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Geometric Methods in Representation Theory

THE PUNCTUAL HILBERT SCHEME: AN INTRODUCTION

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THE PUNCTUAL HILBERT SCHEME: AN INTRODUCTION

JOSÉ BERTIN

Abstract. The punctual Hilbert scheme has been known since the early days of algebraic geometry in EGA style. Indeed it is a very particular case of the Grothendieck's Hilbert scheme which classifies the subschemes of projective space. The general Hilbert scheme is a key object in many geometric constructions, especially in moduli problems. The punctual Hilbert scheme which classifies the 0-dimensional subschemes of fixed degree, roughly finite sets of fat points, while being pathological in most settings, enjoys many interesting properties especially in dimensions at most three. Most interestingly it has been observed in this last decade that the punctual Hilbert scheme, or one of its relatives, the G-Hilbert scheme of Ito-Nakamura, is a convenient tool in many hot topics, as singularities of algebraic varieties, e.g McKay correspondence, enumerative geometry versus Gromov-Witten invariants, combinatorics and symmetric polynomials as in Haiman's work, geometric representation theory (the subject of this school) and many others topics.

The goal of these lectures is to give a self-contained and elementary study of the foundational aspects around the punctual Hilbert scheme, and then to focus on a selected choice of applications motivated by the subject of the summer school, the punctual Hilbert scheme of the affine plane, and an equivariant version of the punctual Hilbert scheme in connection with the A-D-E singularities. As a consequence of our choice some important aspects are not treated in these notes, mainly the cohomology theory, or Nakajima's theory. for which beautiful surveys are already available in the current litterature [24], [43], [47].

Papers with title *something an introduction* are often more difficult to read than *Lectures on something*. One can hope this paper is an exception. I would like to thank M. Brion for discussions and his generous help while preparing these notes.

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1. Preliminary tools

The prototype of problems we are interested in is to describe in some sense the set of ideals of fixed codimension n in the polynomial ring in r variables $k[X_1, \cdots, X_r]$ over a field k assumed algebraically closed to simplify.

In the one variable case, $k[X]$ being a principal ideal domain, an ideal I with $\dim k[X]/I = n$ is of the form $I = (P(X))$ with P monic and $\deg P = n$. These ideals are then parameterized by n parameters, the coefficients of P. In this case the punctual n-Hilbert scheme is an affine space $\mathbb{A}^n_k$. In a different direction, basic linear algebra tells us there is a precise relationship between on one hand the structure of the algebra $A = k[X]/(P)$ and on the other hand properties of the linear map $F \mapsto XF$ from A to A, summarized as follows

$P(X)$	A
without multiple factor	semi-simple
One root $\in k$ with multiplicity > 1	local, nilpotent
non zero discriminant	separable

One of our main goals in these lectures is to extend such a relationship to more general algebras than polynomials in one variable. One of our main theorems, in the two variables case, states that the set of all ideals with codimension n has a natural structure of a smooth algebraic variety of dimension $2n$. So to describe an ideal of codimension n in the polynomial ring $k[X, Y]$, we need exactly $2n$ parameters. Moreover the subset of ideals I with $k[X, Y]/I$ semi-simple is open and dense. The situation dramatically changes if the number of indeterminates is 3 or more. In any case the punctual Hilbert scheme appears to be a very amazing object.

Likewise, if A is a k-algebra (commutative throughout these notes, not necessarily of finite dimension as k-vector space) we can ask about the structure of the set of ideals of A. We shall see in case the dimension of A is finite, that the set of ideals $I \subset A$ with $\dim A/I = n$ is a projective variety, but infortunately in general, a very complicated one.

Throughout this text we fix an arbitrary base field k, not necessarily algebraically closed. In some cases however it will be convenient to assume $k = \overline{k}$, and sometimes the assumption of characteristic zero will be necessary. So in a first lecture the reader may assume $k = \overline{k}$ is a field of characteristic zero.

In this set of lectures, a scheme, or variety, will be mostly a k-scheme, that is a finite type scheme over k. Let us denote $\mathbf{Sch}_k$ the category of k-schemes, and correspondingly $\mathbf{Aff}_k$ the subcategory of affine k-schemes. One knows that $\mathbf{Aff}_k$ is the category opposite to the category $\mathbf{Alg}_k$ of commutative k-algebras of finite type. More generally **Sch** (resp. **Aff**) stands for the category of (locally) noetherian schemes (resp. the category of affine noetherian schemes). If X is a scheme, $\mathbf{Aff}_X$ denotes the category of schemes over X, i.e. of schemes together with a morphism to X. For any $R \in \mathbf{Aff}$, $\operatorname{Spec} R$ stands for the spectrum of A, viewed as usual as a scheme. When $R = k[X_1, \cdots, X_n]/(F_1, \cdots, F_m)$ and $k = \overline{k}$, then $\operatorname{Spec} R \in \mathbf{Aff}_k$ can be thought of as the set $\{x \in k^n, F_1(x) = \cdots = F_m(x) = 0\}$ equipped with the ring of functions R. If X is a scheme, $\mathcal{O}_X$ stands for the sheaf of regular functions on open subsets of X. The stalk of $\mathcal{O}_X$ at a point x will be denoted $\mathcal{O}_{X,x}$ or $\mathcal{O}_x$ if X is fixed. By a point we always mean a closed point.

By an $\mathcal{O}_X$-module (resp. coherent module) we shall mean a quasi-coherent (resp. coherent) sheaf of $\mathcal{O}_X$-modules [34]. Finally a vector bundle, is a coherent $\mathcal{O}_X$-module which is locally free of rank n, i.e. at all $x \in X$ the stalk is a free $\mathcal{O}_{X,x}$- module of rank n. If $X = \operatorname{Spec} A$ the category of $\mathcal{O}_X$-modules is equivalent to the category of A-modules. A locally free module of rank n is a projective module of constant rank n.

We want to point out that the concept of flatness is essential to handle correctly families of objects in algebra or algebraic geometry, for us families of 0-dimensional subschemes, or ideals. We refer to [16], or [45] for the first definitions, and basic results.

Punctual Hilbert schemes will be obtained by glueing together affine schemes. This explains why the first section starts with some comments about this glueing process. Another basic operation that will be used in the sequel is the quotient of a scheme by a finite group action. This operation will be studied in detail in section 1.4.

1.1. Schemes versus representable functors.

1.1.1. *Glueing affine schemes.* One lesson of algebraic geometry in EGA style is that it is often better to think of a scheme $X \in \mathbf{Sch}$ as a contravariant functor, the so-called *functor of points*

$$\underline{X} : \mathbf{Sch} \to \mathbf{Ens} \quad (or,\ \mathbf{Aff} \to \mathbf{Ens}) \tag{1.1}$$

where $\underline{X}(S) = \operatorname{Hom}_{\mathbf{Sch}}(S, X)$. Essentially all the information about the scheme X can be read off the functor of points. It doesn't matter to choose either **Sch** or **Aff**, indeed to reconstruct X from its functor of points, it is sufficient to know $\underline{X}$ on the subcategory **Aff**. In this functorial setting a morphism $f : Y \to X$ can be thought of as a section $f \in \underline{X}(Y)$ or using Yoneda's lemma as a functorial morphism $\underline{Y} \to \underline{X}$. In the sequel we shall use the same letter to denote a scheme and its associated functor.

The functorial view-point as advocated before suggests that to construct a scheme, one has to identify first its functor of points $\mathcal{X}$, and then try to show that this functor is indeed the functors of points of a scheme. This last part which amounts to check $\mathcal{X}$ is representable, is in general not totally obvious. We must list the conditions about the functor $\mathcal{X} = \underline{X}$ expressing that X is the glueing of affine pieces. The first condition comes from restricting $\mathcal{X}$ to the category Open_S of open sets $U \subset S \in \mathbf{Sch}$, the morphisms being the inclusions $U \subset V$. The local character of morphisms implies that $\mathcal{X} : \mathrm{Open}_S \to \mathbf{Ens}$ is not only a presheaf but a Zariski sheaf. We say "Zariski" to keep in mind that the topology used to define the sheaf property is the Zariski topology. In other words if $S = \cup_i U_i$ is an open cover of $S \in \mathbf{Aff}$, the following diagram with obvious arrows is exact

$$h_X(S) \longrightarrow \prod_i h_X(U_i) \rightrightarrows \prod_{i,j} h_X(U_i \cap U_j) \tag{1.2}$$

Let $\mathcal{X}$ be a Zariski sheaf on $\mathbf{Aff}$. We say that $\mathcal{X}$ is *representable* if for some scheme X we have an isomorphism $\xi : \underline{X} \xrightarrow{\sim} \mathcal{X}$. As said before the Yoneda lemma asserts that such a morphism is determined by the single object $\xi(1_X) \in \mathcal{X}(X)$. It is convenient to identify ξ with this object and write $\xi : X \to \mathcal{X}$. In the same way let $F : \mathcal{X} \to \mathcal{Y}$ be a morphism. One says that F is representable if for all $\xi : S \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}} S$, which is a sheaf, is representable.

