigated quite recently, see [E], [EK1,2,3], [M]. Lusztig's description of the canonical basis of **f** in type $A^{(1)}$ in [L1] implies that this basis can be naturally identified with the set of isomorphism classes of the simple objects of a category of modules of the affine Hecke algebras of type A. This identification was mentioned in [G], and it was used in [A]. More precisely, there is a linear isomorphism between **f** and the Grothendieck group of finite dimensional modules of the affine Hecke algebras of type A, and it is proved in [A] that the induction/restriction functors for affine Hecke algebras are given by the action of the Chevalley generators and their transposed operators with respect to some symmetric bilinear form on **f**. In [E], [EK1,2,3] a similar behaviour is conjectured and studied for affine Hecke algebras of type B. Here **f** is replaced by an explicit module ${}^\theta\mathbf{V}(\lambda)$ over an explicit algebra ${}^\theta\boldsymbol{\mathcal{B}}$. First, it is conjectured that ${}^\theta\mathbf{V}(\lambda)$ admits a canonical basis. Next, it is conjectured that this basis is naturally identified with the set of isomorphism classes of the simple objects of a category of modules of the affine Hecke algebras of

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type B. Further, in this identification the branching rules the affine Hecke algebras of type B are given by the ${}^\theta\boldsymbol{\mathcal{B}}$-action on ${}^\theta\mathbf{V}(\lambda)$. The first conjecture has been proved in [E] under the restrictive assumption that $\lambda = 0$. Here we prove the whole set of conjectures. Indeed, our construction is slightly more general, see the appendix.

Roughtly speaking our argument is as follows. In [E] a geometric description of the canonical basis of ${}^\theta\mathbf{V}(0)$ was given. This description is similar to Lusztig's description of the canonical basis of $\mathbf{f}$ via perverse sheaves on the moduli stack of representations of some quiver. It is given in terms of perverse sheaves on the moduli stack of representations of a quiver with involution. First we give a analogue of this for ${}^\theta\mathbf{V}(\lambda)$ for any λ. This yields the existence of a canonical basis ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$ for ${}^\theta\mathbf{V}(\lambda)$ for arbitrary λ. Then we compute explicitely the Ext-algebras between complexes of constructible sheaves naturally attached to quivers with involutions. These complexes enter in a natural way in the definition of ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$. This computation yields a new family of graded algebras ${}^\theta\mathbf{R}_m$ where m is a nonnegative integer. We prove that the algebras ${}^\theta\mathbf{R}_m$ are Morita equivalent to the affine Hecke algebras of type B. Finally we describe ${}^\theta\mathbf{V}(\lambda)$ and the basis ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$ in terms of the Grothendieck group of ${}^\theta\mathbf{R}_m$.

The plan of the paper is the following. Section 1 contains some basic notation for Lusztig's theory of perverse sheaves on the moduli stack of representations of quivers. Section 2 yields similar notation for the case of quivers with involutions. Our setting is more general than in [E], where only the case $\boldsymbol{\Lambda} = 0$ is considered. In Section 3 we introduce the convolution algebra associated with a quiver with involution. The main result of Section 4 is Theorem 4.19 where the polynomial representation of the Ext-algebra $\mathbf{Z}^\delta_{\boldsymbol{\Lambda},\mathbf{V}}$ associated with a quiver with involution is computed. This is a faitful representation. In Section 5 we give the main properties of the graded algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. In Section 6 we introduce the affine Hecke algebra of type B and we prove that it is Morita equivalent to ${}^\theta\mathbf{R}_m$, a specialization of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. Section 7 is a reminder on KLR algebras and on the main result of [VV]. In Section 8 we categorify the module ${}^\theta\mathbf{V}(\lambda)$ from [EK1] using the graded algebra ${}^\theta\mathbf{R}_m$. In Section 9 we prove the isomorphism ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = \mathbf{Z}^\delta_{\boldsymbol{\Lambda},\mathbf{V}}$. This is essential to compare the construction from Section 8 with that in Section 10. In Section 10 we give a categorification of ${}^\theta\mathbf{V}(\lambda)$ "à la Lusztig" in terms of perverse sheaves on the moduli stack of representations of quivers with involution. This is essentially the same construction as in [E]. However, since we need a more general setting than in loc. cit. we have briefly reproduced the main steps of the construction. One of our initial motivations was to give a completely algebraic proof of the conjectures, without any perverse sheaves at all. We still do not know how to do this. The main result of the paper is Theorem 10.19.

The same technic yields similar results for affine Hecke algebras of type D, see [SVV]. Note that the idea to use canonical bases technics to study affine Hecke algebras in non A type is not new, see [L3], [L4]. At the moment we do not know the precise relation between loc. cit. and our approach.

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Contents

0. Notation

0.1. Combinatorics. Given a positive integer m and a tuple $\mathbf{m} = (m_1, m_2, \dots m_r)$ of positive integers we write $\mathfrak{S}_m$ for the symmetric group and $\mathfrak{S}_\mathbf{m}$ for the group $\prod_{l=1}^r \mathfrak{S}_{m_l}$. Set

$$|\mathbf{m}| = \sum_{l=1}^r m_l, \quad \ell_\mathbf{m} = \sum_{l=1}^r \ell_{m_l}, \quad \ell_m = m(m-1)/2.$$

We use the following notation for v-numbers

$$\langle m\rangle = \sum_{l=1}^m v^{m+1-2l}, \quad \langle m\rangle! = \prod_{l=1}^m \langle l\rangle, \quad \left\langle \begin{matrix} m+n \\ n \end{matrix} \right\rangle = \frac{\langle m+n\rangle!}{\langle m\rangle!\langle n\rangle!}, \quad \langle \mathbf{m}\rangle! = \prod_{l=1}^r \langle m_l\rangle!.$$

Given two tuples $\mathbf{m} = (m_1, m_2, \dots m_r)$, $\mathbf{m}' = (m'_1, m'_2, \dots m'_{r'})$ we define the tuple

$$\mathbf{m}\mathbf{m}' = (m_1, m_2, \dots m_r, m'_1, m'_2, \dots m'_{r'}).$$

0.2. Graded modules over graded algebras. Let $\mathbf{k}$ be a field of caracteristic 0. By a graded $\mathbf{k}$-algebra $\mathbf{R} = \bigoplus_i \mathbf{R}_i$ we'll always mean a $\mathbb{Z}$-graded associative $\mathbf{k}$-algebra. Let $\mathbf{R}$**-mod** be the category of finitely generated graded $\mathbf{R}$-modules, $\mathbf{R}$**-mod**^{f} be the full subcategory of finite-dimensional graded modules and $\mathbf{R}$**-proj** be the full subcategory of projective objects. Unless specified otherwise all modules are left modules. We'll abbreviate

$$K(\mathbf{R}) = [\mathbf{R}\text{-}\mathbf{proj}], \quad G(\mathbf{R}) = [\mathbf{R}\text{-}\mathbf{mod}^f].$$

Here $[\mathcal{C}]$ denotes the Grothendieck group of an exact category $\mathcal{C}$. Assume that the $\mathbf{k}$-vector spaces $\mathbf{R}_i$ are finite dimensional for each i. Then $K(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the indecomposable objects in **R-proj**, and $G(\mathbf{R})$ is a free Abelian group with a basis formed by the isomorphism classes of the simple objects in $\mathbf{R}\text{-}\mathbf{mod}^f$. Given an object M of **R-proj** or $\mathbf{R}\text{-}\mathbf{mod}^f$ let $[M]$ denote its class in $K(\mathbf{R})$, $G(\mathbf{R})$ respectively. When there is no risk of confusion we abbreviate $M = [M]$. We'll write $[M : N]$ for the composition multiplicity of the $\mathbf{R}$-module N in the $\mathbf{R}$-module N. Consider the ring $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$. If the grading of $\mathbf{R}$ is bounded below then the $\mathcal{A}$-modules $K(\mathbf{R})$, $G(\mathbf{R})$ are free. Here $\mathcal{A}$ acts on $G(\mathbf{R})$, $K(\mathbf{R})$ as follows

$$vM = M[1], \quad v^{-1}M = M[-1].$$

For any M, N in **R-mod** let

$$\mathrm{hom}_{\mathbf{R}}(M,N) = \bigoplus_d \mathrm{Hom}_{\mathbf{R}}(M, N[d])$$

be the $\mathbb{Z}$-graded $\mathbf{k}$-vector space of all $\mathbf{R}$-module homomorphisms. If $\mathbf{R} = \mathbf{k}$ we'll omit the subscript $\mathbf{R}$ in hom's and in tensor products. As much as possible we'll use the following convention : graded objects are denoted by minuscules and non-graded ones by majuscules. In particular **R-Mod** will denote the category of finitely generated (non-graded) $\mathbf{R}$-modules.

0.3. Constructible sheaves. Given an action of a complex linear algebraic group G on a quasiprojective algebraic variety X over $\mathbb{C}$ we write $\boldsymbol{\mathcal{D}}_G(X)$ for the bounded derived category of complexes of G-equivariant sheaves of $\mathbf{k}$-vector spaces on X. Objects of $\boldsymbol{\mathcal{D}}_G(X)$ are referred to as complexes. If $G = \{e\}$, the trivial group, we abbreviate $\boldsymbol{\mathcal{D}}(X) = \boldsymbol{\mathcal{D}}_G(X)$. For each complexes $\mathcal{L}$, $\mathcal{L}'$ we'll abbreviate

$$\mathrm{Ext}^*_G(\mathcal{L}, \mathcal{L}') = \mathrm{Ext}^*_{\boldsymbol{\mathcal{D}}_G(X)}(\mathcal{L}, \mathcal{L}'), \quad \mathrm{Ext}^*(\mathcal{L}, \mathcal{L}') = \mathrm{Ext}^*_{\boldsymbol{\mathcal{D}}(X)}(\mathcal{L}, \mathcal{L}')$$

if no confusion is possible. The constant sheaf $\mathbf{k}$ on X will be denoted $\mathbf{k}$. For any object $\mathcal{L}$ of $\boldsymbol{\mathcal{D}}_G(X)$ let $H^*_G(X, \mathcal{L})$ be the space of G-equivariant cohomology with coefficients in $\mathcal{L}$. Let $\mathcal{D} \in \boldsymbol{\mathcal{D}}_G(X)$ be the G-equivariant dualizing complex, see [BL, def. 3.5.1]. For each $\mathcal{L}$ we'll abbreviate

$$\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{D})$$

(the internal Hom). Recall that

$$(\mathcal{L}^\vee)^\vee = \mathcal{L}, \quad \mathrm{Ext}^*_G(\mathcal{L}, \mathcal{D}) = H^*_G(X, \mathcal{L}^\vee), \quad \mathrm{Ext}^*_G(\mathbf{k}, \mathcal{L}) = H^*_G(X, \mathcal{L}).$$

We define the space of G-equivariant homology by

$$H^G_*(X, \mathbf{k}) = H^*_G(X, \mathcal{D}).$$

Note that $\mathcal{D} = \mathbf{k}[2d]$ if X is a smooth G-variety of pure dimension d. Consider the following graded $\mathbf{k}$-algebra

$$\mathbf{S}_G = H^*_G(\bullet, \mathbf{k}).$$

The graded $\mathbf{k}$-vector space $H^G_*(X, \mathbf{k})$ has a natural structure of a graded $\mathbf{S}_G$-module. We have

$$H^G_*(\bullet, \mathbf{k}) = \mathbf{S}_G$$

as graded $\mathbf{S}_G$-module. There is a canonical graded $\mathbf{k}$-algebra isomorphism

$$\mathbf{S}_G \simeq \mathbf{k}[\mathfrak{g}]^G.$$

Here the symbol $\mathfrak{g}$ denotes the Lie algebra of G and a G-invariant homogeneous polynomial over $\mathfrak{g}$ of degree d is given the degree $2d$ in $\mathbf{S}_G$.

Fix a morphism of quasi-projective algebraic G-varieties $f : X \to Y$. If f is a proper map there is a direct image homomorphism

$$f_* : H^G_*(X, \mathbf{k}) \to H^G_*(Y, \mathbf{k}).$$

If f is a smooth map of relative dimension d there is an inverse image homomorphism

$$f^* : H^G_i(Y, \mathbf{k}) \to H^G_{i-2d}(X, \mathbf{k}), \quad \forall i.$$

If X has pure dimension d there is a natural homomorphism

$$H^i_G(X, \mathbf{k}) \to H^G_{i-2d}(X, \mathbf{k}).$$

It is invertible if X is smooth. The image of the unit is called the fundamental class of X in $H^G_*(X, \mathbf{k})$. We denote it by $[X]$. If $f : X \to Y$ is the embedding of a G-stable closed subset and $X' \subset X$ is the union of the irreducible components of maximal dimension then the image of $[X']$ by the map f_* is the fundamental class of X in $H^G_*(Y, \mathbf{k})$. It is again denoted by $[X]$.

1. Reminder on quivers and extensions

1.1. Representations of quivers. We assume given a nonempty quiver Γ such that no arrow may join a vertex to itself. Recall that Γ is a tuple $(I, H, h \mapsto h', h \mapsto h'')$ where I is the set of vertices, H is the set of arrows and for each $h \in H$ the vertices $h', h'' \in I$ are the origin and the goal of h respectively. Note that the set I may be infinite. For each $i, j \in I$ we write

$$H_{i,j} = \{h \in H; h' = i, h'' = j\}.$$

We'll abbreviate $i \to j$ for $H_{i,j} \neq \emptyset$, $i \not\to j$ for $H_{i,j} = \emptyset$, and $h : i \to j$ for $h \in H_{i,j}$. Let $h_{i,j}$ be the number of elements in $H_{i,j}$ and set

$$i \cdot j = -h_{i,j} - h_{j,i}, \quad i \cdot i = 2, \quad i \neq j.$$

Let $\boldsymbol{\mathcal{V}}$ be the category of finite-dimensional I-graded $\mathbb{C}$-vector spaces $\mathbf{V} = \bigoplus_{i \in I} \mathbf{V}_i$ with morphisms being linear maps respecting the grading. For each $\nu = \sum_i \nu_i i$ in $\mathbb{N}I$ let $\boldsymbol{\mathcal{V}}_\nu$ be the full subcategory of $\boldsymbol{\mathcal{V}}$ whose objects are those $\mathbf{V}$ such that $\dim(\mathbf{V}_i) = \nu_i$ for all i. We call ν the dimension vector of $\mathbf{V}$. Given an object $\mathbf{V}$ of $\boldsymbol{\mathcal{V}}$ let

$$E_{\mathbf{V}} = \bigoplus_{h \in H} \mathrm{Hom}(\mathbf{V}_{h'}, \mathbf{V}_{h''}).$$

The algebraic group $G_{\mathbf{V}} = \prod_i GL(\mathbf{V}_i)$ acts on $E_{\mathbf{V}}$ by $(g,x) \mapsto y$ where $y_h = g_{h''} x_h g_{h'}^{-1}$, $g = (g_i)$, $x = (x_h)$, and $y = (y_h)$.

Fix a nonzero element ν of $\mathbb{N}I$. Let Y^ν be the set of all pairs $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ where $\mathbf{i} = (i_1, i_2, \dots i_k)$ is a sequence of elements of I and $\mathbf{a} = (a_1, a_2, \dots a_k)$ is a sequence of positive integers such that $\sum_l a_l\, i_l = \nu$. Note that the assignment

$$\mathbf{y} \mapsto (a_1 i_1, a_2 i_2, \dots a_k i_k) \tag{1.1}$$

identifies Y^ν with a set of sequences

$$\nu^1, \nu^2, \dots, \nu^k \in \mathbb{N}I \quad \text{with } \nu = \sum_{l=1}^k \nu^l. \tag{1.2}$$

For each pair $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ as above we'll call $\mathbf{a}$ the multiplicity of $\mathbf{y}$. Let $I^\nu \subset Y^\nu$ be the set of all pairs $\mathbf{y}$ with multiplicity $(1, 1, \dots, 1)$. We'll abbreviate $\mathbf{i}$ for a pair $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ which lies in I^ν. Given a positive integer m we have $\bigsqcup_\nu I^\nu = I^m$, where ν runs over the set of elements ν of $\mathbb{N}I$ with $|\nu| = m$. Here, we write $\nu = \sum_{i\in I} \nu_i i$ and $|\nu| = \sum_i \nu_i$.

1.2. Flags. Let $\nu \in \mathbb{N}I$, $\nu \neq 0$, and assume that $\mathbf{V}$ lies in $\boldsymbol{\mathcal{V}}_\nu$. For each sequence $\mathbf{y} = (\nu^1, \nu^2, \dots \nu^k)$ as in (1.1), (1.2), a flag of type $\mathbf{y}$ in $\mathbf{V}$ is a sequence

$$\phi = (\mathbf{V} = \mathbf{V}^0 \supset \mathbf{V}^1 \supset \cdots \supset \mathbf{V}^k = 0)$$

of I-graded subspace of $\mathbf{V}$ such that for any l the I-graded subspace $\mathbf{V}^{l-1}/\mathbf{V}^l$ belongs to $\boldsymbol{\mathcal{V}}_{\nu^l}$. Let $F_{\mathbf{V},\mathbf{y}}$ be the variety of all flags of type $\mathbf{y}$ in $\mathbf{V}$. The group $G_{\mathbf{V}}$ acts transitively on $F_{\mathbf{V},\mathbf{y}}$ in the obvious way, yielding a smooth projective $G_{\mathbf{V}}$-variety structure on $F_{\mathbf{V},\mathbf{y}}$.

If $x \in E_{\mathbf{V}}$ we say that the flag ϕ is x-stable if $x_h(\mathbf{V}^l_{h'}) \subset \mathbf{V}^l_{h''}$ for all h, l. Let $\widetilde{F}_{\mathbf{V},\mathbf{y}}$ be the variety of all pairs (x, ϕ) such that ϕ is x-stable. Set $d_{\mathbf{y}} = \dim(\widetilde{F}_{\mathbf{V},\mathbf{y}})$. The group $G_{\mathbf{V}}$ acts on $\widetilde{F}_{\mathbf{V},\mathbf{y}}$ by $g : (x, \phi) \mapsto (gx, g\phi)$. The first projection gives a $G_{\mathbf{V}}$-equivariant proper morphism

$$\pi_{\mathbf{y}} : \widetilde{F}_{\mathbf{V},\mathbf{y}} \to E_{\mathbf{V}}.$$

1.3. Ext-algebras. Let $\nu \in \mathbb{N}I$, $\nu \neq 0$, and assume that $\mathbf{V} \in \boldsymbol{\mathcal{V}}_\nu$. We abbreviate $\mathbf{S}_{\mathbf{V}} = \mathbf{S}_{G_{\mathbf{V}}}$. For each sequence $\mathbf{y} \in Y^\nu$ we have the following semisimple complex in $\boldsymbol{\mathcal{D}}_{G_{\mathbf{V}}}(E_{\mathbf{V}})$

$$\mathcal{L}_{\mathbf{y}} = (\pi_{\mathbf{y}})_!(\mathbf{k}), \quad \mathcal{L}^\vee_{\mathbf{y}} = \mathcal{L}_{\mathbf{y}}[2d_{\mathbf{y}}], \quad \mathcal{L}^\delta_{\mathbf{y}} = \mathcal{L}_{\mathbf{y}}[d_{\mathbf{y}}].$$

For $\mathbf{y}, \mathbf{y}'$ in Y^ν we consider the graded $\mathbf{S}_{\mathbf{V}}$-module

$$\mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}'} = \mathrm{Ext}^*_{G_{\mathbf{V}}}(\mathcal{L}^\vee_{\mathbf{y}}, \mathcal{L}^\vee_{\mathbf{y}'}).$$

For $\mathbf{y}, \mathbf{y}', \mathbf{y}''$ in Y^ν the Yoneda composition is a homogeneous $\mathbf{S}_{\mathbf{V}}$-bilinear map of degree zero $\star : \mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}'} \times \mathbf{Z}_{\mathbf{V},\mathbf{y}',\mathbf{y}''} \to \mathbf{Z}_{\mathbf{V},\mathbf{y},\mathbf{y}''}$. The map $\star$ equips the graded $\mathbf{k}$-vector space

$$\mathbf{Z}_{\mathbf{V}} = \bigoplus_{\mathbf{i},\mathbf{i}' \in I^\nu} \mathbf{Z}_{\mathbf{V},\mathbf{i},\mathbf{i}'}$$

with the structure of an associative graded $\mathbf{S}_\mathbf{V}$-algebra with 1. If there is no ambiguity we'll omit the symbol $\star$. We set

$$\mathbf{F}_{\mathbf{V},\mathbf{y}} = \mathrm{Ext}^*_{G_\mathbf{V}}(\mathcal{L}^\vee_\mathbf{x}, \mathcal{D}), \quad \mathbf{F}_\mathbf{V} = \bigoplus_{\mathbf{i}\in I^\nu} \mathbf{F}_{\mathbf{V},\mathbf{i}}.$$

For each $\mathbf{y}, \mathbf{y}'$ in Y^ν the Yoneda product gives a graded $\mathbf{S}_\mathbf{V}$-bilinear map $\mathbf{Z}_{\mathbf{V},\mathbf{x},\mathbf{x}'} \times \mathbf{F}_{\mathbf{V},\mathbf{x}'} \to \mathbf{F}_{\mathbf{V},\mathbf{x}}$. This yields a left graded representation of $\mathbf{Z}_\mathbf{V}$ on $\mathbf{F}_\mathbf{V}$. For each $\mathbf{i} \in I^\nu$ let $1_{\mathbf{V},\mathbf{i}} \in \mathbf{Z}_{\mathbf{V},\mathbf{i},\mathbf{i}}$ denote the identity of $\mathcal{L}_\mathbf{i}$. The elements $1_{\mathbf{V},\mathbf{i}}$ form a complete set of orthogonal idempotents of $\mathbf{Z}_\mathbf{V}$ such that

$$\mathbf{Z}_{\mathbf{V},\mathbf{i},\mathbf{i}'} = 1_{\mathbf{V},\mathbf{i}} \star \mathbf{Z}_\mathbf{V} \star 1_{\mathbf{V},\mathbf{i}'}, \quad \mathbf{F}_{\mathbf{V},\mathbf{i}} = 1_{\mathbf{V},\mathbf{i}} \star \mathbf{F}_\mathbf{V}.$$

We'll change the grading of $\mathbf{Z}_\mathbf{V}$ in the following way. Put

$$\mathbf{Z}^\delta_{\mathbf{V},\mathbf{i},\mathbf{i}'} = \mathrm{Ext}^*_{G_\mathbf{V}}(\mathcal{L}^\delta_\mathbf{i}, \mathcal{L}^\delta_{\mathbf{i}'}), \quad \mathbf{Z}^\delta_\mathbf{V} = \bigoplus_{\mathbf{i},\mathbf{i}'\in I^\nu} \mathbf{Z}^\delta_{\mathbf{V},\mathbf{i},\mathbf{i}'}.$$

The graded $\mathbf{k}$-algebra $\mathbf{Z}^\delta_\mathbf{V}$ depends only on the dimension vector of $\mathbf{V}$. We'll write

$$\mathbf{R}(\Gamma)_\nu = \mathbf{Z}^\delta_\mathbf{V}.$$

This graded $\mathbf{k}$-algebra has been computed explicitely in [VV]. The same result has also been anounced by R. Rouquier. See Section 7 for more details. We set also $I^0 = \{\emptyset\}$, $\mathcal{L}^\delta_\emptyset = \mathbf{k}$ (the constant sheaf over $\{0\}$)

$$\mathbf{R}(\Gamma)_0 = \mathbf{Z}^\delta_{\{0\}} = \mathbf{k}.$$

2. Quivers with involutions

In this section we introduce an analogue of the Ext-algebra $\mathbf{R}(\Gamma)_\nu$ which is associated with quiver with involutions.

2.1. Representations of quivers with involution. Fix a nonempty quiver Γ such that no arrow may join a vertex to itself. An involution θ on Γ is a pair of involutions on I and H, both denoted by θ, such that the following properties hold for each h in H

(a) $\theta(h)' = \theta(h'')$ and $\theta(h)'' = \theta(h')$,

(b) $\theta(h') = h''$ iff $\theta(h) = h$.

We'll always assume that θ has no fixed points in I, i.e., there is no $i \in I$ such that $\theta(i) = i$. To simplify we'll say that θ has no fixed points.

Let ${}^\theta\mathcal{V}$ be the category of finite-dimensional I-graded $\mathbb{C}$-vector spaces $\mathbf{V}$ with a non-degenerate symmetric bilinear form ϖ such that $\mathbf{V}_i$, $\mathbf{V}_j$ are orthogonal if $j \neq \theta(i)$. To simplify we'll say that $\mathbf{V}$ belongs to ${}^\theta\mathcal{V}$ if there is a bilinear form ϖ

such that the pair $(\mathbf{V}, \varpi)$ lies in ${}^\theta\boldsymbol{\mathcal{V}}$. The morphisms in ${}^\theta\boldsymbol{\mathcal{V}}$ are the linear maps which respect the grading and the bilinear form. Let

$$ {}^\theta\mathbb{N}I = \{\nu = \textstyle\sum_i \nu_i \in \mathbb{N}I; \nu_{\theta(i)} = \nu_i, \forall i\}. $$

For each ν in ${}^\theta\mathbb{N}I$ let ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ be the full subcategory of ${}^\theta\boldsymbol{\mathcal{V}}$ consisting of the pairs $(\mathbf{V}, \varpi)$ such that $\mathbf{V}$ lies in $\boldsymbol{\mathcal{V}}_\nu$. Note that $|\nu|$ is an even integer. We'll usually write $|\nu| = 2m$ with $m \in \mathbb{N}$. Given $\mathbf{V}$ in ${}^\theta\boldsymbol{\mathcal{V}}$ and $\boldsymbol{\Lambda}$ in $\boldsymbol{\mathcal{V}}$ we let

$$ {}^\theta E_{\mathbf{V}} = \{x = (x_h) \in E_{\mathbf{V}}; x_{\theta(h)} = -{}^t x_h, \forall h \in H\}, $$

$$ {}^\theta G_{\mathbf{V}} = \{g \in G_{\mathbf{V}}; g_{\theta(i)} = {}^t g_i^{-1}, \forall i \in I\}, $$

$$ {}^\theta E_{\boldsymbol{\Lambda},\mathbf{V}} = {}^\theta E_{\mathbf{V}} \times L_{\boldsymbol{\Lambda},\mathbf{V}}, \quad L_{\boldsymbol{\Lambda},\mathbf{V}} = \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\boldsymbol{\Lambda}, \mathbf{V}). $$

The algebraic groups ${}^\theta G_{\mathbf{V}}$, $G_{\boldsymbol{\Lambda}}$ act on ${}^\theta E_{\mathbf{V}}$, $L_{\boldsymbol{\Lambda},\mathbf{V}}$ in the obvious way.

2.2. Generalities on isotropic flags. Given a finite dimensional $\mathbb{C}$-vector space $\mathbf{W}$ with a non-degenerate symmetric bilinear form ϖ, an *isotropic flag in* $\mathbf{W}$ is a sequence of subspaces

$$ \phi = (\mathbf{W} = \mathbf{W}^{-k} \supset \mathbf{W}^{1-k} \supset \cdots \supset \mathbf{W}^k = 0) $$

such that $(\mathbf{W}^l)^\perp = \mathbf{W}^{-l}$ for any $l = -k, 1-k, \dots, k-1, k$. In particular $\mathbf{W}^0$ is a Lagrangian subspace of $\mathbf{W}$. Let $F(\mathbf{W})$ be the variety of all complete flags in $\mathbf{W}$, and $F(\mathbf{W}, \varpi)$ be the subvariety of all complete isotropic flags, i.e., we require that $\phi = (\mathbf{W}^l)$ is an isotropic flag such that $\mathbf{W}^l$ has the dimension $m - l$. If $\mathbf{W}$ has dimension $2m$ then $F(\mathbf{W}, \varpi)$ has dimension $2\ell_m = m(m-1)$.

2.3. Sequences. Fix a nonzero dimension vector ν in ${}^\theta\mathbb{N}I$. Let ${}^\theta Y^\nu$ be the set of all pairs $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ in Y^ν such that

$$ \mathbf{i} = (i_{1-k}, \dots, i_{k-1}, i_k), \quad \mathbf{a} = (a_{1-k}, \dots, a_{k-1}, a_k), \quad \theta(i_l) = i_{1-l}, \quad a_l = a_{1-l}. $$

As in (1.1) we can identify a pair $\mathbf{y}$ as above with a sequence

$$ \nu^{1-k}, \dots, \nu^{k-1}, \nu^k \in \mathbb{N}I, \quad \theta(\nu^l) = \nu^{1-l}, \quad \sum_l \nu^l = \nu. $$

Let ${}^\theta I^\nu \subset {}^\theta Y^\nu$ be the set of all pairs $\mathbf{y}$ of multiplicity $(1, 1, \dots, 1)$. We'll abbreviate $\mathbf{i} = (\mathbf{i}, \mathbf{a})$ for each pair in ${}^\theta I^\nu$. Note that a sequence in ${}^\theta I^\nu$ contains $|\nu| = 2m$ terms. Unless specified otherwise the entries of a sequence $\mathbf{i}$ in ${}^\theta I^\nu$ will always denoted by

$$ \mathbf{i} = (i_{1-m}, \dots, i_{m-1}, i_m). $$

Finally, we set

$$ {}^\theta I^m = \bigcup_\nu {}^\theta I^\nu, \quad \nu \in {}^\theta\mathbb{N}I, \quad |\nu| = 2m, $$

and we define ${}^\theta Y^m$ in the same way.

2.4. Definition of the map ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{y}}$. Fix $\nu \in {}^\theta\mathbb{N}I$, $\nu \neq 0$, and $\lambda \in \mathbb{N}I$. Fix an object $\mathbf{V}$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. For $\mathbf{y}$ in ${}^\theta Y^\nu$ an *isotropic flag of type* $\mathbf{y}$ in $\mathbf{V}$ is an isotropic flag

$$\phi = (\mathbf{V} = \mathbf{V}^{-k} \supset \mathbf{V}^{1-k} \supset \cdots \supset \mathbf{V}^k = 0)$$

such that $\mathbf{V}^{l-1}/\mathbf{V}^l$ lies in $\boldsymbol{\mathcal{V}}_{\nu^l}$ for each l. We define ${}^\theta F_{\mathbf{V},\mathbf{y}}$ to be the variety of all isotropic flags of type $\mathbf{y}$ in $\mathbf{V}$. Next, we define ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}$ to be the variety of all tuples (x, y, ϕ) satisfying the following conditions :

- $x \in {}^\theta E_\mathbf{V}$ and $\phi \in {}^\theta F_{\mathbf{V},\mathbf{y}}$ is *stable by* x, i.e., $x(\mathbf{V}^l) \subset \mathbf{V}^l$ for each l,
- $y \in L_{\mathbf{\Lambda},\mathbf{V}}$, and $y(\mathbf{\Lambda}) \subset \mathbf{V}^0$.

We set

$$d_{\lambda,\mathbf{y}} = \dim({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}).$$

We have the following formulas.

2.5. Proposition. *For* $\mathbf{i} \in {}^\theta I^\nu$ *we have*

(a) $\dim({}^\theta F_{\mathbf{V},\mathbf{i}}) = \ell_\nu/2$,

(b) $d_{\lambda,\mathbf{i}} = \ell_\nu/2 + \sum_{k<l;\, k+l\neq 1} h_{i_k,i_l}/2 + \sum_{1\leqslant l} \lambda_{i_l}$.

Proof : Fix a subset $J \subset I$ such that $I = J \sqcup \theta(J)$. Set $\mathbf{V}_J = \bigoplus_{j\in J} \mathbf{V}_j$. The assignment $(\mathbf{V}^k) \mapsto (\mathbf{V}^k \cap \mathbf{V}_J)$ takes ${}^\theta F_{\mathbf{V},\mathbf{i}}$ isomorphically onto

$$\prod_{j\in J} F(\mathbf{V}_j).$$

Thus we have

$$\dim({}^\theta F_{\mathbf{V},\mathbf{i}}) = \sum_{j\in J} \ell_{\nu_j} = \sum_{i\in I} \ell_{\nu_i}/2.$$

Next, fix a sequence $\mathbf{i}$ as above and fix a flag $\phi = (\mathbf{V}^k)$ in ${}^\theta F_{\mathbf{V},\mathbf{i}}$. Then we have

$$d_{\lambda,\mathbf{i}} = \ell_\nu/2 + \dim\{x \in {}^\theta E_\mathbf{V};\, x(\mathbf{V}^k) \subset \mathbf{V}^k,\, \forall k\} + \dim\{y \in L_{\mathbf{\Lambda},\mathbf{V}};\, y(\mathbf{\Lambda}) \subset \mathbf{V}^0\}.$$

Finally the discussion in Section 4.9 yields

$$\dim\{x \in {}^\theta E_\mathbf{V};\, x(\mathbf{V}^k) \subset \mathbf{V}^k,\, \forall k\} = \sum_{k<l;\, k+l\neq 1} h_{i_k,i_l}/2,$$

$$\dim\{y \in L_{\mathbf{\Lambda},\mathbf{V}};\, y(\mathbf{\Lambda}) \subset \mathbf{V}^0\} = \sum_{1\leqslant l\leqslant m} \lambda_{i_l}.$$

□

The group ${}^\theta G_\mathbf{V}$ acts transitively on ${}^\theta F_{\mathbf{V},\mathbf{y}}$. It acts also on ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}}$. The first projection gives a ${}^\theta G_\mathbf{V}$-equivariant proper morphism

$${}^\theta\pi_{\mathbf{\Lambda},\mathbf{y}} : {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y}} \to {}^\theta E_{\mathbf{\Lambda},\mathbf{V}}.$$

For a future use we introduce also the obvious projection

$$p : {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_\mathbf{V}, \quad {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} = \coprod_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad {}^\theta F_\mathbf{V} = \coprod_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta F_{\mathbf{V},\mathbf{i}}.$$

2.6. Ext-algebras. Let λ, ν, $\mathbf{\Lambda}$, $\mathbf{V}$ be as above. We abbreviate ${}^\theta\mathbf{S}_\mathbf{V} = \mathbf{S}_{{}^\theta G_\mathbf{V}}$. For $\mathbf{y} \in {}^\theta Y^\nu$ we define the following semisimple complexes in $\boldsymbol{\mathcal{D}}_{{}^\theta G_\mathbf{V}}({}^\theta E_{\mathbf{\Lambda},\mathbf{V}})$

$$ {}^\theta\mathcal{L}_\mathbf{y} = ({}^\theta\pi_{\mathbf{\Lambda},\mathbf{y}})_!(\mathbf{k}), \quad {}^\theta\mathcal{L}^\vee_\mathbf{y} = {}^\theta\mathcal{L}_\mathbf{y}[2d_{\lambda,\mathbf{y}}], \quad {}^\theta\mathcal{L}^\delta_\mathbf{y} = {}^\theta\mathcal{L}_\mathbf{y}[d_{\lambda,\mathbf{y}}]. $$

For $\mathbf{i}, \mathbf{i}'$ in ${}^\theta I^\nu$ we consider the graded ${}^\theta\mathbf{S}_\mathbf{V}$-module

$$ {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = \operatorname{Ext}^*_{{}^\theta G_\mathbf{V}}({}^\theta\mathcal{L}^\vee_\mathbf{i}, {}^\theta\mathcal{L}^\vee_{\mathbf{i}'}). $$

For $\mathbf{i}, \mathbf{i}', \mathbf{i}''$ in ${}^\theta I^\nu$ the Yoneda composition is a homogeneous ${}^\theta\mathbf{S}_\mathbf{V}$-bilinear map of degree zero ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} \times {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}''} \to {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}''}$. It equips the $\mathbf{k}$-vector space

$$ {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i},\mathbf{i}' \in {}^\theta I^\nu} {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} $$

with the structure of a unital associative graded ${}^\theta\mathbf{S}_\mathbf{V}$-algebra. For each $\mathbf{i} \in {}^\theta I^\nu$ we have the graded ${}^\theta\mathbf{S}_\mathbf{V}$-modules

$$ {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = \operatorname{Ext}^*_{{}^\theta G_\mathbf{V}}({}^\theta\mathcal{L}^\vee_{\mathbf{V},\mathbf{i}}, \mathcal{D}), \quad {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i} \in {}^\theta I^\nu} {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}. $$

For each $\mathbf{i}, \mathbf{i}'$ in ${}^\theta I^\nu$ the Yoneda product gives a graded ${}^\theta\mathbf{S}_\mathbf{V}$-bilinear map ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} \times {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \to {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$. This yields a left graded representation of ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$. Our first goal is to compute the graded algebra ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ and the graded representation ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$. For each $\mathbf{i}$ in ${}^\theta I^\nu$ let $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ be the identity of ${}^\theta\mathcal{L}_\mathbf{i}$, regarded as an element of ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}}$. The elements $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ form a complete set of orthogonal idempotents of ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ such that

$$ {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}\, {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}\, 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}, \quad {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}\, {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}. $$

2.7. Remark. Fix a pair $\mathbf{y}$ in ${}^\theta Y^\nu$. Let $\mathbf{i}$ be the sequence of ${}^\theta I^\nu$ obtained by expanding $\mathbf{y}$. We have an isomorphism of complexes in the derived category

$$ {}^\theta\mathcal{L}^\delta_\mathbf{i} = \bigoplus_{w \in \mathfrak{S}_\mathbf{b}} {}^\theta\mathcal{L}^\delta_\mathbf{y}[\ell_\mathbf{b} - 2\ell(w)]. $$

Here $\mathbf{b} = (b_1, \dots, b_m)$ is a sequence such that the multiplicity of $\mathbf{y}$ is

$$ \theta(\mathbf{b})\mathbf{b} := (b_m, \dots, b_2, b_1, b_1, b_2, \dots b_m). $$

We'll abbreviate ${}^\theta\mathcal{L}^\delta_\mathbf{i} = \langle\mathbf{b}\rangle!\, {}^\theta\mathcal{L}^\delta_\mathbf{y}$.

2.8. Shift of the grading. Let λ, ν, $\mathbf{\Lambda}$, $\mathbf{V}$ be as above. We define a new grading on ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ and ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ by

$$ {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'} = \operatorname{Ext}^*_{{}^\theta G_\mathbf{V}}({}^\theta\mathcal{L}^\delta_\mathbf{i}, {}^\theta\mathcal{L}^\delta_{\mathbf{i}'}) = {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}[d_{\lambda,\mathbf{i}} - d_{\lambda,\mathbf{i}'}], $$

$$ {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i},\mathbf{i}' \in {}^\theta I^\nu} {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}, $$

$$ {}^\theta\mathcal{L}_\mathbf{V} = \bigoplus_{\mathbf{i} \in {}^\theta I^\nu} {}^\theta\mathcal{L}_\mathbf{i}, \quad {}^\theta\mathcal{L}^\delta_\mathbf{V} = \bigoplus_{\mathbf{i} \in {}^\theta I^\nu} {}^\theta\mathcal{L}^\delta_\mathbf{i}. $$

We set also ${}^\theta I^0=\{\emptyset\}$, ${}^\theta\mathcal{L}^\delta_\emptyset=\mathbf{k}$, and ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\{0\}}=\mathbf{k}$ as a graded $\mathbf{k}$-algebra. Here $\mathbf{k}$ is regarded as the constant sheaf over $\{0\}$.

3. The convolution algebra

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu\neq 0$ in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Fix an object $(\mathbf{V},\varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. For each sequences $\mathbf{i}$, $\mathbf{i}'$ in ${}^\theta I^\nu$ we set

$$ {}^\theta Z_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}={}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}\times_{{}^\theta E_{\mathbf{\Lambda},\mathbf{V}}}{}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'},\quad {}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}=\coprod_{\mathbf{i},\mathbf{i}'\in{}^\theta I^\nu}{}^\theta Z_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}. $$

the reduced fiber product relative to the maps ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{i}}$, ${}^\theta\pi_{\mathbf{\Lambda},\mathbf{i}'}$. Next we set

$$ {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}=\bigoplus_{\mathbf{i},\mathbf{i}'\in{}^\theta I^\nu}{}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'},\quad {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}=\bigoplus_{\mathbf{i}\in{}^\theta I^\nu}{}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, $$

where

$$ {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'}=H_*^{{}^\theta G_{\mathbf{V}}}({}^\theta Z_{\mathbf{\Lambda},\mathbf{V},\mathbf{i},\mathbf{i}'},\mathbf{k}),\quad {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}=H_*^{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\mathbf{k}). $$

We have

$$ {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}=\mathrm{Ext}^*_{{}^\theta G_{\mathbf{V}}}(\mathbf{k},{}^\theta\mathcal{L}_{\mathbf{i}})=H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta E_{\mathbf{V}},{}^\theta\mathcal{L}_{\mathbf{i}})=H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\mathbf{k}). $$

We have also

$$ H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\mathbf{k})=H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\mathcal{D})[-2d_{\lambda,\mathbf{i}}]={}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}[-2d_{\lambda,\mathbf{i}}]. \tag{3.1} $$

This yields a graded ${}^\theta\mathbf{S}_{\mathbf{V}}$-module isomorphism

$$ {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}={}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}[-2d_{\lambda,\mathbf{i}}]. \tag{3.2} $$

Next, we equip the ${}^\theta\mathbf{S}_{\mathbf{V}}$-module ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ with the convolution product relative to the closed embedding of ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$ into ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}\times{}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$. See [CG, sec. 8.6] for details. We obtain an associative graded ${}^\theta\mathbf{S}_{\mathbf{V}}$-algebra ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ with 1 which acts on the graded ${}^\theta\mathbf{S}_{\mathbf{V}}$-module ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$. The unit is the fundamental class of the closed subvariety ${}^\theta Z^e_{\mathbf{\Lambda},\mathbf{V}}$ of ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$. See Section 4.6 below for the notation. The following is standard, see e.g., [VV].

3.1. Proposition. *(a) The left ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-module ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$ is faithful.*

(b) There is a canonical ${}^\theta\mathbf{S}_{\mathbf{V}}$-algebra isomorphism ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}={}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ such that (3.2) *indentifies the ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ and the ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-action on ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$.*

4. The polynomial representation of the graded algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$. Fix an object $(\mathbf{V}, \varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. The main result of this section is Theorem 4.19 which yields an explicit faithful representation of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$.

4.1. Notations. Let $G = O(\mathbf{V}, \varpi)$ be the orthogonal group, and $F = F(\mathbf{V}, \varpi)$ be the isotropic flag manifold. We can regard F as the (non connected) flag manifold of the (non connected) group G. Next, the group ${}^\theta G_\mathbf{V}$ is canonically identified with a Lévi subgroup of G, i.e., with the subgroup of elements which preserve the decomposition $\mathbf{V} = \bigoplus_i \mathbf{V}_i$. Then ${}^\theta F_\mathbf{V}$ is canonically identified with the closed subvariety of F consisting of all flags which are fixed under the action of the center of ${}^\theta G_\mathbf{V}$. Fix once for all a maximal torus T of ${}^\theta G_\mathbf{V}$. Let $W_\mathbf{V}$ and W be the Weyl groups of the pairs $({}^\theta G_\mathbf{V}, T)$ and (G, T). The canonical inclusion ${}^\theta G_\mathbf{V} \subset G$ yields a canonical inclusion $W_\mathbf{V} \subset W$.

4.2. The root systems. Fix once for all a T-fixed flag $\phi_\mathbf{V}$ in ${}^\theta F_\mathbf{V}$. Let B be the stabilizer of the flag $\phi_\mathbf{V}$ in G. Let Δ be the set of roots of (G, T) and let Δ^+ be the subset of positive roots relative to the Borel subgroup B. We abbreviate $\Delta^- = -\Delta^+$. Let Π be the set of simple roots in Δ^+. We fix once for all one-dimensional T-submodules $\mathbf{D}_{1-m}, \dots, \mathbf{D}_{m-1}, \mathbf{D}_m$ of $\mathbf{V}$ such that

$$\phi_\mathbf{V} = (\mathbf{V}^l), \quad \mathbf{V}^l = \mathbf{D}_{l+1} \oplus \cdots \oplus \mathbf{D}_{m-1} \oplus \mathbf{D}_m.$$

Let $\chi_l \in \mathbf{t}^*$ be the weight of $\mathbf{D}_l$. We have

$$\Delta^+ = \{\chi_k \pm \chi_l;\ 1 \leqslant l < k \leqslant m\},$$

$$\Pi = \{\chi_{l+1} - \chi_l,\ \chi_2 + \chi_1;\ l = 1, 2, \dots, m-1\}.$$

Let $\leqslant$ and ℓ denote the Bruhat order and the length function on W. Note that W is an extended Weyl group of type D_m. In particular we have

$$\ell(w) = 0 \iff w = e, \varepsilon_1,$$

where $\varepsilon_1$is as below, and the set S of simple reflections is given by

$$S = \{s_0, s_1, \dots, s_{m-1}\},$$

with s_k, $k = 0, 1, \dots, m-1$ the reflection with respect to

$$\alpha_0 = \chi_2 + \chi_1, \quad \alpha_1 = \chi_2 - \chi_1, \quad \dots \quad \alpha_{m-1} = \chi_m - \chi_{m-1}.$$

Now we consider the root system of the subgroup ${}^\theta G_\mathbf{V}$ of G. Let ${}^\theta\Delta_\mathbf{V} \subset \Delta$ be the set of roots of $({}^\theta G_\mathbf{V}, T)$ and let ${}^\theta\Delta^+_\mathbf{V}$ be the subset of positive roots relative to the Borel subgroup ${}^\theta B_\mathbf{V} = B \cap {}^\theta G_\mathbf{V}$. We have ${}^\theta\Delta^+_\mathbf{V} = \Delta^+ \cap {}^\theta\Delta_\mathbf{V}$. Note that changing the flag $\phi_\mathbf{V}$ takes ${}^\theta\Delta_\mathbf{V}$ to $w({}^\theta\Delta_\mathbf{V})$ for some w in W. Note also that ${}^\theta G_\mathbf{V}$ is a product of general linear groups (this is due to the fact that θ has no fixed points). Thus we can (and we will) assume that

$${}^\theta\Delta_\mathbf{V} \subset \{\chi_l - \chi_k;\ l \neq k,\ l, k = 1, 2, \dots, m\}.$$

4.3. The wreath product. Let $\mathfrak{S}_m$ be the symmetric group, and $\mathbb{Z}_2 = \{-1, 1\}$. Consider the wreath product $W_m = \mathfrak{S}_m \wr \mathbb{Z}_2$. For each $l = 1, 2, \dots m$ let $\varepsilon_l \in (\mathbb{Z}_2)^m$ be -1 placed at the l-th position. We'll regard ε_l as in element of W_m in the obvious way. There is a unique action of W_m on the set $\{1-m, \dots, m-1, m\}$ such that $\mathfrak{S}_m$ permutes $1, 2, \dots m$ and such that ε_l fixes k if $k \neq l, 1-l$ and switches l and $1-l$. The group W_m acts also on ${}^\theta I^\nu$. Indeed, view a sequence $\mathbf{i}$ as the map

$$\{1-m, \dots, m-1, m\} \to I, \quad l \mapsto i_l.$$

Then we set $w(\mathbf{i}) = \mathbf{i} \circ w^{-1}$ for $w \in W_m$. Thus we get a group isomorphism $W = W_m$ such that $w(\mathbf{D}_l) = \mathbf{D}_{w(l)}$ (as T-modules) for each w, l.

4.4. The W-action on T-fixed flags. The sets F^T and $({}^\theta F_{\mathbf{V}})^T$ consisting of the flags which are fixed by the T-action are equal. Thus the group W acts freely transitively on $({}^\theta F_{\mathbf{V}})^T$. We'll write e for the unit in W. Put

$$\phi_{\mathbf{V},w} = w(\phi_{\mathbf{V}}), \quad \forall w \in W.$$

Thus we have $F^T = \{\phi_{\mathbf{V},w}; w \in W\}$. Let ${}^\theta B_{\mathbf{V},w}$ be the stabilizer of the flag $\phi_{\mathbf{V},w}$ under the ${}^\theta G_{\mathbf{V}}$-action. It is a Borel subgroup of ${}^\theta G_{\mathbf{V}}$ containing T. Let ${}^\theta N_{\mathbf{V},w}$ be the unipotent radical of ${}^\theta B_{\mathbf{V},w}$. Finally, let $\mathbf{i}_w$ be the unique sequence in ${}^\theta I^\nu$ such that $\phi_{\mathbf{V},w}$ lies in ${}^\theta F_{\mathbf{V},\mathbf{i}_w}$. Write

$$\mathbf{i}_e = (i_{1-m}, \dots, i_{m-1}, i_m). \tag{4.1}$$

We have

$$\mathbf{D}_l \subset \mathbf{V}_{i_l}, \quad w^{-1}(\mathbf{i}_e) = \mathbf{i}_w = (i_{w(1-m)}, \dots, i_{w(m-1)}, i_{w(m)}).$$

Let W_ν be the image of the group $W_{\mathbf{V}}$ by the isomorphism $W \to W_m$. It is the parabolic subgroup given by

$$W_\nu = \{w \in W_m; w(\mathbf{i}_e) = \mathbf{i}_e\}.$$

Note that the choices made in Section 4.4 imply that $W_\nu \subset \mathfrak{S}_m$. There is a bijection

$$W_\nu \setminus W_m \to {}^\theta I^\nu, \quad W_\nu w \mapsto \mathbf{i}_w.$$

For each $\mathbf{i}$ in ${}^\theta I^\nu$ we have

$$({}^\theta \widetilde{F}_{\mathbf{V},\mathbf{i}})^T \simeq ({}^\theta F_{\mathbf{V},\mathbf{i}})^T = \{\phi_{\mathbf{V},w}; w \in W_{\mathbf{i}}\}, \quad W_{\mathbf{i}} = \{w \in W; \mathbf{i}_w = \mathbf{i}\}.$$

We'll abbreviate

$${}^\theta F_{\mathbf{V},w} = {}^\theta F_{\mathbf{V},\mathbf{i}_w}, \quad W_w = W_{\mathbf{i}_w}, \quad {}^\theta \pi_{\mathbf{\Lambda},w} = {}^\theta \pi_{\mathbf{\Lambda},\mathbf{i}_w}.$$

We'll also omit the symbol w if $w = e$. For instance we write ${}^\theta B_{\mathbf{V}} = {}^\theta B_{\mathbf{V},e}$ and ${}^\theta N_{\mathbf{V}} = {}^\theta N_{\mathbf{V},e}$. Note that $W_w = W_{\mathbf{V}}\, w$ and that we have an isomorphism of $G_{\mathbf{V}}$-varieties

$${}^\theta G_{\mathbf{V}} / {}^\theta B_{\mathbf{V},w} \to {}^\theta F_{\mathbf{V},w}, \quad g \mapsto g\phi_{\mathbf{V},w}.$$

Note also that ${}^\theta B_{\mathbf{V},w} = {}^\theta B_{\mathbf{V},wu}$ if $\ell(u) = 0$.

4.5. The stratification of ${}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$. The group G acts diagonally on $F \times F$. The action of the subgroup ${}^\theta G_{\mathbf{V}}$ preserves the subset ${}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$. For each w in W let ${}^\theta O^w_{\mathbf{V}}$ be the set of all pairs of flags in ${}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$ which are in relative position w. More precisely, we write

$$ {}^\theta O^w_{\mathbf{V}} = ({}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}) \cap (G\phi_{\mathbf{V},e,w}), \quad \phi_{\mathbf{V},x,y} = (\phi_{\mathbf{V},x}, \phi_{\mathbf{V},y}), \quad \forall x, y \in W. $$

Let ${}^\theta \bar{O}^w_{\mathbf{V}}$ be the Zariski closure of ${}^\theta O^w_{\mathbf{V}}$. For any w, x, y in W we write also

$$ {}^\theta O^w_{\mathbf{V},x,y} = {}^\theta O^w_{\mathbf{V}} \cap ({}^\theta F_{\mathbf{V},x} \times {}^\theta F_{\mathbf{V},y}), \quad {}^\theta \bar{O}^w_{\mathbf{V},x,y} = {}^\theta \bar{O}^w_{\mathbf{V}} \cap ({}^\theta F_{\mathbf{V},x} \times {}^\theta F_{\mathbf{V},y}). $$

We define ${}^\theta P_{\mathbf{V},w,ws}$, $s \in S$, as the smallest parabolic subgroup of ${}^\theta G_{\mathbf{V}}$ containing ${}^\theta B_{\mathbf{V},w}$ and ${}^\theta B_{\mathbf{V},ws}$. The following lemma is standard.

4.6. Lemma. *Let $w, x, y, s, u \in W$.*

(a) The set of T-fixed elements in ${}^\theta O^x_{\mathbf{V}}$ is $\{\phi_{\mathbf{V},w,wx};\, w \in W\}$.

(b) Assume that $\ell(u) = 0$. We have ${}^\theta \bar{O}^u_{\mathbf{V}} = {}^\theta O^u_{\mathbf{V}}$. It is a smooth ${}^\theta G_{\mathbf{V}}$-variety isomorphic to ${}^\theta F_{\mathbf{V}}$. We have ${}^\theta O^u_{\mathbf{V},x,y} = \emptyset$ unless $y = xu$.

(c) Assume that $\ell(s) = 1$. Set $s = s'u$ with $s' \in S$ and $\ell(u) = 0$. We have ${}^\theta \bar{O}^s_{\mathbf{V}} = {}^\theta O^s_{\mathbf{V}} \cup {}^\theta O^u_{\mathbf{V}}$. It is a smooth variety. We have ${}^\theta \bar{O}^s_{\mathbf{V},x,y} = \emptyset$ if $y \neq xs, xu$. Further,

- *either $xs \notin W_{xu}$ and*
$$ {}^\theta F_{\mathbf{V},xs} \neq {}^\theta F_{\mathbf{V},xu}, \quad {}^\theta B_{\mathbf{V},xs} = {}^\theta B_{\mathbf{V},x}, \quad {}^\theta O^u_{\mathbf{V},x,xs} = {}^\theta O^s_{\mathbf{V},x,xu} = \emptyset, $$
- *or $xs \in W_{xu}$ and*
$$ {}^\theta F_{\mathbf{V},xs} = {}^\theta F_{\mathbf{V},xu}, \quad {}^\theta B_{\mathbf{V},xs} \neq {}^\theta B_{\mathbf{V},x}, \quad {}^\theta O^s_{\mathbf{V},x,xs} \simeq {}^\theta O^s_{\mathbf{V},x,xu}, $$
$$ {}^\theta G_{\mathbf{V}} \times_{{}^\theta B_{\mathbf{V},x}} ({}^\theta P_{\mathbf{V},x,xs}/{}^\theta B_{\mathbf{V},x}) \xrightarrow{\sim} {}^\theta \bar{O}^s_{\mathbf{V},x,xu}, \quad (g,h) \mapsto (g\phi_{\mathbf{V},x}, gh\phi_{\mathbf{V},xu}). $$

For a future use let us introduce the following notation. Let q be the obvious projection ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$, and, for each $x \in W$, let ${}^\theta Z^x_{\mathbf{\Lambda},\mathbf{V}}$ be the Zariski closure in ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$ of the locally closed subset $q^{-1}({}^\theta O^x_{\mathbf{V}})$.

4.7. Euler classes in S. Consider the graded $\mathbf{k}$-algebra $\mathbf{S} = \mathbf{S}_T$. The weights $\chi_1, \chi_2, \dots \chi_m$ are algebraically independent generators of $\mathbf{S}$ and they are homogeneous of degree 2. The reflection representation on $\mathfrak{t}$ yields a W-action on $\mathbf{S}$. We have

$$ w(\chi_l) = \chi_{w(l)}, \quad \forall l, w. $$

Now, let M be a finite dimensional representation of $\mathfrak{t}$ and fix a linear form $\lambda \in \mathfrak{t}^*$. Let $M[\lambda] \subset M$ be the weight subspace associated with λ. The *character* of M is the linear form $\mathrm{ch}(M) = \sum_\lambda \dim(M[\lambda])\, \lambda$. Let $\mathrm{eu}(M)$ be the determinant of M, viewed as an element of degree $2\dim(M)$ of $\mathbf{S}$. We'll call $\mathrm{eu}(M)$ the *Euler class of* M. If M is a finite dimensional representation of T let $\mathrm{eu}(M)$ be the Euler class of the differential of M, a module over $\mathfrak{t}$. Now, assume that X is a quasi-projective T-variety and that $x \in X^T$ is a smooth point of X. The cotangent space $T^*_x X$ at x is equipped with a natural representation of T. We'll abbreviate $\mathrm{eu}(X, x) = \mathrm{eu}(T^*_x X)$. We'll be particularly interested by the following elements

$$ \Lambda_w = \mathrm{eu}({}^\theta \widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}, \phi_{\mathbf{V},w}), \quad \Lambda^x_{w,w'} = \mathrm{eu}({}^\theta Z^x_{\mathbf{\Lambda},\mathbf{V}}, \phi_{\mathbf{V},w,w'})^{-1} $$

where $\ell(x) = 0, 1$. Note that Λ_w lies in $\mathbf{S}$ and has the degree $2d_{\lambda,w}$.

4.8. Given $w \in W$ we let ${}^\theta\mathfrak{g}_{\mathbf{V}}$, $\mathfrak{t}$, ${}^\theta\mathfrak{n}_{\mathbf{V},w}$ be the Lie algebras of ${}^\theta G_{\mathbf{V}}$, T, ${}^\theta N_{\mathbf{V},w}$ respectively. We'll denote the flag $\phi_{\mathbf{V},w}$ by

$$\phi_{\mathbf{V},w} = (\mathbf{V} = \mathbf{V}_w^{-m} \supset \cdots \supset \mathbf{V}_w^{m-1} \supset \mathbf{V}_w^m = 0).$$

The ${}^\theta G_{\mathbf{V}}$-action on ${}^\theta E_{\mathbf{\Lambda},\mathbf{V}}$ yields a representation of ${}^\theta B_{\mathbf{V},w}$ on the space

$${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} = \{(x,y) \in {}^\theta E_{\mathbf{\Lambda},\mathbf{V}};\, x(\mathbf{V}_w^l) \subset \mathbf{V}_w^l,\, y(\mathbf{\Lambda}) \subset \mathbf{V}_w^m\}.$$

There is an isomorphism of ${}^\theta G_{\mathbf{V}}$-varieties

$${}^\theta G_{\mathbf{V}} \times_{{}^\theta B_{\mathbf{V},w}} {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} \to {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},w}, \quad (g,x,y) \mapsto (g\phi_{\mathbf{V},w}, gx, gy).$$

Under this isomorphism the map ${}^\theta\pi_{\mathbf{\Lambda},w}$ is identified with the map

$${}^\theta G_{\mathbf{V}} \times_{{}^\theta B_{\mathbf{V},w}} {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} \to {}^\theta E_{\mathbf{\Lambda},\mathbf{V}}, \quad (g,x,y) \mapsto (gx, gy).$$

For each w, w' we write

$${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'} = {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} \cap {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w'},$$

$${}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w'} = {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} / {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'},$$

$${}^\theta\mathfrak{n}_{\mathbf{V},w,w'} = {}^\theta\mathfrak{n}_{\mathbf{V},w} \cap {}^\theta\mathfrak{n}_{\mathbf{V},w'},$$

$${}^\theta\mathfrak{m}_{\mathbf{V},w,w'} = {}^\theta\mathfrak{n}_{\mathbf{V},w} / {}^\theta\mathfrak{n}_{\mathbf{V},w,w'}.$$

We have the following T-module isomorphisms

$${}^\theta\mathfrak{n}_{\mathbf{V},w} = {}^\theta\mathfrak{n}_{\mathbf{V},w,w'} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w,w'},$$

$${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w} = {}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,w'} \oplus {}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w'},$$

$${}^\theta\mathfrak{m}_{\mathbf{V},w,w'} = ({}^\theta\mathfrak{m}_{\mathbf{V},w',w})^*.$$

4.9. Character formulas. As a T-module we have

$$(4.2) \qquad {}^\theta\mathfrak{n}_{\mathbf{V},w} = \bigoplus_\alpha \mathfrak{g}[\alpha], \quad \alpha \in w(\Delta^+) \cap {}^\theta\Delta_{\mathbf{V}}.$$

Recall that $\mathbf{V} = \bigoplus_l \mathbf{D}_l$ as I-graded T-modules, where $l = 1-m, \dots, m-1, m$. We'll use the notation in (4.1). Thus i_l, χ_l are the dimension vector and the character of $\mathbf{D}_l$. Note that

$$h_{i_k,i_l} = h_{i_{1-l},i_{1-k}}, \quad \chi_l = -\chi_{1-l}, \quad \mathbf{D}_{1-l} = \mathbf{D}_l^*.$$

Set $H^0 = \{h \in H;\, h' = \theta(h'')\}$, $H^1 = H \setminus H^0$, and $\lambda = \sum_i \lambda_i\, i$. A short computation yields the following formulas

$$\mathrm{ch}({}^\theta E_{\mathbf{V}}) = \sum_{\chi_l - \chi_k \in \Delta} h_{i_k,i_l}(\chi_l - \chi_k),$$

$$\dim({}^\theta E_{\mathbf{V}}) = \sum_{h \in H^1} \nu_{h'}\nu_{h''}/2 + \sum_{h \in H^0} \nu_{h'}(\nu_{h'} - 1)/2,$$

$$\mathrm{ch}(L_{\mathbf{\Lambda},\mathbf{V}}) = \sum_l \lambda_{i_l}\chi_l.$$

Here the first sum runs over $\alpha \in \Delta$, and for each α we choose one pair (l,k) such that $\alpha = \chi_l - \chi_k$. Recall also that $\mathbf{V}^0 = \bigoplus_{l\geqslant 1} \mathbf{D}_l$. Thus for each w we have also

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}) = \sum_{\chi_l-\chi_k} h_{i_k,i_l}(\chi_l - \chi_k) + \sum_l \lambda_{i_l}\chi_l, \tag{4.3}$$

where the first sum runs over all roots in $w(\Delta^+)$ and the second one over all l in $\{w(1), w(2), \dots, w(m)\}$. Note that (4.3) can be rewriten in the following way

$$\mathrm{ch}({}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}) = \sum_{\chi_l-\chi_k\in\Delta^+} h_{i_{w(k)},i_{w(l)}} w(\chi_l - \chi_k) + \sum_{1\leqslant l\leqslant m} \lambda_{i_{w(l)}} w(\chi_l).$$

4.10. By (4.2) the Euler class $\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w})$ is the product of all roots in ${}^\theta\Delta_{\mathbf{V}} \cap w(\Delta^+)$. Therefore, for each $l = 0, 1, \dots, m-1$ the following hold

(a) either $ws_l \notin W_w$ and we have

$$\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},ws_l}) = \mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w,ws_l}) = \mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},ws_l,w}) = 0,$$

(b) or $ws_l \in W_w$ and we have

$$\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},ws_l}) = -\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w,ws_l}) = -\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},ws_l,w}) = w(\alpha_l).$$

4.11. Formula (4.3) has the following consequences.

(a) First, we have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,w\varepsilon_1}) = w(\chi_1)^{\lambda_{i_{w(1)}}}.$$

(b) Next, assume that $l = 1, 2, \dots, m-1$. We have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,ws_l}) = w(\alpha_l)^{h_{i_{w(l)},i_{w(l+1)}}}.$$

(c) Finally, we have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w,ws_0}) = w(\chi_1)^{\lambda_{i_{w(1)}}} w(\chi_2)^{\lambda_{i_{w(2)}}} w(\alpha_0)^{h_{i_{w(0)},i_{w(2)}}}.$$

4.12. Reduction to the torus. The restriction of functions from ${}^\theta\mathfrak{g}_{\mathbf{V}}$ to $\mathfrak{t}$ gives an isomorphism of graded $\mathbf{k}$-algebras

$${}^\theta\mathbf{S}_{\mathbf{V}} = \mathbf{k}[\chi_1, \chi_2, \dots, \chi_m]^{W_\nu}.$$

The group ${}^\theta G_{\mathbf{V}}$ is a product of several copies of the general linear group. Hence it is connected with a simply connected derived subgroup. If X is a quasi-projective ${}^\theta G_{\mathbf{V}}$-variety then the $\mathbf{S}$-module $H_*^T(X,\mathbf{k})$ is equipped with a canonical $\mathbf{S}$-skewlinear action of the group $W_{\mathbf{V}}$. The forgetful map gives a ${}^\theta\mathbf{S}_{\mathbf{V}}$-module isomorphism

$$H_*^{{}^\theta G_{\mathbf{V}}}(X,\mathbf{k}) \to H_*^T(X,\mathbf{k})^{W_{\mathbf{V}}}.$$

4.13. The W-action and the ${}^\theta\mathbf{S}_\mathbf{V}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$. Fix a tuple $\mathbf{i}$ in ${}^\theta I^\nu$ and an integer $l = 1, 2, \dots, m$. We define $\mathcal{O}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)$ to be the ${}^\theta G_\mathbf{V}$-equivariant line bundle over ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ whose fiber at the flag ϕ is equal to $\mathbf{V}^l/\mathbf{V}^{l-1}$. Assigning to a formal variable $x_\mathbf{i}(l)$ of degree 2 the first equivariant Chern class of $\mathcal{O}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)^{-1}$ we get a graded $\mathbf{k}$-algebra isomorphism

$$\mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots x_\mathbf{i}(m)] = H^*_{{}^\theta G_\mathbf{V}}({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \mathbf{k}).$$

So (3.1), (3.2) yield canonical isomorphisms of graded $\mathbf{k}$-vector spaces

$$\mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots x_\mathbf{i}(m)] = {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}[-2d_{\lambda,\mathbf{i}}] = {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}. \tag{4.4}$$

For a future use we set also

$$x_\mathbf{i}(l) = -x_\mathbf{i}(1-l), \quad l = 1-m, 2-m, \dots, 0.$$

For each w in W_m we set

$$wf(x_\mathbf{i}(1), \dots, x_\mathbf{i}(m)) = f(x_{w(\mathbf{i})}(w(1)), \dots, x_{w(\mathbf{i})}(w(m))).$$

This yields a W_m-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ such that the element w takes ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ to ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},w(\mathbf{i})}$.

The multiplication of polynomials equip both ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ and ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ with an obvious structure of graded $\mathbf{k}$-algebra. For each w in $W_\mathbf{i}$ the pull-back by the inclusion $\{\phi_{\mathbf{V},w}\} \subset {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ yields a graded $\mathbf{k}$-algebra isomorphism

$${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} \to \mathbf{S}, \quad f(-x_\mathbf{i}(1), \dots, -x_\mathbf{i}(m)) \mapsto f(\chi_{w(1)}, \dots \chi_{w(m)}). \tag{4.5}$$

We'll write $w(f)$ for the right hand side. This isomorphism is not canonical, because it depends on the choice of w.

Now, consider the canonical ${}^\theta\mathbf{S}_\mathbf{V}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ coming from the ${}^\theta G_\mathbf{V}$-equivariant cohomology. It can be regarded as a ${}^\theta\mathbf{S}_\mathbf{V}$-action on $\bigoplus_\mathbf{i} \mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots x_\mathbf{i}(m)]$ which is described in the following way. The composition of the obvious projection ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ with the map (4.5) identifies the graded $\mathbf{k}$-algebra of the W_m-invariant polynomials in the $x_\mathbf{i}(l)$'s, with $\mathbf{S}^{W_\mathbf{V}} = {}^\theta\mathbf{S}_\mathbf{V}$. This isomorphism does not depend on the choice of $\mathbf{i}, w$. The ${}^\theta\mathbf{S}_\mathbf{V}$-action on ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ is the composition of this isomorphism and of the multiplication by W_m-invariant polynomials.

4.14. Localization. Let $\mathbf{K}$ be the fraction field of $\mathbf{S}$. Write

$${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}} = H^T_*({}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}, \mathbf{k}),$$

$${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}} \otimes_\mathbf{S} \mathbf{K},$$

$${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}} = H^T_*({}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}, \mathbf{k}),$$

$${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}} \otimes_\mathbf{S} \mathbf{K}.$$

For each w, w' in W let ψ_w be the fundamental class of the singleton $\{\phi_{\mathbf{V},w}\}$ in ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$, and let $\psi_{w,w'}$ be the fundamental class of $\{\phi_{\mathbf{V},w,w'}\}$ in ${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$. Let ψ_w, $\psi_{w,w'}$ denote also the corresponding elements in ${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}}$. We consider also the convolution product

$${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}} \times {}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$$

relative to the inclusion of ${}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}$ in ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}} \times {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$ and the convolution product

$${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}} \times {}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}.$$

Both may be denoted by the symbol $\star$. We'll use the notation in (4.1).

4.15. Proposition. *(a) The* $\mathbf{S}$*-modules* ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$ *and* ${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$ *are free. The* $W_\mathbf{V}$*-action on the* T*-equivariant homology spaces* ${}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$ *defined in Section* 4.12 *is given by* $w(\psi_x) = \psi_{wx}$ *and* $w(\psi_{x,y}) = \psi_{wx,wy}$*. The inclusions* ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}} \subset {}^\theta\mathcal{Z}'_{\mathbf{\Lambda},\mathbf{V}}$ *and* ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}} \subset {}^\theta\mathcal{F}'_{\mathbf{\Lambda},\mathbf{V}}$ *commute with the convolution product.*

(b) The elements ψ_w, $\psi_{w,w'}$ *yield* $\mathbf{K}$*-bases of* ${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V}}$, ${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}}$ *respectively. For each* $\mathbf{i}$ *the map* (4.4) *yields an inclusion of* $\mathbf{k}[x_\mathbf{i}(1),\dots,x_\mathbf{i}(m)]$ *into* ${}^\theta\mathcal{F}''_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *such that*

$$f(-x_\mathbf{i}(1),\dots,-x_\mathbf{i}(m)) \mapsto \sum_{w\in W_\mathbf{i}} w(f)\Lambda_w^{-1}\psi_w.$$

(c) We have $\psi_{w',w}\star\psi_w = \Lambda_w\,\psi_{w'}$ *and* $\psi_{w'',w'}\star\psi_{w',w} = \Lambda_{w'}\,\psi_{w'',w}$.

(d) If $\ell(s)=0,1$ *then* $[{}^\theta Z^s_{\mathbf{\Lambda},\mathbf{V}}] = \sum_{w,w'}\Lambda^s_{w,w'}\,\psi_{w,w'}$ *in* ${}^\theta\mathcal{Z}''_{\mathbf{\Lambda},\mathbf{V}}$.

(e) We have $\Lambda_w = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w})$.

(f) If $\ell(u)=0$ *then* $\Lambda^u_{w,w'}=0$ *if* $w'\neq wu$*, and*

$$\Lambda^e_{w,w} = \Lambda_w^{-1},\quad \Lambda^{\varepsilon_1}_{w,w\varepsilon_1} = (\chi_{w(0)})^{\lambda_{i_{w(1)}}}\Lambda_w^{-1} = (\chi_{w(1)})^{\lambda_{i_{w(0)}}}\Lambda^{-1}_{w\varepsilon_1}.$$

(g) If $k=0,1,\dots,m-1$ *then*

- *either* $ws_k\notin W_w$ *and*

$$\Lambda^{s_k}_{w,ws_k} = \Lambda^{s_k}_{w,w} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w})^{-1},$$

- *or* $ws_k\in W_w$ *and*

$$\Lambda^{s_k}_{w,w} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w}\oplus{}^\theta\mathfrak{m}_{\mathbf{V},w,ws_k})^{-1},$$

$$\Lambda^{s_k}_{w,ws_k} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w}\oplus{}^\theta\mathfrak{m}_{\mathbf{V},ws_k,w})^{-1}.$$

Proof: Parts (a) to (d) are left to the reader. The fiber at $\phi_{\mathbf{V},w}$ of the vector bundle $p: {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}\to{}^\theta F_\mathbf{V}$ is isomorphic to ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w}$ as a T-module. Thus the cotangent space to ${}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$ at the point $\phi_{\mathbf{V},w}$ is isomorphic to ${}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w}\oplus{}^\theta\mathfrak{n}_{\mathbf{V},w}$ as a T-module. This yields (e). Now, observe that the variety ${}^\theta Z^s_{\mathbf{\Lambda},\mathbf{V}}$ is smooth if $\ell(s)\leqslant 1$. First, assume that $\ell(u)=0$. The fiber at $\phi_{\mathbf{V},w,w'}$ of the vector bundle

$$q: {}^\theta Z^u_{\mathbf{\Lambda},\mathbf{V}}\to{}^\theta F_\mathbf{V}\times{}^\theta F_\mathbf{V}$$

is isomorphic to ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,wu}$ as a T-module if $w'=wu$ and it is zero else. Thus we have

$$\begin{aligned}\Lambda^u_{w,wu} &= \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w,wu})\mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w})^{-1}\mathrm{eu}({}^\theta F_\mathbf{V},\phi_{\mathbf{V},w})^{-1},\\ &= \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w,wu})\Lambda_w^{-1}.\end{aligned}$$

Therefore, we have

$$\Lambda^u_{w,wu} = \begin{cases}\Lambda_w^{-1} & \text{if } u=e,\\ (-\chi_{w(1)})^{\lambda_{i_{w(1)}}}\Lambda_w^{-1} & \text{if } u=\varepsilon_1.\end{cases}$$

This yields (f). Finally, let us concentrate on part (g). The fiber at $\phi_{\mathbf{V},w,w'}$ of the vector bundle

$$q : {}^\theta Z^{s_k}_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta F_{\mathbf{V}} \times {}^\theta F_{\mathbf{V}}$$

is isomorphic to ${}^\theta\mathfrak{e}_{\mathbf{\Lambda},\mathbf{V},w,ws_k}$ as a T-module if $w' = w, ws_k$ and it is zero else. Therefore, we have

$$\Lambda^{s_k}_{w,w'} = \begin{cases} \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k})^{-1}\mathrm{eu}({}^\theta\bar{O}^{s_k}_{\mathbf{V}}, \phi_{\mathbf{V},w,w'})^{-1} & \text{if } w' = w, ws_k \\ 0 & \text{else.} \end{cases}$$

Next, if $ws_k \notin W_w$ the cotangent spaces to the variety ${}^\theta\bar{O}^{s_k}_{\mathbf{V}}$ at the points $\phi_{\mathbf{V},w,ws_k}$ and $\phi_{\mathbf{V},w,w}$ are both isomorphic to ${}^\theta\mathfrak{n}_{\mathbf{V},w}$ as T-modules. Finally, if $ws_k \in W_w$ the cotangent spaces to the variety ${}^\theta\bar{O}^{s_k}_{\mathbf{V}}$ at the points $\phi_{\mathbf{V},w,ws_k}$, $\phi_{\mathbf{V},w,w}$ are isomorphic to ${}^\theta\mathfrak{n}_{\mathbf{V},w} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},ws_k,w}$, ${}^\theta\mathfrak{n}_{\mathbf{V},w} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w,ws_k}$ respectively as T-modules.

□

4.16. Description of the ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-action on ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$. For each $k = 0, 1, \ldots, m-1$ let $\sigma_{\mathbf{\Lambda},\mathbf{V}}(k)$ be the fundamental class of ${}^\theta Z^{s_k}_{\mathbf{\Lambda},\mathbf{V}}$ in ${}^\theta\mathcal{Z}^{\leqslant s_k}_{\mathbf{\Lambda},\mathbf{V}}$, and let $\pi_{\mathbf{\Lambda},\mathbf{V}}(1)$ be the fundamental class of ${}^\theta Z^{\varepsilon_1}_{\mathbf{\Lambda},\mathbf{V}}$ in ${}^\theta\mathcal{Z}^{\varepsilon_1}_{\mathbf{\Lambda},\mathbf{V}}$. Next, for each $l = 1, 2, \ldots, m$ the pull-back of the first equivariant Chern class of the line bundle $\bigoplus_{\mathbf{i}} \mathcal{O}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)^{-1}$ by the obvious map

$${}^\theta Z^e_{\mathbf{\Lambda},\mathbf{V}} \to {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V}}$$

belongs to $H^*_{{}^\theta G_{\mathbf{V}}}({}^\theta Z^e_{\mathbf{\Lambda},\mathbf{V}}, \mathbf{k})$. So it yields an element $\varkappa_{\mathbf{\Lambda},\mathbf{V}}(l)$ in ${}^\theta\mathcal{Z}^e_{\mathbf{\Lambda},\mathbf{V}}$. We write

$$\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(k) = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \star \sigma_{\mathbf{\Lambda},\mathbf{V}}(k) \star 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},$$

$$\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(1) = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \star \pi_{\mathbf{\Lambda},\mathbf{V}}(1) \star 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},$$

$$\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}',\mathbf{i}}(l) = 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \star \varkappa_{\mathbf{\Lambda},\mathbf{V}}(l) \star 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}.$$

Now, recall that ${}^\theta\mathcal{Z}^{\leqslant w}_{\mathbf{\Lambda},\mathbf{V}}$ embeds into ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ for each w and that ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ acts on ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$. For any sequence $\mathbf{i} = (i_{1-m}, \ldots, i_{m-1}, i_m)$ and any $l = 1-m, \ldots, m-1, m$ and $k = 1, \ldots, m-1, m$, we set

$$\lambda_{\mathbf{i}}(l) = \lambda_{i_l},$$

$$h_{\mathbf{i}}(k) = \begin{cases} -1 & \text{if } s_k\mathbf{i} = \mathbf{i}, \\ h_{i_k,i_{k+1}} & \text{if } s_k\mathbf{i} \neq \mathbf{i},\ k \neq 0, \\ h_{i_0,i_2} & \text{if } s_0\mathbf{i} \neq \mathbf{i},\ k = 0. \end{cases}$$

4.17. Proposition. *Given $\mathbf{i}, \mathbf{i}', \mathbf{i}''$ in ${}^\theta I^\nu$ and f in ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ the following hold :*

(a) $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \star f = f$ *if* $\mathbf{i} = \mathbf{i}'$ *and* $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'} \star f = 0$ *else.*

(b) $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'',\mathbf{i}'}(l) \star f = 0$ *unless* $\mathbf{i}'' = \mathbf{i}' = \mathbf{i}$ *and* $\varkappa_{\mathbf{\Lambda},\mathbf{V}}(l) \star f = x_{\mathbf{i}}(l)f$.