If this is the case, F is said to be an open immersion (resp. closed immersion, a surjection) if for all ξ as above the projection $\mathcal{X} \times_{\mathcal{Y}} S \to S$ is an open immersion (resp. closed immersion, surjection). The following is the most naïve way to try to represent a functor, but it is sufficient for what follows.

Proposition 1.1. *A Zariski sheaf $\mathcal{X}$ is representable, i.e a scheme X, if and only if: there exist a family morphisms $u_i : U_i \to \mathcal{X}$ such that the following conditions are satisfied*

i) *for any i, $u_i : U_i \to \mathcal{X}$ is an open immersion, in particular $\coprod_i U_i \to \mathcal{X}$ is representable*
ii) *$u : U := \coprod_i U_i \to \mathcal{X}$ is surjective*
iii) *Finally X is separated (so really a scheme), if and only if the graph of the equivalence relation $U \times_{\mathcal{X}} U \hookrightarrow U \times U$ is a closed immersion.*

Proof:
First perform the fiber product

$$\begin{tikzcd}
U_i \arrow[r, "u_i"] & \mathcal{X} \\
U_i \times_{\mathcal{X}} U_j \arrow[u, "v_i"] \arrow[r, "v_j"] & U_j \arrow[u, "u_j"']
\end{tikzcd}$$

so that condition ii) says $U_i \times_{\mathcal{X}} U_j$ is a scheme. Furthermore the arrows v_i, v_j are both open immersions. Let us denote $U_{i,j} \subset U_i$ and $U_{j,i} \subset U_j$ the corresponding open sets. The isomorphism $U_i \times_{\mathcal{X}} U_j \xrightarrow{\sim} U_{i,j}$ together with the corresponding one with U_{ji}, yields an isomorphism $\theta_{j,i}$, viz.

$$\begin{tikzcd}
& U_i \times_{\mathcal{X}} U_j \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\
U_{i,j} \arrow[rr, dotted, "\theta_{j,i}", "\sim"'] & & U_{j,i}
\end{tikzcd} \tag{1.3}$$

The associativity of the fiber product quickly yields the following cocycle condition

$$\theta_{k,j|U_{j,i}\cap U_{j,k}}\theta_{j,i|U_{i,j}\cap U_{i,k}} = \theta_{k,i|U_{k,j}\cap U_{k,i}}, \quad \theta_{i,j}\theta_{j,i} = 1_{U_{i,j}} \tag{1.4}$$

Now the scheme X is obtained by glueing the $U_i's$ along the common open sets $U'_{i,j}s$ by means of the glueing isomorphisms $\theta_{i,j}$.

We now see u_i as section of $\mathcal{X}$ over U_i (Yoneda's lemma). Then if one uses the same notation for U_i and its image into X, it is easily seen that u_i and u_j are equal on $U_i \cap U_j$. Since $\mathcal{X}$ is a Zariski sheaf, this defines a global section $u \in \mathcal{X}(X)$, thus a morphism $u : X \to \mathcal{X}$. The result is that u is an isomorphism of sheaves. One must check that for all $S \in \mathbf{Sch}$, one has

$$u(S) : \mathrm{Hom}(S, X) \xrightarrow{\sim} \mathcal{X}(S)$$

Keep in mind that $u(S)$ is the map $f : (S \to X) \mapsto f^*(u) \in \mathcal{X}(S)$. First, let us check the injectivity, that is if $f, g \in \mathrm{Hom}(S, X)$, then $f^*(u) = g^*(u) \iff f = g$. We have the equality in a set-theoretical sense. Indeed, it suffices to check this when $S = \mathrm{Spec} k = \{s\}$. Suppose that $f(s) \in U_i$, $g(s) \in U_j$. Then the hypothesis means that we have a commutative diagram

$$\begin{tikzcd}
 & & U_i \arrow[r] & \mathcal{X} \\
S \arrow[urr] \arrow[r, dotted] \arrow[rrr, bend right, "g"'] & U_I \times_{\mathcal{X}} U_j \arrow[ur] \arrow[rr] & & U_j \arrow[u]
\end{tikzcd} \tag{1.5}$$

therefore we can fill in the dotted arrow, meaning $f(s) \in U_{ij}$, $g(s) \in U_{ji}$, and $g(s) = \theta_{ji}(f(s))$. Thus $f(s) = g(s)$. But now the equality $f = g$ is also true in a scheme sense. Indeed restricting to $S_i = f^{-1}(U_i) = g^{-1}(U_i)$, since U_i is a subfunctor of F, this yields $f = g$ on S_i. In turn we get finally $f = g$.

Now let's check the surjectivity. Let $f : S \to \mathcal{X}$ be an S-section of $\mathcal{X}$. We must check this section locally (in the Zariski sense) lifts to X. This immediately follows from the cartesian square

$$\begin{tikzcd}
U_i \arrow[r] & \mathcal{X} \\
S_i = U_i \times_F S \arrow[u] \arrow[r] & S \arrow[u, "f"']
\end{tikzcd}$$

whichs says that the restriction to S_i lifts.

To complete the proof that $\mathcal{X}$ is a scheme X, we need to check that X is separated, meaning the diagonal $X \hookrightarrow X \times X$ is closed, therefore a closed immersion. Let us write the cartesian diagram

$$\begin{tikzcd}
U \times U \arrow[r] & X \times X \\
U \times_X U \arrow[u] \arrow[r] & X \arrow[u]
\end{tikzcd}$$

where the right vertical arrow is the diagonal map. If X is separated, then its diagonal is closed, thus the immersion $U \times_X U \hookrightarrow U \times U$ is closed. Conversely, denoting $\Delta \subset X \times X$ the diagonal, the previous construction shows that $U_i \times U_j \cap \Delta$ is exactly the graph of the glueing morphism θ_{ji} that is, the image of $U_i \times_X U_j$. Then Δ being closed means that for all (i, j) the graph of θ_{ji} is closed, equivalently $U \times_X U$ is closed in $U \times U$.

□

It is known that the functor of points X of a scheme remains a sheaf for finer topologies of **Sch** than the Zariski topology. Descent theory shows the functor X is a sheaf not only for the etale topology, but also for the fppf topology, the faithfully flat and finite presentation topology. We have no need for this in what follows.

To end this section, let us remind the so-called *valuative criterion* of separatedness (resp. of properness) ([34], Theorem 4.7). In the setting of Proposition 1.1 condition iii) holds true if and only if for any discrete valuation ring A with fraction field K, and any pair of morphisms $f, g : \operatorname{Spec} R \longrightarrow X$, if $f = g$ at the generic point, then $f = g$. In other terms the map

$$\operatorname{Hom}(\operatorname{Spec} A, X) \longrightarrow \operatorname{Hom}(\operatorname{Spec} K, X) \tag{1.6}$$

is injective. Furthermore (1.6) is surjective if and only if X is proper. This will be used to check the Hilbert scheme is separated and complete.

1.2. **Affine spaces, Projective spaces and Grassmaniann s.** A first example of scheme given by its functor of points is the **affine** space $\mathbb{A}^n = \operatorname{Spec} k[X_1, \cdots, X_n]$, namely

$$\operatorname{Hom}(\operatorname{Spec} A, \mathbb{A}^n) = \operatorname{Hom}_{k-alg}(k[X_1, \cdots, X_n], A) = A^n \tag{1.7}$$

More generally let $\mathcal{E}$ be a quasi coherent sheaf over X, with dual $\mathcal{E}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$. The functor over $\mathbf{Sch}_X$ given by

$$(f : S \to X) \mapsto \operatorname{Hom}_{\mathcal{O}_S}(f^*(\mathcal{E}), \mathcal{O}_S) \tag{1.8}$$

is represented by the scheme $\operatorname{Spec}(\mathcal{S}ym(\mathcal{E}))$ which is natively a scheme over X [34].