(c) $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'',\mathbf{i}'}(1) \star f = 0$ *unless* $\mathbf{i}' = \mathbf{i}$, $\mathbf{i}'' = \varepsilon_1\mathbf{i}$ *and*

$$\pi_{\mathbf{\Lambda},\mathbf{V}}(1) \star f = x_{\varepsilon_1\mathbf{i}}(1)^{\lambda_{\varepsilon_1\mathbf{i}}(1)}\varepsilon_1(f).$$

(d) $\sigma_{\mathbf{V},\mathbf{i}'',\mathbf{i}'}(k)\star f=0$ *unless* $\mathbf{i}'=\mathbf{i}$ *and* $\mathbf{i}''=s_k\mathbf{i}$ *or* $\mathbf{i}$, *and we have*

- *if* $s_k\mathbf{i}=\mathbf{i}$ *and* $k\neq 0$ *then*

$$\sigma_{\mathbf{V}}(k)\star f=(x_{\mathbf{i}}(k)-x_{\mathbf{i}}(k+1))^{h_{\mathbf{i}}(k)}(s_k(f)-f),$$

- *if* $s_0\mathbf{i}=\mathbf{i}$ *then*

$$\sigma_{\mathbf{V}}(0)\star f=(x_{\mathbf{i}}(0)-x_{\mathbf{i}}(2))^{h_{\mathbf{i}}(0)}x_{\mathbf{i}}(1)^{\lambda_{\mathbf{i}}(1)}x_{\mathbf{i}}(2)^{\lambda_{\mathbf{i}}(2)}(s_0(f)-f),$$

- *if* $s_k\mathbf{i}\neq\mathbf{i}$ *and* $k\neq 0$ *then*

$$\sigma_{\mathbf{V},s_k\mathbf{i},\mathbf{i}}(k)\star f=(x_{s_k\mathbf{i}}(k)-x_{s_k\mathbf{i}}(k+1))^{h_{s_k\mathbf{i}}(k)}s_k(f),$$
$$\sigma_{\mathbf{V},\mathbf{i},\mathbf{i}}(k)\star f=(x_{\mathbf{i}}(k)-x_{\mathbf{i}}(k+1))^{h_{\mathbf{i}}(k)}f$$

- *if* $s_0\mathbf{i}\neq\mathbf{i}$ *then*

$$\sigma_{\mathbf{V},s_0\mathbf{i},\mathbf{i}}(0)\star f=(x_{s_0\mathbf{i}}(0)-x_{s_0\mathbf{i}}(2))^{h_{s_0\mathbf{i}}(0)}x_{s_0\mathbf{i}}(1)^{\lambda_{s_0\mathbf{i}}(1)}x_{s_0\mathbf{i}}(2)^{\lambda_{s_0\mathbf{i}}(2)}s_0(f),$$
$$\sigma_{\mathbf{V},\mathbf{i},\mathbf{i}}(0)\star f=(x_{\mathbf{i}}(0)-x_{\mathbf{i}}(2))^{h_{\mathbf{i}}(0)}x_{\mathbf{i}}(1)^{\lambda_{\mathbf{i}}(1)}x_{\mathbf{i}}(2)^{\lambda_{\mathbf{i}}(2)}f.$$

Proof : Parts (a), (b) are left to the reader. Proposition 4.15(b), or, equivalently, the map (4.5), yields

$$f(-x_{\mathbf{i}}(1),\dots,-x_{\mathbf{i}}(m))\,\psi_w=f(\chi_{w(1)},\dots,\chi_{w(m)})\,\psi_w,\quad\forall w\in W_{\mathbf{i}}.$$

Further we have $\varepsilon_1(\mathbf{i}_w)=\mathbf{i}_{w\varepsilon_1}$. Therefore, part (c) follows from the following computation

$$[{}^\theta Z^{\varepsilon_1}_{\mathbf{\Lambda},\mathbf{V}}]\star\psi_w=\Lambda^{\varepsilon_1}_{w\varepsilon_1,w}\Lambda_w\psi_{w\varepsilon_1}=(\chi_{w\varepsilon_1(0)})^{\lambda_{i_{w\varepsilon_1(1)}}}\psi_{w\varepsilon_1},$$

where $i_{w\varepsilon_1(l)}$ is the l-th component of the sequence $\mathbf{i}_{w\varepsilon_1}$. Let us concentrate on (d). The first claim is obvious, because ${}^\theta Z^{s_k}_{w,w'}=\emptyset$ unless $w'=w,ws_k$ and $\mathbf{i}_{ws_k}=s_k\mathbf{i}_w$. Now, given $\mathbf{i}'=\mathbf{i}$ or $s_k\mathbf{i}$ we must compute the linear operator

$$(4.6)\qquad {}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}}\to{}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}'},\quad f\mapsto\sigma_{\mathbf{V},\mathbf{i}',\mathbf{i}}(k)\star f.$$

By (4.4) we have a $\mathbf{k}$-vector space isomorphism

$${}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}}=\mathbf{k}[x_{\mathbf{i}}(1),x_{\mathbf{i}}(2),\dots x_{\mathbf{i}}(m)],$$

and Proposition 4.15(b) yields an embedding

$${}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}}\to\bigoplus_{w\in W_{\mathbf{i}}}\mathbf{K}\psi_w,\quad f(-x_{\mathbf{i}}(1),\dots,-x_{\mathbf{i}}(m))\mapsto\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda_w^{-1}\psi_w.$$

Under this inclusion the map (4.6) is of the following form

$$\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda_w^{-1}\psi_w\mapsto\sum_{w'\in W_{\mathbf{i}'}}g_{w'}\psi_{w'},\quad g_{w'}=\sum_{w\in W_{\mathbf{i}}}w(f)\Lambda^{s_k}_{w',w}.$$

We claim that the rhs is the image of a polynomial g in ${}^\theta\mathcal{F}_{\mathbf{V},\mathbf{i}'}$ that we'll compute explicitly. The polynomial g is completely determined by the following relations

$$g_{w'} = w'(g)\Lambda_{w'}^{-1}, \quad \forall w' \in W_{\mathbf{i}'}. \tag{4.7}$$

In the rest of the proof we'll fix w, w' in the following way

$$w \in W_{\mathbf{i}}, \quad w' \in W_{\mathbf{i}'}, \quad w' = w \text{ or } ws_k.$$

In particular we have $\mathbf{i} = \mathbf{i}_w$, $\mathbf{i}' = \mathbf{i}_{w'}$, and $\mathbf{i}' = \mathbf{i}$ or $s_k\mathbf{i}$.

(i) First, assume that $s_k\mathbf{i} = \mathbf{i}$. Then $\mathbf{i}' = \mathbf{i}$, $w's_k \in W_{w'}$, and we have

$$g_{w'} = w'(f)\Lambda^{s_k}_{w',w'} + w's_k(f)\Lambda^{s_k}_{w',w's_k}.$$

Note that Section 4.10 and Proposition 4.15 yield

$$\Lambda_{w'} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w'} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w'}),$$

$$\Lambda^{s_k}_{w',w'} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w'} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w',w's_k})^{-1},$$

$$\Lambda^{s_k}_{w',w's_k} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w'} \oplus {}^\theta\mathfrak{m}_{\mathbf{V},w's_k,w'})^{-1},$$

$$\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w',w's_k}) = -\mathrm{eu}({}^\theta\mathfrak{m}_{\mathbf{V},w's_k,w'}) = w'(\alpha_k).$$

In particular, we have

$$\Lambda^{s_k}_{w',w'} = \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})w'(\alpha_k)^{-1}\Lambda^{-1}_{w'} = -\Lambda^{s_k}_{w',w's_k}.$$

Therefore we obtain

$$g_{w'} = w'(f - s_k(f))\,\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})w'(\alpha_k)^{-1}\Lambda^{-1}_{w'}.$$

Now, assume that $k \neq 0$. There is no arrow joining $i_{w'(k)}$ and $i_{w'(k+1)}$, because $i_{w'(k)} = i_{w'(k+1)}$. Thus 4.11 yields

$$\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k}) = 1.$$

Hence

$$\begin{aligned} g_{w'} &= w'(f - s_k(f))w'(\alpha_k)^{-1}\Lambda^{-1}_{w'} \\ &= w'(g)\Lambda^{-1}_{w'}, \\ g &= (f - s_k(f))\alpha_k^{-1}. \end{aligned}$$

Next, assume that $k = 0$. There is no arrow joining $i_{w'(0)}$ and $i_{w'(2)}$. Thus 4.11 yields

$$\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_0}) = (-\chi_{w'(1)})^{\lambda_{i_{w'(1)}}}(-\chi_{w'(2)})^{\lambda_{i_{w'(2)}}}.$$

Therefore we have

$$\begin{aligned} g_{w'} &= w'(f - s_0(f))(-\chi_{w'(1)})^{\lambda_{i_{w'(1)}}}(-\chi_{w'(2)})^{\lambda_{i_{w'(2)}}}w'(\alpha_0)^{-1}\Lambda^{-1}_{w'} \\ &= w'(g)\Lambda^{-1}_{w'}, \\ g &= (f - s_0(f))(-\chi_1)^{\lambda_{i_{w'(1)}}}(-\chi_2)^{\lambda_{i_{w'(2)}}}\alpha_0^{-1}. \end{aligned}$$

(*ii*) Finally, assume that $s_k\mathbf{i} \neq \mathbf{i}$, i.e., that $ws_k \notin W_w$. First, note that Section 4.10 and Proposition 4.15 yield

$$\mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},ws_k}) = \mathrm{eu}({}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\Lambda_w = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\Lambda_{ws_k} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},ws_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w}),$$

$$\Lambda^{s_k}_{w,w} = \mathrm{eu}({}^\theta\mathfrak{e}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k} \oplus {}^\theta\mathfrak{n}_{\mathbf{V},w})^{-1} = \Lambda^{s_k}_{ws_k,w}.$$

In particular, we have

$$\Lambda^{s_k}_{ws_k,w}\Lambda_{ws_k} = \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},ws_k,w}),$$

$$\Lambda^{s_k}_{w,w}\Lambda_w = \mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w,ws_k}).$$

Next, one of the two following alternatives holds :

- either $\mathbf{i}' = s_k\mathbf{i}$, $w' = ws_k$ and

$$\begin{aligned} g_{w'} &= w's_k(f)\Lambda^{s_k}_{w',w's_k} \\ &= w's_k(f)(\Lambda^{s_k}_{ws_k,w}\Lambda_{ws_k})\Lambda^{-1}_{w'} \\ &= w's_k(f)\,\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})\,\Lambda^{-1}_{w'}. \end{aligned}$$

- or $\mathbf{i}' = \mathbf{i}$, $w' = w$ and

$$\begin{aligned} g_{w'} &= w'(f)\Lambda^{s_k}_{w',w'} \\ &= w'(f)(\Lambda^{s_k}_{w,w}\Lambda_w)\Lambda^{-1}_{w'} \\ &= w'(f)\,\mathrm{eu}({}^\theta\mathfrak{d}^*_{\mathbf{\Lambda},\mathbf{V},w',w's_k})\,\Lambda^{-1}_{w'}. \end{aligned}$$

Now we consider the cases $k \neq 0$ and $k = 0$. First, assume that $k \neq 0$. By 4.11 we have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w',w's_k}) = w'(\alpha_k)^{h_{i_{w'(k)},i_{w'(k+1)}}}.$$

Thus (4.7) holds with

$$g = s_k(f)(-\alpha_k)^{h_{i_{w'(k)},i_{w'(k+1)}}}$$

in the first case and with

$$g = f(-\alpha_k)^{h_{i_{w'(k)},i_{w'(k+1)}}}$$

in the second one. Next, assume that $k = 0$. By 4.11 we have

$$\mathrm{eu}({}^\theta\mathfrak{d}_{\mathbf{\Lambda},\mathbf{V},w',w's_0}) = w'(\chi_1)^{\lambda_{i_{w'(1)}}}\,w'(\chi_2)^{\lambda_{i_{w'(2)}}}\,w'(\alpha_0)^{h_{i_{w'(0)},i_{w'(2)}}}.$$

Thus (4.7) holds with

$$g = s_0(f)(-\chi_1)^{\lambda_{i_{w'(1)}}}\,(-\chi_2)^{\lambda_{i_{w'(2)}}}\,(-\alpha_0)^{h_{i_{w'(0)},i_{w'(2)}}}$$

in the first case and with

$$g = f(-\chi_1)^{\lambda_{i_{w'(1)}}}\,(-\chi_2)^{\lambda_{i_{w'(2)}}}\,(-\alpha_0)^{h_{i_{w'(0)},i_{w'(2)}}}$$

in the second one.

□

4.18. From ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$ to ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$. The action of

$$1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\quad \varkappa_{\mathbf{\Lambda},\mathbf{V}}(l),\quad \sigma_{\mathbf{\Lambda},\mathbf{V}}(k),\quad \pi_{\mathbf{\Lambda},\mathbf{V}}(1),$$

yields linear operators in $\mathrm{End}({}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}})$. Recall that ${}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}$ is a faithful left ${}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}$-module and that there are canonical isomorphisms

$${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}},\quad {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} = {}^\theta\mathcal{F}_{\mathbf{\Lambda},\mathbf{V}}.$$

Thus the graded left ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$-module ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}$ is also faithful. Recall also that

$${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}} = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\quad {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}} = \mathbf{k}[x_\mathbf{i}(1), x_\mathbf{i}(2), \dots, x_\mathbf{i}(m)].$$

We obtain the following.

4.19. Theorem. *The graded* $\mathbf{k}$*-algebra* ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$ *is isomorphic to a graded* $\mathbf{k}$*-subalgebra of* $\mathrm{End}({}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}})$ *which contains the linear operators*

$$1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}},\quad \varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l),\quad \sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k),\quad \pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1),$$

$$\mathbf{i}\in{}^\theta I^\nu,\quad k=0,1,\dots,m-1,\quad l=1,2,\dots,m,$$

defined as follows :

(*a*) $1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *is the projection to* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *relatively to* $\bigoplus_{\mathbf{i}'\neq\mathbf{i}}{}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$,

(*b*) $\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l) = 0$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$ *if* $\mathbf{i}'\neq\mathbf{i}$*, and it acts by multiplication by* $x_\mathbf{i}(l)$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$,

(*c*) $\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k) = 0$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$ *if* $\mathbf{i}'\neq\mathbf{i}$*, and it takes a polynomial* f *in* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *to*

$$\begin{array}{ll}(x_\mathbf{i}(k)-x_\mathbf{i}(k+1))^{h_\mathbf{i}(k)}(s_k(f)-f) & \text{if } s_k\mathbf{i}=\mathbf{i}, k\neq 0,\\ (x_\mathbf{i}(0)-x_\mathbf{i}(2))^{h_\mathbf{i}(0)}x_\mathbf{i}(1)^{\lambda_\mathbf{i}(1)}x_\mathbf{i}(2)^{\lambda_\mathbf{i}(2)}(s_0(f)-f) & \text{if } s_k\mathbf{i}=\mathbf{i}, k=0,\\ (x_{s_k\mathbf{i}}(k)-x_{s_k\mathbf{i}}(k+1))^{h_{s_k\mathbf{i}}(k)}s_k(f) & \text{if } s_k\mathbf{i}\neq\mathbf{i}, k\neq 0,\\ (x_{s_0\mathbf{i}}(0)-x_{s_0\mathbf{i}}(2))^{h_{s_0\mathbf{i}}(0)}x_{s_0\mathbf{i}}(1)^{\lambda_{s_0\mathbf{i}}(1)}x_{s_0\mathbf{i}}(2)^{\lambda_{s_0\mathbf{i}}(2)}s_0(f) & \text{if } s_k\mathbf{i}\neq\mathbf{i}, k=0,\end{array}$$

(*d*) $\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1) = 0$ *on* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}'}$ *if* $\mathbf{i}'\neq\mathbf{i}$*, and it takes a polynomial* f *in* ${}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}$ *to*

$$x_{\varepsilon_1\mathbf{i}}(1)^{\lambda_{\varepsilon_1\mathbf{i}}(1)}\varepsilon_1(f).$$

The degrees of these operators are given by the following formulas

$$\begin{aligned}&\deg(1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}) = 0,\\ &\deg(\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)) = 2,\\ &\deg(\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1)) = 2\lambda_{\varepsilon_1\mathbf{i}}(1),\\ &\deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(0)) = 2h_{s_0\mathbf{i}}(0)+2\lambda_{s_0\mathbf{i}}(1)+2\lambda_{s_0\mathbf{i}}(2),\\ &\deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k)) = 2h_{s_k\mathbf{i}}(k)\quad \text{if } k\neq 0.\end{aligned}$$

4.20. Shift of the grading. We have

$$ {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V},s_k\mathbf{i},\mathbf{i}} = {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V},s_k\mathbf{i},\mathbf{i}}[d_{\lambda,s_k\mathbf{i}} - d_{\lambda,\mathbf{i}}]. $$

Recall that $h_{\theta(i),j} = h_{\theta(j),i}$ for each i, j. Hence, an easy computation yields

$$ d_{\lambda,s_k\mathbf{i}} - d_{\lambda,\mathbf{i}} = \begin{cases} h_{s_k\mathbf{i}}(k) - h_{\mathbf{i}}(k) & \text{if } k \neq 0, \\ h_{s_0\mathbf{i}}(0) - h_{\mathbf{i}}(0) + \lambda_{s_0\mathbf{i}}(2) + \lambda_{s_0\mathbf{i}}(1) - \lambda_{\mathbf{i}}(2) - \lambda_{\mathbf{i}}(1) & \text{if } k = 0, \end{cases} $$

$$ d_{\lambda,\varepsilon_1\mathbf{i}} - d_{\lambda,\mathbf{i}} = \lambda_{\varepsilon_1\mathbf{i}}(1) - \lambda_{\mathbf{i}}(1). $$

Therefore the grading of ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ is given by the following rules :

$$ \deg(1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}) = 0, $$

$$ \deg(\varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l)) = 2, $$

$$ \deg(\pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1)) = \lambda_{\mathbf{i}}(0) + \lambda_{\mathbf{i}}(1), $$

$$ \deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(0)) = -i_0 \cdot i_2 + \lambda_{\mathbf{i}}(-1) + \lambda_{\mathbf{i}}(0) + \lambda_{\mathbf{i}}(1) + \lambda_{\mathbf{i}}(2), $$

$$ \deg(\sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k)) = -i_k \cdot i_{k+1} \quad \text{if } k \neq 0. $$

5. The graded k-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector $\nu \neq 0$ in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$.

5.1. Definition of the graded k-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. Assume that $m > 0$. We define a graded **k**-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ with 1 generated by $1_\mathbf{i}$, $\varkappa_l$, σ_k, π_1 with $\mathbf{i} \in {}^\theta I^\nu$, $k = 1, \dots, m-1$, $l = 1, 2, \dots, m$, modulo the following defining relations†

(a) $1_\mathbf{i}\, 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}, \quad \sigma_k 1_\mathbf{i} = 1_{s_k\mathbf{i}} \sigma_k, \quad \varkappa_l 1_\mathbf{i} = 1_\mathbf{i} \varkappa_l, \quad \pi_1 1_\mathbf{i} = 1_{\varepsilon_1\mathbf{i}} \pi_1,$

(b) $\varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l, \quad \pi_1 \varkappa_l = \varkappa_{\varepsilon_1(l)} \pi_1,$

(c) $\sigma_k^2 1_\mathbf{i} = Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_k) 1_\mathbf{i}, \quad \pi_1^2 1_\mathbf{i} = \varkappa_0^{\lambda_\mathbf{i}(0)} \varkappa_1^{\lambda_\mathbf{i}(1)} 1_\mathbf{i},$

(d) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $k \neq k' \pm 1, \quad \pi_1 \sigma_k = \sigma_k \pi_1$ if $k \neq 1,$

(e) $(\sigma_1\pi_1)^2 1_\mathbf{i} = (\pi_1\sigma_1)^2 1_\mathbf{i} +$

$$ +\delta_{i_0,i_2}(-1)^{\lambda_{i_2}} \frac{\varkappa_0^{\lambda_{i_1}+\lambda_{i_2}} - \varkappa_2^{\lambda_{i_1}+\lambda_{i_2}}}{\varkappa_0 - \varkappa_2} \sigma_1 1_\mathbf{i} - \delta_{i_0,i_2}\delta_{i_1,i_2}(-1)^{\lambda_{i_2}} \frac{\varkappa_0^{2\lambda_{i_1}} - \varkappa_2^{2\lambda_{i_1}}}{\varkappa_0^2 - \varkappa_2^2} 1_\mathbf{i}, $$

†We thank M. Kashiwara who indicate us an error in a previous version of the relations

(f) $(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k)1_{\mathbf{i}} =$

$$= \delta_{i_k,i_{k+2}} \frac{Q_{i_k,i_{k+1}}(\varkappa_{k+1},\varkappa_k) - Q_{i_k,i_{k+1}}(\varkappa_{k+1},\varkappa_{k+2})}{\varkappa_k - \varkappa_{k+2}} 1_{\mathbf{i}},$$

$$(g)\ \ (\sigma_k\varkappa_l - \varkappa_{s_k(l)}\sigma_k)1_{\mathbf{i}} = \begin{cases} -1_{\mathbf{i}} & \text{if } l=k,\ i_k = i_{k+1}, \\ 1_{\mathbf{i}} & \text{if } l = k+1,\ i_k = i_{k+1}, \\ 0 & \text{else.} \end{cases}$$

Here $\delta_{i,j}$ is the Kronecker symbol, $\varkappa_{1-l} = -\varkappa_l$, and

$$Q_{i,j}(u,v) = \begin{cases} (-1)^{h_{i,j}}(u-v)^{-i\cdot j} & \text{if } i \neq j, \\ 0 & \text{else.} \end{cases} \tag{5.1}$$

We'll abbreviate $\sigma_{\mathbf{i},k} = \sigma_k 1_{\mathbf{i}}$, $\varkappa_{\mathbf{i},l} = \varkappa_l 1_{\mathbf{i}}$, and $\pi_{\mathbf{i},1} = \pi_1 1_{\mathbf{i}}$. The grading on ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is given by the following rules :

$$\deg(1_{\mathbf{i}}) = 0,$$

$$\deg(\varkappa_{\mathbf{i},l}) = 2,$$

$$\deg(\pi_{\mathbf{i},1}) = \lambda_{\mathbf{i}}(0) + \lambda_{\mathbf{i}}(1),$$

$$\deg(\sigma_{\mathbf{i},k}) = -i_k \cdot i_{k+1}.$$

If $\nu = 0$ we set ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = \mathbf{k}$ as a graded $\mathbf{k}$-algebra. Let ω be the unique anti-involution of the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ which fixes $1_{\mathbf{i}}$, $\varkappa_l$, σ_k, π_1.

5.2. Remarks. (a) We may set $\sigma_0 = \pi_1\sigma_1\pi_1$. We have

$$\deg(\sigma_0 1_{\mathbf{i}}) = -i_0 \cdot i_2 + \lambda_{s_0\mathbf{i}}(1) + \lambda_{s_0\mathbf{i}}(2) + \lambda_{\mathbf{i}}(1) + \lambda_{\mathbf{i}}(2).$$

(b) We may also set $\pi_l = \sigma_{l-1}\dots\sigma_2\sigma_1\pi_1\sigma_1\sigma_2\dots\sigma_{l-1}$. We have

$$\deg(\pi_l 1_{\mathbf{i}}) = -(i_1 + i_2 + \cdots + i_{l-1}) \cdot (i_l + i_{1-l}) + \lambda_{\mathbf{i}}(l) + \lambda_{\mathbf{i}}(1-l).$$

5.3. The polynomial representation and the PBW theorem. Given any objects $\mathbf{V}$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and $\boldsymbol{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$ we abbreviate

$${}^\theta\mathbf{F}_\nu = {}^\theta\mathbf{F}_{\boldsymbol{\Lambda},\mathbf{V}}, \quad {}^\theta\mathbf{F}_{\mathbf{i}} = {}^\theta\mathbf{F}_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}, \quad {}^\theta\mathbf{S}_\nu = {}^\theta\mathbf{S}_{\mathbf{V}}.$$

5.4. Proposition. *There is an unique graded $\mathbf{k}$-algebra morphism ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} \to \mathrm{End}({}^\theta\mathbf{F}_\nu)$ such that, for each $\mathbf{i} \in {}^\theta I^\nu$, $k = 0,1,\dots,m-1$, $l = 1,2,\dots,m$, we have*

$$1_{\mathbf{i}} \mapsto 1_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}, \quad \varkappa_{\mathbf{i},l} \mapsto \varkappa_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}(l), \quad \sigma_{\mathbf{i},k} \mapsto \sigma_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}(k), \quad \pi_{\mathbf{i},1} \mapsto \pi_{\boldsymbol{\Lambda},\mathbf{V},\mathbf{i}}(1).$$

Proof : The defining relations of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ are checked by a direct computation. Let us (only) give a few indications concerning the relation 5.1(e). We have

$$\sigma_1 1_{\mathbf{i}} = (\varkappa_1 - \varkappa_2)^{h_{i_2,i_1}}(s_1 - \delta_{i_1,i_2})1_{\mathbf{i}}, \quad \pi_1 1_{\mathbf{i}} = \varkappa_1^{\lambda_{i_0}}\varepsilon_1 1_{\mathbf{i}}.$$

This yields

$$\sigma_1\pi_1 1_{\mathbf{i}} = (\varkappa_1-\varkappa_2)^{h_{i_2,i_0}}(s_1-\delta_{i_0,i_2})\varkappa_1^{\lambda_{i_0}}\varepsilon_1 1_{\mathbf{i}},$$

$$\pi_1\sigma_1 1_{\mathbf{i}} = \varkappa_1^{\lambda_{i_{-1}}}\varepsilon_1(\varkappa_1-\varkappa_2)^{h_{i_2,i_1}}(s_1-\delta_{i_1,i_2})1_{\mathbf{i}}.$$

Therefore we have

$$(\sigma_1\pi_1)^2 1_{\mathbf{i}} = (\varkappa_1-\varkappa_2)^{h_{i_2,i_1}}(s_1-\delta_{i_1,i_2})\varkappa_1^{\lambda_{i_{-1}}}\varepsilon_1(\varkappa_1-\varkappa_2)^{h_{i_2,i_0}}(s_1-\delta_{i_0,i_2})\varkappa_1^{\lambda_{i_0}}\varepsilon_1 1_{\mathbf{i}},$$

$$(\pi_1\sigma_1)^2 1_{\mathbf{i}} = \varkappa_1^{\lambda_{i_0}}\varepsilon_1(\varkappa_1-\varkappa_2)^{h_{i_2,i_0}}(s_1-\delta_{i_0,i_2})\varkappa_1^{\lambda_{i_{-1}}}\varepsilon_1(\varkappa_1-\varkappa_2)^{h_{i_2,i_1}}(s_1-\delta_{i_1,i_2})1_{\mathbf{i}}.$$

Hence we have

$$(\sigma_1\pi_1)^2 1_{\mathbf{i}} = (\varkappa_1-\varkappa_2)^{h_{i_2,i_1}}(\varkappa_0-\varkappa_2)^{h_{i_2,i_0}}A,$$

$$(\pi_1\sigma_1)^2 1_{\mathbf{i}} = (\varkappa_1-\varkappa_2)^{h_{i_2,i_1}}(\varkappa_0-\varkappa_2)^{h_{i_2,i_0}}B,$$

where

$$A = (s_1-\delta_{i_1,i_2})\varkappa_1^{\lambda_{i_{-1}}}\varepsilon_1(s_1-\delta_{i_0,i_2})\varkappa_1^{\lambda_{i_0}}\varepsilon_1 1_{\mathbf{i}},$$

$$B = \varkappa_1^{\lambda_{i_0}}\varepsilon_1(s_1-\delta_{i_0,i_2})\varkappa_1^{\lambda_{i_{-1}}}\varepsilon_1(s_1-\delta_{i_1,i_2})1_{\mathbf{i}}.$$

If $i_0\neq i_2$ it is easy to see that $A=B$. If $i_0=i_2$ a direct computation yields

$$B-A = \big(\varkappa_2^{\lambda_{i_1}}(-\varkappa_2)^{\lambda_{i_2}} - \varkappa_1^{\lambda_{i_2}}(-\varkappa_1)^{\lambda_{i_1}}\big)s_1.$$

The rest of the computation is left to the reader.

□

The **k**-algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is a left graded ${}^\theta\mathbf{F}_\nu$-module such that $x_{\mathbf{i}}(l)$ acts by the left multiplication with the element $\varkappa_{\mathbf{i},l}$ for each $l=1,2,\dots,m$. To unburden the notation we may write $\varkappa_{\mathbf{i},l}=x_{\mathbf{i}}(l)$. The following convention is important.

From now on we'll regard W_m as a Weyl group of type B_m, rather than an extended Weyl group of type D_m as in Section 4.2.

For each w in W_m we choose a reduced decomposition $\dot w$ of w. By the observation above $\dot w$ is a decomposition of the following form

$$w = s_{k_1}s_{k_2}\cdots s_{k_r},\quad 0<k_1,k_2,\dots,k_r\leqslant m,\quad s_m=\varepsilon_1.$$

We define an element $\sigma_{\dot w}$ in ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ by the following formula

$$\sigma_{\dot w} = \sum_{\mathbf{i}} 1_{\mathbf{i}}\sigma_{\dot w},\quad 1_{\mathbf{i}}\sigma_{\dot w} = \begin{cases} 1_{\mathbf{i}} & \text{if } r=0 \\ 1_{\mathbf{i}}\sigma_{k_1}\sigma_{k_2}\cdots\sigma_{k_r} & \text{else,}\end{cases} \tag{5.2}$$

where we have set $\sigma_m=\pi_1$. Observe that $\sigma_{\dot w}$ may depend on the choice of the reduced decomposition $\dot w$.

5.5. Proposition. *The* $\mathbf{k}$*-algebra* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *is a free (left or right)* ${}^\theta\mathbf{F}_\nu$*-module with basis* $\{\sigma_{\dot w};\, w\in W_m\}$*. Its rank is* $2^m m!$*. The operator* $1_\mathbf{i}\sigma_{\dot w}$ *is homogeneous and its degree is independent of the choice of the reduced decomposition* $\dot w$*.*

Proof : The algebra ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is filtered with $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$ in degree 0 and $\sigma_{\mathbf{i},k}$, $\pi_{\mathbf{i},1}$ in degree 1. The *Nil Hecke algebra* of type B_m is the $\mathbf{k}$-algebra ${}^\theta\mathbf{NH}_m$ generated by $\bar\pi_1,\bar\sigma_1,\bar\sigma_2,\dots,\bar\sigma_{m-1}$ with relations

$$\bar\sigma_k\bar\sigma_{k'}=\bar\sigma_{k'}\bar\sigma_k \text{ if } |k-k'|>1,\quad \bar\pi_1\bar\sigma_k=\bar\sigma_k\bar\pi_1 \text{ if } k\neq 1,\quad (\bar\pi_1\bar\sigma_1)^2=(\bar\sigma_1\bar\pi_1)^2,$$

$$\bar\sigma_{k+1}\bar\sigma_k\bar\sigma_{k+1}=\bar\sigma_k\bar\sigma_{k+1}\bar\sigma_k,\quad \bar\pi_1^2=\bar\sigma_k^2=0.$$

We can form the semidirect product ${}^\theta\mathbf{F}_\nu\rtimes{}^\theta\mathbf{NH}_m$, which is generated by $1_\mathbf{i}$, $\bar\varkappa_l$, $\bar\pi_1,\bar\sigma_k$ with the relations above and

$$\bar\sigma_k\bar\varkappa_l=\bar\varkappa_{s_k(l)}\bar\sigma_k,\quad \bar\pi_1\bar\varkappa_l=\bar\varkappa_{\varepsilon_1(l)}\bar\pi_1\ ,\quad \bar\varkappa_l\bar\varkappa_{l'}=\bar\varkappa_{l'}\bar\varkappa_{l'}.$$

We have a surjective $\mathbf{k}$-algebra morphism

$${}^\theta\mathbf{F}_\nu\rtimes{}^\theta\mathbf{NH}_m\to\mathrm{gr}\,{}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu},\quad 1_\mathbf{i}\mapsto 1_\mathbf{i},\quad \bar\varkappa_l\mapsto\varkappa_l,\quad \bar\pi_1\mapsto\pi_1,\quad \bar\sigma_k\mapsto\sigma_k.$$

Thus the elements $\sigma_{\dot w}$ with $w\in W_m$ generate ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ as a ${}^\theta\mathbf{F}_\nu$-module. We must prove that they yield indeed a basis of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. This is rather clear, since the images of these elements in $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ under the polynomial representation are independent over ${}^\theta\mathbf{F}_\nu$ (by Galois theory).

□

Let ${}^\theta\mathbf{F}'_\nu=\bigoplus_\mathbf{i}{}^\theta\mathbf{F}'_\mathbf{i}$, where ${}^\theta\mathbf{F}'_\mathbf{i}$ is the localization of the ring ${}^\theta\mathbf{F}_\mathbf{i}$ with respect to the multiplicative system generated by

$$\{\varkappa_{\mathbf{i},l}\pm\varkappa_{\mathbf{i},l'};\,1\leqslant l\neq l'\leqslant m\}\cup\{\varkappa_{\mathbf{i},l};\,l=1,2,\dots,m\}.$$

5.6. Corollary. *The polynomial representation of* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *on* ${}^\theta\mathbf{F}_\nu$ *is faithful. The inclusion of* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *into* $\mathrm{End}({}^\theta\mathbf{F}_\nu)$ *yields an isomorphism of* ${}^\theta\mathbf{F}'_\nu$*-algebras from* ${}^\theta\mathbf{F}'_\nu\otimes_{{}^\theta\mathbf{F}_\nu}{}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *to* ${}^\theta\mathbf{F}'_\nu\rtimes W_m$*, such that for each* $\mathbf{i}$ *and each* $l=1,2,\dots,m$*,* $k=1,2,\dots,m-1$ *we have*

$$\begin{aligned}&1_\mathbf{i}\mapsto 1_\mathbf{i},\\ &\varkappa_{\mathbf{i},l}\mapsto\varkappa_l 1_\mathbf{i},\\ &\pi_{\mathbf{i},1}\mapsto\varkappa_1^{\lambda_{i_0}}\varepsilon_1 1_\mathbf{i},\\ &\sigma_{\mathbf{i},k}\mapsto\begin{cases}(\varkappa_k-\varkappa_{k+1})^{-1}(s_k-1)1_\mathbf{i} & \text{if } i_k=i_{k+1},\\ (\varkappa_k-\varkappa_{k+1})^{h_{i_{k+1},i_k}}s_k 1_\mathbf{i} & \text{if } i_k\neq i_{k+1}.\end{cases}\end{aligned}\tag{5.3}$$

Restricting the ${}^\theta\mathbf{F}_\nu$-action on ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ to the subalgebra ${}^\theta\mathbf{S}_\nu$ of ${}^\theta\mathbf{F}_\nu$ we get a structure of graded ${}^\theta\mathbf{S}_\nu$-algebra on ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$.

5.7. Proposition. *(a)* ${}^\theta\mathbf{S}_\nu$ *is isomorphic to the center of* ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$.

(b) ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *is a free graded module over* ${}^\theta\mathbf{S}_\nu$ *of rank* $(2^m m!)^2$.

Proof : First we prove (a). Recall that

$$ {}^\theta\mathbf{S}_\nu = \mathbf{k}[\chi_1, \chi_2, \dots, \chi_m]^{W_\nu} = \Big(\bigoplus_{\mathbf{i}} \mathbf{k}[\varkappa_1, \varkappa_2, \dots, \varkappa_m]1_{\mathbf{i}}\Big)^{W_m}. $$

Given a sequence $\mathbf{i}$ in ${}^\theta I^\nu$ the assignment $x \mapsto x1_{\mathbf{i}}$ embeds ${}^\theta\mathbf{S}_\nu$ as a central subalgebra of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. We must check that this map surjects onto the center of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. This follows from Corollary 5.6. Part (b) follows from (a) and Proposition 5.5.

□

In Section 9 we'll prove the following theorem.

5.8. Theorem. *For any* $\nu \in {}^\theta\mathbb{N}I$, $\lambda \in \mathbb{N}I$ *there is an unique graded* ${}^\theta\mathbf{S}_{\mathbf{V}}$*-algebra isomorphism*

$$ \Psi : {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} \to {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}} $$

which intertwines the representations of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ *and* ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ *on* ${}^\theta\mathbf{F}_\nu$.

5.9. Examples. (a) If $m = 0$ then ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = \mathbf{k}$ by definition.

(b) Assume that $m = 1$. Fix a vertex i in I and set $\nu = i + \theta(i)$. We have ${}^\theta I^\nu = \{\mathbf{i}, \theta(\mathbf{i})\}$ with $\mathbf{i} = i\theta(i)$ and $\theta(\mathbf{i}) = \theta(i)i$. We have

$$ {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} = (\mathbf{k}[\varkappa_1] \oplus \pi_1\mathbf{k}[\varkappa_1])1_{\mathbf{i}} \oplus (\mathbf{k}[\varkappa_1] \oplus \pi_1\mathbf{k}[\varkappa_1])1_{\theta(\mathbf{i})}, $$

$$ \pi_1\varkappa_1 1_{\mathbf{i}} = -\varkappa_1\pi_1 1_{\mathbf{i}}, \quad \pi_1\varkappa_1 1_{\theta(\mathbf{i})} = -\varkappa_1\pi_1 1_{\theta(\mathbf{i})}, $$

$$ \pi_1^2 1_{\theta(\mathbf{i})} = (-1)^{\lambda_i}\varkappa_1^{\lambda_i+\lambda_{\theta(i)}} 1_{\theta(\mathbf{i})}, \quad \pi_1^2 1_{\mathbf{i}} = (-1)^{\lambda_{\theta(i)}}\varkappa_1^{\lambda_i+\lambda_{\theta(i)}} 1_{\mathbf{i}}. $$

The inclusion ${}^\theta\mathbf{S}_\nu \subset {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$ is given by

$$ \mathbf{k}[\chi] \to {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}, \quad \chi \mapsto (\varkappa_1 1_{\mathbf{i}}, 0, -\varkappa_1 1_{\theta(\mathbf{i})}, 0). $$

6. Affine Hecke algebras of type B

6.1. Affine Hecke algebras of type B. Fix p, q in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the extended affine Hecke algebra $\mathbf{H}_m$ of type B_m as follows (we use the same convention as in [Mc]). If $m > 0$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra generated by

$$ T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \dots, m-1, \quad l = 1, 2, \dots, m $$

satisfying the following defining relations :

(a) $X_l X_{l'} = X_{l'} X_l$,

(*b*) $(T_0T_1)^2 = (T_1T_0)^2$, $T_kT_{k-1}T_k = T_{k-1}T_kT_{k-1}$ if $k \neq 0, 1$, and $T_kT_{k'} = T_{k'}T_k$ if $|k - k'| \neq 1$,

(*c*) $T_0X_1^{-1}T_0 = X_1$, $T_kX_kT_k = X_{k+1}$ if $k \neq 0$, and $T_kX_l = X_lT_k$ if $l \neq k, k+1$,

(*d*) $(T_k - p)(T_k + p^{-1}) = 0$ if $k \neq 0$, and $(T_0 - q)(T_0 + q^{-1}) = 0$.