Another very familiar example for the sequel is the functor of points of the **projective** space $\mathbb{P}^n$. This is the contravariant functor (see for example [34])

$$\mathbb{P}^n(S) = \{(\mathcal{L}, \varphi : \mathcal{O}_S^{n+1} \to \mathcal{L})\} /_{\cong} \tag{1.9}$$

where $\mathcal{L}$ is a line bundle, and φ is onto. Here φ, φ' are identified if there is an isomorphism $\psi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ with $\varphi' = \psi\varphi$. In this definition the closed points of $\mathbb{P}^n$ are the hyperplanes of k^{n+1}. The set of lines is the dual projective space $\mathbb{P}^{n\vee}$.

If $e_0, \cdots, e_n$ stands for the canonical basis of the free $\mathcal{O}_S$-module $\mathcal{O}_S^{n+1}$, then the subfunctor given by imposing the condition that $\varphi(e_i)$ generates $\mathcal{L}$ is readily seen to be open and representable by an affine space $\mathbb{A}^n$. Furthermore these subfunctors yield a covering of $\mathbb{P}^n$. The resulting geometric space $\mathbb{P}^n$ can be built by means of the operation Proj [34].

More generally the **Grassmann scheme** $\mathbf{G}_{n,r}$ classifying the vector subspaces of rank $r \in [1, n-1]$ of k^n is the scheme representing the functor

$$S \mapsto \{\varphi : \mathcal{O}_S^n \to E\}/_{\cong} \tag{1.10}$$

where E is a locally free sheaf of rank $n-r$, φ is onto, and the equivalence relation is as before. The punctual Hilbert functor to be defined is as we shall see a refinement of the Grassmann functor. The construction of the Hilbert scheme below incidently will give a proof of the existence of the Grassmann scheme. More precisely if we choose $W \subset k^n$ a subspace of dimension $n-r$, then the subfunctor of the Grassmann functor whose objects are the complementary subspaces of W is open, and easily shown to be representable by an affine space. These open subfunctors yield a cover. If we take the determinant of E e.g the top exterior power, namely

$$\wedge^{n-r}\varphi : \wedge^{n-r}(\mathcal{O}_S^n) = \mathcal{O}_S^{\binom{n}{r}} \to \wedge^{n-r}E \tag{1.11}$$

we get a point of $\mathbb{P}^{\binom{n}{n-r}}$. It can be shown this yields a closed embedding, the so-called Plücker embedding $\mathbf{G}_{n,r} \hookrightarrow \mathbf{P}^{\binom{n}{r}}$. The equations that describe the image are the so-called Plücker equations [45].

It will be useful for us to generalize slightly this construction. Let $\mathcal{F}$ be a quasi-coherent sheaf on a scheme X. For a fixed integer $r \geq 1$, let us define a contravariant functor $\mathcal{G}_r(\mathcal{F})$ over $\mathbf{Sch}_X$ as follows

$$\mathcal{G}_r(\mathcal{F})(f : S \to X) = \{f^*(\mathcal{F}) \xrightarrow{\alpha\,\text{onto}} \mathcal{E}\} \,/ \cong \tag{1.12}$$

where $\mathcal{E}$ is a locally free sheaf over S of rank r, and $\cong$ means up to isomorphism of the target.

Proposition 1.2. *The functor $\mathcal{G}_r(\mathcal{F})$ is representable i.e. is a scheme $\mathbf{G}_r(\mathcal{F})$ over X.*

Proof:
If U is an open subset of X, then the functor $\mathcal{G}_r(\mathcal{F}_{|U}) = \mathcal{G}_r(\mathcal{F}) \times_X U$ is clearly an open subfunctor of $\mathcal{G}_r(\mathcal{F})$. We can assume from now that $X = \operatorname{Spec} R$ is affine, and we may work entirely in the category of R-algebras not necessarily of finite type, indeed even not noetherian. Then $\mathcal{F}$ is given by a R-module F (of finite type or not). Let $\beta : R^r \to F$ be any linear map. We define a subfunctor $\mathcal{G}_{r,\beta}(F)$ of $\mathcal{G}_r(F)$ be requiring that $\alpha \circ (\beta \otimes 1)$ is an isomorphism. Equivalently the sections over the R-algebra B of this subfunctor are the B-linear maps

$$\alpha : F \otimes_R B \to B^r$$

such that $\alpha \circ (\beta \otimes 1) = id$. If non empty, this subfunctor is readily seen to be representable by an affine scheme. Let us assume the module F is given by a presentation

$$R^{(I)} \xrightarrow{\Phi} R^{(J)} \to F \to 0 \tag{1.13}$$

where Φ can be seen as a matrix (a_{ij}) with entries in R. We can lift β to $R^{(J)}$, and identify this map with a matrix (β_{kj}) with entries in R. Then the scheme that represents the previous subfunctor is

$$\operatorname{Spec} R[(T_j^k)_{j\in J, 1\le k\le r}]/(\cdots, \sum_j a_{ij}T_j^k, \sum_j \beta_{lj}T_j^k - \delta_{kl}, \cdots) \tag{1.14}$$

the spectrum of the quotient of a polynomial ring with perhaps infinitely many variables, by the ideal generated by the obvious relations. Now it is very easy to check the subfunctor $\mathcal{G}_{r,\beta}(F)$ where β runs over the linear maps $\mathrm{R}^r \to F$ is a covering family by open subfunctors. In turn, the conclusion follows from the general recipe (1.1).

□

1.3. **Quotient by a finite group.**

1.3.1. *The construction.* It is very important in algebraic geometry to be able to perform a quotient of a scheme by a group action. The case of interest for us is a quotient X/G of a scheme X endowed with an action of a finite group G of order denoted by $|G|$. Say G is reductive if $|G| \neq 0$ in k. Despite our general philosophy, the functor of points of the quotient X/G being rather complicated*, our construction of X/G will be purely geometric, relying on classical invariant theory. It should be noted the scheme X/G can have an eccentric behaviour if G is non reductive. For this reason in the next two sections G will be assumed to be reductive i.e any G-module is semi-simple. Presently G is arbitrary.

Let us start with some comments and definitions about actions of groups. Assume G acts on X. A G-stable subscheme is a subscheme $Y \subset X$ such that for all $g \in G$ the morphism $g\imath : Y \to X$ ($\imath =$ inclusion) factors through Y. If this is the case there is a well-defined isomorphism $g : Y \xrightarrow{\sim} Y$, induced by $g \in G$, defining an action of G on Y. If G acts on X, there is an obvious induced action of G on the set $\operatorname{Hom}(X,Y)$, viz. $g.f = gf^{-1}$ ($g \in G, f \in \operatorname{Hom}(X,Y)$). We say that f is G-invariant if $g.f = f$ for all $g \in G$. An important case is when $X = \operatorname{Spec} R$, where R is a finitely generated k-algebra, then an action of G on X translates into a left action of G on R. Let us write the action of $g \in G$ on R as ga instead of $(g^{-1})^*(a)$.

In order to ensure X/G is really a scheme, one must assume the action of G on X is admissible, meaning there is a cover of X by affine G-invariant open subsets. This condition is fulfilled under a mild restriction.

Lemma 1.3. *Let the finite group G acts on a quasi-projective scheme X. Then the action is admissible.*

Proof:
For any point $x \in X$, the finite set Gx must be contained in an affine open set, say U. This is readily seen from the quasi-projectivity assumption. Now the scheme is separated, so the finite intersection $\bigcap_{g\in G} gU$ must be affine, contains x, and is obviously G-invariant.

□

Recall G is arbitrary e.g not necessarily reductive. The result below sumarizes the key facts about the quotient scheme X/G.

*what is natural is the functor of points of the quotient stack $[X/G]$.