If $m = 0$ then $\mathbf{H}_0 = \mathbf{k}$, the trivial $\mathbf{k}$-algebra. Note that $\mathbf{H}_1$ is the $\mathbf{k}$-algebra generated by T_0, $X_1^{\pm 1}$ with the defining relations

$$T_0X_1^{-1}T_0 = X_1, \quad (T_0 - q)(T_0 + q^{-1}) = 0.$$

6.2. Intertwiners and blocks of $\mathbf{H}_m$. We define

$$\mathbf{A} = \mathbf{k}[X_1^{\pm 1}, X_2^{\pm 1}, \ldots, X_m^{\pm 1}], \quad \mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative set generated by

$$1 - X_lX_{l'}^{\pm 1}, \quad 1 - p^2X_lX_{l'}^{\pm 1}, \quad 1 - X_l^2, \quad 1 - q^2X_l^{\pm 2}, \quad l \neq l'.$$

For $k = 0, \ldots, m-1$ the intertwiner φ_k in $\mathbf{H}'_m$ is given by the following formulas

$$\varphi_k - 1 = \frac{X_k - X_{k+1}}{pX_k - p^{-1}X_{k+1}}\,(T_k - p) \quad \text{if } k \neq 0, \tag{6.1}$$
$$\varphi_0 - 1 = \frac{X_1^{-2} - 1}{qX_1^{-2} - q^{-1}}\,(T_0 - q).$$

The group W_m acts on $\mathbf{A}'$ as follows

$$(s_ka)(X_1, \ldots, X_m) = a(X_1, \ldots, X_{k+1}, X_k, \ldots, X_m),$$

$$(\varepsilon_1a)(X_1, \ldots, X_m) = a(X_1^{-1}, X_2, \ldots, X_m).$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes W_m \to \mathbf{H}'_m, \quad s_k \mapsto \varphi_k, \quad \varepsilon_1 \mapsto \varphi_0, \quad k \neq 0.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2 = \mathbb{Z} \rtimes \{-1, 1\}$ acts on $\mathbf{k}^\times$ by $(n, \varepsilon) : z \mapsto z^\varepsilon p^{2n}$. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m$-$\mathbf{Mod}_I$ the category of all finitely generated $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \ldots, X_m$ is locally finite and all the eigenvalues belong to I. We associate to the set I the quiver Γ with set of vertices I and with one arrow $i \to p^2i$ whenever i, p^2i lie in I. We equip Γ with an involution θ such that $\theta(i) = i^{-1}$ for each vertex i and such that θ takes the arrow $i \to p^2i$ to the arrow $p^{-2}i^{-1} \to i^{-1}$. *We'll assume that the set I does not contain* 1 *nor* -1 *and that* $p \neq 1, -1$. Thus the involution θ has no fixed points and no arrow may join a vertex of Γ to itself.

6.3. Remark. We may assume that either I is a $\mathbb{Z}$-orbit or I contains at least one of $\pm q$, see the discussion in [EK1]. Thus, we can assume that one of the following two cases holds :

(*a*) I is a $\mathbb{Z}$-orbit which does not contain 1, -1, q, $-q$. So either $I = \{p^n; n \in \mathbb{Z}_{\mathrm{odd}}\}$ or $I = \{-p^n; n \in \mathbb{Z}_{\mathrm{odd}}\}$. Then Γ is of type A_∞ if p has infinite order and Γ is of type $\mathrm{A}_r^{(1)}$ if p^2 is a primitive r-th root of unity.

(*b*) $q \in I$ (the case $-q \in I$ is similar) and $-1, 1 \notin I$. Then we have $I = \{qp^{2n}; n \in \mathbb{Z}\} \cup \{q^{-1}p^{2n}; n \in \mathbb{Z}\}$ with $q^2 \neq p^{4n}$ for all $n \in \mathbb{Z}$. Thus Γ is of type A_∞, $\mathrm{A}_\infty \times \mathrm{A}_\infty$, $\mathrm{A}_r^{(1)}$, or $\mathrm{A}_r^{(1)} \times \mathrm{A}_r^{(1)}$.

6.4. $\mathbf{H}_m$-modules versus ${}^\theta\mathbf{R}_m$-modules. Given an element λ of $\mathbb{N}I$ we define the graded $\mathbf{k}$-algebra

$$ {}^\theta\mathbf{R}_{I,\lambda,m}=\bigoplus_\nu {}^\theta\mathbf{R}_{I,\lambda,\nu},\quad {}^\theta\mathbf{R}_{I,\lambda,\nu}={}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu},\quad {}^\theta I^m=\coprod_\nu {}^\theta I^\nu, $$

where ν runs over the set of all dimension vectors in ${}^\theta\mathbb{N}I$ such that $|\nu|=2m$. When there is no risk of confusion we abbreviate ${}^\theta\mathbf{R}_m={}^\theta\mathbf{R}_{I,\lambda,m}$. Note that the $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$ may not have 1, because the set I may be infinite, and that ${}^\theta\mathbf{R}_0=\mathbf{k}$ as a graded $\mathbf{k}$-algebra. From now on, unless specified otherwise we'll set

$$ \lambda=\sum_i i,\quad i\in I\cap\{q,-q\}. \tag{6.2} $$

Given sequences

$$ \mathbf{i}=(i_{1-m},\dots,i_{m-1},i_m),\quad \mathbf{i}'=(i'_1,\dots,i'_{m'-1},i'_{m'}), $$

we define a sequence $\theta(\mathbf{i}')\mathbf{i}\mathbf{i}'$ in as follows

$$ \theta(\mathbf{i}')\mathbf{i}\mathbf{i}'=(\theta(i'_{m'}),\dots,\theta(i'_1),i_{1-m},\dots,i_m,i'_1,\dots,i'_{m'}). $$

Let ν, ν' be dimension vectors in ${}^\theta\mathbb{N}I$ and $\mathbb{N}I$ respectively such that $|\nu|=2m$, $|\nu'|=m'$, and $m+m'=m''$. We define an idempotent in ${}^\theta\mathbf{R}_{m''}$ by

$$ 1_{\nu,\nu'}=\sum_{\mathbf{i},\mathbf{i}'}1_{\theta(\mathbf{i}')\mathbf{i}\mathbf{i}'},\quad \mathbf{i}\in{}^\theta I^\nu,\ \mathbf{i}'\in I^{\nu'}. $$

Given $\nu'_1,\nu'_2,\dots,\nu'_r$ in $\mathbb{N}I$ we define $1_{\nu,\nu'_1,\dots,\nu'_r}$ in the same way. Finally, for any graded ${}^\theta\mathbf{R}_{m''}$-module M we set

$$ 1_{m,\nu'}M=\bigoplus_{\mathbf{i},\mathbf{i}'}1_{\theta(\mathbf{i}')\mathbf{i}\mathbf{i}'}M,\quad \mathbf{i}\in{}^\theta I^m,\ \mathbf{i}'\in I^{\nu'}. \tag{6.3} $$

If M is a right graded ${}^\theta\mathbf{R}_{m''}$-module se define $M1_{m,\nu'}$ in the same way.

We define ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0$ as the category of all finitely generated ${}^\theta\mathbf{R}_m$-modules such that the elements $\varkappa_1,\varkappa_2,\dots,\varkappa_m$ act locally nilpotently. Let ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0^f$ and $\mathbf{H}_m$-$\mathbf{Mod}_I^f$ be the full subcategories of finite dimensional modules in ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0$ and $\mathbf{H}_m$-$\mathbf{Mod}_I$ respectively.

6.5. Theorem. *We have an equivalence of categories*

$$ {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0\to\mathbf{H}_m\text{-}\mathbf{Mod}_I,\quad M\mapsto M $$

which is given by

(a) X_l acts on $1_\mathbf{i}M$ by $i_l^{-1}e^{\varkappa_l}$ for each $l=1,2,\dots,m$,

(b) T_k acts on $1_\mathbf{i}M$ as follows for each $k=1,2,\dots,m-1$,

$$ \frac{(pe^{\varkappa_k}-p^{-1}e^{\varkappa_{k+1}})(\varkappa_k-\varkappa_{k+1})}{e^{\varkappa_k}-e^{\varkappa_{k+1}}}\sigma_k+p \qquad \text{if } i_k=i_{k+1}, $$

$$ \frac{e^{\varkappa_k}-e^{\varkappa_{k+1}}}{(p^{-1}e^{\varkappa_k}-pe^{\varkappa_{k+1}})(\varkappa_k-\varkappa_{k+1})}\sigma_k-\frac{(p^2-1)e^{\varkappa_{k+1}}}{p^{-1}e^{\varkappa_k}-pe^{\varkappa_{k+1}}} \qquad \text{if } i_k=p^2i_{k+1}, $$

$$ \frac{pi_{k+1}e^{\varkappa_k}-p^{-1}i_ke^{\varkappa_{k+1}}}{i_{k+1}e^{\varkappa_k}-i_ke^{\varkappa_{k+1}}}(\sigma_k-1)+p \qquad \text{if } i_k\neq i_{k+1},p^2i_{k+1}, $$

(*c*) *T_0 acts on $1_{\mathbf{i}}M$ as follows*

$$\begin{aligned} &\frac{1-e^{2\varkappa_1}}{(q^{-1}-qe^{2\varkappa_1})\varkappa_1}\pi_1 - \frac{(q^2-1)e^{2\varkappa_1}}{q^{-1}-qe^{2\varkappa_1}} && \text{if } i_1=\pm q^{-1}, \\ &\frac{q-q^{-1}i_1^{-2}e^{2\varkappa_1}}{1-i_1^{-2}e^{2\varkappa_1}}(\pi_1-1)+q && \text{if } i_1\neq \pm q^{-1}. \end{aligned}$$

Proof : First, recall that $\pm 1\notin I$ and that $p\neq\pm 1$. Observe also that (6.2) yields

$$i_1=\pm q^{-1} \iff i_0=\pm q \iff \lambda_{i_0}=1,$$

$$i_1\neq\pm q^{-1} \iff i_0\neq\pm q \iff \lambda_{i_0}=0.$$

The functor above is well defined by formulas (5.3) and (6.1). A quasi-inverse functor $\mathbf{H}_m\text{-}\mathbf{Mod}_I \to {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0$ such that $M\mapsto M$ is given by the following rules

(*a*) $1_{\mathbf{i}}M=\{m\in M;\,(i_lX_l-1)^r m=0,\ r\gg 0\}$,

(*b*) $\varkappa_l$ acts on $1_{\mathbf{i}}M$ by $\log(i_lX_l)$ for $l=1,2,\dots,m$,

(*c*) σ_k acts on $1_{\mathbf{i}}M$ as follows for each $k=1,2,\dots,m-1$,

$$\begin{aligned} &\frac{X_k-X_{k+1}}{(pX_k-p^{-1}X_{k+1})\log(X_k/X_{k+1})}(T_k+p) && \text{if } i_k=i_{k+1}, \\ &\frac{\log(p^2X_k/X_{k+1})}{pX_k-p^{-1}X_{k+1}}\Big((X_k-X_{k+1})T_k+(p-p^{-1})X_{k+1}\Big) && \text{if } i_k=p^2 i_{k+1}, \\ &\frac{X_k-X_{k+1}}{pX_k-p^{-1}X_{k+1}}(T_k-p)+1 && \text{if } i_k\neq i_{k+1}, p^2 i_{k+1}, \end{aligned}$$

(*d*) π_1 acts on $1_{\mathbf{i}}M$ as follows

$$\begin{aligned} &\frac{(1-X_1^2)\log(q^{-2}X_1^2)}{2q(1-q^{-2}X_1^2)}T_0+\frac{(1-q^{-2})\log(q^{-2}X_1^2)}{2(1-q^{-2}X_1^2)}X_1^2 && \text{if } i_1=\pm q^{-1}, \\ &\frac{1-X_1^2}{q-q^{-1}X_1^2}(T_0-q)+1 && \text{if } i_1\neq\pm q^{-1}. \end{aligned}$$

□

6.6. Corollary. *There is an equivalence of categories*

$$\Psi : {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f \to \mathbf{H}_m\text{-}\mathbf{Mod}_I^f,\quad M\mapsto M.$$

6.7. Example. Let $m=1$. Using Example 5.9(*b*) it is easy to check that the 1-dimensional ${}^\theta\mathbf{R}_m$-modules are labelled by $\{i\in I\,;\,\lambda_i+\lambda_{\theta(i)}\neq 0\}$, and that the irreducible 2-dimensional ${}^\theta\mathbf{R}_m$-modules are labelled by $\{i\in I\,;\,\lambda_i+\lambda_{\theta(i)}=0\}/\theta$. Now, observe that $\lambda_i+\lambda_{\theta(i)}\neq 0$ iff $i=q,-q,q^{-1}$, or $-q^{-1}$. Finally, the 1-dimensional objects in $\mathbf{H}_m\text{-}\mathbf{Mod}_I$ are given by

(*a*) $X_1=T_0=i^{-1}$ with $i\in I\cap\{-q,q^{-1}\}$, (*b*) $X_1 = -T_0 = i^{-1}$ with $i \in I \cap \{q, -q^{-1}\}$,

and the irreducible 2-dimensional objects in $\mathbf{H}_m$-$\mathbf{Mod}_I$ are given by

(*c*) $X_1 = \begin{pmatrix} i & 0 \\ 0 & i^{-1} \end{pmatrix}$, $T_0 = \begin{pmatrix} -i^2a/b & a^2 - b^2/i^2 \\ -i^2/b^2 & a/b \end{pmatrix}$ with $a = q - q^{-1}$, $b = 1 - i^2$, and $i \neq q, -q, q^{-1}, -q^{-1}$.

6.8. Induction and restriction of $\mathbf{H}_m$-modules. For $i \in I$ we define functors

$$E_i : \mathbf{H}_m\text{-}\mathbf{Mod}_I^f \to \mathbf{H}_{m-1}\text{-}\mathbf{Mod}_I^f, \tag{6.4}$$

$$F_i : \mathbf{H}_m\text{-}\mathbf{Mod}_I^f \to \mathbf{H}_{m+1}\text{-}\mathbf{Mod}_I^f,$$

where $E_iM \subset M$ is the generalized $\theta(i)$-eigenspace of the X_m-action, and where

$$F_iM = \mathrm{Ind}_{\mathbf{H}_m \otimes \mathbf{k}[X_{m+1}^{\pm 1}]}^{\mathbf{H}_{m+1}}(M \otimes \mathbf{k}_i).$$

Here $\mathbf{k}_i$ is the 1-dimensional representation of $\mathbf{k}[X_{m+1}^{\pm 1}]$ defined by $X_{m+1} \mapsto \theta(i)$.

7. Global bases of $\mathbf{f}$ and projective graded modules of KLR algebras

This section is a reminder on KLR algebras. Most of the results here are due to [KL1]. Although we are essentially concerned by KLR algebras of type A, everything here holds true in any type.

7.1. Definition of the graded k-algebra $\mathbf{R}_m$. Fix a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset $I \subset \mathbf{k}^\times$ as in Section 6.2. Let Γ be the corresponding quiver. For each integer $m \geqslant 0$ we put

$$\mathbf{R}(I)_m = \bigoplus_\nu \mathbf{R}(I)_\nu, \quad \mathbf{R}(I)_\nu = \mathbf{R}(\Gamma)_\nu,$$

where ν runs over the set of all dimension vectors in $\mathbb{N}I$ such that $|\nu| = m$. When there is no risk of confusion we abbreviate $\mathbf{R}_m = \mathbf{R}(I)_m$. The graded $\mathbf{k}$-algebra $\mathbf{R}_m$ is generated by elements $1_\mathbf{i}$, $\varkappa_{\mathbf{i},l}$, $\sigma_{\mathbf{i},k}$ with $\mathbf{i} \in I^m$, $l = 1, 2, \dots, m$ and $k = 1, 2, \dots, m-1$ satisfying the following defining relations

(*a*) $1_\mathbf{i} 1_{\mathbf{i}'} = \delta_{\mathbf{i},\mathbf{i}'} 1_\mathbf{i}, \quad \sigma_{\mathbf{i},k} = 1_{s_k\mathbf{i}} \sigma_{\mathbf{i},k} 1_\mathbf{i}, \quad \varkappa_{\mathbf{i},l} = 1_\mathbf{i} \varkappa_{\mathbf{i},l} 1_\mathbf{i}$,

(*b*) $\varkappa_l \varkappa_{l'} = \varkappa_{l'} \varkappa_l$,

(*c*) $\sigma_k^2 1_\mathbf{i} = Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_k) 1_\mathbf{i}$,

(*d*) $\sigma_k \sigma_{k'} = \sigma_{k'} \sigma_k$ if $|k - k'| > 1$,

(*e*) $(\sigma_{k+1}\sigma_k\sigma_{k+1} - \sigma_k\sigma_{k+1}\sigma_k) 1_\mathbf{i} =$

$$= \begin{cases} \dfrac{Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_k) - Q_{i_k,i_{k+1}}(\varkappa_{k+1}, \varkappa_{k+2})}{\varkappa_k - \varkappa_{k+2}} 1_\mathbf{i} & \text{if } i_k = i_{k+2}, \\ 0 & \text{else,} \end{cases}$$

$$(f)\ \ (\sigma_k\varkappa_{k'}-\varkappa_{s_k(k')}\sigma_k)1_{\mathbf{i}}=\begin{cases}-1_{\mathbf{i}} & \text{if } k'=k,\ i_k=i_{k+1},\\ 1_{\mathbf{i}} & \text{if } k'=k+1,\ i_k=i_{k+1},\\ 0 & \text{else.}\end{cases}$$

Here $Q_{i,j}(u,v)$ is as in (5.1). The grading on $\mathbf{R}_m$ is given by the following rules : $1_{\mathbf{i}}$ has the degree 0, $\varkappa_{\mathbf{i},l}$ has the degree 2, and $\sigma_{\mathbf{i},k}$ has the degree $-i_k\cdot i_{k+1}$. Given any element a in $1_{\mathbf{i}}\mathbf{R}_m 1_{\mathbf{i}'}$ we write $\varkappa_k a$ for $\varkappa_{\mathbf{i},k}a$, etc. Note that the $\mathbf{k}$-algebra $\mathbf{R}_m$ may not have 1, because the set I may be infinite. If $m=0$ we have $\mathbf{R}_m=\mathbf{k}$ as a graded $\mathbf{k}$-algebra.

Let ω be the unique anti-involution of the graded $\mathbf{k}$-algebra $\mathbf{R}_m$ given by

$$\omega:\ 1_{\mathbf{i}},\ \varkappa_l,\ \sigma_k\mapsto 1_{\mathbf{i}},\ \varkappa_l,\ \sigma_k.$$

Note that $Q_{i,j}(u,v)=Q_{j,i}(v,u)$. Hence there is an unique involution τ of the graded $\mathbf{k}$-algebra $\mathbf{R}_m$ such that

$$\tau:\ 1_{\mathbf{i}},\ \varkappa_l,\ \sigma_k\mapsto 1_{w_0(\mathbf{i})},\ \varkappa_{m+1-l},\ -\sigma_{m-k},$$

where w_0 is the longest element in $\mathfrak{S}_m$. Finally, we have $Q_{i,j}(u,v)=Q_{\theta(i),\theta(j)}(-u,-v)$. Hence there is an unique involution

$$\iota:\ 1_{\mathbf{i}},\ \varkappa_l,\ \sigma_k\mapsto 1_{\theta(\mathbf{i})},\ -\varkappa_l,\ -\sigma_k.$$

We define

$$\kappa=\iota\circ\tau=\tau\circ\iota. \tag{7.1}$$

7.2. The Grothendieck groups of $\mathbf{R}_m$. The graded $\mathbf{k}$-algebra $\mathbf{R}_m$ is finite dimensional over its center, a commutative graded $\mathbf{k}$-subalgebra. Therefore any simple object of $\mathbf{R}_m$**-mod** is finite-dimensional and there is a finite number of simple modules in $\mathbf{R}_m$**-mod**. The Abelian group $G(\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of $\mathbf{R}_m$**-mod**. The Abelian group $K(\mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both are free $\mathcal{A}$-modules where v shifts the grading by 1. We define

$$\mathbf{K}_I=\bigoplus_{m\geqslant 0}\mathbf{K}_{I,m},\quad \mathbf{K}_{I,m}=K(\mathbf{R}_m),$$

$$\mathbf{G}_I=\bigoplus_{m\geqslant 0}\mathbf{G}_{I,m},\quad \mathbf{G}_{I,m}=G(\mathbf{R}_m).$$

The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$$\mathbf{K}_I\times\mathbf{G}_I\to\mathcal{A},\quad \langle P:M\rangle=\operatorname{gdim}\hom_{\mathbf{R}}(P,M).$$

Now, fix integers $m,m',m''\geqslant 0$ with $m''=m+m'$. Given sequences $\mathbf{i}\in I^m$ and $\mathbf{i}'\in I^{m'}$ we write $\mathbf{i}''=\mathbf{i}\mathbf{i}'$. We abbreviate

$$\mathbf{R}_{m,m'}=\mathbf{R}_m\otimes\mathbf{R}_{m'}.$$

There is an unique inclusion of graded **k**-algebras

$$\begin{aligned}
&\mathbf{R}_{m,m'} \to \mathbf{R}_{m''},\\
&1_{\mathbf{i}} \otimes 1_{\mathbf{i}'} \mapsto 1_{\mathbf{i}''},\\
&\varkappa_{\mathbf{i},l} \otimes 1_{\mathbf{i}'} \mapsto \varkappa_{\mathbf{i}'',l},\\
&1_{\mathbf{i}} \otimes \varkappa_{\mathbf{i}',l} \mapsto \varkappa_{\mathbf{i}'',m+l},\\
&\sigma_{\mathbf{i},k} \otimes 1_{\mathbf{i}'} \mapsto \sigma_{\mathbf{i}'',k},\\
&1_{\mathbf{i}} \otimes \sigma_{\mathbf{i}',k} \mapsto \sigma_{\mathbf{i}'',m+k}.
\end{aligned} \tag{7.2}$$

We have a triple of adjoint functors $(\phi_!, \phi^*, \phi_*)$ where

$$\phi^* \,:\, \mathbf{R}_{m''}\text{-}\mathbf{mod} \to \mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod}$$

is the restriction and $\phi_!$, ϕ_* are given by

$$\phi_! \,:\, \begin{cases} \mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to \mathbf{R}_{m''}\text{-}\mathbf{mod}, \\ (M, M') \mapsto \mathbf{R}_{m''} \otimes_{\mathbf{R}_{m,m'}} (M \otimes M'), \end{cases}$$

$$\phi_* \,:\, \begin{cases} \mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to \mathbf{R}_{m''}\text{-}\mathbf{mod}, \\ (M, M') \mapsto \hom_{\mathbf{R}_{m,m'}}(\mathbf{R}_{m''}, M \otimes M'). \end{cases}$$

The functor ϕ^* is exact, it commutes with the shift of the grading, and it takes finite dimensional modules to finite dimensional modules. Thus $\phi_!$ takes projectives to projectives and it commutes with the shift of the grading. By [KL1, prop. 2.16] the right graded $\mathbf{R}_{m,m'}$-module $\mathbf{R}_{m''}$ is free of finite rank. Thus $\phi_!$ is exact and it takes finite dimensional modules to finite dimensional ones. For the same reason the left graded $\mathbf{R}_{m,m'}$-module $\mathbf{R}_{m''}$ is free of finite rank. Thus ϕ_* is exact and ϕ^* takes projective modules to projective ones. To summarize, the functors $\phi_!$, ϕ^*, ϕ_* are exact, they commute with the shift of the grading, they take projective modules to projective ones and finite dimensional modules to finite dimensional ones. Taking the sum over all m, m' we get an $\mathcal{A}$-bilinear map

$$\phi_! \,:\, \mathbf{K}_I \times \mathbf{K}_I \to \mathbf{K}_I.$$

In the same way we define also an $\mathcal{A}$-linear map

$$\phi^* \,:\, \mathbf{K}_I \to \mathbf{K}_I \otimes_{\mathcal{A}} \mathbf{K}_I.$$

From now on, to unburden the notation we may abbreviate $\mathbf{R} = \mathbf{R}_m$, hopping it will not create any confusion. Recall the anti-automorphism ω from the previous section. Consider the duality

$$\mathbf{R}\text{-}\mathbf{proj} \to \mathbf{R}\text{-}\mathbf{proj}, \quad P \mapsto P^\sharp = \hom_{\mathbf{R}}(P, {}^\theta\mathbf{R}),$$

with the action and the grading given by

$$(xf)(p) = f(p)\omega(x), \quad (P^\sharp)_i = \hom(P_i, \mathbf{k}).$$

The duality on **R-mod** yields an $\mathcal{A}$-antilinear map

$$\mathbf{K}_I \to \mathbf{K}_I, \quad P \mapsto P^\sharp.$$

The $\mathcal{A}$-module $\mathbf{K}_I$ comes equipped with a symmetric bilinear form

$$\mathbf{K}_I \times \mathbf{K}_I \to \mathcal{A}, \quad (P : Q) = \mathrm{gdim}(P^\omega \otimes_{\mathbf{R}} Q).$$

Finally, we equip $\mathbf{K}_I \otimes_{\mathcal{A}} \mathbf{K}_I$ with the algebra structure such that

$$(P \otimes Q : P' \otimes Q') \mapsto v^{-\mu \cdot \nu'} \phi_!(P : P') \otimes \phi_!(Q : Q'),$$

and with the $\mathcal{A}$-antilinear map such that

$$P \otimes Q \mapsto (P \otimes Q)^\sharp = P^\sharp \otimes Q^\sharp.$$

7.3. Proposition. *The map $\phi_!$ turns $\mathbf{K}_I$ into an associative $\mathcal{A}$-algebra with 1, and it commutes with the duality $\sharp$. The map ϕ^* turns $\mathbf{K}_I$ into a coassociative $\mathcal{A}$-coalgebra with 1, and it is an algebra homomorphism $\mathbf{K}_I \to \mathbf{K}_I \otimes_{\mathcal{A}} \mathbf{K}_I$.*

Proof : See [KL1]. The last claim follows from a version of the Mackey's induction-restriction theorem.

□

7.4. The projective graded $\mathbf{R}_m$-module $\mathbf{R}_\mathbf{y}$. Fix an element ν in $\mathbb{N}I$ with $|\nu| = m$. For $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ in Y^ν we define an object $\mathbf{R}_\mathbf{y}$ in $\mathbf{R}_m$-**proj** as follows.

- (a) If $\mathbf{i} = i^m$, $i \in I$, and $\mathbf{a} = m$ then we set $\mathbf{R}_\mathbf{y} = \mathbf{F}_\nu[\ell_m]$. As a left graded $\mathbf{R}_\nu$-modules we have a canonical isomorphism $\mathbf{R}_\nu = \bigoplus_{w \in \mathfrak{S}_m} \mathbf{R}_\mathbf{y}[2\ell(w) - \ell_m]$. We choose once for all an idempotent $1_\mathbf{y}$ in $\mathbf{R}_m$ such that $\mathbf{R}_\mathbf{y} = (\mathbf{R}_m 1_\mathbf{y})[\ell_m]$.
- (b) If $\mathbf{i} = (i_1, \dots, i_k)$ and $\mathbf{a} = (a_1, \dots a_k)$ we define the idempotent $1_\mathbf{y}$ as the image of the element $\bigotimes_{l=1}^k 1_{(i_l)^{a_l}, a_l}$ by the inclusion of graded $\mathbf{k}$-algebras $\bigotimes_{l=1}^k \mathbf{R}_{a_l i_l} \subset \mathbf{R}_\nu$ in (7.2). Then we set $\mathbf{R}_\mathbf{y} = (\mathbf{R}_\nu 1_\mathbf{y})[\ell_\mathbf{a}]$.

The graded module $\mathbf{R}_\mathbf{y}$ satisfies the following properties.

- (c) Let $\mathbf{i}' \in I^\nu$ be the sequence obtained by expanding the pair $\mathbf{y} = (\mathbf{i}, \mathbf{a})$. We have the following formula in $\mathbf{R}_m$-**proj**

$$\mathbf{R}_{\mathbf{i}'} = \mathbf{R}1_{\mathbf{i}'} = \bigoplus_{w \in \mathfrak{S}_\mathbf{a}} \mathbf{R}_\mathbf{y}[2\ell(w) - \ell_\mathbf{a}] =: \langle \mathbf{a} \rangle ! \, \mathbf{R}_\mathbf{y}.$$

- (d) Given $\mathbf{y} = (\mathbf{i}, \mathbf{a}) \in Y_\nu$ and $\mathbf{y}' = (\mathbf{i}', \mathbf{a}') \in Y_{\nu'}$ we set $\mathbf{y}\mathbf{y}' = (\mathbf{i}\mathbf{i}', \mathbf{a}\mathbf{a}')$. We have an isomorphism of graded $\mathbf{R}_{\nu''}$-modules $\phi_!(\mathbf{R}_\mathbf{y} \otimes \mathbf{R}_{\mathbf{y}'}) = \mathbf{R}_{\mathbf{y}''}$.
- (e) We have $\mathbf{R}_\mathbf{y}[d]^\sharp = \mathbf{R}_\mathbf{y}[-d]$ in $\mathbf{R}_m$-**proj** for each integer d. This follows from part (c) because the graded module $\mathbf{R}_{\mathbf{i}'}$ is selfdual, i.e., it is fixed by the duality functor.

7.5. Examples. (a) The graded $\mathbf{k}$-algebra $\mathbf{R}_1$ is generated by elements 1_i, $\varkappa_i$, $i \in I$, satisfying the defining relations $1_i\,1_{i'} = \delta_{i,i'}1_i$ and $\varkappa_i = 1_i\varkappa_i 1_i$. We have

$$\mathbf{R}_i = 1_i\mathbf{R}_1 = \mathbf{R}_1 1_i, \quad \mathrm{top}(\mathbf{R}_i) = \mathbf{R}_i/(\varkappa_i) =: \mathbf{L}_i.$$

(b) Let $\mathcal{B} = \mathbb{Z}((v))$ and let $\mathcal{B}I^m$ be the free $\mathcal{B}$-module with basis I^m. The *character* of any finitely generated graded $\mathbf{R}_m$-module M is the element of $\mathcal{B}I^m$ given by

$$\mathrm{ch}(M) = \sum_{\mathbf{i}} \mathrm{gdim}(1_{\mathbf{i}}M)\,\mathbf{i}.$$

Let $\mathbf{L}_{mi} = \mathrm{top}(\mathbf{R}_{mi})$, where $\mathbf{R}_{mi}$ is regarded as a graded $\mathbf{R}_m$-module. We have

$$\mathrm{ch}(\mathbf{R}_{mi}) \in v^{-\ell_m}\mathbb{Z}[[v^2]]\,i^m, \quad \mathrm{ch}(\mathbf{L}_{mi}) = \langle m\rangle!\,i^m.$$

Further the graded $\mathbf{R}_{m-1}\otimes\mathbf{R}_1$-module $\mathbf{L}_{mi}$ has a filtration by graded submodules such that the associated graded is isomorphic to $[m]\,\mathbf{L}_{(m-1)i}\otimes\mathbf{L}_i$, and the socle of $\mathbf{L}_{mi}$ as a graded $\mathbf{R}_{m-n}\otimes\mathbf{R}_n$-module is equal to $\mathbf{L}_{(m-n)i}\otimes\mathbf{L}_{ni}$ for each $m \geqslant n \geqslant 0$.

7.6. Categorification of the global bases of f. Set $\mathcal{K} = \mathbb{Q}(v)$. Let $\mathbf{f}$ be the $\mathcal{K}$-algebra generated by elements θ_i, $i \in I$, with the defining relations

$$\sum_{a+b=1-i\cdot j} (-1)^a\theta_i^{(a)}\theta_j\theta_i^{(b)} = 0, \quad i \neq j, \quad \theta_i^{(a)} = \theta_i^a/\langle a\rangle!, \quad a \geqslant 0. \tag{7.3}$$

We have a weight decomposition $\mathbf{f} = \bigoplus_{\nu\in\mathbb{N}I}\mathbf{f}_\nu$. We equip the tensor square of $\mathbf{f}$ with the $\mathcal{K}$-algebra structure such that

$$(x\otimes y)(x'\otimes y') = v^{-\mu\cdot\nu'}xx'\otimes yy', \quad x\in\mathbf{f}_\nu,\ x'\in\mathbf{f}_{\nu'},\ y\in\mathbf{f}_\mu,\ y'\in\mathbf{f}_{\mu'}.$$

Consider the $\mathcal{K}$-algebra homomorphism such that

$$r : \mathbf{f}\to\mathbf{f}\otimes\mathbf{f}, \quad \theta_i \mapsto \theta_i\otimes 1 + 1\otimes\theta_i, \quad \forall i.$$

The $\mathcal{K}$-algebra $\mathbf{f}$ comes equipped with a bilinear form $(\bullet : \bullet)$ which is uniquely determined by the following conditions

(a) $(1:1) = 1$,

(b) $(\theta_i : \theta_j) = \delta_{i,j}(1-v^2)^{-1}$ for all $i,j\in I$,

(c) $(x : yy') = (r(x) : y\otimes y')$ for all x, y, y',

(d) $(xx' : y) = (x\otimes x' : r(y))$ for all x, x', y.

This bilinear form is symmetric and non-degenerate. Let ${}_{\mathcal{A}}\mathbf{f}$ be the $\mathcal{A}$-submodule of $\mathbf{f}$ generated by all products of the elements $\theta_i^{(a)}$ with $a\in\mathbb{Z}_{\geqslant 0}$ and $i\in I$. We set ${}_{\mathcal{A}}\mathbf{f}_\nu$ equal to ${}_{\mathcal{A}}\mathbf{f}\cap\mathbf{f}_\nu$. The element θ_i lies in ${}_{\mathcal{A}}\mathbf{f}_i$ for each $i\in I$. Let $\theta^i \in {}_{\mathcal{A}}\mathbf{f}_i^*$ be the element dual to θ_i. For each pair $\mathbf{y} = (\mathbf{i},\mathbf{a})$ in Y^ν with $\mathbf{i} = (i_1,\dots,i_k)$, $\mathbf{a} = (a_1,\dots a_k)$ we write $\theta_{\mathbf{y}} = \theta_{i_1}^{(a_1)}\theta_{i_2}^{(a_2)}\cdots\theta_{i_k}^{(a_k)}$.

Let $\mathbf{G}^{\mathrm{low}}$ be the *canonical basis (=the lower global basis)* of $\mathbf{f}$. It is a $\mathcal{A}$-basis of ${}_{\mathcal{A}}\mathbf{f}$. The *upper global basis of* $\mathbf{f}$ is the $\mathcal{K}$-basis $\mathbf{G}^{\mathrm{up}}$ which is dual to $\mathbf{G}^{\mathrm{low}}$ with respect to the inner product $(\bullet : \bullet)$. We may regard $\mathbf{G}^{\mathrm{up}}$ as a $\mathcal{K}$-basis of $\mathbf{f}^*$. Similarly, we may regard ${}_{\mathcal{A}}\mathbf{f}^*$ as an $\mathcal{A}$-submodule of $\mathbf{f}$ via $\langle\bullet:\bullet\rangle$. Let $B(\infty)$ be the set of irreducible (non graded) $\mathbf{R}$-modules such that the elements $\varkappa_1,\varkappa_2,\dots,\varkappa_m$ act nilpotently.

7.7. Theorem. *(a) There is an unique $\mathcal{A}$-algebra isomorphism $\gamma : {}_{\mathcal{A}}\mathbf{f} \to \mathbf{K}_I$ which intertwines r and ϕ^*, and such that $\gamma(\theta_{\mathbf{y}}) = \mathbf{R}_{\mathbf{y}}$ for each $\mathbf{y}$.*

(b) We have $\mathbf{G}^{\mathrm{low}} = \{G^{\mathrm{low}}(b);\, b \in B(\infty)\}$, where $\gamma(G^{\mathrm{low}}(b))$ is the unique selfdual indecomposable projective graded module whose top is isomorphic to b. The map γ takes the bilinear form $(\bullet : \bullet)$ and the involution $\bar{\bullet}$ on ${}_{\mathcal{A}}\mathbf{f}$ to the bilinear form $(\bullet : \bullet)$ and the involution $\bullet^\sharp$ on $\mathbf{K}_I$.

(c) The transpose map $\mathbf{G}_I \to {}_{\mathcal{A}}\mathbf{f}^*$ takes the $\mathcal{A}$-basis of $\mathbf{G}_I$ of the selfdual simple objects to $\mathbf{G}^{\mathrm{up}}$. We have ${}^t\gamma(\mathbf{L}_i) = \theta^i$ for all $i \in I$, and $\mathbf{G}^{\mathrm{up}} = \{G^{\mathrm{up}}(b);\, b \in B(\infty)\}$ with $G^{\mathrm{up}}(b) = {}^t\gamma\,\mathrm{top}\,\gamma\, G^{\mathrm{low}}(b)$.

Proof : Claim (a), and the second part of (b) are due to [KL1, prop. 3.4]. The first part of (b) is due to [VV] (the same result has also been anounced by R. Rouquier). Part (c) follows form (b). For instance, the last claim in (c) is as proved as follows. Let $\langle \bullet : \bullet \rangle$ denote both the Cartan pairing and the canonical pairing

$$ {}_{\mathcal{A}}\mathbf{f} \times {}_{\mathcal{A}}\mathbf{f}^* \to \mathcal{A}. $$

Then we have

$$ \langle \mathbf{b}' : {}^t\gamma\,\mathrm{top}\,\gamma(\mathbf{b}) \rangle = \langle \gamma(\mathbf{b}') : \mathrm{top}\,\gamma(\mathbf{b}) \rangle = \delta_{\mathbf{b},\mathbf{b}'}. $$

□

8. Global bases of ${}^\theta\mathbf{V}(\lambda)$ and projective graded ${}^\theta\mathbf{R}$-modules

Given an integer $m \geqslant 0$ we consider the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$ introduced in Section 6.4.

8.1. The Grothendieck groups of ${}^\theta\mathbf{R}_m$. The graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_m$ is free of finite rank over its center, a commutative graded $\mathbf{k}$-subalgebra. Therefore any simple object of ${}^\theta\mathbf{R}_m$**-mod** is finite-dimensional and there is a finite number of isomorphism classes of simple modules in ${}^\theta\mathbf{R}_m$**-mod**. The Abelian group $G({}^\theta\mathbf{R}_m)$ is free with a basis formed by the classes of the simple objects of ${}^\theta\mathbf{R}_m$**-mod**. The Abelian group $K({}^\theta\mathbf{R}_m)$ is free with a basis formed by the classes of the indecomposable projective objects. Both are free $\mathcal{A}$-modules where v shifts the grading by 1. We consider the following $\mathcal{A}$-modules

$$ {}^\theta\mathbf{K}_I = \bigoplus_{m \geqslant 0} {}^\theta\mathbf{K}_{I,m}, \quad {}^\theta\mathbf{K}_{I,m} = K({}^\theta\mathbf{R}_m), $$

$$ {}^\theta\mathbf{G}_I = \bigoplus_{m \geqslant 0} {}^\theta\mathbf{G}_{I,m}, \quad {}^\theta\mathbf{G}_{I,m} = G({}^\theta\mathbf{R}_m). $$

From now on, to unburden the notation we may abbreviate ${}^\theta\mathbf{R} = {}^\theta\mathbf{R}_m$, hopping it will not create any confusion. For any M, N in ${}^\theta\mathbf{R}$**-mod** we set

$$ (M : N) = \mathrm{gdim}(M^\omega \otimes_{{}^\theta\mathbf{R}} N), \quad \langle M : N \rangle = \mathrm{gdim}\,\mathrm{hom}_{{}^\theta\mathbf{R}}(M, N). $$

The Cartan pairing is the perfect $\mathcal{A}$-bilinear form

$$ {}^\theta\mathbf{K}_I \times {}^\theta\mathbf{G}_I \to \mathcal{A}, \quad \langle P : M\rangle = \mathrm{gdim}\,\hom_{{}^\theta\mathbf{R}}(P, M). $$

First, we concentrate on the $\mathcal{A}$-module ${}^\theta\mathbf{G}_I$. Consider the duality

$$ {}^\theta\mathbf{R}\text{-}\mathbf{mod}^f \to {}^\theta\mathbf{R}\text{-}\mathbf{mod}^f, \quad M \mapsto M^\flat = \hom(M, \mathbf{k}), $$

with the action and the grading given by

$$ (xf)(m) = f(\omega(x)m), \quad (M^\flat)_i = \hom(M_{-i}, \mathbf{k}). $$

This duality functor yields an $\mathcal{A}$-antilinear map

$$ {}^\theta\mathbf{G}_I \to {}^\theta\mathbf{G}_I, \quad M \mapsto M^\flat. $$

Let ${}^\theta B(\lambda)$ denote the set of irreducible (non graded) ${}^\theta\mathbf{R}$-modules in ${}^\theta\mathbf{R}\text{-}\mathbf{Mod}_0^f$. We can now define the upper global basis of ${}^\theta\mathbf{G}_I$ as follows. The proof is given in Section 8.25.

8.2. Proposition/Definition. *For each $b \in {}^\theta B(\lambda)$ there is a unique selfdual irreducible graded ${}^\theta\mathbf{R}$-module ${}^\theta G^{\mathrm{up}}(b)$ which is isomorphic to b as a (non graded) ${}^\theta\mathbf{R}$-module. We set ${}^\theta G^{\mathrm{up}}(0) = 0$ and ${}^\theta\mathbf{G}^{\mathrm{up}}(\lambda) = \{{}^\theta G^{\mathrm{up}}(b);\, b \in {}^\theta B(\lambda)\}$. Hence ${}^\theta\mathbf{G}^{\mathrm{up}}(\lambda)$ is a $\mathcal{A}$-basis of ${}^\theta\mathbf{G}_I$.*

Now, we concentrate on the $\mathcal{A}$-module ${}^\theta\mathbf{K}_I$. We equip ${}^\theta\mathbf{K}_I$ with the symmetric $\mathcal{A}$-bilinear form

$$ {}^\theta\mathbf{K}_I \times {}^\theta\mathbf{K}_I \to \mathcal{A}, \quad (M, N) \mapsto (M : N). \tag{8.1} $$

Consider the duality

$$ {}^\theta\mathbf{R}\text{-}\mathbf{proj} \to {}^\theta\mathbf{R}\text{-}\mathbf{proj}, \quad P \mapsto P^\sharp = \hom_{{}^\theta\mathbf{R}}(P, {}^\theta\mathbf{R}), $$

with the action and the grading given by

$$ (xf)(p) = f(p)\omega(x), \quad (P^\sharp)_i = \hom(P_i, \mathbf{k}). $$

This duality functor yields an $\mathcal{A}$-antilinear map

$$ {}^\theta\mathbf{K}_I \to {}^\theta\mathbf{K}_I, \quad P \mapsto P^\sharp. $$

Let $\mathcal{K} \to \mathcal{K}$, $f \mapsto \bar{f}$ be the unique involution such that $\bar{v} = v^{-1}$.

8.3. Definition. *For each b in ${}^\theta B(\lambda)$ let ${}^\theta G^{\mathrm{low}}(b)$ be the unique indecomposable graded module in ${}^\theta\mathbf{R}\text{-}\mathbf{proj}$ whose top is isomorphic to ${}^\theta G^{\mathrm{up}}(b)$. We set ${}^\theta G^{\mathrm{low}}(0) = 0$ and ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda) = \{{}^\theta G^{\mathrm{low}}(b);\, b \in {}^\theta B(\lambda)\}$, a $\mathcal{A}$-basis of ${}^\theta\mathbf{K}_I$.*

8.4. Proposition. *(a) We have* $\langle {}^\theta G^{\mathrm{low}}(b) : {}^\theta G^{\mathrm{up}}(b')\rangle = \delta_{b,b'}$ *for each* b, b' *in* ${}^\theta B(\lambda)$.

(b) We have $\langle P^\sharp : M\rangle = \overline{\langle P : M^\flat\rangle}$ *for each* P, M.

(c) We have ${}^\theta G^{\mathrm{low}}(b)^\sharp = {}^\theta G^{\mathrm{low}}(b)$ *for each* b *in* ${}^\theta B(\lambda)$.

Proof : Part (a) is obvious because we have

$$\langle {}^\theta G^{\mathrm{low}}(b) : {}^\theta G^{\mathrm{up}}(b')\rangle = \mathrm{gdim}\,\mathrm{hom}_{{}^\theta\mathbf{R}}({}^\theta G^{\mathrm{low}}(b), \mathrm{top}\,{}^\theta G^{\mathrm{low}}(b')) = \delta_{b,b'}.$$

Part (c) is a consequence of (b). Finally (b) is proved as follows

$$\begin{aligned}\langle P^\sharp : M\rangle &= \mathrm{gdim}\,\mathrm{hom}_{{}^\theta\mathbf{R}}(\mathrm{hom}_{{}^\theta\mathbf{R}}(P, {}^\theta\mathbf{R})^\omega, M),\\ &= \mathrm{gdim}\,(P^\omega \otimes_{{}^\theta\mathbf{R}} M),\\ &= \overline{\mathrm{gdim}\,\mathrm{hom}(P^\omega \otimes_{{}^\theta\mathbf{R}} M, \mathbf{k})},\\ &= \overline{\mathrm{gdim}\,\mathrm{hom}_{{}^\theta\mathbf{R}}(P, M^\flat)},\\ &= \overline{\langle P : M^\flat\rangle}.\end{aligned}$$

Here, the second equality holds because P is a projective graded module and the fourth one is adjointness of $\otimes$ and Hom, see e.g., [CuR,(2.19)].

□

8.5. Example. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Set ${}^\theta\mathbf{L}_{\mathbf{i}} = \mathrm{top}({}^\theta\mathbf{R}_{\mathbf{i}})$. We have ${}^\theta\mathbf{R}_0 = \mathbf{k}$. The global bases are given by the following. First, the weight zero parts are given by ${}^\theta\mathbf{G}^{\mathrm{low}}_0(\lambda) = {}^\theta\mathbf{G}^{\mathrm{up}}_0(\lambda) = \{\mathbf{k}\}$. Next, let us consider the weight ν parts.

- If $\lambda_i + \lambda_{\theta(i)} \neq 0$ then ${}^\theta\mathbf{G}^{\mathrm{low}}_\nu(\lambda) = \{{}^\theta\mathbf{R}_{\mathbf{i}}, {}^\theta\mathbf{R}_{\theta(\mathbf{i})}\}$ and ${}^\theta\mathbf{G}^{\mathrm{up}}_\nu(\lambda) = \{{}^\theta\mathbf{L}_{\mathbf{i}}, {}^\theta\mathbf{L}_{\theta(\mathbf{i})}\}$.
- If $\lambda_i + \lambda_{\theta(i)} = 0$ then ${}^\theta\mathbf{G}^{\mathrm{low}}_\nu(\lambda) = \{{}^\theta\mathbf{R}_{\mathbf{i}}\}$, ${}^\theta\mathbf{G}^{\mathrm{up}}_\nu(\lambda) = \{{}^\theta\mathbf{L}_{\mathbf{i}}\}$, ${}^\theta\mathbf{R}_{\mathbf{i}} = {}^\theta\mathbf{R}_{\theta(\mathbf{i})}$, and ${}^\theta\mathbf{L}_{\mathbf{i}} = {}^\theta\mathbf{L}_{\theta(\mathbf{i})}$.

8.6. Definition of the operators e_i, f_i, e'_i, f'_i**.** First, let us introduce the following notation for a future use. Given integers $m, m', n, n' \geqslant 0$ such that

$$m + m' = n + n' = m'',$$

let $D_{m,m'}$ be the set of minimal representative in $W_{m''}$ of the cosets in

$$W_{m,m'} \setminus W_{m''}, \quad W_{m,m'} = W_m \times \mathfrak{S}_{m'}.$$

Write also

$$D_{m,m';n,n'} = D_{m,m'} \cap (D_{n,n'})^{-1}.$$

For each element w of $D_{m,m';n,n'}$ we set

$$W(w) = W_{m,m'} \cap w(W_{n,n'})w^{-1}.$$

We abbreviate

$${}^\theta\mathbf{R}_{m,m'} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_{m'}.$$

More generally, for any positive integers $m'_1, \dots, m'_r$ we define the graded $\mathbf{k}$-algebra ${}^\theta\mathbf{R}_{m,m'_1,m'_2,\dots,m'_r}$ in the same way. There is an unique inclusion of graded $\mathbf{k}$-algebras

$$
\begin{aligned}
&{}^\theta\mathbf{R}_{m,m'} \to {}^\theta\mathbf{R}_{m''},\\
&1_{\mathbf{i}} \otimes 1_{\mathbf{i}'} \mapsto 1_{\mathbf{i}''},\\
&1_{\mathbf{i}} \otimes \varkappa_{\mathbf{i}',l} \mapsto \varkappa_{\mathbf{i}'',m+l},\\
&1_{\mathbf{i}} \otimes \sigma_{\mathbf{i}',k} \mapsto \sigma_{\mathbf{i}'',m+k},\\
&\varkappa_{\mathbf{i},l} \otimes 1_{\mathbf{i}'} \mapsto \varkappa_{\mathbf{i}'',l},\\
&\pi_{\mathbf{i},1} \otimes 1_{\mathbf{i}'} \mapsto \pi_{\mathbf{i}'',1},\\
&\sigma_{\mathbf{i},k} \otimes 1_{\mathbf{i}'} \mapsto \sigma_{\mathbf{i}'',k},
\end{aligned}
\tag{8.2}
$$

where, given $\mathbf{i} \in {}^\theta I^m$ and $\mathbf{i}' \in I^{m'}$, we have set $\mathbf{i}'' = \theta(\mathbf{i}')\mathbf{i}\mathbf{i}'$, a sequence in ${}^\theta I^{m''}$.

8.7. Lemma. *The graded ${}^\theta\mathbf{R}_{m,m'}$-module ${}^\theta\mathbf{R}_{m''}$ is free of rank $2^{m'}\binom{m''}{m}$.*

Proof : Set $\nu'' = \theta(\nu') + \nu + \nu'$, where ν, ν' are vector dimensions in ${}^\theta\mathbb{N}I$, $\mathbb{N}I$ respectively, such that $|\nu| = 2m$ and $|\nu'| = m'$. For each w in $D_{m,m'}$ we have the element $\sigma_{\dot{w}}$ in ${}^\theta\mathbf{R}_{m''}$ defined in (5.2). Using filtered/graded arguments it is easy to see that

$$
{}^\theta\mathbf{R}_{m''} = \bigoplus_{w \in D_{m,m'}} {}^\theta\mathbf{R}_{m,m'}\sigma_{\dot{w}}.
$$

□

Now, we consider the triple of adjoint functors $(\psi_!, \psi^*, \psi_*)$ where

$$
\psi^* : {}^\theta\mathbf{R}_{m''}\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod}
$$

is the restriction and $\psi_!$, ψ_* are given by

$$
\psi_! : \begin{cases} {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_{m''}\text{-}\mathbf{mod}, \\ (M, M') \mapsto {}^\theta\mathbf{R}_{m''} \otimes_{{}^\theta\mathbf{R}_{m,m'}} (M \otimes M'), \end{cases}
$$

$$
\psi_* : \begin{cases} {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \times \mathbf{R}_{m'}\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_{m''}\text{-}\mathbf{mod}, \\ (M, M') \mapsto \mathrm{hom}_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, M \otimes M'). \end{cases}
$$

The same discussion as for the triple $(\phi_!, \phi^*, \phi_*)$ implies that $\psi_!$, ψ^*, ψ_* are exact, they commute with the shift of the grading, they take projective modules to projective ones and finite dimensional modules to finite dimensional ones. The functors $\psi_!$, ψ_* yield $\mathcal{A}$-bilinear maps

$$
\mathbf{K}_I^\theta \times \mathbf{K}_I \to {}^\theta\mathbf{K}_I, \quad {}^\theta\mathbf{G}_I \times \mathbf{G}_I \to {}^\theta\mathbf{G}_I.
$$

The functor ψ^* yields maps in the inverse direction. For any graded ${}^\theta\mathbf{R}_m$-module M we write

$$
f_i M = {}^\theta\mathbf{R}_{m+1} 1_{m,i} \otimes_{{}^\theta\mathbf{R}_m} M, \quad e'_i M = 1_{m-1,i} M. \tag{8.3}
$$

Here the symbol $1_{m,i}$ is as in (6.3). Note that $f_i M$ is a graded ${}^\theta\mathbf{R}_{m+1}$-module, while $e'_i M$ is a graded ${}^\theta\mathbf{R}_{m-1}$-module.

8.8. Definition. *Let e'_i, f_i be the $\mathcal{A}$-linear operators on ${}^\theta\mathbf{K}_I$ given by the formula (8.3). Let e_i, f'_i be the $\mathcal{A}$-linear operators on ${}^\theta\mathbf{G}_I$ which are the transpose of f_i, e'_i with respect to the Cartan pairing. We will also write $e_i = (1-v^2)\,e'_i$ and $f_i = (1-v^2)\,f'_i$.*

8.9. Lemma. *(a) For each $M \in {}^\theta\mathbf{R}_m\text{-}\mathbf{mod}^f$ we have*

$$e_iM = 1_{m-1,i}M, \quad f'_iM = \hom_{{}^\theta\mathbf{R}_{m,1}}({}^\theta\mathbf{R}_{m+1}, M \otimes \mathbf{R}_i).$$

(b) For each $M, M'' \in {}^\theta\mathbf{R}\text{-}\mathbf{mod}$ and $M' \in \mathbf{R}\text{-}\mathbf{mod}$ we have

$$(\psi_!(M, M') : M'') = (M \otimes M' : \psi^*(M'')), \quad (e'_iM : M'') = (M : f_iM'').$$

(c) We have $f_i(P)^\sharp = f_i(P^\sharp)$ for each $P \in {}^\theta\mathbf{R}\text{-}\mathbf{proj}$.

(d) We have $e_i(M)^\flat = e_i(M^\flat)$ for each $M \in {}^\theta\mathbf{R}\text{-}\mathbf{mod}^f$.

Proof : For each M in ${}^\theta\mathbf{R}\text{-}\mathbf{mod}^f$ and each P in ${}^\theta\mathbf{R}\text{-}\mathbf{proj}$ we have

$$\begin{aligned}
\hom_{{}^\theta\mathbf{R}}(f_iP, M) &= \hom_{{}^\theta\mathbf{R}}(\psi_!(P, \mathbf{R}_i), M) \\
&= \hom_{{}^\theta\mathbf{R}\otimes\mathbf{R}}(P \otimes \mathbf{R}_i, \psi^*M) \\
&= \hom_{{}^\theta\mathbf{R}}(P, 1_{m-1,i}M), \\
\hom_{{}^\theta\mathbf{R}}(e'_iP, M) &= \hom_{{}^\theta\mathbf{R}}(1_{m-1,i}P, M) \\
&= \hom_{{}^\theta\mathbf{R}\otimes\mathbf{R}}(\psi^*P, M \otimes \mathbf{R}_i) \\
&= \hom_{{}^\theta\mathbf{R}}(P, \psi_*(M, \mathbf{R}_i)).
\end{aligned}$$

This proves part (a). Part (b) follows from the following identities

$$\begin{aligned}
(\psi_!(M, M') : M'') &= \mathrm{gdim}\Big((M^\omega \otimes M'^\omega) \otimes_{{}^\theta\mathbf{R}_{m,m'}} \psi^*(M'')\Big), \\
&= (M \otimes M' : \psi^*(M'')).
\end{aligned}$$

Part (c) follows from the following identities

$$\begin{aligned}
f_i(P)^\sharp &= \hom_{{}^\theta\mathbf{R}_m}\big({}^\theta\mathbf{R}_m \otimes_{{}^\theta\mathbf{R}_{m-1,1}} (P \otimes \mathbf{R}_i),\ {}^\theta\mathbf{R}_m\big), \\
&= \hom_{{}^\theta\mathbf{R}_{m-1,1}}\big(P \otimes \mathbf{R}_i,\ {}^\theta\mathbf{R}_m\big), \\
&= {}^\theta\mathbf{R}_m \otimes_{{}^\theta\mathbf{R}_{m-1,1}} \hom_{{}^\theta\mathbf{R}_{m-1,1}}\big(P \otimes \mathbf{R}_i,\ {}^\theta\mathbf{R}_{m-1,1}\big), \\
&= {}^\theta\mathbf{R}_m \otimes_{{}^\theta\mathbf{R}_{m-1,1}} \big(\hom_{{}^\theta\mathbf{R}_{m-1}}(P,\ {}^\theta\mathbf{R}_{m-1}) \otimes \mathbf{R}_i\big), \\
&= f_i(P^\sharp).
\end{aligned}$$

Here the second equality is Frobenius reciprocity and the third one follows from Lemma 8.7, see e.g., [CuR, (2.29)]. Part (d) follows from (c) and Proposition 8.4(b).

□

8.10. Shuffles, projectives, and characters. For each sequence $\mathbf{i}$ in ${}^\theta I^m$ we define a projective graded module in ${}^\theta\mathbf{R}_m$**-proj** by setting ${}^\theta\mathbf{R}_\mathbf{i} = {}^\theta\mathbf{R}_m 1_\mathbf{i}$. More generally, for $\mathbf{y} \in {}^\theta Y^m$ we define an object ${}^\theta\mathbf{R}_\mathbf{y}$ of ${}^\theta\mathbf{R}_m$**-proj** as follows. Write

$$\mathbf{y} = (\theta(\mathbf{j})\mathbf{j}, \theta(\mathbf{b})\mathbf{b}), \quad \mathbf{j} \in I^m, \quad \mathbf{b} \in \mathbb{Z}^m.$$

Define the idempotent $1_\mathbf{y}$ as the image of the idempotent $1_{(\mathbf{j},\mathbf{b})}$ by the inclusion $\mathbf{R}_m \subset {}^\theta\mathbf{R}_m$ given by setting m, m', m'' equal to $0, m, m$ in (8.2). Then set

$${}^\theta\mathbf{R}_\mathbf{y} = ({}^\theta\mathbf{R}_m 1_\mathbf{y})[\ell_\mathbf{b}].$$

The graded module ${}^\theta\mathbf{R}_\mathbf{y}$ satisfies the same properties as the projective graded $\mathbf{R}_m$-modules introduced in Section 7.4. In particular ${}^\theta\mathbf{R}_\mathbf{y}$ is selfdual, and if the sequence $\mathbf{i}$ in ${}^\theta I^m$ is the expansion of the pair $\mathbf{y}$ then we have

$$ {}^\theta\mathbf{R}_\mathbf{i} = \langle\mathbf{b}\rangle!\, {}^\theta\mathbf{R}_\mathbf{y}. \tag{8.4}$$

Further, for each $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ in ${}^\theta Y^m$ and $\mathbf{y}' = (\mathbf{i}', \mathbf{a}')$ in $Y^{m'}$ we have

$$\psi_!({}^\theta\mathbf{R}_\mathbf{y} \otimes \mathbf{R}_{\mathbf{y}'}) = {}^\theta\mathbf{R}_{\mathbf{y}''}, \quad \mathbf{y}'' = (\theta(\mathbf{i}')\mathbf{i}\mathbf{i}', \theta(\mathbf{a}')\mathbf{a}\mathbf{a}').$$

8.11. Definition. *A shuffle of a pair of sequences* $(\mathbf{i}, \mathbf{i}')$ *in* ${}^\theta I^m \times I^{m'}$ *is a sequence* $\mathbf{i}''$ *in* ${}^\theta I^{m''}$ *together with a subsequence of* $\mathbf{i}''$ *isomorphic to* $\mathbf{i}$ *and such that the complementary subsequence is equal to* $\theta(\mathbf{i}')\mathbf{i}'$ *modulo* θ.

Let $Sh(\mathbf{i}, \mathbf{i}')$ be the set of shuffles of $\mathbf{i}$, $\mathbf{i}'$. The assignment $w \mapsto w^{-1}(\theta(\mathbf{i}')\mathbf{i}\mathbf{i}')$ gives a bijection from $D_{m,m'}$ to $Sh(\mathbf{i}, \mathbf{i}')$. To a shuffle $\mathbf{i}''$ in $Sh(\mathbf{i}, \mathbf{i}')$ associated with an element w of $D_{m,m'}$ we assign the following degree

$$\deg(\mathbf{i}, \mathbf{i}'; \mathbf{i}'') = \deg(\sigma_{\dot{w}} 1_{\mathbf{i}''}).$$

This degree does not depend of the choice of the reduced decomposition $\dot{w}$ of w. Let ${}^\theta\mathcal{B}I^m$ be the free $\mathcal{B}$-module with basis ${}^\theta I^m$.

8.12. Definition. *For any finitely generated graded* ${}^\theta\mathbf{R}_m$*-module* M *we define the character of* M *as the element of* ${}^\theta\mathcal{B}I^m$ *given by* $\mathrm{ch}(M) = \sum_\mathbf{i} \mathrm{gdim}(1_\mathbf{i} M)\,\mathbf{i}$.

For any f in ${}^\theta\mathcal{B}I^m$ we write $f = \sum_\mathbf{i} f(\mathbf{i})\,\mathbf{i}$. Then, we define

$$(f \circledast g)(\mathbf{i}'') = \sum_{\mathbf{i},\mathbf{i}'} v^{\deg(\mathbf{i},\mathbf{i}';\mathbf{i}'')} f(\mathbf{i}) g(\mathbf{i}'), \quad \forall f \in {}^\theta\mathcal{B}I^m,\ \forall g \in \mathcal{B}I^{m'}.$$

The sum is over all ways to represent $\mathbf{i}''$ as a shuffle of $\mathbf{i}$ and $\mathbf{i}'$.

8.13. Proposition. *For any* $M \in {}^\theta\mathbf{R}_m$**-mod** *and any* $M' \in \mathbf{R}_{m'}$**-mod** *we have*

$$\mathrm{ch}(\psi_!(M, M')) = \mathrm{ch}(M) \circledast \mathrm{ch}(M').$$

Proof: We have $\mathrm{ch}(M) = \sum_\mathbf{i} ({}^\theta\mathbf{R}_\mathbf{i}, M)\,\mathbf{i}$. Thus the proposition follows from Lemma 8.9(b) and the following formula

$$\psi^*({}^\theta\mathbf{R}_{\mathbf{i}''}) = \bigoplus_{\mathbf{i},\mathbf{i}'} {}^\theta\mathbf{R}_\mathbf{i} \otimes \mathbf{R}_{\mathbf{i}'}[\deg(\mathbf{i}, \mathbf{i}'; \mathbf{i}'')],$$

where the sum runs over all sequences $\mathbf{i}'$ such that $\mathbf{i}''$ lies in $Sh(\mathbf{i}, \mathbf{i}')$, which is a consequence of the Mackey's induction-restriction theorem. The details are left to the reader, see e.g., the proof of Theorem 8.30 below.

□

8.14. Proposition. *We have*

$$f_i({}^\theta\mathbf{R}_\mathbf{i}) = {}^\theta\mathbf{R}_{\theta(i)\mathbf{i}i}, \quad e'_i({}^\theta\mathbf{R}_\mathbf{i}) = (1-v^2)^{-1}\bigoplus_{\mathbf{i}'} {}^\theta\mathbf{R}_{\mathbf{i}'}[\deg(\mathbf{i}', i; \mathbf{i})].$$

The sum runs over all sequences in ${}^\theta I^{m-1}$ *such that* $\mathbf{i}$ *lies in* $Sh(\mathbf{i}', i)$*, and* $(1-v^2)^{-1} = \sum_{r\geqslant 0} v^{2r}$.

Proof : Left to the reader.

□

8.15. Example. Set $\nu = i+\theta(i)$ and $\mathbf{i} = i\theta(i)$. We compute e_j, f_j, e'_j, f'_j, and ch. We have

$$f_i(\mathbf{k}) = {}^\theta\mathbf{R}_{\theta(\mathbf{i})}, \quad e'_j({}^\theta\mathbf{R}_\mathbf{i}) = \begin{cases} (1-v^2)^{-1}v^{\lambda_i+\lambda_{\theta(i)}}\,\mathbf{k} & \text{if } j=i, \\ (1-v^2)^{-1}\mathbf{k} & \text{if } j=\theta(i), \\ 0 & \text{else.} \end{cases}$$

We have ${}^\theta\mathbf{R}_{0,1} = \mathbf{R}_1$, and the inclusion in (8.2) yields ${}^\theta\mathbf{R}_1 = \mathbf{R}_1 \oplus \mathbf{R}_1\pi_1$. We get

$$\psi_*(\mathbf{k}, \mathbf{R}_i) = \hom_{\mathbf{R}_1}({}^\theta\mathbf{R}_1, \mathbf{R}_i) = {}^\theta\mathbf{R}_\mathbf{i}[\lambda_i + \lambda_{\theta(i)}],$$

$$\psi_!(\mathbf{k}, \mathbf{R}_i) = {}^\theta\mathbf{R}_1 \otimes_{\mathbf{R}_1} \mathbf{R}_i = {}^\theta\mathbf{R}_{\theta(\mathbf{i})}.$$

In particular, we have

- If $\lambda_i + \lambda_{\theta(i)} \neq 0$ then $e_j({}^\theta\mathbf{L}_{\theta(\mathbf{i})}) = \mathbf{k}$ if $j=i$ and 0 else, and $f'_i(\mathbf{k}) = (1-v^2)^{-1}(v^{\lambda_i+\lambda_{\theta(i)}}\,{}^\theta\mathbf{L}_\mathbf{i} + {}^\theta\mathbf{L}_{\theta(\mathbf{i})})$. Further $\mathrm{ch}({}^\theta\mathbf{L}_{\theta(\mathbf{i})}) = \theta(\mathbf{i})$ and $\mathrm{ch}({}^\theta\mathbf{L}_\mathbf{i}) = \mathbf{i} + \theta(\mathbf{i})$.
- If $\lambda_i + \lambda_{\theta(i)} = 0$ then $e_j({}^\theta\mathbf{L}_{\theta(\mathbf{i})}) = \mathbf{k}$ if $j = i, \theta(i)$ and 0 else, and $f'_i(\mathbf{k}) = (1-v^2)^{-1}\,{}^\theta\mathbf{L}_\mathbf{i}$. Further $\mathrm{ch}({}^\theta\mathbf{L}_\mathbf{i}) = \mathbf{i}$.

8.16. Induction of $\mathbf{H}_m$-modules versus induction of ${}^\theta\mathbf{R}_m$-modules. Recall the functors E_i, F_i on $\mathbf{H}\text{-}\mathbf{Mod}_I^f$ defined in (6.4). We have also the functors

$$\Psi : {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f \to \mathbf{H}_m\text{-}\mathbf{Mod}_I^f, \quad \mathbf{for} : {}^\theta\mathbf{R}_m\text{-}\mathbf{mod}^f \to {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f,$$

where **for** is the forgetting of the grading. Finally we define functors

$$E_i : {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f \to {}^\theta\mathbf{R}_{m-1}\text{-}\mathbf{Mod}_0^f, \quad E_iM = 1_{m-1,i}M,$$
$$F_i : {}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f \to {}^\theta\mathbf{R}_{m+1}\text{-}\mathbf{Mod}_0^f, \quad F_iM = \psi_!(M, \mathbf{L}_i). \tag{8.5}$$

8.17. Proposition. *There are canonical isomorphisms of functors*

$$E_i\circ\Psi = \Psi\circ E_i, \quad F_i\circ\Psi = \Psi\circ F_i, \quad E_i\circ\mathbf{for} = \mathbf{for}\circ e_i, \quad F_i\circ\mathbf{for} = \mathbf{for}\circ f_{\theta(i)}.$$

Proof : Recall that $\mathbf{k}_i$ is the 1-dimensional $\mathbf{k}[X_{m+1}^{\pm 1}]$-module such that $X_{m+1} \mapsto \theta(i)$, that $\mathbf{L}_i$ is the 1-dimensional $\mathbf{R}_1$-module such that $1_i \mapsto 1$ and $\varkappa_i \mapsto 0$, and that

Ψ identifies X_{m+1} and the element $1 \otimes \theta(i)e^{\varkappa_i}$ in ${}^\theta\mathbf{R}_{m,1}$. The first two isomorphisms are obvious consequences of (6.4), (8.5), because E_iM is the generalized $\theta(i)$-eigenspace of M with respect to the action of X_{m+1}, and F_iM is induced from the $\mathbf{H}_m \otimes \mathbf{k}[X_{m+1}^{\pm 1}]$-module $M \otimes \mathbf{k}_i$. The third isomorphism follows from (8.5) and Lemma 8.9(a). Now, we concentrate on the last isomorphism. Lemma 8.9(a) yields

$$f_{\theta(i)}M = \psi_*(M, \mathbf{L}_{\theta(i)}).$$

For any graded $\mathbf{R}$-module N let N^κ be equal to N, with the $\mathbf{R}$-action twisted by the involution κ in (7.1). Note that $\mathbf{L}_i^\kappa = \mathbf{L}_{\theta(i)}$. Therefore the proposition follows from the following lemma.

8.18. Lemma. *For each $M \in {}^\theta\mathbf{R}_m$-$\mathbf{mod}^f$ and $N \in \mathbf{R}_{m'}$-$\mathbf{mod}^f$ there is an isomorphism of (non-graded) ${}^\theta\mathbf{R}_{m''}$-modules $\psi_!(M,N) = \psi_*(M, N^\kappa)$.*

Proof: Recall that ${}^\theta\mathbf{R}_{m,m'} = {}^\theta\mathbf{R}_m \otimes \mathbf{R}_{m'}$. The involution $\kappa : \mathbf{R}_{m'} \to \mathbf{R}_{m'}$ in (7.1) yields an involution of ${}^\theta\mathbf{R}_{m,m'}$. Let us denote it by κ again. Let ${}^\theta\mathbf{R}^\kappa_{m,m'}$ be the $({}^\theta\mathbf{R}_{m,m'}, {}^\theta\mathbf{R}_{m,m'})$-bimodule which is equal to ${}^\theta\mathbf{R}_{m,m'}$ as a right ${}^\theta\mathbf{R}_{m,m'}$-module, and such that the left ${}^\theta\mathbf{R}_{m,m'}$-action is twisted by κ. It is enough to prove that there is an isomorphism of (non-graded) $({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}_{m,m'})$-bimodules

$${}^\theta\mathbf{R}_{m''} \to \hom_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}^\kappa_{m,m'}).$$

The bimodule structure on the rhs is given by

$$(xfy)(z) = f(zx)y, \quad x, z \in {}^\theta\mathbf{R}_{m''},\ y \in {}^\theta\mathbf{R}_{m,m'}.$$

Lemma 8.7 yields an isomorphism

$${}^\theta\mathbf{R}_{m''} = \bigoplus_{w \in D_{m,m'}} {}^\theta\mathbf{R}_{m,m'}\sigma_{\dot w}$$

of graded ${}^\theta\mathbf{R}_{m,m'}$-modules. The longest double coset representative in $D_{m,m';m,m'}$ is the coset of the involution $u \in W_{m''}$ given by

$$u = w_0'\,\varepsilon_{m+1}\ldots\varepsilon_{m''},$$

with w_0' the longest element of $\mathfrak{S}_{m'}$. There is an unique morphism of $({}^\theta\mathbf{R}_{m,m'}, {}^\theta\mathbf{R}_{m,m'})$-bimodules

$$h : {}^\theta\mathbf{R}_{m,m'} \to \hom_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}^\kappa_{m,m'}),$$

taking 1 to the map

$$y\,\sigma_{\dot w} \mapsto \kappa(y)\,\delta_{w,u}, \quad y \in {}^\theta\mathbf{R}_{m,m'}, \quad w \in D_{m,m';m,m'}.$$

Since the rhs is a left ${}^\theta\mathbf{R}_{m''}$-module, by Frobenius reciprocity h yields a morphism of $({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}_{m,m'})$-bimodules

$${}^\theta\mathbf{R}_{m''} \to \hom_{{}^\theta\mathbf{R}_{m,m'}}({}^\theta\mathbf{R}_{m''}, {}^\theta\mathbf{R}^\kappa_{m,m'})[-\deg(\sigma_{\dot u})].$$

This map is invertible. The proof is the same as in [M, sec. 3], [LV, thm. 2.2]. □

8.19. Proposition. *(a) The functor Ψ yields an isomorphism of Abelian groups*

$$\bigoplus_{m\geqslant 0}[{}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f] = \bigoplus_{m\geqslant 0}[\mathbf{H}_m\text{-}\mathbf{Mod}_I^f].$$

The functors E_i, F_i yield endomorphisms of both sides which are intertwined by Ψ.

(b) The forgetful functor **for** *factors to a group isomorphism*

$${}^\theta\mathbf{G}_I/(v-1) = \bigoplus_{m\geqslant 0}[{}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0^f].$$

Proof : Claim (a) follows from Corollary 6.6 and Proposition 8.17. Claim (b) follows from Proposition 8.2.

□

8.20. The crystal operators on ${}^\theta\mathbf{G}_I$ and ${}^\theta B(\lambda)$. Fix a vertex i in I. For each irreducible graded ${}^\theta\mathbf{R}$-module M we define

$$\tilde e_i(M) = \mathrm{soc}\,(e_iM), \quad \tilde f_i(M) = \mathrm{top}\,\psi_!(M,\mathbf{L}_i), \quad \varepsilon_i(M) = \max\{n\geqslant 0; e_i^nM\neq 0\}.$$

For each positive integers $m\geqslant n$ we consider the functor

$$\Delta_{ni} : {}^\theta\mathbf{R}_m\text{-}\mathbf{mod} \to {}^\theta\mathbf{R}_{m-n}\text{-}\mathbf{mod}\times\mathbf{R}_{ni}\text{-}\mathbf{mod}, \quad M\mapsto 1_{m-n,ni}M.$$

Given an irreducible graded ${}^\theta\mathbf{R}_m$-module M we have

$$\varepsilon_i(M) = \max\{n\geqslant 0; \Delta_{ni}(M)\neq 0\}, \quad e_iM = \Delta_iM.$$

8.21. Proposition. *Let M be an irreducible graded ${}^\theta\mathbf{R}_m$-module and n be an integer $\geqslant 0$. Set $\varepsilon = \varepsilon_i(M)$, $M^+ = \psi_!(M,\mathbf{L}_{ni})$ and $M^- = \Delta_{ni}(M)$.*

(a) If $\varepsilon = 0$ then $\Delta_{ni}(M^+) = M\otimes\mathbf{L}_{ni}$, $\mathrm{top}(M^+)$ is irreducible, $\varepsilon_i(\mathrm{top}(M^+)) = n$, all other composition factors L of M^+ have $\varepsilon_i(L) < n$.

(b) If $\varepsilon\geqslant n$ then any irreducible submodule of M^- is of the form $N\otimes\mathbf{L}_{ni}$ with $\varepsilon_i(N) = \varepsilon - n$. If $\varepsilon = n$ then M^- is irreducible. If $\varepsilon\geqslant n$ then $\mathrm{soc}(M^-)$ is irreducible. In particular $\tilde e_i(M)$ is irreducible if $\varepsilon\neq 0$ and 0 else. Finally we have $\mathrm{soc}(M^-) = \tilde e_i^n(M)\otimes\mathbf{L}_{ni}$.

(c) $\mathrm{top}(M^+)$ is irreducible, $\varepsilon_i(\mathrm{top}(M^+)) = \varepsilon + n$, and all other composition factors L of M^+ have $\varepsilon_i(L) < \varepsilon + n$. In particular $\tilde f_i(M)$ is irreducible. Finally we have $\mathrm{top}(M^+) = \tilde f_i^n(M)$.

Proof : Part (a) is the analogue of [K, lem. 5.1.3], [KL1, lem. 3.7]. More precisely, note first that we have

$$\mathrm{ch}(\Delta_{ni}(M^+)) = \sum_{\mathbf{i}}\mathrm{gdim}(1_{\theta(i^n)\mathbf{i}i^n}M^+)\,\theta(i^n)\mathbf{i}i^n. \tag{8.6}$$

Hence, since $\varepsilon = 0$ Proposition 8.13 implies that

$$\dim(\Delta_{ni}(M^+)) = \dim(M\otimes\mathbf{L}_{ni}).$$

Since $\Delta_{ni}(M^+)$ contains a copy of $M \otimes \mathbf{L}_{ni}$, we get the first claim of (a). By Frobenius reciprocity, a copy of $M \otimes \mathbf{L}_{ni}$, possibly with a grading shift, appears as a submodule of $\Delta_{ni}(M')$ for any nonzero quotient $M^+ \to M'$. Since

$$\Delta_{ni}(M^+) = M \otimes \mathbf{L}_{ni},$$

this implies that $\mathrm{top}(M^+)$ is irreducible with $\varepsilon_i(\mathrm{top}(M^+)) \geqslant n$, that

$$\Delta_{ni}(M^+) = \Delta_{ni}(\mathrm{top}(M^+)),$$

and that $\Delta_{ni}(L) = 0$ for all other composition factors L of M^+. Finally we have $\varepsilon_i(\mathrm{top}(M^+)) = n$, because $\varepsilon = 0$.