Proposition 1.4. *Assume the finite group G act admissibly on a scheme X.*

i) *There exists a scheme X/G together with a G-invariant morphism $\pi : X \to X/G$ such that any G-invariant morphism $f : X \to Y$ ($Y \in$ **Sch**) factors (uniquely) through π. More precisely $h \mapsto h\pi$ defines a functorial isomorphism $\mathrm{Hom}(X/G, Y) \cong \mathrm{Hom}(X, Y)^G$. As a consequence $(X/G, \pi)$ is unique up to a unique isomorphism.*

ii) *The morphism $\pi : X \to X/G$ is finite and surjective. Furthermore π induces a bijection between the points of X/G and the G-orbits of points of X.*

iii) *For any open set $V \subset X/G$ we have $V = \pi^{-1}(V)/G$, in particular the topology of X/G is the quotient topology. Furthermore the natural map $\pi^* : \mathcal{O}_{X/G} \xrightarrow{\sim} \mathcal{O}_X^G$ is an isomorphism; we say $(X/G, \pi)$ is a geometric quotient.*

iv) *Let $S \to X/G$ be a flat morphism, then under the natural action (on the left) of G on $X \times_{X/G} S$, we have the base change property $(X \times_{X/G} S)/G = S$. The result holds true for any base change $S \to X/G$ assuming G to be reductive, for instance k of characteristic zero.*

v) *If X is a normal variety (integral with integrally closed local rings [34]), then so is X/G.*

Proof:

i) Suppose first $X = \mathrm{Spec}\, R$, the spectrum of a finitely generated algebra. The claim is that $X/G = \mathrm{Spec}\, R^G$, where R^G denote the subring of invariant elements of R. It is a classical and important result going back to Gordan, Hilbert and Emmy Noether, that R^G is a finitely generated k-algebra [14]. Now let us denote $\pi : \mathrm{Spec}\, R \to \mathrm{Spec}\, R^G$ the morphism dual to the inclusion $R^G \hookrightarrow R$. It is easy to check the equality $R_f^G = (R^G)_f$ for any $f \in R^G$, more generally $(R \otimes_R A)^G = A$ whenever A is a flat R^G-algebra. Indeed if we see R^G as the kernel of the R^G-linear map

$$0 \to R^G \to R \xrightarrow{\prod_{g \in G} g^*} R^{|G|}$$

then tensoring over R^G this sequence with A yields the exact sequence

$$0 \to A \to R \otimes_{R^G} A \xrightarrow{\prod_{g \in G} g^* \otimes 1} (R \otimes_{R^G} A)^{|G|}$$

showing $A = (R \otimes_{R^G} A)^G$.

Let now $f : \mathrm{Spec}\, R \to Y$ be a morphism. If $Y = \mathrm{Spec}\, A$ is affine, then $f \in \mathrm{Hom}^G(X, Y)$ means that the comorphism $f^* : A \to R$ maps A into R^G, leading to $h : \mathrm{Spec}\, R^G \to \mathrm{Spec}\, A$. Thus i) becomes obvious in this case. Note the result is also clear for any Y if the image of f lies in an affine open subset of Y. In the general case it is readily seen using iii), to be proved below, that we can choose a covering

$$\mathrm{Spec}\, R^G = \bigcup_{i=1}^m \mathrm{Spec}\, (R^G)_{f_i}$$

i.e. a partition of unity $\sum_i R^G f_i = R^G$, such that $f(\mathrm{Spec}\, (R^G)_{f_i})$ is contained in an affine open set of Y. This yields a well-defined morphism

$$h_i : V_i = \mathrm{Spec}\, (R^G)_{f_i} \longrightarrow Y \tag{1.15}$$

such that $f_{|U_i} = h_i \pi_{|U_i}$ where $U_i = \operatorname{Spec} R_{f_i} = \pi^{-1}(V_i)$. On the intersection $V_i \cap V_j$ the two morphisms f_i and f_j coincide as a consequence of the unicity as shown in the first part of the proof. Then we can glue together the morphisms h_i to get $h : \operatorname{Spec} R^G \to Y$ with $h\pi = f$. The proof of i) is complete.

Under the same hypothesis, that is X affine, it is not difficult to check ii) and iii). For any $a \in R$ the polynomial

$$P(T) = \prod_{g \in G} (T - ga) = T^{|G|} - (\sum_{g \in G} ga) T^{|G|-1} + \cdots + (-1)^{|G|} \prod_{g \in G} ga \tag{1.16}$$

has its coefficients in R^G, thus a is integral over R^G, and since R is a finitely generated k-algebra, a standard argument shows R is a finitely generated R^G-module. This shows $\pi : X \to X/G$ is finite. To check the surjectivity notice a finite morphism is closed [34], but π is clearly dominant, so onto. Let now $\mathbf{Q} \in \operatorname{Spec} R^G$ be a prime ideal. Let us choose $\mathbf{P} \in \operatorname{Spec} R$ over $\mathbf{Q}$. The claim is that $\pi^{-1}(\mathbf{Q})$ is the orbit $G\mathbf{P}$. Let $\mathbf{P} = \mathbf{P}_1, \cdots, \mathbf{P}_d$ denote the distinct points of $G\mathbf{P}$. It is a standard consequence of the finiteness of R as R^G-module that if $i \neq j$ then $\mathbf{P}_i \not\subseteq \mathbf{P}_j$. Take $a \in \mathbf{P}_i \, (i > 1)$ so that the norm $\prod_{g \in G} ga$ is in $R^G \cap \mathbf{P}_i = \mathbf{Q}$. Therefore $\prod_{g \in G} ga \in \mathbf{P}_1$, thus for some $g \in G$, $ga \in \mathbf{P}_1$. This yields the inclusion

$$\mathbf{P}_i \subset \cup_{g \in G} g\mathbf{P}_1$$

The prime avoidance lemma [16] then shows $\mathbf{P}_i \subset g\mathbf{P}_1$ for some $g \in G$, and this implies the equality $\mathbf{P}_i = g\mathbf{P}_1$ as expected.

Let $Z \subset \operatorname{Spec} R$ be a G-stable closed subset. The previous discussion yields the equality $Z = \pi^{-1}(\pi(Z)$. As a consequence if $U \subset \operatorname{Spec} R$ is a G-invariant open subset, since π is closed we have that $\pi(U)$ is open and $U = \pi^{-1}\pi(U)$. The fact that $\pi^* : \mathcal{O}_{X/G} \xrightarrow{\sim} \mathcal{O}_X^G$ is an isomorphism is clear from the construction.

Let us pass to the general case where X is no longer assumed to be affine. From our hypothesis, there is a cover of X by (finitely many) affine open G-invariant subsets $X = \cup_i U_i$. Thus the quotient $V_i = U_i/G$ exists as shown by the previous part, with quotient map $\pi : U_i \to V_i$. The intersection $U_i \cap U_j$ is a G-invariant open affine subset of U_i, thus $V_{i,j} = \pi_i(U_i \cap U_j)$ is open in V_i and as shown before $V_{i,j} \cong (U_i \cap U_j)/G$. Similarly we get an open set $V_{j,i} \subset V_j$ and an isomorphism $V_{j,i} \cong (U_i \cap U_j)/G$. Finally this yields a uniquely defined isomorphism $\theta_{j,i} : V_{i,j} \cong V_{j,i}$ making the diagram commutative

$$\begin{tikzcd} & V_i \cap V_j \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\ V_{i,j} \arrow[rr, dotted, "\theta_{j,i}", "\sim"'] & & V_{j,i} \end{tikzcd} \tag{1.17}$$

We can glue together the affine schemes V_i along the open subsets $U_{i,j}$ by means of the $\theta'_{i,j}s$ to get a scheme Y together with a morphism $\pi : X \to Y$. The construction shows that V_i is an open subset of Y, and $\pi^{-1}(V_i) = U_i$. It is readily seen that (Y, π) is a categorical quotient of X by G in the sense of i).

The assertions ii) and iii) are also clear from the previous step, in particular $\pi^* : \mathcal{O}_{X/G} \xrightarrow{\sim} \mathcal{O}_X^G$ is an isomorphism, since it is so on an affine open cover.

iv) The fact that performing a quotient X/G commutes with a flat base change $S \to X/G$ is immediately reduced to the affine case. Property i) gives us a canonical morphism $(X \times_{X/G} S)^G \to S$. Over an affine open subset $V \subset S$ such that its image in X/G lies in an affine open subset, we know this morphism is an isomorphism, the conclusion follows. The proof is completed.

v) It is an elementary fact that R^G is a normal ring whenever R is normal [16].

□

Without further assumption assertion iv) can be wrong. As an example take $X = \operatorname{Spec} k[X, Y]$ the affine plane over a field of characteristic two. Let $G = \{1, \sigma\}$ be the group of order two where $\sigma(X) = Y, \sigma(Y) = X$. Then $k[X, Y]^G = k[XY, X + Y]$ is a polynomial ring. As base change we take

$$k[XY, X + Y] \to k[XY, X + Y]/(X + Y) = k[X^2]$$

then $k[X, Y]/(X + Y) = k[X]$. But now G acts trivially on $k[X]$, so $k[X]^G \neq k[X^2]$. It can be proved that in any case the morphism $(X \times_{X/G} S)/G \to S$ is purely inseparable (universally bijective) [5]. However under the reductivity assumption things work better.