Now we prove (b). The first claim is the analogue of [K, lem. 5.1.2]. Indeed, any irreducible submodule of M^- is of the form $N \otimes \mathbf{L}_{in}$ with N irreducible. We have $\varepsilon_i(N) \leqslant \varepsilon - n$ by definition of ε_i. For the reverse inequality, Frobenius reciprocity and the irreducibility of M imply that M is a quotient of $\psi_!(N, \mathbf{L}_{ni})$. So applying the exact functor $\Delta_{\varepsilon i}$ we see that $\Delta_{\varepsilon i}(M)$ is a quotient of $\Delta_{\varepsilon i}\psi_!(N, \mathbf{L}_{ni})$. In particular

$$\Delta_{\varepsilon i}\psi_!(N, \mathbf{L}_{ni}) \neq 0.$$

By Proposition 8.13 and (8.6) we have also $\Delta_{(\varepsilon-n)i}(N) \neq 0$. Thus $\varepsilon_i(N) = \varepsilon - n$. The second claim of (b) is the analogue of [K, lem. 5.1.4]. Indeed, if $\varepsilon = n$ then any irreducible submodule of M^- is of the form $N \otimes \mathbf{L}_{ni}$ with $\varepsilon_i(N) = 0$. Once again Frobenius reciprocity and the irreducibility of M imply that M is a quotient of $\psi_!(N, \mathbf{L}_{ni})$. Hence M^- is a quotient of $\Delta_{ni}\psi_!(N, \mathbf{L}_{ni})$. But the later is isomorphic to $N \otimes \mathbf{L}_{ni}$ by (a). Next, the third claim of (b) is the analogue of [K, lem. 5.1.6], [KL1, prop. 3.10]. Indeed, suppose that $N \otimes \mathbf{L}_{ni} \subset \mathrm{soc}(M^-)$. Then $\varepsilon_i(N) = \varepsilon - n$ by the first part of (b). Thus N contributes a non-trivial submodule to $\Delta_{\varepsilon i}(M)$. But $\Delta_{\varepsilon i}(M)$ is an irreducible graded ${}^\theta\mathbf{R}_{m-\varepsilon,\varepsilon}$-module by the second part of (b). Thus the socle of $\Delta_{\varepsilon i}(M)$ as a graded ${}^\theta\mathbf{R}_{m-\varepsilon,\varepsilon-n,n}$-module is $N \otimes \mathbf{L}_{(\varepsilon-n)i} \otimes \mathbf{L}_{ni}$ by Example 7.5. Hence $\mathrm{soc}(M^-)$ must equal $N \otimes \mathbf{L}_{ni}$. Finally, the last claim of (b) is the analogue of [K, lem. 5.2.1(i)], [KL1, lem. 3.13]. Indeed, note first that if $n > \varepsilon$ then

$$\mathrm{soc}(M^-) = \tilde{e}_i^n(M) = 0.$$

Assume now that $\varepsilon \geqslant n$. Observe that

$$\tilde{e}_i(M) = \mathrm{soc}(\Delta_i(M))$$

is irreducible or zero by the third part of (b). Hence $\tilde{e}_i(M) \otimes \mathbf{L}_i$ is a submodule of $\Delta_i(M)$. Applying this n times we deduce that $\tilde{e}_i^n(M) \otimes (\mathbf{L}_i)^{\otimes n}$ is a submodule of $\Delta_{ni}(M)$ as a graded ${}^\theta\mathbf{R}_{m-n,1^n}$-module. Hence $\tilde{e}_i^n(M) \otimes \mathbf{L}_{ni}$ is a submodule of $\Delta_{ni}(M)$ by Frobenius reciprocity.

Finally, we prove (c). It is the analogue of [K, lem. 5.2.1(ii)], [KL1, lem. 3.13]. Indeed, by (b) the graded module $\Delta_{\varepsilon i}(M)$ is of the form $N \otimes \mathbf{L}_{ni}$, with N irreducible such that $\varepsilon_i(N) = 0$. Thus, by Frobenius reciprocity M is a quotient of $\psi_!(N, \mathbf{L}_{\varepsilon i})$. So the transitivity of induction implies that M^+ is a quotient of $\psi_!(N, \mathbf{L}_{(\varepsilon+n)i})$. Hence all claims except the last one follow from (a). Finally, by exactness of the induction $\tilde{f}_i^n(M)$ is a quotient of M^+, hence they are equal by simplicity of the top.

□

For each irreducible module b in ${}^\theta B(\lambda)$ we define

(8.7) $$\tilde{E}_i(b) = \mathrm{soc}(E_i b), \quad \tilde{F}_i(b) = \mathrm{top}(F_i b), \quad \varepsilon_i(b) = \max\{n \geqslant 0; E_i^n b \neq 0\}.$$

Hence we have $\mathbf{for} \circ \tilde{e}_i = \tilde{E}_i \circ \mathbf{for}$, $\mathbf{for} \circ \tilde{f}_i = \tilde{F}_i \circ \mathbf{for}$, and $\varepsilon_i(M) = \varepsilon_i(\mathbf{for}\, M)$.

8.22. Proposition. *For each b, b' in ${}^\theta B(\lambda)$ we have*

(a) $\tilde F_i(b) \in {}^\theta B(\lambda)$,

(b) $\tilde E_i(b) \in {}^\theta B(\lambda) \cup \{0\}$,

(c) $\tilde F_i(b) = b' \iff \tilde E_i(b') = b$,

(d) $\varepsilon_i(b) = \max\{n \geqslant 0; \tilde E_i^n(b) \neq 0\}$,

(e) $\varepsilon_i(\tilde F_i(b)) = \varepsilon_i(b) + 1$,

(f) if $\tilde E_i(b) = 0$ *for all* i *then* $b = \mathbf{k}$.

Proof : Parts (a), (b), (d), (e) and (f) are immediate consequences of Proposition 8.21. Part (c) is proved as in [K, lem. 5.2.3]. More precisely, let M, N be irreducible graded modules. By Proposition 8.21(c) we have

$$\begin{aligned}\tilde f_i(M) = N &\iff \mathrm{Hom}_{{}^\theta\mathbf{R}}(\psi_!(M, \mathbf{L}_i), N) \neq 0\\ &\iff \mathrm{Hom}_{{}^\theta\mathbf{R}}(M \otimes \mathbf{L}_i, e_i(N)) \neq 0\\ &\iff \mathrm{Hom}_{{}^\theta\mathbf{R}}(M \otimes \mathbf{L}_i, \mathrm{soc}(e_i N)) \neq 0\\ &\iff M = \tilde e_i(N).\end{aligned}$$

Note that the proposition can also be deduced from [M, Section 4] and Proposition 8.17. □

8.23. Proposition. *There are elements $f_{b,b'}$ in $\mathcal{A}$ such that*

$$f_i {}^\theta G^{\mathrm{low}}(b) = \langle \varepsilon_i(b) + 1\rangle {}^\theta G^{\mathrm{low}}(\tilde F_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'), \quad \forall b \in {}^\theta B(\lambda),$$

where b' runs over the set of elements of ${}^\theta B(\lambda)$ such that $\varepsilon_i(b') > \varepsilon_i(b) + 1$.

Proof : We claim that there are elements $f_{b',b}$ in $\mathcal{A}$ such that

$$e_i {}^\theta G^{\mathrm{up}}(b) = \langle \varepsilon_i(b)\rangle {}^\theta G^{\mathrm{up}}(\tilde E_i b) + \sum_{b'} f_{b',b} {}^\theta G^{\mathrm{up}}(b'), \tag{8.8}$$

where b' runs over the elements of ${}^\theta B(\lambda)$ with $\varepsilon_i(b') < \varepsilon_i(b) - 1$. Thus we have

$$\begin{aligned}f_i {}^\theta G^{\mathrm{low}}(b) &= \langle \varepsilon_i(\tilde F_i b)\rangle {}^\theta G^{\mathrm{low}}(\tilde F_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'),\\ &= \langle \varepsilon_i(b) + 1\rangle {}^\theta G^{\mathrm{low}}(\tilde F_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'),\end{aligned}$$

where $b' \in {}^\theta B(\lambda)$ with $\varepsilon_i(b) + 1 < \varepsilon_i(b')$. Now, let us prove (8.8). This is the analogue of [K, lem. 5.5.1(i)]. Now, fix an irreducible ${}^\theta\mathbf{R}_m$-module b. Set $\varepsilon = \varepsilon_i(b)$, $M = {}^\theta G^{\mathrm{up}}(b)$ and $N = {}^\theta G^{\mathrm{up}}(\tilde E_i^\varepsilon b)$. We can assume that $\varepsilon > 0$, because else (8.8) is obvious. Note that $\varepsilon_i(N) = 0$ by Proposition 8.22. By Frobenius reciprocity and Proposition 8.21(b) there is a short exact sequence of graded modules

$$0 \to R \to \psi_!(N, \mathbf{L}_{\varepsilon i}) \to M \to 0.$$

Applying the functor e_i we obtain the following exact sequence of graded modules

$$0 \to e_i R \to e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) \to e_i M \to 0.$$

Note that

$$e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) = 1_{m-1,i}{}^\theta\mathbf{R}_m 1_{m-\varepsilon,\varepsilon i} \otimes_{{}^\theta\mathbf{R}_{m-\varepsilon,\varepsilon}} (N \otimes \mathbf{L}_{\varepsilon i}).$$

Note also that

$$D_{m-1,1;\,m-\varepsilon,\varepsilon} = \{e, x, y\},$$

$$x = s_{m-1} \dots s_{m-\varepsilon+1} s_{m-\varepsilon}, \quad y = s_{m-1} \dots s_{m-\varepsilon+1} \varepsilon_{m-\varepsilon+1},$$

$$W(e) = W_{m-\varepsilon,\varepsilon-1,1}, \quad W(x) = W_{m-\varepsilon-1,1,\varepsilon}, \quad W(y) = W_{m-\varepsilon,\varepsilon-1,1}.$$

We can filter the graded bimodule $1_{m-1,i}{}^\theta\mathbf{R}_m 1_{m-\varepsilon,\varepsilon i}$ as in the Mackey induction-restriction theorem. Compare Lemma 8.31 below and the references there. Since $e_i(N) = 0$ the contribution of x to $e_i\psi_!(N, \mathbf{L}_{\varepsilon i})$ is zero. Since $1_\nu \mathbf{L}_{\varepsilon i} = 0$ if $\nu \neq \varepsilon i$ the contribution of y to $e_i\psi_!(N, \mathbf{L}_{\varepsilon i})$ is also zero. Thus e is the only element of $D_{m-1,1;\,m-\varepsilon,\varepsilon}$ which has a nonzero contribution. This yields

$$e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) = {}^\theta\mathbf{R}_{m-1,1} \otimes_{\mathbf{R}'} (N \otimes \mathbf{L}_{\varepsilon i}), \quad \mathbf{R}' = {}^\theta\mathbf{R}_{m-\varepsilon,\varepsilon-1,1}.$$

By Example 7.5 the graded $\mathbf{R}_{\varepsilon-1,1}$-module $\mathbf{L}_{\varepsilon i}$ has a filtration by graded submodules whose associated graded is isomorphic to

$$\langle \varepsilon \rangle \mathbf{L}_{(\varepsilon-1)i} \otimes \mathbf{L}_i.$$

Therefore, up to some filtration, we have

$$e_i \psi_!(N, \mathbf{L}_{\varepsilon i}) = \langle \varepsilon \rangle \psi_!(N, \mathbf{L}_{(\varepsilon-1)i}) \otimes \mathbf{L}_i.$$

Now, by Proposition 8.21$(a),(c)$ we have

$$\operatorname{top} \psi_!(N, \mathbf{L}_{(\varepsilon-1)i}) = \tilde{f}_i^{\varepsilon-1}(N) = \tilde{e}_i(M)$$

and all other composition factors L of $\psi_!(N, \mathbf{L}_{(\varepsilon-1)i})$ have $\varepsilon_i(L) < \varepsilon - 1$. Moreover, by Proposition 8.21(a) all composition factors L of R have $\varepsilon_i(L) < \varepsilon$. Thus, by Proposition 8.21(b) all composition factors of $e_i(R)$ are of the form $L \otimes \mathbf{L}_i$ with $\varepsilon_i(L) < \varepsilon - 1$. Therefore, we obtain

$$e_i(M) = \langle \varepsilon \rangle\, \tilde{e}_i(M) + \sum_r f_r N_r, \quad f_r \in \mathcal{A},$$

where N_r is an irreducible graded module with $\varepsilon_i(N_r) < \varepsilon - 1$.

□

8.24. Example. Set $\nu = i + \theta(i)$ and $\mathbf{i} = i\theta(i)$. Let us compute $\tilde{e}_j$ and ε_j.

- If $\lambda_i + \lambda_{\theta(i)} \neq 0$ then $\tilde{e}_j({}^\theta\mathbf{L}_\mathbf{i}) = \mathbf{k}$ if $j = \theta(i)$ and 0 else. We have $\varepsilon_j({}^\theta\mathbf{L}_\mathbf{i}) = 1$ if $j = \theta(i)$ and 0 else.
- If $\lambda_i + \lambda_{\theta(i)} = 0$ then $\tilde{e}_j({}^\theta\mathbf{L}_\mathbf{i}) = \mathbf{k}$ if $j = i, \theta(i)$ and 0 else. We have $\varepsilon_j({}^\theta\mathbf{L}_\mathbf{i}) = 1$ if $j = i, \theta(i)$ and 0 else.

8.25. Proof of Proposition 8.2. First, we prove the following.

8.26. Proposition. *The character map* $\mathrm{ch} : {}^\theta\mathbf{G}_{I,m} \to {}^\theta\mathcal{B}I^m$ *is injective.*

Proof : The proof is similar to that of [K, thm. 5.3.1]. We must prove that the characters of the irreducible graded modules are linearly independent. We proceed by induction on m, the case $m = 0$ being trivial. Suppose $m > 0$ and there is a non-trivial $\mathcal{A}$-linear dependence

$$\sum_M c_M \,\mathrm{ch}(M) = 0. \tag{8.9}$$

We'll show by downward induction on $\varepsilon_i(M)$ that $c_M = 0$. If $\varepsilon_i(M) = m$ then $\Delta_{mi}(M) = 0$ except if $M = \mathbf{L}_{mi}$. Then, applying Δ_{mi} to (8.9) and using (8.6) we deduce that the coefficient of $\mathbf{L}_{mi}$ is zero. Now, assume that $\varepsilon_i(M) = k < m$ and that we have shown that $c_N = 0$ for all N with $\varepsilon_i(N) > k$. Applying Δ_{ki} to (8.9) we get

$$\sum_N c_N \,\mathrm{ch}(\Delta_{ki} N) = 0,$$

where N runs over all irreducible graded modules with $\varepsilon_i(N) = k$. For such a graded module N we have $\Delta_{ki}(N) = \tilde{e}_i^k(N) \otimes \mathbf{L}_{ki}$ by Proposition 8.21(b). Further $\tilde{e}_i^k(N) \neq \tilde{e}_i^k(N')$ if $N \neq N'$ by Proposition 8.22(c). So we conclude by the induction hypothesis.

□

We can now prove Proposition 8.2. Forgetting the grading takes irreducible graded ${}^\theta\mathbf{R}_m$-modules to irreducible modules, and any irreducible module in ${}^\theta\mathbf{R}$-$\mathbf{Mod}_0^f$ comes from an irreducible graded module in ${}^\theta\mathbf{R}$-$\mathbf{mod}^f$ which is unique up to isomorphism and up to grading shift, see e.g., [NV, thm. 4.4.4(v), thm. 9.6.8]. Thus it is enough to prove that for any irreducible graded module M there is an integer d such that $M[d]$ is selfdual. This is proved as in [KL1, p. 342]. More precisely, for any graded module M in ${}^\theta\mathbf{R}$-$\mathbf{mod}^f$ we have $\mathrm{ch}(M^\flat) = \mathrm{ch}(M)$ modulo $(v-1)$. Thus if M is irreducible then we have $M^\flat = M[d]$ for some integer d. We must prove that d is even. We have

$$\mathrm{gdim}(1_\mathbf{i} M^\flat) = \overline{\mathrm{gdim}(1_\mathbf{i} M)}.$$

It is enough to prove the following.

8.27. Lemma. *If M is irreducible then for each* $\mathbf{i}$ *we have*

$$\mathrm{gdim}(1_\mathbf{i} M) \in v\mathbb{Z}[v^2, v^{-2}] \cup \mathbb{Z}[v^2, v^{-2}].$$

Proof : Indeed, we'll prove that this identity holds for the projective module $M = {}^\theta\mathbf{R}_\mathbf{j}$ where $\mathbf{j}$ is any sequence in ${}^\theta I^m$. This implies our claim. Set $\mathbf{j} = (j_{1-m}, \dots, j_{m-1}, j_m)$. Proposition 8.13 yields

$$\mathrm{ch}({}^\theta\mathbf{R}_\mathbf{j}) = \mathrm{ch}({}^\theta\mathbf{R}_{j_1}) \circledast \mathrm{ch}(\mathbf{R}_{j_2}) \circledast \cdots \circledast \mathrm{ch}(\mathbf{R}_{j_m}).$$

Examples 5.9, 7.5(a) yield

$$\mathrm{ch}({}^\theta\mathbf{R}_{j_1}) = (1-v^2)^{-1}\big(j_1\theta(j_1) + v^{\lambda_{j_1}+\lambda_{\theta(j_1)}}\theta(j_1)j_1\big),$$

$$\mathrm{ch}(\mathbf{R}_{j_k}) = (1-v^2)^{-1}\, j_k.$$

So it is enough to check that for each reflection w of W_m which fixes the sequence $\mathbf{j}$ the degree of $\sigma_{\dot w}1_{\mathbf{j}}$ is even. The lemma reduces to the following computation. Fix distinct indices k, l such that $j_k = j_l$. If $1 \leqslant k < l$ and $\dot w = s_k \dots s_{l-2}s_{l-1}s_{l-2}\dots s_k$, or if $1 \leqslant 1-k < l$ and $\dot w = s_{-k}\cdots s_1\varepsilon_1 s_1 \dots s_{l-1}\dots s_1\varepsilon_1 s_1\dots s_{-k}$ then $\deg(\sigma_{\dot w}1_{\mathbf{j}})$ is even.

□

8.28. The algebra ${}^\theta\mathcal{B}$ and its representation in ${}^\theta\mathbf{V}(\lambda)$. Following [EK1,2,3] we define a $\mathcal{K}$-algebra ${}^\theta\mathcal{B}$ as follows.

8.29. Definition. *Let ${}^\theta\mathcal{B}$ be the $\mathcal{K}$-algebra generated by e_i, f_i and invertible elements t_i, $i \in I$, satisfying the following defining relations*

(a) $t_it_j = t_jt_i$ *and* $t_{\theta(i)} = t_i$ *for all* i, j,

(b) $t_ie_jt_i^{-1} = v^{i\cdot j+\theta(i)\cdot j}e_j$ *and* $t_if_jt_i^{-1} = v^{-i\cdot j-\theta(i)\cdot j}f_j$ *for all* i, j,

(c) $e_if_j = v^{-i\cdot j}f_je_i + \delta_{i,j} + \delta_{\theta(i),j}t_i$ *for all* i, j,

(d) *formula* (7.3) *holds with* $\theta_i = e_i$, *or with* $\theta_i = f_i$.

We can now construct a representation of ${}^\theta\mathcal{B}$ as follows. The $\mathcal{K}$-vector space

$${}^\theta\mathbf{V}(\lambda) = \mathcal{K}\otimes_{\mathcal{A}}{}^\theta\mathbf{K}_I$$

is equipped with $\mathcal{K}$-linear operators e_i, e'_i, f_i and with a $\mathcal{K}$-bilinear form in the obvious way. Let ϕ_λ be the class of $\mathbf{k}$ in ${}^\theta\mathbf{K}_I$, where $\mathbf{k}$ is regarded as the trivial module over the ring ${}^\theta\mathbf{R}_0$. Let λ be as in (6.2).

8.30. Theorem. *(a) The operators e_i, f_i define a representation of ${}^\theta\mathcal{B}$ on ${}^\theta\mathbf{V}(\lambda)$. The ${}^\theta\mathcal{B}$-module ${}^\theta\mathbf{V}(\lambda)$ is irreducible and for each $i \in I$ we have*

$$e_i\phi_\lambda = 0,\quad t_i\phi_\lambda = v^{\lambda_i+\lambda_{\theta(i)}}\phi_\lambda,\quad \{x\in{}^\theta\mathbf{V}(\lambda);\ e_jx = 0,\ \forall j\} = \mathbf{k}\,\phi_\lambda.$$

(b) The symmetric bilinear form on ${}^\theta\mathbf{V}(\lambda)$ is non-degenerate. We have $(\phi_\lambda : \phi_\lambda) = 1$ and $(e'_ix : y) = (x : f_iy)$ for all i and all $x, y \in {}^\theta\mathbf{V}(\lambda)$.

(c) There is a unique $\mathcal{K}$-antilinear map ${}^\theta\mathbf{V}(\lambda)\to{}^\theta\mathbf{V}(\lambda)$ such that $P\mapsto P^\sharp$ for all graded projective module P. It is the unique $\mathcal{K}$-antilinear map such that $\phi_\lambda^\sharp = \phi_\lambda$ and $(f_ix)^\sharp = f_i(x^\sharp)$ for all $x \in {}^\theta\mathbf{V}(\lambda)$.

Proof : For each i in I we define the $\mathcal{A}$-linear operator t_i on ${}^\theta\mathbf{K}_I$ by setting

$$t_iP = v^{\lambda_i+\lambda_{\theta(i)}-\nu\cdot(i+\theta(i))}P,\quad \forall P\in{}^\theta\mathbf{R}_\nu\text{-}\mathbf{proj}.$$

We must prove that the operators e_i, f_i, and t_i satisfy the relations of ${}^\theta\mathcal{B}$. All relations are obvious except (c). To check it we need a version of the Mackey's induction-restriction theorem. Note that we have

$$D_{m,1;m,1} = \{e, s_m, \varepsilon_{m+1}\},$$

$$W(e) = W_{m,1},\quad W(s_m) = W_{m-1,1,1},\quad W(\varepsilon_{m+1}) = W_{m,1}.$$

8.31. Lemma. *Fix i, j in I. Let μ, ν in ${}^\theta\mathbb{N}I$ be such that $\nu+i+\theta(i)=\mu+j+\theta(j)$. Put $|\nu|=|\mu|=2m$. The graded $({}^\theta\mathbf{R}_{m,1},{}^\theta\mathbf{R}_{m,1})$-bimodule $1_{\nu,i}{}^\theta\mathbf{R}_{m+1}1_{\mu,j}$ has a filtration by graded bimodules whose associated graded is isomorphic to :*

(a) $({}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_i)\oplus\Big(({}^\theta\mathbf{R}_m1_{\nu',i}\otimes\mathbf{R}_i)\otimes_{\mathbf{R}'}(1_{\nu',i}{}^\theta\mathbf{R}_m\otimes\mathbf{R}_i)\Big)[d]$ *if* $j=i$,

(b) $({}^\theta\mathbf{R}_\nu\otimes\mathbf{R}_{\theta(i)})[d']\oplus\Big(({}^\theta\mathbf{R}_m1_{\nu',\theta(i)}\otimes\mathbf{R}_i)\otimes_{\mathbf{R}'}(1_{\nu',i}{}^\theta\mathbf{R}_m\otimes\mathbf{R}_{\theta(i)})\Big)[d]$ *if* $j=\theta(i)$,

(c) $\Big(({}^\theta\mathbf{R}_m1_{\nu',j}\otimes\mathbf{R}_i)\otimes_{\mathbf{R}'}(1_{\nu',i}{}^\theta\mathbf{R}_m\otimes\mathbf{R}_j)\Big)[d]$ *if* $j\neq i,\theta(i)$.

Here we have set $\nu'=\nu-j-\theta(j)$, $\mathbf{R}'={}^\theta\mathbf{R}_{m-1,1,1}$, $d=\deg(\sigma_m1_{\nu',i,j})$, and $d'=\deg(\pi_{m+1}1_{\nu,\theta(i)})$

The proof is similar to the proof of [M, thm. 1], [KL1, prop. 2.18]. It is left to the reader. Note that we have the following formulas, see Remark 5.2,

$$\deg(\pi_{m+1}1_{\nu,\theta(i)})=\lambda_i+\lambda_{\theta(i)}-\nu\cdot(i+\theta(i))/2,\quad \deg(\sigma_m1_{\nu',i,j})=-i\cdot j.$$

Now, recall that

$$f_jP={}^\theta\mathbf{R}_{m+1}1_{m,j}\otimes_{{}^\theta\mathbf{R}_{m,1}}(P\otimes\mathbf{R}_1),\quad e'_iP=1_{m-1,i}P,$$

where $1_{m-1,i}P$ is regarded as a ${}^\theta\mathbf{R}_{m-1}$-module. Therefore we have

$$e'_if_jP=1_{m,i}{}^\theta\mathbf{R}_{m+1}1_{m,j}\otimes_{{}^\theta\mathbf{R}_{m,1}}(P\otimes\mathbf{R}_1),$$
$$f_je'_iP={}^\theta\mathbf{R}_m1_{m-1,j}\otimes_{{}^\theta\mathbf{R}_{m-1,1}}(1_{m-1,i}P\otimes\mathbf{R}_1).$$

Therefore, up to some filtration we have the following identities

(a) $e'_if_iP=P\otimes\mathbf{R}_i+f_ie'_iP[-2]$,

(b) $e'_if_{\theta(i)}P=P\otimes\mathbf{R}_{\theta(i)}[\lambda_i+\lambda_{\theta(i)}-\nu\cdot(i+\theta(i))/2]+f_{\theta(i)}e'_iP[-i\cdot\theta(i)]$,

(c) $e'_if_jP=f_je'_iP[-i\cdot j]$ if $i\neq j,\theta(j)$.

The first claim of part (a) of the theorem follows because

$$(1-v^2)\mathrm{gdim}(\mathbf{R}_i)=1.$$

Note that ${}^\theta\mathbf{V}(\lambda)$ is generated by ϕ_λ as a ${}^\theta\boldsymbol{\mathcal{B}}$-module by Lemma 8.32 below. The rest of (a) is standard, see [EK2, prop. 4.2(i)].

Part (b) of the theorem follows from [EK2, prop. 4.2(ii)] and Lemma 8.9(b).

Finally, for part (c) of the theorem it is enough to check that $(f_iP)^\sharp=f_i(P^\sharp)$ for any graded module P in ${}^\theta\mathbf{R}$**-proj**. By Lemma 8.32 below we may assume that $P={}^\theta\mathbf{R}_\mathbf{y}$ for some $\mathbf{y}$. Finally, by (8.3) we can also assume that $\mathbf{y}=\mathbf{i}$ lies in ${}^\theta I^\nu$. Then the claim follows from the formulas in Proposition 8.14.

□

8.32. Lemma. *Any object of ${}^\theta\mathbf{R}_m$-**proj** is of the form ${}^\theta\mathbf{R}_m\otimes_{\mathbf{R}_m}P$ for some P in $\mathbf{R}_m$-**proj**. The $\mathcal{A}$-module ${}^\theta\mathbf{K}_{I,m}$ is spanned by the ${}^\theta\mathbf{R}_\mathbf{y}$'s with $\mathbf{y}$ in ${}^\theta Y^m$.*

Proof : An easy induction using Proposition 8.23 implies that for each b in ${}^\theta B(\lambda)$ and for each integer $a \geqslant 1$ we have we have

$$f_i^{(a)}\,{}^\theta G^{\mathrm{low}}(b) = \left\langle \begin{matrix} \varepsilon_i(b)+a \\ a \end{matrix} \right\rangle {}^\theta G^{\mathrm{low}}(\tilde{F}_i^a b) + \sum_{b'} f_{b,b'}\,{}^\theta G^{\mathrm{low}}(b'),$$

where b' runs over the set of elements of ${}^\theta B(\lambda)$ such that $\varepsilon_i(b') > \varepsilon_i(b)+a$ and $f_{b,b'}$ lies in $\mathcal{A}$. Thus the claim follows by Proposition 8.22$(e),(c)$.

□

8.33. Remark. Note that ${}^\theta\mathbf{V}(\lambda)$ is equal to the ${}^\theta\boldsymbol{\mathcal{B}}$-module $V_\theta(\lambda+\theta(\lambda))$ in [EK1, prop. 2.5].

9. Presentation of the graded k-algebra ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector ν in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$. Fix an object $(\mathbf{V},\varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. In this section we give a proof of Theorem 5.8. By Theorem 4.19 and Corollary 5.6 there is a unique injective graded **k**-algebra homomorphism

$$\Phi : {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu} \to {}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}},$$

$$1_{\mathbf{i}} \mapsto 1_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}, \quad \varkappa_{\mathbf{i},l} \mapsto \varkappa_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(l), \quad \sigma_{\mathbf{i},k} \mapsto \sigma_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(k), \quad \pi_{\mathbf{i},1} \mapsto \pi_{\mathbf{\Lambda},\mathbf{V},\mathbf{i}}(1),$$

$$\mathbf{i} \in {}^\theta I^\nu, \quad k = 1,\dots,m-1, \quad l = 1,2,\dots,m.$$

We must prove that Φ is a surjective map. Note that both algebras have 1, because the set ${}^\theta I^\nu$ is finite. Since the grading does not matter, we can replace ${}^\theta\mathbf{Z}^\delta_{\mathbf{\Lambda},\mathbf{V}}$ by ${}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}$. To unburden the notation we abbreviate

$${}^\theta\mathbf{F}_\nu = {}^\theta\mathbf{F}_{\mathbf{\Lambda},\mathbf{V}}, \quad {}^\theta\mathbf{Z}_\nu = {}^\theta\mathbf{Z}_{\mathbf{\Lambda},\mathbf{V}}, \quad {}^\theta\mathcal{Z}_\nu = {}^\theta\mathcal{Z}_{\mathbf{\Lambda},\mathbf{V}}, \quad {}^\theta Z_\nu = {}^\theta Z_{\mathbf{\Lambda},\mathbf{V}}, \quad etc.$$

9.1. The filtration of ${}^\theta\mathbf{Z}_\nu$. Recall that W_m is regarded as a Coxeter group of type B_m as in Section 5.3. Let $\leqslant$ and ℓ be the corresponding Bruhat order and length function. They differ from the order and length function introduced in Section 4.2. We hope this will not create any confusion. For a future use set

$$\Delta^+ = \Delta^+_s \sqcup \Delta^+_l, \quad \Delta^+_s = \{\chi_k \pm \chi_l;\ 1 \leqslant l < k \leqslant m\}, \quad \Delta^+_l = \{\chi_1,\chi_2,\dots,\chi_m\}.$$

The set of simple reflections of W_m is $\{s_1,\dots,s_m\}$ with $s_m = \varepsilon_1$. We can now introduce a filtration of ${}^\theta Z_\nu$. We define

$${}^\theta O^{\leqslant x}_\nu = \bigcup_{w\leqslant x} {}^\theta O^w_\nu, \quad {}^\theta Z^{\leqslant x}_\nu = q^{-1}({}^\theta O^{\leqslant x}_\nu), \quad {}^\theta\mathcal{Z}^{\leqslant x}_\nu = H^{{}^\theta G_\nu}_*({}^\theta Z^{\leqslant x}_\nu, \mathbf{k}).$$

Let ${}^\theta\mathbf{Z}^{\leqslant x}_\nu$ be the image of ${}^\theta\mathcal{Z}^{\leqslant x}_\nu$ by the isomorphism ${}^\theta\mathbf{Z}_\nu = {}^\theta\mathcal{Z}_\nu$ in Proposition 3.1(b).

9.2. Lemma. *(a) The set* ${}^\theta Z_\nu^{\leqslant x}$ *is closed. The variety* ${}^\theta Z_\nu^x$ *is smooth if* $\ell(x)=1$.

(b) The direct image by the inclusion ${}^\theta Z_\nu^{\leqslant x}\subset {}^\theta Z_\nu$ *is an injection* ${}^\theta \mathcal{Z}_\nu^{\leqslant x}\to {}^\theta\mathcal{Z}_\nu$.

(c) The convolution product maps ${}^\theta\mathcal{Z}_\nu^{\leqslant x}\times {}^\theta\mathcal{Z}_\nu^{\leqslant y}$ *into* ${}^\theta\mathcal{Z}_\nu^{\leqslant xy}$ *for each* x, y *such that* $\ell(xy)=\ell(x)+\ell(y)$.

(d) The unit of ${}^\theta\mathcal{Z}_\nu$ *lies in* ${}^\theta\mathcal{Z}_\nu^e$.

Proof : To avoid confusions, let $\leqslant_D$ and ℓ_D be the Bruhat order and length function introduced in Section 4.2. The claims in the lemma are well-known if we replace $\leqslant$, ℓ by $\leqslant_D$, ℓ_D respectively. Therefore, it is enough to prove the following : assume that ${}^\theta O^v_{\nu,x,y}$ and ${}^\theta O^w_{\nu,x,y}$ are non empty. Then ${}^\theta O^v_{\nu,x,y}\subset {}^\theta\bar O^w_{\nu,x,y}$ iff $v\leqslant w$. Up to the action of a well-chosen diagonal element we may assume that $x=e$. We can also assume that y is minimal in the coset $W_\nu y$. Since ${}^\theta O^v_{\nu,e,y}$ and ${}^\theta O^w_{\nu,e,y}$ are non empty, we have $v,w\in W_\nu y$. Finally, on the coset $W_\nu y$ the orders $\leqslant$, $\leqslant_D$ are the same because $W_\nu\subset\mathfrak{S}_m$, see e.g., the last remark in Section 4.2.

□

9.3. PBW theorem for ${}^\theta\mathbf{Z}_\nu$. View ${}^\theta\mathbf{F}_\nu$ as a graded (commutative) **k**-algebra as in Section 4.13. The graded **k**-algebra ${}^\theta\mathbf{Z}_\nu$ has a natural structure of graded ${}^\theta\mathbf{F}_\nu$-module such that $x_{\mathbf{i}}(l)$ acts as $\varkappa_{\mathbf{i}}(l)$ for each $l=1,2,\dots,m$ under the isomorphism in Theorem 4.19. Recall that ${}^\theta\mathbf{Z}_\nu={}^\theta\mathcal{Z}_\nu$. The following is immediate, see e.g., [CG, sec. 5.5].

9.4. Lemma. *We have* ${}^\theta\mathbf{Z}_\nu^{\leqslant x}=\bigoplus_{w\leqslant x}{}^\theta\mathbf{F}_\nu\,[{}^\theta Z_\nu^w]$ *for each* x. *In particular* ${}^\theta\mathbf{Z}_\nu$ *is a free graded* ${}^\theta\mathbf{F}_\nu$*-module of rank* $2^m m!$.

The map Φ is a graded ${}^\theta\mathbf{F}_\nu$-module homomorphism. For each x we set

$$\mathbf{R}^{\leqslant x}=\sum_{w\leqslant x}{}^\theta\mathbf{F}_\nu\,1_\nu\sigma_{\dot w},$$

where $\sigma_{\dot w}$ is defined as in (5.2). It is a graded ${}^\theta\mathbf{F}_\nu$-submodule of ${}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}$. We abbreviate $\mathbf{R}^e=\mathbf{R}^{\leqslant e}$. The proof of the surjectivity of Φ consists of two steps. First we prove that $\Phi(\mathbf{R}^{\leqslant x})\subset{}^\theta\mathbf{Z}_\nu^{\leqslant x}$. Then we prove that this inclusion is an equality.

9.5. Step 1. Since Φ is a ${}^\theta\mathbf{F}_\nu$-module homomorphism it is enough to prove that $\Phi(\sigma_{\dot x})$ lies in ${}^\theta\mathbf{Z}_\nu^{\leqslant x}$. By an easy induction on the length of x it is enough to observe that we have

$$\Phi(1)\in{}^\theta\mathbf{Z}_\nu^e,\quad \Phi(\sigma_k)\subset{}^\theta\mathbf{Z}_\nu^{\leqslant s_k},\quad k=1,2,\dots,m.$$

9.6. Step 2. Note that ${}^\theta\mathbf{Z}_\nu^e$ is the free ${}^\theta\mathbf{F}_\nu$-module of rank one generated by $[{}^\theta Z_\nu^e]$. Therefore we have

$$\Phi(\mathbf{R}^e)={}^\theta\mathbf{F}_\nu\,[{}^\theta Z_\nu^e]={}^\theta\mathbf{Z}_\nu^e.$$

To complete the proof of Step 2 we are reduced to prove the following.

9.7. Lemma. *If* $\ell(s_kw)=\ell(w)+1$ *and* $k=1,2,\dots m$, *then we have the following formula in* ${}^\theta\mathbf{Z}_\nu^{\leqslant s_kw}/{}^\theta\mathbf{Z}_\nu^{<s_kw}$ *:*

$$[{}^\theta Z_\nu^{s_k}]\star[{}^\theta Z_\nu^w]=[{}^\theta Z_\nu^{s_kw}].$$

Proof : By Lemmas 9.2(c) and 9.4 there is an unique element c in ${}^\theta\mathbf{F}_\nu$ such that

$$[{}^\theta Z_\nu^{s_k}]\star[{}^\theta Z_\nu^{w}]=c\star[{}^\theta Z_\nu^{s_kw}]\ \text{in}\ {}^\theta\mathbf{Z}_\nu^{\leqslant s_kw}/{}^\theta\mathbf{Z}_\nu^{<s_kw}.$$

Let us prove that $c=1$. For each x,y,z there is a unique element $\Lambda^x_{y,z}$ in $\mathbf{K}$ such that

$$[{}^\theta Z_\nu^{x}]=\sum_{y,z}\Lambda^x_{y,z}\psi_{y,z},$$

see Section 4.14. Since $\phi_{\nu,y,yx}$ is a smooth point of ${}^\theta Z^x_\nu$ we have also

$$\Lambda^x_{y,yx}=\mathrm{eu}({}^\theta Z^x_\nu,\phi_{\nu,y,yx})^{-1}.$$

Hence, in the expansion of the element $[{}^\theta Z_\nu^{s_kw}]$ in the $\mathbf{K}$-basis $(\psi_{y,z})$ the coordinate along the vector ψ_{x,xs_kw} is equal to

$$\Lambda^{s_kw}_{x,xs_kw}=\mathrm{eu}({}^\theta Z^{s_kw}_\nu,\phi_{\nu,x,xs_kw})^{-1}.$$

On the other hand, since $\Lambda^w_{x,xs_kw}=0$ and

$$[{}^\theta Z_\nu^{s_k}]=\sum_x(\Lambda^{s_k}_{x,x}\psi_{x,x}+\Lambda^{s_k}_{x,xs_k}\psi_{x,xs_k}),$$

the coordinate of $[{}^\theta Z_\nu^{s_k}]\star[{}^\theta Z_\nu^{w}]$ along ψ_{x,xs_kw} is equal to

$$\Lambda^{s_k}_{x,xs_k}\Lambda^{w}_{xs_k,xs_kw}\Lambda_{xs_k}=\mathrm{eu}({}^\theta Z^{s_k}_\nu,\phi_{\nu,x,xs_k})^{-1}\mathrm{eu}({}^\theta Z^{w}_\nu,\phi_{\nu,xs_k,xs_kw})^{-1}\Lambda_{xs_k}.$$

Thus we must check that

$$\mathrm{eu}({}^\theta Z^{s_k}_\nu,\phi_{\nu,x,xs_k})\,\mathrm{eu}({}^\theta Z^{w}_\nu,\phi_{\nu,xs_k,xs_kw})=\mathrm{eu}({}^\theta Z^{s_kw}_\nu,\phi_{\nu,x,xs_kw})\,\Lambda_{xs_k}.$$

This follows from the lemma below.

9.8. Lemma. *(a) For each x,y in W we have*

$$\mathrm{eu}({}^\theta O^y_\nu,\phi_{\nu,x,xy})=\mathrm{eu}({}^\theta\mathfrak{n}_{\nu,x}\oplus{}^\theta\mathfrak{m}_{\nu,xy,x}),$$

$$\mathrm{eu}({}^\theta Z^y_\nu,\phi_{\nu,x,xy})=\mathrm{eu}({}^\theta O^y_\nu,\phi_{\nu,x,xy})\,\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,x,xy}),$$

$$\Lambda_x=\mathrm{eu}({}^\theta Z^e_\nu,\phi_{\nu,x,x})=\mathrm{eu}({}^\theta F_\nu,\phi_{\nu,x})\,\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,x}).$$

(b) For each w,x,y in W such that $\ell(xy)=\ell(x)+\ell(y)$ we have

$$\mathrm{eu}({}^\theta O^{xy}_\nu,\phi_{\nu,w,wxy})\,\mathrm{eu}({}^\theta F_\nu,\phi_{\nu,wx})=\mathrm{eu}({}^\theta O^x_\nu,\phi_{\nu,w,wx})\,\mathrm{eu}({}^\theta O^y_\nu,\phi_{\nu,wx,wxy}),$$

$$\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,w,wxy}\oplus{}^\theta\mathfrak{e}^*_{\nu,wx})=\mathrm{eu}({}^\theta\mathfrak{e}^*_{\nu,w,wx}\oplus{}^\theta\mathfrak{e}^*_{\nu,wx,wxy}).$$

Proof : Part (a) is left to the reader. Compare Proposition 4.15 where similar formulas are given. Let us prove (b). Clearly we can assume $w=e$. Set

$$\Delta(y)^-=y(\Delta^+)\cap\Delta^-,\quad \Delta(y)^+=y(\Delta^-)\cap\Delta^+.$$

Recall that

$$\ell(xy) = \ell(x) + \ell(y) \Rightarrow \begin{cases} \Delta(xy)^- = \Delta(x)^- \sqcup x(\Delta(y)^-), \\ \Delta(xy)^+ = \Delta(x)^+ \sqcup x(\Delta(y)^+). \end{cases}$$

For each x, y in W the T-module ${}^\theta\mathfrak{m}_{\nu,xy,x}$ is the sum of the root subspaces whose weight belong to the set $x(\Delta(y)^-) \cap {}^\theta\Delta_\nu$. Thus, by (a), the first claim follows from the equality

$$\Delta(xy)^- \cap {}^\theta\Delta_\nu = \big(\Delta(x)^- \sqcup x(\Delta(y)^-)\big) \cap {}^\theta\Delta_\nu.$$

Now, let us concentrate on the second claim. Let Ξ be the set of weights of the T-module ${}^\theta E_{\mathbf{V}}$, and set

$$S_{x,xy} = x(\Delta_s^+ \cap y(\Delta_s^+)), \quad L_{x,xy} = x(\Delta_l^+ \cap y(\Delta_l^+)).$$

The character of the T-module ${}^\theta\mathfrak{e}_{\nu,x,xy}$ is

$$\sum \alpha + \sum \lambda_{i_l}\chi_l, \quad \alpha \in S_{x,xy} \cap \Xi, \quad \chi_l \in L_{x,xy}.$$

See Section 4.9 for details. Let

$$S = S_{e,xy} \sqcup x(\Delta_s^+), \quad S' = S_{e,x} \sqcup x(S_{e,y}),$$
$$L = L_{e,xy} \sqcup x(\Delta_l^+), \quad L' = L_{e,x} \sqcup x(L_{e,y}).$$

The character of the T-module ${}^\theta\mathfrak{e}_{\nu,e,xy} \oplus {}^\theta\mathfrak{e}_{\nu,x}$ is

$$\sum \alpha + \sum \lambda_{i_l}\chi_l, \quad \alpha \in S \cap \Xi, \quad \chi_l \in L,$$

while the character of the T-module ${}^\theta\mathfrak{e}_{\nu,e,x} \oplus {}^\theta\mathfrak{e}_{\nu,x,xy}$ is

$$\sum \alpha + \sum \lambda_{i_l}\chi_l, \quad \alpha \in S' \cap \Xi, \quad \chi_l \in L'.$$

Now, a short computation yields

$$S = S' \iff \Delta_s^+ \cap \Delta(xy)^+ = \Delta_s^+ \cap \big(\Delta(x)^+ \sqcup x(\Delta(y)^+)\big),$$
$$L = L' \iff \Delta_l^+ \cap \Delta(xy)^+ = \Delta_l^+ \cap \big(\Delta(x)^+ \sqcup x(\Delta(y)^+)\big).$$

□

10. Perverse sheaves on ${}^\theta\mathbf{E}_{\mathbf{\Lambda},\mathbf{V}}$ and the global bases of ${}^\theta\mathbf{V}(\lambda)$

Fix a quiver Γ with set of vertices I and set of arrows H. Fix an involution θ on Γ. Assume that Γ has no 1-loops and that θ has no fixed points. Fix a dimension vector ν in ${}^\theta\mathbb{N}I$ and a dimension vector λ in $\mathbb{N}I$. Set $|\nu| = 2m$. Fix an object $(\mathbf{V}, \varpi)$ in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$ and an object $\mathbf{\Lambda}$ in $\boldsymbol{\mathcal{V}}_\lambda$. To unburden the notation we'll abbreviate

$$ {}^\theta G_\nu = {}^\theta G_{\mathbf{V}}, \quad {}^\theta\mathbf{R}_\nu = {}^\theta\mathbf{R}(\Gamma)_{\lambda,\nu}. $$

10.1. Perverse sheaves on ${}^\theta E_\nu$. First, we must generalize the setting in Section 2. We define *an orientation of* I to be a partition $I = \Omega \sqcup \bar{\Omega}$. Fix an orientation. For each I-graded vector space $\mathbf{W}$ we write $\mathbf{W}_\Omega = \bigoplus_{i\in\Omega} \mathbf{W}_i$. Now, we define

$$ L_{\mathbf{\Lambda},\mathbf{V},\Omega} = \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{\Lambda}_\Omega, \mathbf{V}_\Omega) \oplus \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{V}_{\bar\Omega}, \mathbf{V}_{\bar\Omega}), \quad {}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega} = {}^\theta E_{\mathbf{V}} \times L_{\mathbf{\Lambda},\mathbf{V},\Omega}. $$

An element of ${}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega}$ is a triplet (x, y, z) with

$$ x \in {}^\theta E_{\mathbf{V}}, \quad y \in \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{\Lambda}_\Omega, \mathbf{V}_\Omega), \quad z \in \mathrm{Hom}_{\boldsymbol{\mathcal{V}}}(\mathbf{V}_{\bar\Omega}, \mathbf{\Lambda}_{\bar\Omega}). $$

For each $\mathbf{y}$ in ${}^\theta Y^\nu$ we define also

$$ {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y},\Omega} = \{(x, y, z, \phi) \in {}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega} \times {}^\theta F_{\mathbf{V},\mathbf{y}}\,;\, \phi = (\mathbf{V}^l),\, x(\mathbf{V}^l) \subset \mathbf{V}^l, $$
$$ y(\mathbf{\Lambda}) \subset \mathbf{V}^0,\, z(\mathbf{V}^0) = 0\}. $$

To unburden the notation we'll abbreviate

$$ {}^\theta E_{\nu,\Omega} = {}^\theta E_{\mathbf{\Lambda},\mathbf{V},\Omega}, \quad {}^\theta F_{\mathbf{y}} = {}^\theta F_{\mathbf{V},\mathbf{y}}, \quad {}^\theta\widetilde{F}_{\mathbf{y},\Omega} = {}^\theta\widetilde{F}_{\mathbf{\Lambda},\mathbf{V},\mathbf{y},\Omega}. $$

We define the semisimple complex ${}^\theta\mathcal{L}_{\mathbf{y},\Omega}$ over ${}^\theta E_{\nu,\Omega}$ as the direct image of the constant sheaf over ${}^\theta\widetilde{F}_{\mathbf{y},\Omega}$ by the obvious projection. We define ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$ as the set of isomorphism classes of simple perverse sheaves over ${}^\theta E_{\nu,\Omega}$ which appear as a direct factor of ${}^\theta\mathcal{L}_{\mathbf{y},\Omega}[d]$ for some $\mathbf{y} \in {}^\theta Y^\nu$ and $d \in \mathbb{Z}$. Next, we define ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}$ as the full subcategory of $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\mathbf{V}}}({}^\theta E_{\nu,\Omega})$ consisting of the objects which are isomorphic to finite direct sums of $\mathcal{L}[d]$ with $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$ and $d \in \mathbb{Z}$. When there is no risk of confusion we abbreviate ${}^\theta\boldsymbol{\mathcal{P}} = {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$, ${}^\theta\boldsymbol{\mathcal{Q}} = {}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}$, and ${}^\theta\mathcal{L}_{\mathbf{y}} = {}^\theta\mathcal{L}_{\mathbf{y},\Omega}$.

10.2. Example. Let Γ, θ, and λ be as in Sections 6.2, 6.4. Set $\bar\Omega = \emptyset$, and $\nu = i + \theta(i)$ for some $i \in I$. We have ${}^\theta E_{\nu,\Omega} = L_i \times L_{\theta(i)}$ with $L_j = \mathrm{Hom}(\mathbf{\Lambda}_j, \mathbf{V}_j)$, ${}^\theta I^\nu = \{\mathbf{i}, \theta(\mathbf{i})\}$ with $\mathbf{i} = i\theta(i)$, and

$$ {}^\theta\widetilde{F}_{\mathbf{i},\Omega} = \{(\mathbf{V} \supset \mathbf{V}_{\theta(i)} \supset 0)\} \times L_{\theta(i)}, \quad {}^\theta\widetilde{F}_{\theta(\mathbf{i}),\Omega} = \{(\mathbf{V} \supset \mathbf{V}_i \supset 0)\} \times L_i. $$

Therefore the following holds

- if $\lambda_i + \lambda_{\theta(i)} \neq 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{L_i}[\lambda_i], \mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}]\}$, ${}^\theta\mathcal{L}^\delta_{\theta(\mathbf{i})} = \mathbf{k}_{L_i}[\lambda_i]$, and ${}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}]$,
- if $\lambda_i + \lambda_{\theta(i)} = 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{\{0\}}\}$ and ${}^\theta\mathcal{L}^\delta_{\theta(\mathbf{i})} = {}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \mathbf{k}_{\{0\}}$.

10.3. Multiplication of complexes. Set $\nu'' = \nu + \nu' + \theta(\nu')$ with $\nu' \in \mathbb{N}I$. Fix $\mathbf{V}' \in \boldsymbol{\mathcal{V}}_{\nu'}$ and $\mathbf{V}'' \in {}^\theta\boldsymbol{\mathcal{V}}_{\nu''}$. Let T be the set of triples (V, γ, γ') where

- V is an I-graded subspace of $\mathbf{V}''$ such that $\mathbf{V}''/V \in \boldsymbol{\mathcal{V}}_{\nu'}$ and $V^\perp \subset V$,
- $\gamma : \mathbf{V} \to V/V^\perp$ is an isomorphism in ${}^\theta\boldsymbol{\mathcal{V}}_\nu$,
- $\gamma' : \mathbf{V}' \to \mathbf{V}''/V$ is an isomorphism in $\boldsymbol{\mathcal{V}}_{\nu'}$.

We consider the following diagram

$$
{}^\theta E_{\nu,\Omega} \times E_{\nu'} \xleftarrow{p_1} {}^\theta E_{1,\Omega} \xrightarrow{p_2} {}^\theta E_{2,\Omega} \xrightarrow{p_3} {}^\theta E_{\nu'',\Omega}.
$$

Here ${}^\theta E_{2,\Omega}$ is the variety of tuples (x, y, z, V) where

- V is an I-graded subspace of $\mathbf{V}''$ such that $\mathbf{V}''/V \in \boldsymbol{\mathcal{V}}_{\nu'}$ and $V^\perp \subset V$,
- $(x, y, z) \in {}^\theta E_{\nu'',\Omega}$, $y(\mathbf{\Lambda}) \subset V$, $x(V) \subset V$, and $z(V^\perp) = 0$,

and ${}^\theta E_{1,\Omega}$ is the variety of tuples $(x, y, z, V, \gamma, \gamma')$ where

- $(V, \gamma, \gamma') \in T$,
- $(x, y, z, V) \in {}^\theta E_{2,\Omega}$.

Finally the maps are given by

- $p_1(x, y, z, V, \gamma, \gamma') = (x_\gamma, y_\gamma, z_\gamma, x_{\gamma'})$,
- $p_2(x, y, z, V, \gamma, \gamma') = (x, y, z, V)$,
- $p_3(x, y, z, V) = (x, y, z)$,

where

- $x_\gamma = \gamma^{-1} \circ (x|_{V/V^\perp}) \circ \gamma$,
- $x_{\gamma'} = (\gamma')^{-1} \circ (x|_{\mathbf{V}''/V}) \circ \gamma'$,
- $y_\gamma = \gamma^{-1} \circ y$.
- $z_\gamma = z \circ \gamma$.

The group ${}^\theta G_{\nu''}$ acts on ${}^\theta E_{1,\Omega}$, ${}^\theta E_{2,\Omega}$ and the maps p_2, p_3 are ${}^\theta G_{\nu''}$-equivariant. Note that p_1 is a smooth map with connected fibers, that p_2 is a principal bundle, and that p_3 is proper. Therefore, for any complexes $\mathcal{E} \in \boldsymbol{\mathcal{D}}_{{}^\theta G_\nu}({}^\theta E_{\nu,\Omega})$ and $\mathcal{E}' \in \boldsymbol{\mathcal{D}}_{G_{\nu'}}(E_{\nu'})$ there is a unique complex $\mathcal{E}_2 \in \boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{2,\Omega})$ such that

$$
p_1^*(\mathcal{E} \boxtimes \mathcal{E}') = p_2^*(\mathcal{E}_2).
$$

Then, we define a complex $\mathcal{E}'' = \varphi_!(\mathcal{E}, \mathcal{E}')$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{\nu'',\Omega})$ by

$$
\mathcal{E}'' = (p_3)_!(\mathcal{E}_2).
$$

Now, let $\nu' = i$. Hence $E_{\nu'} = 0$. Let $\mathcal{L}_i = \mathbf{k}_{E_{\nu'}}$, the trivial complex over $E_{\nu'}$.

10.4. Definition. *Set $\nu' = i$. For $\mathcal{E}$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_\nu}({}^\theta E_{\nu,\Omega})$ we define the complex*

$$
\underline{f_i}(\mathcal{E}) = \varphi_!(\mathcal{E}, \mathcal{L}_i)[b_{\nu,i}], \quad b_{\nu,i} = \nu_i + \sum_j \nu_j \, h_{i,j} + \lambda_{\Omega,\theta(i)} + \lambda_{\bar{\Omega},i}.
$$

10.5. Proposition. *(a)* $\underline{f_i}$ *yields a functor* ${}^\theta\boldsymbol{\mathcal{Q}} \to {}^\theta\boldsymbol{\mathcal{Q}}$.

(b) $\underline{f_i}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}) = {}^\theta\mathcal{L}^\delta_{i\mathbf{i}\theta(i)}$ *for each* $\mathbf{i}$ *in* ${}^\theta I^\nu$.

Proof : A standard computation yields

$$\varphi_!({}^\theta\mathcal{L}_{\mathbf{y}}, \mathcal{L}_i) = {}^\theta\mathcal{L}_{i\mathbf{y}\theta(i)}, \quad \mathbf{y} \in {}^\theta Y^\nu.$$

See [E, prop. 4.11], [L2, sec. 9.2.6-7]. This implies (a). The same computation as in Proposition 2.5 yields

$$d_{\mathbf{i}} = \ell_\nu/2 + \sum_{k<l;\, k+l\neq 1} h_{i_k,i_l}/2 + \sum_{1\leqslant l}(\lambda_{\Omega,i_l} + \lambda_{\bar\Omega,\theta(i_l)}), \tag{10.1}$$

where $\lambda_{\Omega,i} = \lambda_i$ if $i \in \Omega$ and 0 else. Thus we have

$$d_{\lambda,i\mathbf{i}\theta(i)} - d_{\lambda,\mathbf{i}} = b_{\nu,i}.$$

Part (b) follows from this equality.

□

10.6. Restriction of complexes. Set $\nu'' = \nu + \nu' + \theta(\nu')$ with $\nu' \in \mathbb{N}I$. Fix $\mathbf{V}' \in \boldsymbol{\mathcal{V}}_{\nu'}$, $\mathbf{V}'' \in {}^\theta\boldsymbol{\mathcal{V}}_{\nu''}$, and fix a triple $(V, \gamma, \gamma') \in T$. Consider the diagram

$${}^\theta E_{\nu,\Omega} \times E_{\nu'} \xleftarrow{\kappa} {}^\theta E_{3,\Omega} \xrightarrow{\iota} {}^\theta E_{\nu'',\Omega}.$$

Here we have set

- ${}^\theta E_{3,\Omega} = \{(x,y,z) \in {}^\theta E_{\nu'',\Omega}\,;\, x(V) \subset V,\, y(\boldsymbol{\Lambda}) \subset V,\, z(V^\perp) = 0\}$,
- $\kappa(x,y,z) = (x_\gamma, y_\gamma, z_\gamma, x_{\gamma'})$,
- $\iota(x,y,z) = (x,y,z)$.

For any $\mathcal{E}''$ in $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{\nu'',\Omega})$ we define a complex in $\boldsymbol{\mathcal{D}}_{{}^\theta G_\nu \times G_{\nu'}}({}^\theta E_{\nu,\Omega} \times E_{\nu'})$ by

$$\varphi^*(\mathcal{E}'') = \kappa_!\iota^*(\mathcal{E}'').$$

10.7. Definition. *Set* $\nu' = i$. *For any* $\mathcal{E}''$ *in* $\boldsymbol{\mathcal{D}}_{{}^\theta G_{\nu''}}({}^\theta E_{\nu'',\Omega})$ *we define*

$$\underline{e_i}(\mathcal{E}'') = \varphi^*(\mathcal{E}'')[a_{\nu,i}], \quad a_{\nu,i} = -\nu_i + \sum_j \nu_j\, h_{i,j} + \lambda_{\Omega,\theta(i)} + \lambda_{\bar\Omega,i}.$$

10.8. Proposition. *(a)* $\underline{e_i}$ *yields a functor from* ${}^\theta\boldsymbol{\mathcal{Q}} \to {}^\theta\boldsymbol{\mathcal{Q}}$.

(b) $\underline{e_i}({}^\theta\mathcal{L}_{\mathbf{y}}) = \bigoplus_k {}^\theta\mathcal{L}_{\mathbf{y}_k}[-2d_k]$ *for some integer* d_k. *The sum runs over all* k *such that* $i_k = i$, *and*

$$\mathbf{y}_k = (\mathbf{i}, \mathbf{a}^{(k)}), \quad \mathbf{a}^{(k)} = (a^{(k)}_l), \quad a^{(k)}_l = a_l - \delta_{l,k} - \delta_{l,1-k}.$$

(c) $\underline{e_i}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}) = \bigoplus_{\mathbf{i}'} {}^\theta\mathcal{L}^\delta_{\mathbf{i}'}[\deg(\mathbf{i}', \theta(i); \mathbf{i})]$, *where* $\mathbf{i}'$ *runs over all sequences such that* $\mathbf{i}$ *lies in* $Sh(\mathbf{i}', \theta(i))$.

Proof: Part (a) follows from (b). Parts (b) and (c) are analogues of [E, prop. 4.11(ii)] (which itself is an analogue of [L2, sec. 9.2.6]), where the case $\lambda = 0$ is considered. Our proof is similar. Assume that $\nu' = i$. Hence we have $E_{\nu'} = \{0\}$. We define

$$
\begin{gathered}
{}^\theta\widetilde{F}_{3,\Omega} = \{(x,y,z,\phi) \in {}^\theta\widetilde{F}_{\mathbf{y},\Omega}\,;\,(x,y,z) \in {}^\theta E_{3,\Omega}\},\\
{}^\theta F_3^{(k)} = \{\phi = (\mathbf{V}^l) \in {}^\theta F_{\mathbf{y}}\,;\,\mathbf{V}^k \subset V,\, \mathbf{V}^{k-1} \not\subset V\},\\
{}^\theta\widetilde{F}^{(k)}_{3,\Omega} = \{(x,y,z,\phi) \in {}^\theta\widetilde{F}_{3,\Omega};\,\phi \in {}^\theta F_3^{(k)}\}.
\end{gathered}
$$

Note that ${}^\theta\widetilde{F}_{3,\Omega} = \bigcup_k {}^\theta\widetilde{F}^{(k)}_{3,\Omega}$ is a partition into locally closed subsets. Let $\mathbf{y}_k$ be as above. Consider the map

$$
f_k : {}^\theta\widetilde{F}^{(k)}_{3,\Omega} \to {}^\theta\widetilde{F}_{\mathbf{y}_k,\Omega}, \quad (x,y,z,\phi) \mapsto (x_\gamma, y_\gamma, z_\gamma, \phi_\gamma),
$$

where ϕ_γ is the flag whose l-th term is equal to

$$
\gamma^{-1}\big((V \cap \mathbf{V}^l + V^\perp)/V^\perp\big).
$$

We get the following diagram, whose right square is Cartesian

$$
\begin{array}{ccccccc}
{}^\theta\widetilde{F}_{\mathbf{y}_k,\Omega} & \xleftarrow{f_k} & {}^\theta\widetilde{F}^{(k)}_{3,\Omega} & \longrightarrow & {}^\theta\widetilde{F}_{3,\Omega} & \longrightarrow & {}^\theta\widetilde{F}_{\mathbf{y},\Omega} \\
\downarrow & & & & \downarrow & & \downarrow \\
{}^\theta E_{\nu,\Omega} & & \xleftarrow{\quad\kappa\quad} & & {}^\theta E_{3,\Omega} & \xrightarrow{\iota} & {}^\theta E_{\nu'',\Omega}.
\end{array}
$$

It is easy to prove that f_k is an affine bundle. Let $d_k = d_{f_k}$ be its relative dimension. A standard argument using the diagram above yields

$$
\varphi^*({}^\theta\mathcal{L}_{\mathbf{y}}) = \kappa_!\iota^*({}^\theta\mathcal{L}_{\mathbf{y}}) = \bigoplus_{i_k = i} {}^\theta\mathcal{L}_{\mathbf{y}_k}[-2d_k].
$$

This proves (b). Now, we concentrate on (c). Assume that $\mathbf{y} = \mathbf{i}$ lies in ${}^\theta I^{\nu''}$. Therefore we have

$$
\mathbf{i} = (i_{-m}, i_{1-m}, \ldots, i_{m+1}), \quad k = -m, 1-m, \ldots, m+1.
$$

First, we compute explicitly the integer d_k. The map ${}^\theta F_3^{(k)} \to {}^\theta F_{\mathbf{y}_k}$, $\phi \mapsto \phi_\gamma$ is an affine bundle of relative dimension

$$
\sharp\{l\,;\,-m \leqslant l < k,\, i_l = i\}.
$$

Further, for each tuple (x,y,z,ϕ) in ${}^\theta\widetilde{F}_{\mathbf{i}_k,\Omega}$ and for each $\phi' \in {}^\theta F_3^{(k)}$ such that $\phi'_\gamma = \phi$, the space of tuples (x',y',z') in ${}^\theta E_{3,\Omega}$ such that (x',y',z',ϕ') lies in ${}^\theta\widetilde{F}_{3,\Omega}$ and $\kappa(x',y',z') = (x,y,z)$ has the dimension

$$
\sum_{k < l \leqslant m+1} h_{i,i_l} + \delta_{k \leqslant 0}\,(\lambda_{\Omega,\theta(i)} + \lambda_{\bar\Omega,i} - h_{i,\theta(i)}).
$$

See [E, prop. 4.11(ii)] for details. Therefore we have

$$d_k = \sum_{k<l\leqslant m+1} h_{i,i_l} + \delta_{k\leqslant 0}\,(\lambda_{\Omega,\theta(i)} + \lambda_{\bar\Omega,i} - h_{i,\theta(i)}) + \sharp\{l\,;\,-m\leqslant l<k,\, i_l=i\}.$$

Next, (10.1) yields

$$d_{\lambda,\mathbf{i}} - d_{\lambda,\mathbf{i}_k} = \nu_i + \sum_{-m\leqslant l<k} h_{i_l,i} + \sum_{k<l\leqslant m+1} h_{i,i_l} + \delta_{k\geqslant 1}(\lambda_{\Omega,i} + \lambda_{\bar\Omega,\theta(i)} - h_{\theta(i),i}) +$$
$$+\delta_{k\leqslant 0}(\lambda_{\bar\Omega,i} + \lambda_{\Omega,\theta(i)} - h_{i,\theta(i)}).$$

Finally we have

$$a_{\nu,i} = -\nu_i + \sum_{-m\leqslant l\leqslant m+1} h_{i,i_l} - h_{i,\theta(i)} + \lambda_{\Omega,\theta(i)} + \lambda_{\bar\Omega,i}.$$

Therefore we get

$$a_{\nu,i} + d_{\lambda,\mathbf{i}} - d_{\lambda,\mathbf{i}_k} - 2d_k = -\sum_{-m\leqslant l<k} i\cdot i_l + \delta_{k\geqslant 1}(i\cdot\theta(i) + \lambda_i + \lambda_{\theta(i)}).$$

On the other hand $\deg(\mathbf{i}_k,\theta(i)\,;\mathbf{i})$ is the degree of the homogeneous element $\sigma_{\dot w}1_{\mathbf{i}}$, where $\dot w$ is a reduced decomposition of an element w of W_{m+1} such that $w(\mathbf{i}) = i\,\mathbf{i}_k\theta(i)$. If $k\leqslant 0$ then we can choose $\dot w = s_m s_{m-1}\dots s_{1-k}$ and we get

$$\deg(\mathbf{i}_k,\theta(i)\,;\mathbf{i}) = -\sum_{-m\leqslant l<k} i\cdot i_l.$$

If $k\geqslant 1$ then we can choose $\dot w = s_m s_{m-1}\dots s_1\varepsilon_1 s_1\dots s_{k-1}$ and we get

$$\begin{aligned}\deg(\mathbf{i}_k,\theta(i)\,;\mathbf{i}) &= -\sum_{-m\leqslant l\leqslant 0} i\cdot i_l + i\cdot\theta(i) + \lambda_i + \lambda_{\theta(i)} - \sum_{1\leqslant l<k} i\cdot i_l,\\ &= -\sum_{-m\leqslant l<k} i\cdot i_l + i\cdot\theta(i) + \lambda_i + \lambda_{\theta(i)}.\end{aligned}$$

The proposition is proved.

□

10.9. Example. Let Γ, θ, λ, ν, and Ω be as in Example 10.2. Let $\mathbf{k}$ denote the unique element of ${}^\theta\boldsymbol{\mathcal{P}}_{0,\Omega}$. We have

$$\{0\} \xleftarrow{\kappa} L_{\theta(i)} \xrightarrow{\iota} L_i\times L_{\theta(i)}\ .$$

We obtain

$$a_{\nu,i} = \lambda_{\theta(i)},\quad \underline{e_i}(\mathbf{k}_{L_i}[\lambda_i]) = \mathbf{k}[\lambda_i + \lambda_{\theta(i)}],\quad \underline{e_i}(\mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}]) = \mathbf{k},$$
$$b_{\nu,i} = \lambda_{\theta(i)},\quad \underline{f_i}(\mathbf{k}) = \mathbf{k}_{L_{\theta(i)}}[\lambda_{\theta(i)}].$$

10.10. A key estimate. First, let us introduce the following notation. For any complex of constructible sheaves $\mathcal{L}$ and any integer d we'll write $v^d\mathcal{L}$ for the shifted complex $\mathcal{L}[d]$.

10.11. Proposition. *For each $i \in I$ there are maps*

$$\underline{\varepsilon}_i : {}^\theta\boldsymbol{\mathcal{P}} \cup \{0\} \to \mathbb{Z}_{\geqslant 0}, \quad \underline{\tilde{E}}_i : {}^\theta\boldsymbol{\mathcal{P}} \to {}^\theta\boldsymbol{\mathcal{P}} \cup \{0\}, \quad \underline{\tilde{F}}_i : {}^\theta\boldsymbol{\mathcal{P}} \to {}^\theta\boldsymbol{\mathcal{P}}$$

such that for each $\mathcal{L}$ in ${}^\theta\boldsymbol{\mathcal{P}}$ the following hold

(a) we have $\underline{\varepsilon}_i(\underline{\tilde{E}}_i(\mathcal{L})) = \underline{\varepsilon}_i(\mathcal{L}) - 1$ and

$$\underline{e_i}(\mathcal{L}) = v^{1-\underline{\varepsilon}_i(\mathcal{L})}\, \underline{\tilde{E}}_i(\mathcal{L}) + \sum_{\mathcal{L}'} e_{\mathcal{L},\mathcal{L}'}\, {}^\theta\mathcal{L}',$$

$$\mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}, \quad \underline{\varepsilon}_i(\mathcal{L}') \geqslant \underline{\varepsilon}_i(\mathcal{L}), \quad e_{\mathcal{L},\mathcal{L}'} \in v^{1-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v],$$

(b) we have $\underline{\varepsilon}_i(\underline{\tilde{F}}_i(\mathcal{L})) = \underline{\varepsilon}_i(\mathcal{L}) + 1$ and

$$\underline{f_i}(\mathcal{L}) = \langle \underline{\varepsilon}_i(\mathcal{L}) + 1\rangle\, \underline{\tilde{F}}_i(\mathcal{L}) + \sum_{\mathcal{L}'} f_{\mathcal{L},\mathcal{L}'}\mathcal{L}',$$

$$\mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}, \quad \underline{\varepsilon}_i(\mathcal{L}') > \underline{\varepsilon}_i(\mathcal{L}) + 1, \quad f_{\mathcal{L},\mathcal{L}'} \in v^{2-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v],$$

(c) we have

$$\underline{\varepsilon}_i(0) = 0, \quad \underline{\tilde{E}}_i(\mathcal{L}) \neq 0 \text{ if } \underline{\varepsilon}_i(\mathcal{L}) > 0,$$

$$\underline{\tilde{E}}_i\underline{\tilde{F}}_i(\mathcal{L}) = \mathcal{L}, \quad \underline{\tilde{F}}_i\underline{\tilde{E}}_i(\mathcal{L}) = \mathcal{L} \text{ if } \underline{\tilde{E}}_i(\mathcal{L}) \neq 0,$$

(d) if $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}$ is such that $\underline{\varepsilon}_i(\mathcal{L}) = 0$ for all i, then $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{0,\Omega}$,

(e) the elements of ${}^\theta\boldsymbol{\mathcal{P}}$ are selfdual.

Proof : We'll prove the proposition for any quiver $\Gamma = (I, H)$ with an involution θ such that Γ has no 1-loops and θ has no fixed points. The estimates in (a), (b) are analogue of [E, thm. 5.3], where they are proved under the assumption $\lambda = 0$. Our proof is essentially the same. Fix a vertex i. First, we can assume that

- i is a sink of Γ,
- $i \in \Omega$,
- $\theta(i) \in \bar{\Omega}$.

More precisely we have the following lemma. Its proof is left to the reader. It is proved as in [E, thm. 4.19], [L2], using Fourier transforms.

10.12. Lemma. *Let $(\Gamma^{(1)}, \theta^{(1)})$, $(\Gamma^{(2)}, \theta^{(2)})$ be two quivers with involutions without fixed points. Assume that $\Gamma^{(1)}$, $\Gamma^{(2)}$ have the set of vertices I and that they have the same set of unoriented arrows. Let $\Omega^{(1)}$, $\Omega^{(2)}$ be two orientations of I. There is an equivalence of categories ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(1)}} \to {}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(2)}}$ which commutes with the functors $\underline{f_i}$, $\underline{e_i}$ and with the Verdier duality. The categories ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(1)}}$ and ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega^{(2)}}$ are relative to the quivers $\Gamma^{(1)}$, $\Gamma^{(2)}$ respectively. This equivalence yields a bijection ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega^{(1)}} \to {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega^{(2)}}$.*

Then, for each integer $r \geqslant 0$ let ${}^\theta E_{\nu,\Omega,\geqslant r}$ be the closed subset of ${}^\theta E_{\nu,\Omega}$ consisting of the triples (x,y,z) such that there is a I-graded subspace $W \subset \mathbf{V}$ of codimension vector ri such that

$$x(W) \subset W, \quad y(\mathbf{\Lambda}) \subset W, \quad z(W^\perp) = 0.$$

Then, we set

$${}^\theta E_{\nu,\Omega,r} = {}^\theta E_{\nu,\Omega,\geqslant r} \setminus {}^\theta E_{\nu,\Omega,\geqslant r+1}, \quad {}^\theta E_{\nu,\Omega,\leqslant r} = {}^\theta E_{\nu,\Omega} \setminus {}^\theta E_{\nu,\Omega,\geqslant r+1}.$$

Finally we set $\underline{\varepsilon}_i(0) = 0$ and for $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}$ we define

$$\underline{\varepsilon}_i(\mathcal{L}) = \max\{r\,;\, \mathrm{sup}(\mathcal{L}) \subset {}^\theta E_{\nu,\Omega,\geqslant r}\}.$$

Set $\nu' = i$ and consider the diagram

$${}^\theta E_{\nu,\Omega} \xleftarrow{p_1} {}^\theta E_{1,\Omega} \xrightarrow{p_2} {}^\theta E_{2,\Omega} \xrightarrow{p_3} {}^\theta E_{\nu'',\Omega}.$$

Under restriction it yields the diagram

$${}^\theta E_{\nu,\Omega,r} \longleftarrow {}^\theta E_{1,\Omega,r} \longrightarrow {}^\theta E_{2,\Omega,r+1} \longrightarrow {}^\theta E_{\nu'',\Omega,r+1},$$

where

$${}^\theta E_{1,\Omega,r} = p_1^{-1}({}^\theta E_{\nu,\Omega,r}), \quad {}^\theta E_{2,\Omega,r+1} = p_3^{-1}({}^\theta E_{\nu'',\Omega,r+1}).$$

We have also ${}^\theta E_{1,r} = p_2^{-1}({}^\theta E_{2,r+1})$, and the map ${}^\theta E_{2,r+1} \to {}^\theta E_{\nu'',r+1}$ is a $\mathbb{P}^r$-bundle. Finally, we set $p = p_3 p_2$ and we define ${}^\theta E_{2,\Omega,\leqslant r}$ and ${}^\theta E_{1,\Omega,\leqslant r}$ in the obvious way.

Now, we concentrate on (b). Fix a simple perverse sheaf $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$. Set $\varepsilon = \underline{\varepsilon}_i(\mathcal{L})$. The maps p_1, p_2 are smooth with connected fibers of dimension d_{p_1}, d_{p_2} such that

$$b_{\nu,i} = d_{p_1} - d_{p_2}. \tag{10.2}$$

Thus, there is a unique simple ${}^\theta G_{\nu''}$-equivariant perverse sheaf $\mathcal{L}_2$ on ${}^\theta E_{2,\Omega}$ with

$$p_1^*(\mathcal{L})[b_{\nu,i}] = p_2^*(\mathcal{L}_2).$$

We have

$$\underline{f_i}(\mathcal{L}) = (p_3)_!(\mathcal{L}_2).$$

The complex $\underline{f_i}(\mathcal{L})$ is supported on ${}^\theta E_{\nu'',\Omega,\varepsilon+1}$. Further, the restriction of $\mathcal{L}_2$ to ${}^\theta E_{2,\Omega,\leqslant\varepsilon+1}$ is supported on ${}^\theta E_{2,\Omega,\varepsilon+1}$, and it is constant along the fibers of p_3 by ${}^\theta G_{\nu''}$-equivariance. Thus

$$\mathcal{L}_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon+1}} = p_3^*(\mathcal{L}'')[\varepsilon]$$

for some simple perverse sheaf $\mathcal{L}''$ on ${}^\theta E_{\nu'',\Omega,\leq\varepsilon+1}$. Let $\mathcal{L}_0$ be the minimal perverse extension of $\mathcal{L}''$ to ${}^\theta E_{\nu'',\Omega}$. Since $\underline{f_i}(\mathcal{L})$ is semi-simple, we get

$$\underline{f_i}(\mathcal{L}) = \langle\varepsilon+1\rangle\,\mathcal{L}_0 + \sum_{\mathcal{L}'} f_{\mathcal{L},\mathcal{L}'}\mathcal{L}',$$

$$\mathcal{L}_0, \mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu'',\Omega}, \quad \underline{\varepsilon}_i(\mathcal{L}_0) = \varepsilon+1, \quad \underline{\varepsilon}_i(\mathcal{L}') > \varepsilon+1.$$

Let us that $f_{\mathcal{L},\mathcal{L}'}$ lies in $v^{2-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v]$. Write

$$\underline{f_i}(\mathcal{L}) = \bigoplus_{\mathcal{L}'} \mathcal{L}' \otimes M_{\mathcal{L}'},$$

where $M_{\mathcal{L}'}$ is a complex of $\mathbf{k}$-vector spaces. Set $\varepsilon' = \underline{\varepsilon}_i(\mathcal{L}')$. We have

$$\mathrm{R}\mathcal{H}om(\mathcal{L}', \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}} \otimes M^*_{\mathcal{L}'} \subset \mathrm{R}\mathcal{H}om((p_3)_!(\mathcal{L}_2), \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}}.$$

On the other hand, since p_3 restricts to a $\mathbb{P}^{\varepsilon'-1}$-bundle ${}^\theta E_{2,\Omega,\varepsilon'} \to {}^\theta E_{\nu'',\Omega,\varepsilon'}$, we have

$$\mathrm{R}\mathcal{H}om((p_3)_!(\mathcal{L}_2), \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}} = (p_3)_*\mathrm{R}\mathcal{H}om(\mathcal{L}_2, \mathcal{L}'[\varepsilon'-1])|_{{}^\theta E_{2,\leqslant\varepsilon'}}[\varepsilon'-1].$$

Since $\mathcal{L}'[\varepsilon'-1]|_{{}^\theta E_{2,\leqslant\varepsilon'}}$ is a perverse sheaf the complex

$$\mathrm{R}\mathcal{H}om(\mathcal{L}_2, \mathcal{L}'[\varepsilon'-1])|_{{}^\theta E_{2,\leqslant\varepsilon'}}$$

is concentrated in degrees $\geqslant 0$. Its 0-th cohomology group is zero because $\mathcal{L}_2$ and $\mathcal{L}'[\varepsilon'-1]$ are simple and non isomorphic. Thus the complex

$$\mathrm{R}\mathcal{H}om((p_3)_!(\mathcal{L}_2), \mathcal{L}')|_{{}^\theta E_{\nu'',\leqslant\varepsilon'}}$$

is concentrated in degrees $> 1-\varepsilon'$. This implies the estimate we want.