Proposition 1.5. *Suppose G is a reductive finite group acting effectively on $X \in$ **Sch**. For any base change $S \to X/G$ ($S \in$ **Sch**), the canonical morphism $(X \times_{X/G} S)^G \to S$ is an isomorphism.*

Proof:
Under the reductivity assumption, the embedding $\mathcal{O}_{X/G} \hookrightarrow \pi_*(\mathcal{O}_X)$ admits a nice retraction, the average operator (or Reynolds operator)

$$R_G(a) = \frac{1}{|G|} \sum_{g \in G} g.a \tag{1.18}$$

This is standard and easy to see. More precisely a section a of $\pi_*(\mathcal{O}_X)$ is G-invariant iff $R_G(a) = a$. As in the proof before it is sufficient to check the base change property in the affine case, so assume $X = \operatorname{Spec} R$, $S = \operatorname{Spec} A$, with a base change morphism $R^G \to A$. Since, via to the operator R_G, R^G is a direct summand of R, the morphism $A \to R \otimes_{R^G} A$ is into. Now R_G extends to $R \otimes_{R^G} A$, viz. $R_G(x \otimes a) = aR_G(x)$ as a projector onto A. Thus $A = (R \otimes_{R^G} A)^G$.

□

Example 1.1. *(***ADE singularities***)*

The following example is very popular, and a cornerstone of many subjects. Let G be a finite subgroup of $\mathbf{SU}_2(\mathbb{C})$, equivalently of $\mathbf{SL}_2(\mathbb{C})$. As it is known, such G is one of the so-called binary polyedral groups, i.e. fits into one of the conjugacy classes

name	group	order	type
cyclic	$C_n\,(n \geq 2)$	n	A_{n-1}
binary diedral	$\tilde{D}_n\,(n \geq 2)$	$2n$	D_{n+2}
binary tetraedral	$\tilde{T}$	24	E_6
binary octaedral	$\tilde{O}$	48	E_7
binary icosaedral	$\tilde{I}$	120	E_8

One can show that the corresponding quotient surface $\mathbb{C}^2/G$ is embedded in the 3-dimensional affine space, thus described by an equation $f(x,y,z) = 0$, see below (for a proof see: [15], [48], [59])

A_n	$x^2 + y^2 + z^{n+1}$
D_{n+2}	$x^2 + y^2 z + z^{n+1}$
E_6	$x^2 + y^3 + z^4$
E_7	$x^2 + y^3 + yz^3$
E_8	$x^2 + y^3 + z^5$

From now on G is a reductive group, and $k = \overline{k}$ Let us return to our general setting

$\pi : X \to Y = X/G$. There is no loss of generality to assume that G acts faithfully. We want to understand the local structure of Y at some closed point y. Let us choose $x \in \pi^{-1}(y) = \{x = x_1, \cdots, x_m\}$. If H stands for the stabilizer group of x, then $m = |G/H|$ (1.4, ii)). Since the object we are interested in is the local ring $\mathcal{O}_{Y,y}$, or even the complete local ring $\hat{\mathcal{O}}_{Y,y}$ [16], it is useful to perform the flat base change $\mathrm{Spec}\,\hat{\mathcal{O}}_{Y,y} \to Y$. Then the scheme $X \times_Y \mathrm{Spec}\,\hat{\mathcal{O}}_{Y,y}$ is finite over $\mathrm{Spec}\,\hat{\mathcal{O}}_{Y,y}$, thus of the form

$$X \times_Y \mathrm{Spec}\,\hat{\mathcal{O}}_{Y,y} = \mathrm{Spec}\,B$$

where B is an $\hat{\mathcal{O}}_{Y,y}$ algebra finitely generated as a module, thus a complete semi-local ring. A classical structure theorem [16] yields for A

$$A = \prod_{g \in G/H} \hat{\mathcal{O}}_{X,gx} = \prod_{i=1}^{m} \hat{\mathcal{O}}_{X,x_i} = \mathbf{Ind}_H^G \hat{\mathcal{O}}_{X,x} \tag{1.19}$$

where $\mathbf{Ind}_H^G(W)$ stands for the induced G-module of the H-module W, in the sense of representation theory of finite groups. From this we get the following result:

Proposition 1.6. *We have $\hat{\mathcal{O}}_{Y,y} \cong \hat{\mathcal{O}}_{X,x}^H$. The morphism π is etale over y iff $H = 1$.*

Proof:
The first point follows easily from the structure of A as a G-module (1.19). For the second point it is known that π is etale over y iff π is etale at x iff $\pi_x^* : \hat{\mathcal{O}}_{Y,y} \longrightarrow \hat{\mathcal{O}}_{X,x}$ is an isomorphism. But clearly this precisely means that $H = 1$.

$\square$

Let $x \in X$ be a closed point with stabilizer H. Assuming x is a smooth point, if we choose a parameter system $(x_1, \cdots, x_n)$ at x, then $\hat{\mathcal{O}}_{X,x} = k[[X_1, \cdots, X_n]]$ a ring of formal power series. It is not difficult to check that the action of G can be linearized, as G is reductive. This means there is no loss of generality to assume $H \subset \mathbf{GL}_n(k)$, with the obvious action on the coordinates. In general the precise description of the ring $\hat{\mathcal{O}}_{Y,y} = k[[X_1, \cdots, X_n]]^H$ can be a difficult task.

The group G acts faithfully on X, this implies that the stabilizer of a general point $x \in X$ is trivial, so $\pi : X \to Y$ is generically etale. Denoting $R_\pi \subset X$ the locus of point with non trivial stabilizer, then clearly R_π is closed, it is called the *ramification locus* of π. Its image $B_\pi = \pi(R_\pi)$ is called the *branch locus*.

Corollary 1.7. *Under the previous hypothesis assume G acts freely on X, then $\pi : X \to X/G$ is etale. Furthermore X smooth $\Longleftrightarrow X/G$ smooth.*

Beside the quotient X/G previously studied, we are also interested in the fixed point subset. This subset needs to be defined in a schematic sense. To this end, we define a contravariant functor $\mathbf{Sch} \to \mathrm{Ens}$ by

$$S \mapsto \mathrm{Hom}_G(S, X) \tag{1.20}$$

where $\mathrm{Hom}_G(S, X)$ denotes the set of G-invariants morphisms, the action of G on S being the trivial one. This functor is representable, in other words:

Proposition 1.8. *Let us assume the action of G on X is admissible (i.e. X quasi-projective). Then, there is is a closed subscheme $X^G \subset X$, such that*

(1) *The action of G on X^G is trivial,*
(2) *If $f : S \to X$ is any G-invariant morphism, then f factors uniquely through X^G.*

In particular the closed points of X^G are the fixed (closed) points of X.

Proof:
(sketch) Due to our assumption, we may assume $X = \mathrm{Spec}\, A$ affine. The coaction of G on A will be denoted $(g, a) \mapsto ga$. It is readily seen that the answer to our representability problem is

$$X^G = \mathrm{Spec}\, A_G, \quad A_G = A/\langle ga - a\rangle_{g \in G, a \in A} \tag{1.21}$$

where $\langle ga - a\rangle$ stands for the ideal generated by the elements of the indicated form.

$\square$

We now assume that the action of G on X is faithful, and X is connected. Let $x \in X$ be a closed point. The stabilizer $H = G_x$ of x acts in an obvious way on the local ring $\mathcal{O}_{X,x}$, therefore on the cotangent vector space $T^*_{X,x} = \mathcal{M}_x/\mathcal{M}_x^2$. This defines a linear representation of H in $T^*_{X,x}$. Recall G is reductive.

Lemma 1.9. *The representation $G_x \longrightarrow \mathbf{GL}(T^*_{X,x})$ is faithful.*

Proof:
Since G is reductive, the surjection $\mathcal{M}_x \to V = \mathcal{M}_x/\mathcal{M}_x^2$ splits in the category of G-modules, therefore we can find a G-invariant subspace $V \subset \mathcal{M}_x$, suth that the restriction map $V \subset \mathcal{M}_x \to V = \mathcal{M}_x/\mathcal{M}_x^2$ is bijective. Thus if $g \in G$ acts trivially on V, it is easy to see that then g acts trivially on $A/\mathcal{M}_x^{k+1}$ for any $k \geq 1$. Since A is separated for the $\mathcal{M}$-adic-topology, this in turn yields $g = 1$.