Next, we prove (a). Fix a triple (V,γ,γ') in T. Observe that the hypothesis on Γ, Ω, i implies that for each (x,y,z,W,ρ,ρ') in ${}^\theta E_{1,\Omega}$ we have $x(W^\perp) = z(W^\perp) = 0$, $x(\mathbf{V}), y(\mathbf{\Lambda}) \subset W$, z is completely determined by its restriction to W, and y is completely determined by its composition with the projection $\mathbf{V} \to \mathbf{V}/W^\perp$. Hence x, y, z are completely determined by x_ρ, y_ρ, z_ρ. Therefore κ is an isomorphism. Consider the diagram

(10.3)
$$\begin{array}{ccccc}
{}^\theta E_{\nu,\Omega} & \overset{\kappa}{=\!=} & {}^\theta E_{3,\Omega} & \xrightarrow{\iota} & {}^\theta E_{\nu'',\Omega} \\
& \nwarrow^{p_1} & \downarrow{s} & \nearrow_{p} & \\
& & {}^\theta E_{1,\Omega}, & &
\end{array}$$

where

$$\kappa(x,y,z) = (x_\gamma, y_\gamma, z_\gamma), \quad s(x,y,z) = (x,y,z,V,\gamma,\gamma'),$$

$$p_1(x,y,z,W,\rho,\rho') = (x_\rho, y_\rho, z_\rho), \quad p(x,y,z,W,\rho,\rho') = (x,y,z).$$

The left square is Cartesian. Fix a simple perverse sheaf $\mathcal{L}$ in ${}^\theta\boldsymbol{\mathcal{P}}_{\nu'',\Omega}$. Set $\varepsilon = \underline{\varepsilon}_i(\mathcal{L})$. We'll assume that $\varepsilon > 0$ (the case $\varepsilon = 0$ is left to the reader). We have

$$\underline{e_i}(\mathcal{L}) = \kappa_! s^* p^*(\mathcal{L})[a_{\nu,i}].$$

The restriction $\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}}$ is a simple ${}^\theta G_{\nu''}$-equivariant perverse sheaf supported on ${}^\theta E_{\nu'',\Omega,\varepsilon}$. Let d_p, d_s be the relative dimension of the maps p, s. Since p restricts to a smooth map ${}^\theta E_{1,\Omega,\varepsilon-1} \to {}^\theta E_{\nu'',\Omega,\varepsilon}$, the complex

$$\mathcal{L}_1 = p^*(\mathcal{L})[d_p]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon-1}}$$

is again a simple ${}^\theta G_{\nu''}$-equivariant perverse sheaf. It is constant along the fibers of p_1 by ${}^\theta G_{\nu''}$-equivariance. Thus

$$\begin{aligned}\mathcal{L}'' &= \kappa_! s^* p^*(\mathcal{L})[d_p + d_s]|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon-1}} \\ &= \underline{e_i}(\mathcal{L})[d_p + d_s - a_{\nu,i}]|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon-1}}\end{aligned}$$

is a simple perverse sheaf over ${}^\theta E_{\nu,\Omega,\leqslant\varepsilon-1}$. Using (10.2) we get

$$d_p + d_s = d_{p_2} + \varepsilon - 1 - d_{p_1}, \quad d_{p_1} - d_{p_2} = b_{\nu,i}, \quad b_{\nu,i} = \nu_i, \quad a_{\nu,i} = -\nu_i.$$

Therefore, we have

$$d_p + d_s - a_{\nu,i} = \varepsilon - 1.$$

Let $\mathcal{L}_0$ be the minimal perverse extension of $\mathcal{L}''$ to ${}^\theta E_{\nu,\Omega,}$. Since $\underline{e_i}(\mathcal{L})$ is semi-simple we get

$$\begin{gathered}\underline{e_i}(\mathcal{L}) = v^{1-\varepsilon}\,\mathcal{L}_0 + \sum_{\mathcal{L}'} e_{\mathcal{L},\mathcal{L}'}\mathcal{L}' \\ \mathcal{L}_0, \mathcal{L}' \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}, \quad \underline{\varepsilon}_i(\mathcal{L}_0) = \varepsilon - 1, \quad \underline{\varepsilon}_i(\mathcal{L}') \geqslant \varepsilon.\end{gathered} \tag{10.4}$$

Now, one proves that $e_{\mathcal{L},\mathcal{L}'}$ lies in $v^{1-\underline{\varepsilon}_i(\mathcal{L}')}\mathbb{Z}[v]$ as in [E, thm. 5.3]. More precisely, since $p_1^*\underline{e_i}(\mathcal{L})$ and $p^*(\mathcal{L})[a_{\nu,i}]$ are constant along the fibers of p_1 and since

$$\underline{e_i}(\mathcal{L}) = \kappa_! s^* p^*(\mathcal{L})[a_{\nu,i}],$$

we have

$$p_1^*\underline{e_i}(\mathcal{L}) = p^*(\mathcal{L})[-b_{\nu,i}]. \tag{10.5}$$

On the other hand, we have

$$p_1^*\underline{e_i}(\mathcal{L}) = \bigoplus_{\mathcal{L}''} p_1^*(\mathcal{L}'') \otimes M_{\mathcal{L}''},$$

where the graded $\mathbf{k}$-vector space $M_{\mathcal{L}''}$ is the multiplicity space of the simple perverse sheaf $\mathcal{L}'' \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$ in $\underline{e_i}(\mathcal{L})$. Let $\mathcal{L}''_2$ be the perverse sheaf over ${}^\theta E_{2,\Omega}$ such that

$$p_1^*(\mathcal{L}'')[b_{\nu,i}] = p_2^*(\mathcal{L}''_2).$$

We obtain

$$\bigoplus_{\mathcal{L}''} \mathcal{L}''_2 \otimes M_{\mathcal{L}''} = p_3^*(\mathcal{L}).$$

Now, let $\mathcal{L}'$ be as in (10.4). Set $\varepsilon' = \underline{\varepsilon_i}(\mathcal{L}')$. We have

$$\begin{aligned}\bigoplus_{\mathcal{L}''}\mathrm{RHom}(\mathcal{L}_2''|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}},\mathcal{L}_2'|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}})\otimes M^*_{\mathcal{L}''} =\\ &= \mathrm{RHom}(p_3^*(\mathcal{L})|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}},\mathcal{L}_2'|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon'+1}})\\ &= \mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}},(p_3)_!(\mathcal{L}_2')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}})\\ &= \mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}},\underline{f_i}(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}})\\ &= \langle\varepsilon'+1\rangle\,\mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}},\underline{\tilde F}_i(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}),\end{aligned}$$

where the last equality follows from part (b). The complex

$$\mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}},\underline{\tilde F}_i(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}}),$$

is concentrated in degrees $\geqslant 1$, because the perverse sheaves $\mathcal{L}'$ and $\underline{\tilde F}_i(\mathcal{L}')$ are simple and distincts. Thus the complex

$$\mathrm{RHom}(\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}},\underline{f_i}(\mathcal{L}')|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon'+1}})$$

is concentrated in degrees $\geqslant 1-\varepsilon'$. Choosing $\mathcal{L}'=\mathcal{L}''$ we get that $M^*_{\mathcal{L}'}$ is also concentrated in degrees $\geqslant 1-\varepsilon'$. Therefore

$$e_i'(\mathcal{L}) = \bigoplus_{\mathcal{L}''}\mathcal{L}''\otimes M_{\mathcal{L}''} = \bigoplus_{\mathcal{L}''}\bigoplus_{d\in\mathbb{Z}} v^{-d}\mathcal{L}''\otimes M_{\mathcal{L}'',d},$$

with $M_{\mathcal{L}'',d}=0$ unless $-d\geqslant 1-\varepsilon'$. We are done.

Now, we concentrate on (c). The second claim in (c) is obvious. Now, we prove that $\underline{\tilde E}_i\underline{\tilde F}_i(\mathcal{L})=\mathcal{L}$ for $\mathcal{L}$ in ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega}$. Recall the diagram (10.3). Set $\varepsilon=\underline{\varepsilon_i}(\mathcal{L})$ and take a simple perverse sheaf $\mathcal{L}_2$ on ${}^\theta E_{2,\Omega}$ such that

$$p_1^*(\mathcal{L})[b_{\nu,i}] = p_2^*(\mathcal{L}_2),\quad (p_3)_!(\mathcal{L}_2)=\underline{f_i}(\mathcal{L}).$$

We have

$$(p_3)_!(\mathcal{L}_2)|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon+1}} = \langle\varepsilon+1\rangle\,\underline{\tilde F}_i(\mathcal{L})|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon+1}}.$$

On the other hand, since

$$\mathcal{L}_2|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon+1}} = p_3^*(\underline{\tilde F}_i\mathcal{L})[\varepsilon]|_{{}^\theta E_{2,\Omega,\leqslant\varepsilon+1}},$$

we have

$$p^*(\underline{\tilde F}_i\mathcal{L})|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon+1}} = p_2^*(\mathcal{L}_2)[-\varepsilon]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon+1}} = p_1^*(\mathcal{L})[b_{\nu,i}-\varepsilon]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon+1}}.$$

Therefore we have also

$$\begin{aligned}\underline{e_i}(\underline{\tilde F}_i\mathcal{L})|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon+1}} &= \kappa_! s^* p^*(\underline{\tilde F}_i\mathcal{L})[a_{\nu,i}]|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon+1}}\\ &= \mathcal{L}[-\varepsilon]|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon+1}}.\end{aligned}$$

Therefore $\underline{\tilde{E}}_i\underline{\tilde{F}}_i(\mathcal{L}) = \mathcal{L}$. Finally, fix $\mathcal{L} \in {}^\theta\boldsymbol{\mathcal{P}}_{\nu'',\Omega}$ such that $\underline{\varepsilon}_i(\mathcal{L}) > 0$ and let us prove that $\underline{\tilde{F}}_i\underline{\tilde{E}}_i(\mathcal{L}) = \mathcal{L}$. Write $\varepsilon = \underline{\varepsilon}_i(\mathcal{L})$. By (10.5) we have

$$p_1^*\underline{e_i}(\mathcal{L}) = p^*(\mathcal{L})[a_{\nu,i}].$$

Hence we have also

$$p_1^*(\underline{\tilde{E}}_i\mathcal{L})[-a_{\nu,i}]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon-1}} = p^*(\mathcal{L})[\varepsilon-1]|_{{}^\theta E_{1,\Omega,\leqslant\varepsilon-1}}.$$

Since $p_3^*(\mathcal{L})[\varepsilon-1]|_{{}^\theta E_{2,\Omega,\varepsilon}}$ is a simple perverse sheaf, we have

$$\underline{f_i}(\underline{\tilde{E}}_i\mathcal{L})|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}} = (p_3)_!p_3^*(\mathcal{L})[\varepsilon-1]|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}} = \langle\varepsilon\rangle\,\mathcal{L}|_{{}^\theta E_{\nu'',\Omega,\leqslant\varepsilon}}.$$

This implies that $\underline{\tilde{F}}_i\underline{\tilde{E}}_i(\mathcal{L}) = \mathcal{L}$.

Next, (d) is obvious. If $\nu \neq 0$ we choose $\mathbf{y}$, d such that $\mathcal{L}[d]$ is a direct summand of ${}^\theta\mathcal{L}_{\mathbf{y}}$. We may assume that $\mathbf{y} = (\mathbf{i}, \mathbf{a})$ with $a_1 > 0$. Then $\underline{\varepsilon}_{i_1}(\mathcal{L}) > 0$ by (b) and Proposition 10.5(b).

Finally, we prove (e) by descending induction on ν. Any element in ${}^\theta\boldsymbol{\mathcal{P}}_{0,\Omega}$ is selfdual. Assume that $\nu > 0$. By part (d) there is i such that $\underline{\varepsilon}_i(\mathcal{L}) > 0$. Set $\varepsilon = \underline{\varepsilon}_i(\mathcal{L})$. We prove that $\mathcal{L}$ is selfdual by descending induction on ε. By parts (b), (c) we have

$$\underline{f_i}(\underline{\tilde{E}}_i\mathcal{L}) = \langle\varepsilon\rangle\,\mathcal{L} + \sum_{\mathcal{L}'} f_{\mathcal{L},\mathcal{L}'}\mathcal{L}', \quad \underline{\varepsilon}_i(\mathcal{L}') > \varepsilon.$$

The perverse sheaf $\underline{\tilde{E}}_i\mathcal{L}$ is selfdual by the induction hypothesis on ν. It is easy to see that $\underline{f_i}$ commutes with the Verdier duality. Hence the lhs is also selfdual. We have

$$\underline{f_i}(\underline{\tilde{E}}_i\mathcal{L})|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon}} = \langle\varepsilon\rangle\,\mathcal{L}|_{{}^\theta E_{\nu,\Omega,\leqslant\varepsilon}}.$$

Since $\mathcal{L}$ is the minimal extension of its restriction to ${}^\theta E_{\nu,\Omega,\leqslant\varepsilon}$, it is selfdual.

□

Let $K({}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega})$ be the Abelian group with one generator $[\mathcal{L}]$ for each isomorphism class of objects of ${}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}$ and with relations $[\mathcal{L}]+[\mathcal{L}'] = [\mathcal{L}'']$ whenever $\mathcal{L}''$ is isomorphic to $\mathcal{L}\oplus\mathcal{L}'$. To unburden the notation we'll abbreviate

$$K({}^\theta\boldsymbol{\mathcal{Q}}) = \bigoplus_\nu K({}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega}), \quad \mathcal{L} = [\mathcal{L}].$$

Note that $K({}^\theta\boldsymbol{\mathcal{Q}})$ is a free $\mathcal{A}$-module such that $v\mathcal{L} = \mathcal{L}[1]$ and $v^{-1}\mathcal{L} = \mathcal{L}[-1]$. Further the Verdier duality yields an $\mathcal{A}$-antilinear map $K({}^\theta\boldsymbol{\mathcal{Q}}) \to K({}^\theta\boldsymbol{\mathcal{Q}})$.

10.13. Corollary. *The $\mathcal{A}$-module $K({}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega})$ is spanned by $\{{}^\theta\mathcal{L}^\delta_{\mathbf{y}};\ \mathbf{y}\in{}^\theta Y^\nu\}$.*

Proof : The corollary is proved as in Lemma 8.32, using Proposition 10.11 instead of Propositions 8.22, 8.23.

□

10.14. Example. Let Γ, θ, λ, ν be as in Example 10.2, and set $\Omega = \{i\}$. We have ${}^\theta E_{\nu,\Omega} = L_i \times L^*_{\theta(i)}$, ${}^\theta E_{\nu,\Omega,0} = {}^\theta E_{\nu,\Omega} \setminus \{0\}$, and ${}^\theta E_{\nu,\Omega,1} = \{0\}$. We have also

- if $\lambda_i + \lambda_{\theta(i)} \neq 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{{}^\theta E_{\nu,\Omega}}[\lambda_i + \lambda_{\theta(i)}], \mathbf{k}_{\{0\}}\}$, and

$$\underline{\varepsilon}_i(\mathbf{k}_{{}^\theta E_{\nu,\Omega}}[\lambda_i + \lambda_{\theta(i)}]) = 0, \quad \underline{\varepsilon}_i(\mathbf{k}_{\{0\}}) = 1, \quad \underline{\tilde{F}}_i(\mathbf{k}) = \mathbf{k}_{\{0\}},$$

- if $\lambda_i + \lambda_{\theta(i)} = 0$ then ${}^\theta\boldsymbol{\mathcal{P}}_{\nu,\Omega} = \{\mathbf{k}_{\{0\}}\}$, and

$$\underline{\varepsilon}_i(\mathbf{k}_{\{0\}}) = 1, \quad \underline{\tilde{F}}_i(\mathbf{k}) = \mathbf{k}_{\{0\}}.$$

10.15. Comparison of the crystals. We choose Γ, θ and λ as in Sections 6.2, 6.4, and we set $\Omega = I$. We define a functor

$$\mathbf{Y} : {}^\theta\boldsymbol{\mathcal{Q}}_{\nu,\Omega} \to {}^\theta\mathbf{R}_\nu\textbf{-mod}, \quad \mathbf{Y}(\mathcal{L}) = \bigoplus_{\mathbf{i}\in{}^\theta I^\nu} \mathrm{Ext}^*_{{}^\theta G_\nu}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}, \mathcal{L}).$$

The functor $\mathbf{Y}$ is additive and it commutes with the shift (the shift of complexes in the lhs and the shift of the grading in the rhs).

10.16. Proposition. *(a) $\mathbf{Y}$ takes ${}^\theta\boldsymbol{\mathcal{Q}}$ to ${}^\theta\mathbf{R}$-**proj**, and ${}^\theta\mathcal{L}^\delta_{\mathbf{y}}$ to ${}^\theta\mathbf{R}_{\mathbf{y}}$. It maps ${}^\theta\boldsymbol{\mathcal{P}}$ bijectively to the set of selfdual indecomposable projective graded modules.*

(b) $\mathbf{Y}$ yields an $\mathcal{A}$-module isomorphism $K({}^\theta\boldsymbol{\mathcal{Q}}) \to {}^\theta\mathbf{K}_I$ which maps ${}^\theta\boldsymbol{\mathcal{P}}$ bijectively onto ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$. It commutes with the duality. We have

$$e_i \circ \mathbf{Y} = \mathbf{Y} \circ \underline{e_{\theta(i)}}, \quad f_i \circ \mathbf{Y} = \mathbf{Y} \circ \underline{f_{\theta(i)}}.$$

Proof : The first claim of (a) is obvious. If the sequence $\mathbf{i}$ of ${}^\theta I^\nu$ is the expansion of the pair $\mathbf{y}$ in ${}^\theta Y^\nu$ then we have

$${}^\theta\mathbf{R}_{\mathbf{i}} = \langle\mathbf{b}\rangle!\, {}^\theta\mathbf{R}_{\mathbf{y}}, \quad {}^\theta\mathcal{L}^\delta_{\mathbf{i}} = \langle\mathbf{b}\rangle!\, {}^\theta\mathcal{L}^\delta_{\mathbf{y}},$$

where $\mathbf{b}$ is a sequence such that the multiplicity of $\mathbf{y}$ is $\theta(\mathbf{b})\mathbf{b}$. See Remark 2.7 and (8.3). Therefore to prove the second claim of (a) it is enough to observe that we have $\mathbf{Y}({}^\theta\mathcal{L}^\delta_{\mathbf{i}}) = {}^\theta\mathbf{R}^\delta_{\mathbf{i}}$. The same proof as in [VV, sec. 4.7] implies that $\mathbf{Y}$ yields an injection from ${}^\theta\boldsymbol{\mathcal{P}}$ to the set of indecomposable projective graded modules. Since $\mathbf{Y}({}^\theta\mathcal{L}^\delta_{\mathbf{y}}) = {}^\theta\mathbf{R}^\delta_{\mathbf{y}}$ and both sides are selfdual, the functor $\mathbf{Y}$ takes the elements of ${}^\theta\boldsymbol{\mathcal{P}}$ to selfdual graded modules, see Sections 2.6 and 8.10. Part (a) is proved. Next, the first claim of (b) follows from Proposition 8.4(c) and Corollary 10.13. Finally the last claim of (b) follows from Propositions 10.5(b), 10.8(c) and Proposition 8.14.

□

Recall the set ${}^\theta\mathbf{G}^{\mathrm{low}}(\lambda)$ introduced in Definition 8.3. For each b in ${}^\theta B(\lambda)$ let ${}^\theta\mathcal{L}(b)$ denote the unique element in ${}^\theta\boldsymbol{\mathcal{P}}$ such that

$$\mathbf{Y}({}^\theta\mathcal{L}(b)) = {}^\theta G^{\mathrm{low}}(b). \tag{10.6}$$

Hence we have ${}^\theta\boldsymbol{\mathcal{P}} = \{{}^\theta\mathcal{L}(b);\, b \in {}^\theta B(\lambda)\}$. We'll set also ${}^\theta\mathcal{L}(0) = 0$. Combining Propositions 8.23 and 10.11 we can now compare the crystal $({}^\theta B(\lambda), \tilde{E}_i, \tilde{F}_i, \varepsilon_i)$ from Proposition 8.22 with the crystal $({}^\theta\boldsymbol{\mathcal{P}}, \underline{\tilde{E}}_i, \underline{\tilde{F}}_i, \underline{\varepsilon}_i)$ from Proposition 10.11.

10.17. Proposition. *For each $i \in I$ and each $b \in {}^\theta B(\lambda)$ we have*

$$\underline{\tilde{E}}_i({}^\theta\mathcal{L}(b)) = {}^\theta\mathcal{L}(\tilde{E}_{\theta(i)}b), \quad \underline{\tilde{F}}_i({}^\theta\mathcal{L}(b)) = {}^\theta\mathcal{L}(\tilde{F}_{\theta(i)}b), \quad \underline{\varepsilon}_i({}^\theta\mathcal{L}(b)) = \varepsilon_{\theta(i)}(b).$$

Proof : We can regard $\underline{\varepsilon}_i$, $\underline{\tilde{E}}_i$, and $\underline{\tilde{F}}_i$ as maps

$$\underline{\varepsilon}_i : {}^\theta B(\lambda) \cup \{0\} \to \mathbb{Z}_{\geqslant 0}, \quad \underline{\tilde{E}}_i : {}^\theta B(\lambda) \to {}^\theta B(\lambda) \cup \{0\}, \quad \underline{\tilde{F}}_i : {}^\theta B(\lambda) \to {}^\theta B(\lambda).$$

Propositions 10.11(b), 10.16(b) yield

$$f_{\theta(i)} {}^\theta G^{\mathrm{low}}(b) = \langle \underline{\varepsilon}_i(b) + 1\rangle \, {}^\theta G^{\mathrm{low}}(\underline{\tilde{F}}_i b) + \sum_{b'} f_{b,b'} {}^\theta G^{\mathrm{low}}(b'), \quad \underline{\varepsilon}_i(b') > \underline{\varepsilon}_i(b) + 1.$$

Taking the transpose, Definition 8.8 and Proposition 8.4(a) yield

$$e_{\theta(i)} {}^\theta G^{\mathrm{up}}(b) = \langle \underline{\varepsilon}_i(b)\rangle \, {}^\theta G^{\mathrm{up}}(\underline{\tilde{E}}_i b) + \sum_{b'} f_{b',b} {}^\theta G^{\mathrm{up}}(b'), \quad \underline{\varepsilon}_i(b') < \underline{\varepsilon}_i(b) - 1.$$

Now, recall that

$$\varepsilon_{\theta(i)}(b) = \max\{n \geqslant 0; e^n_{\theta(i)} {}^\theta G^{\mathrm{up}}(b) \neq 0\}, \quad \underline{\varepsilon}_i(\underline{\tilde{E}}_i b) = \underline{\varepsilon}_i(b) - 1.$$

Thus, using Proposition 8.17 and (8.7) we get $\underline{\varepsilon}_i = \varepsilon_{\theta(i)}$. Then, comparing the formulas above with Proposition 8.23 we get $\underline{\tilde{F}}_i = \tilde{F}_{\theta(i)}$. Finally, Proposition 8.22(c) and 10.11(c) yield $\underline{\tilde{E}}_i = \tilde{E}_{\theta(i)}$.

□

10.18. The global bases of ${}^\theta\mathbf{V}(\lambda)$. Since the operators e_i, f_i on ${}^\theta\mathbf{V}(\lambda)$ satisfy the relations $e_i f_i = v^{-2} f_i e_i + 1$, we can define the modified root operators $\tilde{\mathbf{e}}_i$, $\tilde{\mathbf{f}}_i$ on the ${}^\theta\boldsymbol{\mathcal{B}}$-module ${}^\theta\mathbf{V}(\lambda)$ as follows. For each u in ${}^\theta\mathbf{V}(\lambda)$ we write

$$u = \sum_{n \geqslant 0} f_i^{(n)} u_n \text{ with } e_i u_n = 0,$$

$$\tilde{\mathbf{e}}_i(u) = \sum_{n \geqslant 1} f_i^{(n-1)} u_n, \quad \tilde{\mathbf{f}}_i(u) = \sum_{n \geqslant 0} f_i^{(n+1)} u_n.$$

Let $\mathcal{R} \subset \mathcal{K}$ be the set of functions which are regular at $v = 0$. Let ${}^\theta\mathbf{L}(\lambda)$ be the $\mathcal{R}$-submodule of ${}^\theta\mathbf{V}(\lambda)$ spanned by the elements $\tilde{\mathbf{f}}_{i_1} \dots \tilde{\mathbf{f}}_{i_l}(\phi_\lambda)$ with $l \geqslant 0$, $i_1, \dots, i_l \in I$. We can now apply [EK3, thm. 4.1, cor. 4.4] as in [E, Section 2.3]. Together with Propositions 10.15 and 10.16 this yields the following, which is the main result of the paper.

10.19. Theorem. *(a) We have*

$${}^\theta\mathbf{L}(\lambda) = \bigoplus_{b \in {}^\theta B(\lambda)} \mathcal{R}\, {}^\theta G^{\mathrm{low}}(b), \quad \tilde{\mathbf{e}}_i({}^\theta\mathbf{L}(\lambda)) \subset {}^\theta\mathbf{L}(\lambda), \quad \tilde{\mathbf{f}}_i({}^\theta\mathbf{L}(\lambda)) \subset {}^\theta\mathbf{L}(\lambda),$$

$$\tilde{\mathbf{e}}_i {}^\theta G^{\mathrm{low}}(b) = {}^\theta G^{\mathrm{low}}(\tilde{E}_i b) \bmod v\, {}^\theta\mathbf{L}(\lambda), \quad \tilde{\mathbf{f}}_i {}^\theta G^{\mathrm{low}}(b) = {}^\theta G^{\mathrm{low}}(\tilde{F}_i b) \bmod v\, {}^\theta\mathbf{L}(\lambda).$$

(b) The assignment $b \mapsto {}^\theta G^{\mathrm{low}}(b) \bmod v\,{}^\theta\mathbf{L}(\lambda)$ *yields a bijection from* ${}^\theta B(\lambda)$ *to the subset of* ${}^\theta\mathbf{L}(\lambda)/v{}^\theta\mathbf{L}(\lambda)$ *consisting of the* $\tilde{\mathbf{f}}_{i_1}\dots\tilde{\mathbf{f}}_{i_l}(\phi_\lambda)$*'s. Further* ${}^\theta G^{\mathrm{low}}(b)$ *is the unique element* x *of* ${}^\theta\mathbf{V}(\lambda)$ *satisfying the following conditions*

$$x^\sharp = x, \quad x = {}^\theta G^{\mathrm{low}}(b) \bmod v\,{}^\theta\mathbf{L}(\lambda).$$

(c) For each b, b' *in* ${}^\theta B(\lambda)$ *let* $E_{i,b,b'}, F_{i,b,b'} \in \mathcal{A}$ *be the coefficients of* ${}^\theta G^{\mathrm{low}}(b')$ *in* $e_{\theta(i)}{}^\theta G^{\mathrm{low}}(b)$, $f_i{}^\theta G^{\mathrm{low}}(b)$ *respectively. Then we have*

$$E_{i,b,b'}|_{v=1} = [F_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b')) : \Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b))],$$

$$F_{i,b,b'}|_{v=1} = [E_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b')) : \Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b))].$$

Proof : Part (a) follows from [EK3, thm. 4.1, cor. 4.4], [E, Section 2.3], Proposition 10.17 and (10.6). The first claim in (b) follows from (a). The second one is obvious. Part (c) follows from Proposition 8.17. More precisely, by duality we can regard $E_{i,b,b'}$, $F_{i,b,b'}$ as the coefficients of ${}^\theta G^{\mathrm{up}}(b)$ in $f_{\theta(i)}{}^\theta G^{\mathrm{up}}(b')$, $e_i{}^\theta G^{\mathrm{up}}(b')$ respectively. Therefore, by Proposition 8.17 we can regard $E_{i,b,b'}|_{v=1}$, $F_{i,b,b'}|_{v=1}$ as the coefficients of $\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b))$ in $F_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b'))$, $E_i\Psi\mathbf{for}({}^\theta G^{\mathrm{up}}(b'))$ respectively.

□

A. Appendix

A large part of the statements above generalizes to affine Hecke algebras of type C. We use the convention in [Mc].

A.1. Affine Hecke algebras of type C. Fix p, q_0, q_1 in $\mathbf{k}^\times$. For any integer $m \geqslant 0$ we define the extended affine Hecke algebra $\mathbf{H}_m$ of type C_m as follows. If $m > 0$ then $\mathbf{H}_m$ is the $\mathbf{k}$-algebra generated by

$$T_k, \quad X_l^{\pm 1}, \quad k = 0, 1, \dots, m-1, \quad l = 1, 2, \dots, m$$

satisfying the following defining relations :

- (a) $X_l X_{l'} = X_{l'} X_l$,
- (b) $(T_0T_1)^2 = (T_1T_0)^2$, $T_kT_{k-1}T_k = T_{k-1}T_kT_{k-1}$ if $k \neq 0, 1$, and $T_kT_{k'} = T_{k'}T_k$ if $|k-k'| \neq 1$,
- (c) $T_0X_1^{-1} - X_1T_0 = (q_1^{-1} - q_0)X_1 + (q_0q_1^{-1} - 1)$, $T_kX_kT_k = X_{k+1}$ if $k \neq 0$, and $T_kX_l = X_lT_k$ if $l \neq k, k+1$,
- (d) $(T_k - p)(T_k + p^{-1}) = 0$ if $k \neq 0$, and $(T_0 - q_0)(T_0 + q_1^{-1}) = 0$.

If $m = 0$ then $\mathbf{H}_0 = \mathbf{k}$, the trivial $\mathbf{k}$-algebra.

A.2. Remark. The affine Hecke algebra of type B_m is equal to $\mathbf{H}_m/(q_0 - q, q_1 - q)$.

A.3. Intertwiners and blocks of $\mathbf{H}_m$. We define

$$\mathbf{A}' = \mathbf{A}[\Sigma^{-1}], \quad \mathbf{H}'_m = \mathbf{A}' \otimes_{\mathbf{A}} \mathbf{H}_m,$$

where Σ is the multiplicative system generated by

$$X_{l'}^{\pm 1} - X_l, \quad X_{l'}^{\pm 1} - p^2 X_l, \quad 1 - X_l^2, \quad 1 + q_0 X_l^{\pm 1}, \quad 1 - q_1 X_l^{\pm 1}, \quad l \neq l'.$$

For $k = 0, \dots, m-1$ the intertwiner φ_k in $\mathbf{H}'_m$ is given by the following formulas

$$(A.1) \qquad \begin{aligned} \varphi_k - 1 &= \frac{X_k - X_{k+1}}{pX_k - p^{-1}X_{k+1}}(T_k - p) && \text{if } k \neq 0, \\ \varphi_0 - 1 &= q_1 \frac{X_1^2 - 1}{(X_1 + q_0)(X_1 - q_1)}(T_0 - q_0). \end{aligned}$$

There is an isomorphism of $\mathbf{A}'$-algebras

$$\mathbf{A}' \rtimes W_m \to \mathbf{H}'_m, \quad s_k \mapsto \varphi_k, \quad \varepsilon_1 \mapsto \varphi_0, \quad k \neq 0.$$

The semi-direct product group $\mathbb{Z} \rtimes \mathbb{Z}_2$ acts on $\mathbf{k}^\times$ as in Section 6.2. Given a $\mathbb{Z} \rtimes \mathbb{Z}_2$-invariant subset I of $\mathbf{k}^\times$ we denote by $\mathbf{H}_m$-$\mathbf{Mod}_I$ the category of all $\mathbf{H}_m$-modules such that the action of $X_1, X_2, \dots, X_m$ is locally finite and all the eigenvalues belong to I. We associate to the set I a quiver Γ with an involution θ as in loc. cit. Finally, we assume that

$$1, -1 \notin I, \quad p, q_0, q_1 \neq 1, -1.$$

Next, we define the element λ of $\mathbb{N}I$ as

$$(A.2) \qquad \lambda = \sum_i i, \quad i \in I \cap \{-q_0, q_1\},$$

and we define ${}^\theta\mathbf{R}_m$ and ${}^\theta\mathbf{R}_m$-$\mathbf{Mod}_0$ as in Section 6.4. Note that if $q_0 = q_1 = q$ then λ is the same as in (6.2).

A.4. Theorem. *There is an equivalence of categories*

$${}^\theta\mathbf{R}_m\text{-}\mathbf{Mod}_0 \to \mathbf{H}_m\text{-}\mathbf{Mod}_I, \quad M \mapsto M$$

which is given by

(a) X_l *acts on* $1_{\mathbf{i}}M$ *by* $i_l^{-1}e^{\varkappa_l}$ *for each* $l = 1, 2, \dots, m$,

(b) T_k *acts on* $1_{\mathbf{i}}M$ *as follows for each* $k = 1, 2, \dots, m-1$,

$$\begin{aligned} &\frac{(pe^{\varkappa_k} - p^{-1}e^{\varkappa_{k+1}})(\varkappa_k - \varkappa_{k+1})}{e^{\varkappa_k} - e^{\varkappa_{k+1}}}\sigma_k + p && \text{if } i_k = i_{k+1}, \\ &\frac{e^{\varkappa_k} - e^{\varkappa_{k+1}}}{(p^{-1}e^{\varkappa_k} - pe^{\varkappa_{k+1}})(\varkappa_k - \varkappa_{k+1})}\sigma_k - \frac{(p^2-1)e^{\varkappa_{k+1}}}{p^{-1}e^{\varkappa_k} - pe^{\varkappa_{k+1}}} && \text{if } i_k = p^2 i_{k+1}, \\ &\frac{pi_{k+1}e^{\varkappa_k} - p^{-1}i_k e^{\varkappa_{k+1}}}{i_{k+1}e^{\varkappa_k} - i_k e^{\varkappa_{k+1}}}(\sigma_k - 1) + p && \text{if } i_k \neq i_{k+1}, p^2 i_{k+1}, \end{aligned}$$

(c) *T_0 acts on $1_{\mathbf{i}}M$ as follows*

$$\begin{cases} \dfrac{(e^{\varkappa_1}-1)^2}{(q_1e^{2\varkappa_1}-q_1^{-1})\varkappa_1^2}\pi_1+\dfrac{(q_0q_1-1)e^{2\varkappa_1}+2e^{\varkappa_1}}{q_1e^{2\varkappa_1}-q_1^{-1}} & \text{if } i_1=-q_0^{-1}=q_1^{-1}, \\ \dfrac{(q_1e^{\varkappa_1}+q_0)(e^{\varkappa_1}-1)}{(q_1^2e^{2\varkappa_1}-1)\varkappa_1}\pi_1+\dfrac{(q_0q_1-1)q_1e^{2\varkappa_1}+(q_1-q_0)e^{\varkappa_1}}{q_1^2e^{2\varkappa_1}-1} & \text{if } i_1=q_1^{-1}\neq -q_0^{-1}, \\ \dfrac{q_0(q_0e^{\varkappa_1}+q_1)(e^{\varkappa_1}-1)}{q_1(q_0^2e^{2\varkappa_1}-1)\varkappa_1}\pi_1+\dfrac{(q_0q_1-1)q_0^2e^{2\varkappa_1}-(q_1-q_0)q_0e^{\varkappa_1}}{q_1(q_0^2e^{2\varkappa_1}-1)} & \text{if } i_1=-q_0^{-1}\neq q_1^{-1}, \\ \dfrac{(e^{\varkappa_1}+q_0i_1)(e^{\varkappa_1}-q_1i_1)}{q_1(e^{2\varkappa_1}-i_1^2)}(\pi_1-1)+q_0 & \text{if } i_1\neq -q_0^{-1},q_1^{-1}. \end{cases}$$

Proof : Formula $(A.2)$ yields

$$\begin{cases} \lambda_{i_0}=2 & \text{if } i_1=-q_0^{-1}=q_1^{-1}, \\ \lambda_{i_0}=0 & \text{if } i_1\neq -q_0^{-1},q_1^{-1}, \\ \lambda_{i_0}=1 & \text{else.} \end{cases}$$

The proof is the same as the proof Theorem 6.5, using (5.3) and (A.1).

□

Now, all the statements in Section 8 generalizes. The proof is straightforward and is left to the reader. In particular, if ${}^\theta\mathbf{K}_I$ denotes the Grothendieck group of the category ${}^\theta$**R-proj**, then we have a canonical isomorphism

$${}^\theta\mathbf{V}(\lambda)=\mathcal{K}\otimes_{\mathcal{A}}{}^\theta\mathbf{K}_I,$$

where ${}^\theta\mathbf{V}(\lambda)$ the same ${}^\theta\boldsymbol{\mathcal{B}}$-module as in Theorem 8.30, with λ given by $(A.2)$ instead of (6.2). Theorem 10.19 generalizes as well.

Index of notation

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Université de Cergy-Pontoise, UMR CNRS 8088, F-95000 Cergy-Pontoise,
E-mail address: `michela.varagnolo@math.u-cergy.fr`

Département de Mathématiques, Université Paris 7, 175 rue du Chevaleret, F-75013 Paris,
E-mail address: `vasserot@math.jussieu.fr`