$\square$

Exercise 1.1. *Let $x \in X^G$. Under the previous assumptions, show that $(T^*_{X,x})^G$ is the cotangent space of X^G at x.*

The setting we are interested in is the case X smooth. Then we have the interesting wellknown result*:

Theorem 1.10. *The fixed point subscheme X^G is smooth (perhaps not connected), and if $x \in X^G$, we have $T_{X^G,x} = T^G_{X,x}$.*

Proof:
The problem is local at x, it amounts to check that $\mathcal{O}^G_{X,x}$ is a regular local ring. Via the same argument as in lemma 1.9, we can find a G-invariant subspace $V \subset \mathcal{M}_x$, suth that the restriction map $V \subset \mathcal{M}_x \to \mathcal{M}_x/\mathcal{M}_x^2$ is bijective. If we set $V_G := V/\langle ga - a\rangle_{g\in G, a\in V}$, then it well known and easy to see that $V = V^G \oplus V_G$. Now a short calculation yields $\mathcal{O}_{X^G,x} = \mathcal{O}_{X,x}/\langle V_G\rangle$. Therefore $\mathcal{O}_{X^G,x}$ is the quotient of $\mathcal{O}_{X,x}$ by a subset of a system of parameters, which in turn means that $\mathcal{O}_{X^G,x}$ it is regular [16]. $\square$

Assuming X smooth, we can investigate more precisely the structure of the ramification locus of the quotient $\pi : X \to X/G$. Let $H \subset G$ be a subgroup. Denote X^H the subset of points fixed by H, and let Δ_H be the subset of points with stabilizer exactly H. Then

$$\Delta^H = X^H - \bigcup_{H\subset K, H\neq K} X^K \tag{1.22}$$

As shown by 1.10 X^H is a smooth closed subscheme, and $R = \sqcup_{1\neq H}\Delta_H$ is a stratification of R by locally closed smooth subvarieties.

Proposition 1.11. *Let Δ be an irreducible component of codimension one of the ramification locus R_π. Define $I_\Delta = \{g \in G,\ g = 1 \,\text{on}\, \Delta\}$. Then I_Δ is cyclic and $\neq 1$.*

Proof:
We have $\Delta \subset \bigcup_{1\neq H} X^H$, which in turn implies $\Delta \subset X^H$ for some H. Clearly $H \subset I_\Delta$, thus $I_\Delta \neq 1$. Notice Δ is a connected component of X^{I_Δ}, in particular Δ is smooth. Let P be the generic point of Δ. The local ring $\mathcal{O} = \mathcal{O}_{X,P}$ is a discrete valuation ring. Let $\mathcal{M} = (t)$ be the maximal ideal. The residue field $k(\mathcal{O})$ is the function field

*A more general result with G an affine reductive group, due to Fogarty, is true of Δ, in particular I_Δ acts trivially on $k(\mathcal{O})$. For $\sigma \in I_\Delta$ we can write $\sigma(t) = a_\sigma t$ for $a_\sigma \in \mathcal{O} - (t)$. It is elementary to check that

$$a_{\tau\sigma} = a_\tau \tau(a_\sigma)$$

Denoting $\overline{a}_\sigma$ the residue class of a_σ in $k(P)$, we see that $\sigma \in I_\Delta \mapsto \overline{a}_\sigma$ is a group morphism, thus its image is cyclic. We must check this morphism is injective. This can be deduced from Proposition 1.8, but we prefer to give an adhoc argument. Let J the kernel of this morphism, and assume $J \neq 1$. If $\sigma \in J$, then $a_\sigma = t + b_\sigma t^k$ where $k \geq 2$ can be choosen such that for some $\sigma \in J$, $b_\sigma \notin (t)$. It is easily seen that for $\tau, \sigma \in J$,

$$b_{\tau\sigma} = b_\tau + \tau(b_\sigma)$$

As a consequence $\sigma \mapsto \bar{b}_\sigma \in k(P)$ is a morphism from J to the additive group $k(P)$ which in turn yields $e\bar{b}_\sigma$ if $e = |I_\Delta|$. But $e \neq 0$ in $k(P)$, so we get a contradiction, and finally $J = 1$. □The subgroup I_Δ (also denoted I_P) is called the

inertia subgroup along the divisor Δ. The order $e(\Delta)$ of I_Δ is the *inertia index* at Δ. The notations being as before, let us denote Δ' (resp. P') the image of Δ (resp. P) in X/G. In some cases it is convenient to write $e(\Delta) = e(\Delta/\Delta')$ to refer precisely in which setting the inertia index is defined. The following result is standard:

Proposition 1.12. *Under the previous assumptions the order of the stabilizer of Δ (or P) in G is*

$$|G_P| = e(\Delta)[k(P) : k(P')] \tag{1.23}$$

equivalently the extension $k(P)/k(P')$ is galois with group G_P/I_Δ. Furthermore if $\mathcal{M}_P$ (resp. $\mathcal{M}_{P'}$) denote the maximal ideal of $\mathcal{O}_P$ (resp. $\mathcal{O}_{P'}$), then $\mathcal{M}_{P'}\mathcal{O}_P = \mathcal{M}_P^{e(\Delta)}$.

Proof:
The problem is purely local at P', so after base change we may assume that $X = \mathrm{Spec}\overline{\mathcal{O}}_{P'}$ the normalisation of $\mathcal{O}_{P'}$ in $k(X)$. We may even by base change from $\mathcal{O}_{P'}$ to the complete local ring $\hat{\mathcal{O}}_{P'}$ assume $A' = \overline{\mathcal{O}}_{P'}$ is complete, which in turn yields

$$\overline{\mathcal{O}}_{P'} \otimes_{\mathcal{O}_{P'}} A' = \prod_{g \in G/G_P} \hat{\mathcal{O}}_{g(p)} \tag{1.24}$$

Furthermore if $A = \hat{\mathcal{O}}_P$ with maximal ideal $\mathcal{M}$, then $A^{G_P} = A'$. By the assumption of reductivity, we also have $(A \otimes_{A'} k(P'))^{G_P} = k(P')$ which in turn yields $k(P)^{G_P} = k(P')$. As a consequence $k(P)/k(P')$ is galois with group G_P/I_P. Define the integer ν by $\mathcal{M}'A = \mathcal{M}^\nu$, then using the filtration

$$\mathcal{M}^\nu \subset \mathcal{M}^{\nu-1} \subset \cdots \subset \mathcal{M} \subset A$$

we get
(1.25)

$$|G_P| = \dim_{k(P')} A \otimes_{A'} k(P') = \sum_{j=1}^{\nu} \dim_{k(P')} \mathcal{M}^{j-1}/\mathcal{M}^j = \nu[k(P) : k(P')] = \nu[G_P : I_P]$$

Thus $e = \nu$ as required. v □

Let us assume that G acts freely on X, and that as before X is smooth. Recall a rational p-form of X is an object which in terms of a given system of parameters $(U; x_1, \cdots, x_n)$ of X has an expression

$$\omega = \sum_{1 \le i_1 < \cdots < i_p \le n} f_{i_1, \cdots, i_n} dx_{i_1} \wedge \cdots \wedge dx_{i_p} \quad (f_{i_1, \cdots, i_p} \in k(X))$$

If $(V; y_1, \cdots, y_n)$ is another system of local coordinates, and if

$$\omega = \sum_{j_1 < \cdots < j_p} g_{j_1, \cdots, j_p} dy_{j_1} \wedge \cdots \wedge dy_{j_p}$$

then the local expressions for ω are related on $U \cap V$ by

$$g_{j_1, \cdots, j_p} dy_{j_1} = \sum_{i_1 < \cdots < i_p} f_{i_1, \cdots, i_n} dx_{i_1} \frac{\partial (x_{i_1} \cdots x_{i_p})}{\partial (y_{j_1} \cdots y_{j_p})} \tag{1.26}$$

The p-form is regular if all $f_{i_1, \cdots, i_p}$ are regular on their open sets of definition. If $f : X \to Y$ is a morphism, and if X and Y are smooth, the pull-back of a regular p-form ω of Y is defined as usual. Choose $(U; x_1, \cdots, x_n)$ and $(V; y_1, \cdots, y_n)$ local coordinates systems on X respectively Y, with $f(U) \subset V$. Let $\omega = \sum_{j_1 < \cdots < j_p} g_{j_1, \cdots, j_p} dy_{j_1} \wedge \cdots \wedge dy_{j_p}$, then the local expression of $f^*(\omega)$ is

$$f^*(\omega) = \sum_{i_1 < \cdots < i_p} f^*(g_{j_1, \cdots, j_p}) \frac{\partial (y_{j_1} \cdots y_{j_p})}{\partial (x_{i_1} \cdots x_{i_p})} \tag{1.27}$$

Let $\Omega^p_{X/k}$ (resp. $(\Omega^p_{X/k})^G$) be the vector space of regular p-forms o(resp. G-invariant regular p-forms) on X. Let $\pi : X \to Y$ be the quotient morphism. Then π is etale and hence:

Proposition 1.13. *The map $\eta \mapsto \pi^*(\eta)$ yields an isomorphism $\Omega^p_{Y/k} \xrightarrow{\sim} (\Omega^p_{X/k})^G$.*

Proof:
Since π is etale, a coordinate system for Y lifts to a coordinate system for X. Therefore we can write locally an invariant p-form on X, as

$$\omega = \sum_{j_1 < \cdots < j_p} g_{j_1, \cdots, j_p} d\pi^*(y_{j_1}) \wedge \cdots \wedge d\pi^*(y_{j_p}) \tag{1.28}$$

where $(y_1, \cdots, y_n)$ is a coordinate system on Y. The G-invariance of ω amounts to the invariance of the coefficients $g_{j_1, \cdots, j_p}$. Then we can write $g_{j_1, \cdots, j_p} = \pi^*(f_{j_1, \cdots, j_p})$ with $f_{j_1, \cdots, j_p}$ a unique rational function on Y, regular on a suitable open chart. This shows that $\omega = \pi^(\eta)$, with $\eta = \sum_{j_1 < \cdots < j_p} f_{j_1, \cdots, j_p} dy_{j_1} \wedge \cdots \wedge dy_{j_p}$. $\square$

The result that follows is a weak form of the important *purity of the branch locus*:

Proposition 1.14. *Let $\pi : X \to Y$ be a finite surjective morphism between two smooth varieties of dimension n. Then either π is etale, or the branch locus if purely of dimension $n - 1$.*

Proof:
In this proof it is necessary to understand the ramification locus not as subset, but as a closed subcheme locally of the form $\{f = 0\}$, i.e. a divisor. To see this, let $\pi(p) = q$, and let us choose a system of local parameters $x_1, \cdots, x_n$ near p, and $y_1, \cdots, y_n$ near q. Then the local equation of R_π at p is

$$\frac{\partial(y_1 \cdots y_n)}{\partial(x_1 \cdots x_n)} \tag{1.29}$$

It is easy to check this is a consistent definition. Since π etale at p is equivalent to f being a unit at p, the conclusion follows. $\square$

1.3.2. *Groups generated by pseudo-reflections.* In general if a finite group G acts on a smooth scheme $X \in \mathbf{Sch}_k$, the quotient X/G will be singular, due to the existence of fixed points. It is useful to understand precisely when X/G is singular at a point $y = \pi(x)$. The problem is local, see (1.19), thus we may assume that $X = \operatorname{Spec} \mathcal{O}_{X,x}$, and that G is a finite group which acts faithfully on X, i.e. on $A = \mathcal{O}_{X,x}$. Then $X/G = \operatorname{Spec} A^G$. Let $A' = A^G$ be the invariant subring. This is a local ring with maximal ideal $\mathcal{M}'$, furthermore A is finitely generated over A'. Clearly the action of G on A yields a representation of G on the cotangent space $V = \mathcal{M}/\mathcal{M}^2$ of A. Since regularity is preserved if we pass to the associated complete local ring, finally we may assume that A, and then A', is local and complete.

Recall a pseudo-reflection σ of V is a diagonalizable automorphism of finite order such that $\operatorname{rk}(\sigma - 1) = 1$. The main result of this subsection due to Chevalley-Shephard-Todd, is:

Theorem 1.15. *Under the previous assumptions, the following conditions are equivalent:*

The image of G in $\mathbf{GL}(V)$ is generated by pseudo-reflections.

ii) *The invariant ring $A' = A^G$ is regular.*

iii) *The ring A is flat over A'.*

Proof:
We are going to prove that $i) \Longrightarrow iii) \Longrightarrow ii) \Longrightarrow i)$. Starting with the assumption i), we must check that $\operatorname{Tor}_1^{A'}(k, A) = 0$, or equivalently from the Nakayama's lemma that $\Sigma := \operatorname{Tor}_1^{A'}(k, A)/\mathcal{M}\operatorname{Tor}_1^{A'}(k, A) = 0$. Clearly G acts on the vector space Σ. We check this action is indeed trivial. Due to our hypothesis, it suffices to check that any pseudo-reflection $\sigma \in G$ acts trivially. Suppose $\sigma \in G$ acts as a pseudo-reflection on V. Then as in lemma 1.9 we can choose $v \in \mathcal{M} - \mathcal{M}^2$ such that $\sigma(v) = \zeta v$ for some root of unity ζ, and $\sigma = Id$ on A/Av. Thus we can write $\sigma - 1 = \varphi.v$ where $\varphi : A \to A$ is A'-linear. From this decomposition it is clear that $\sigma = Id$ on Σ, which in turn yields that G acts trivially on Σ. The group G being reductive, the map

$$(\operatorname{Tor}_1^{A'}(k, A))^G \longrightarrow (\operatorname{Tor}_1^{A'}(k, A)/\mathcal{M}\operatorname{Tor}_1^{A'}(k, A))^G = \operatorname{Tor}_1^{A'}(k, A)/\mathcal{M}\operatorname{Tor}_1^{A'}(k, A)$$

is surjective. But $(\operatorname{Tor}_1^{A'}(k, A))^G = \operatorname{Tor}_1^{A'}(k, A^G) = \operatorname{Tor}_1^{A'}(k, A') = 0$, thus finally $\operatorname{Tor}_1^{A'}(k, A) = 0$, and A is a flat A'-module.

Suppose now that A is a regular local ring faithfully flat over A', with $\dim A = \dim A' = n$, then A' is also regular. Indeed let M' be a finitely generated A'-module. We can choose a resolution of M'

$$0 \to N' \to L_{n-1} \to \cdots \to L_0 \to M' \to 0$$

where $L_0, \cdots, L_{n-1}$ are finitely generated and free. By the flatness hypothesis we get a resolution of the A-module $M' \otimes_{A'} A$

$$0 \to N' \otimes_{A'} A \to L_{n-1} \otimes_{A'} A \to \cdots \to L_0 \otimes_{A'} A \to M' \otimes_{A'} A \to 0$$

Since A is regular of dimension n, the A-module $N' \otimes_{A'} A$ must be free, which in turn implies that N' is free. This shows that A' is of finite homological dimension, thus regular [16].

We now check ii) $\Longrightarrow$ i). Let G_0 be the normal subgroup generated by the pseudo-reflections of G. We know from the first part that $A_0 = A^{G_0}$ is regular. From the weak form of the purity of the branch locus (Proposition 1.14) we know the quotient $X \to X/G$ must be ramified along a divisor through x, equivalently , there must exist a height one prime ideal P of A with inertia index $e > 1$. We saw that P must be generated by an element $t = x_1$, part of system of local coordinates of A (Proposition 1.11), and that the inertia subgroup I_P is cyclic with generator acting as a pseudo-reflection on V. Thus unless $G = G_P = 1$, we have $G_0 \neq 1$. The same remark also shows that the inertia index of P in both extensions A/A_0 and A/A' are the same. The inertia index being multiplicative under composite extensions (this is readily seen from Proposition 1.11) we see the extension A_0/A' is non ramified, i.e. etale, in codimension one. The purity of the branch locus (1.14) forces the equality $A' = A_0$, which in turn yields $G = G_0$.

□

The previous result has a well known equivalent in the graded case. Let G be a subgroup of $\mathbf{GL}(V)$, for some vector space over an algebraically closed field of characteristic prime to $|G|$ with $\dim V = r$. Let $S = \mathbf{S}(V)$ be the symmetric algebra of V, i.e. a polynomial algebra, and let $R = S^G$ be the graded subalgebra of invariant polynomials. Then the following are equivalent:

i) G is generated by pseudo reflections
ii) R is regular, i.e. a graded polynomial algebra
iii) S is a free R-module

Then under one of these conditions we have $R = k[z_1, \cdots, z_r]$ for some homogeneous elements $z_1, \cdots, z_r \in S$, and if $\deg z_i = d_i$, we have $|G| = d_1 \cdots d_r$ and $\sum_{i=1}^{r}(d_i - 1)$ is the number of pseudo reflections contained in G.

1.3.3. *Symmetric powers.* Throughout this section $k = \overline{k}$. Let X be a quasi-projective scheme. For any integer $n \geq 2$, the symmetric group $\mathbf{S}_n$ acts in an obvious way on X^n, viz.

$$\sigma(x_1, \cdots, x_n) = (x_{\sigma^{-1}(1)}, \cdots, x_{\sigma^{-1}(n)}) \tag{1.30}$$

Definition 1.16. *The n-symmetric power of X, denoted $X^{(n)}$ is the quotient scheme $X^n/\mathbf{S}_n$.*

Let $\pi_n : X^n \to X^{(n)}$ denote the canonical morphism. Notice the quotient exists due the quasi-projectivity assumption. The closed points of $X^{(n)}$ correspond to the $\mathbf{S}_n$ orbits in X^n, that is, to unordered set of points $(x_1, \cdots, x_n)$, $x_i \in X$. Such a set of n unordered points of X is called a 0-*cycle of degree* n of X. Recall the group of 0-cycles on X, denoted $Z_0(X)$ is the free abelian group on all (closed) points of X. Thus a 0-cycle is a formal finite sum $z = \sum_{i=1}^r n_i x_i$ where the x_i are points of X, and $n_i \in \mathbb{Z}$. The sum $\sum_i n_i$ is the *degree* of z, and z is *effective* if for all i, we have $n_i \geq 0$.

Therefore the points of $X^{(n)}$ can be identified with the effective 0-cycles of degree n on X.

It is easy to detect the ramification locus of π_n. Indeed let $x = (x_1, \cdots, x_n) \in X^n$ be a sequence of n points of X. Since we are viewing x as a 0-cycle, we may label the $x_i's$ such that

$$x_1 = \cdots = x_{k_1} \neq x_{k_1+1} = \cdots = x_{k_1+k_2} \neq \cdots \neq x_{k_1+\cdots+k_{r-1}+1} = \cdots = x_n \tag{1.31}$$

This means the $x_i's$ take r distinct values $x_1, x_{k_1+1}, \cdots, x_{k_1+\cdots+k_{r-1}+1}$ with multiplicities $k_1, \cdots, k_r \geq 1$, where $k_1 + \cdots + k_r = n$. A slightly different, but convenient notation will be $x = \sum_{i=1}^r k_i x_i$ where the r points x_i are pairwise distinct. Since we may further permute the $x_i's$, there is no loss of generality to assume $k_1 \geq k_2 \geq \cdots \geq k_r$, i.e. $(k_1, \cdots, k_r)$ is a *partition* of n. It is classical to denote a partition by a greek letter, say

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots), \sum_i \lambda_i = n$$

The *length* of λ is the greatest integer r such that $\lambda_r > 0$. Let us denote $X_\lambda^{(n)} \subset X^{(n)}$ the locus of 0-cycles of *type* λ, i.e. the λ-stratum. Finally let $\Delta \subset X^n$ be the *big diagonal*, i.e the locus of $(x_1, \cdots, x_n) \in X^n$ such that for some $i \neq j$ we have $x_i = x_j$. With these notations in mind, we have two elementary facts:

Lemma 1.17. *i) The stabilizer of* $x = \sum_{i=1}^r k_i x_i$ *of type* $\lambda = (k_1 \geq \cdots \geq k_r)$ *is* $H = \mathbf{S}_\lambda := \mathbf{S}_{k_1} \times \cdots \times \mathbf{S}_{k_r} \subset \mathbf{S}_n$.
ii) The morphism $X_*^r = X^r - \Delta \hookrightarrow X^{(n)}$, $(x_1, \cdots, x_r) \mapsto \sum_{i=1}^r k_i x_i$ *is an isomorphism onto the locally closed subset* $X_\lambda^{(n)}$.

Proof:
i) is clear. For ii) it is readily seen that the λ-stratum is a locally closed subset. Indeed one may view the morphism $(x_1, \cdots, x_r) \in X_*^r \to \sum_i k_i x_i \in X^{(n)}$ as $\pi_n \imath_\lambda$ where

$$\imath_\lambda(x_1, \cdots, x_r) = (\overbrace{x_1, \cdots, x_1}^{k_1}, \cdots, \overbrace{x_r \cdots, x_r}^{k_r})$$

But clearly X_*^r embeds into X^n via $\imath_\lambda$. Now the subgroup $\mathbf{S}_\lambda$ acts trivially on this subscheme, thus proving the restiction of π_n is an embedding. The result follows. $\square$

In particular the ramification locus of π_n is Δ the *big diagonal.* The branch locus is the set of 0-cycles with at least one $k_1 > 1$, i.e. $\cup_{\lambda, k_1>1} X_\lambda^{(n)}$.

Example 1.2. *(Viete's morphism)*

Take $X = \mathbb{A}^1$ the affine line. Then $X^n = \mathbb{A}^n$. We identify a point of $\mathbb{A}^n$, say $(\lambda_1, \cdots, \lambda_n)$ with the polynomial

$$P(T) = \prod_{i=1}^{n}(T - \lambda_i) = T^n + a_1 T^{n-1} + \cdots + a_n \tag{1.32}$$

where $a_i = (-1)^i \sigma_i(\lambda_1, \cdots, \lambda_n)$, σ_i being the elementary symmetric functions. The Viete morphism is $V : \mathbb{A}^n \longrightarrow \mathbb{A}^n$, $V(\lambda_1, \cdots, \lambda_n) = (a_1, \cdots, a_n)$. By the main theorem on symmetric polynomials V is the same as π_n, i.e. $(\mathbb{A}^1)^{(n)} \cong \mathbb{A}^n$.

Returning to the general setting, our aim is to give a complete description of the complete local ring $\hat{\mathcal{O}}_{X^{(n)},x}$ at any 0-cycle $x \in X^{(n)}$. Let $x = \sum_{i=1}^r k_i x_i = \pi_n(x_1, \cdots, x_1, x_2, \cdots, x_r)$ as before. Lemma 1.6 yields

$$\hat{\mathcal{O}}_{X^{(n)},x} \;\cong\; \hat{\mathcal{O}}^H_{X^n,(x_1,\cdots,x_1,x_2,\cdots,x_r)} \tag{1.33}$$

But now using Lemma 1.17, the right hand-side can be identified with the completed tensor product

$$\hat{\mathcal{O}}^{\mathbf{S}_{k_1}}_{X^{k_1},(x_1,\cdots,x_1)} \hat{\otimes} \cdots \hat{\otimes}\, \hat{\mathcal{O}}^{\mathbf{S}_{k_r}}_{X^{k_r},(x_r,\cdots,x_r)} \tag{1.34}$$

Thus the knowledge of the ring $\hat{\mathcal{O}}_{X^{(n)},x}$ amounts to understand this ring in the special case where $r = 1$. That is, for a totally degenerated 0-cycle $\sum_{i=1}^n x$ ($x \in X$). Let $d = \dim X$ be the dimension of X. There is a classical answer to this last question in the case, X smooth. Indeed, let us choose uniformizing parameters* $(t_1, \cdots, t_d)$ for X at the point x. Working with n copies of X, we shall denote by $(t_{i,j})_{1 \le i \le n}$ the previous local coordinates but on the i-th copy of X. Thus we can view the entries of the $n \times d$ matrix $|t_{i,j}|$ as a system of local coordinates for X^n at the point $(x, x, \cdots, x)$. The symmetric group $\mathbf{S}_n$ permutes the factors, and acts on the local coordinates according to the rule

$$\sigma t_{i,j} = t_{\sigma^{-1}(i),j} \tag{1.35}$$

Finally, we are going to describe the ring on simultaneous, or vector invariants, of $\mathbf{S}_n$ acting diagonally on the polynomial ring (or ring of formal power series)

$$k[T_1, \cdots, T_d]^{\otimes n} = k[T_{i,j}]_{1 \le i \le n; 1 \le j \le d}$$

which can be seen as the coordinate ring $k[V^{\oplus n}]$, $\dim V = d$.

To generate invariant polynomials, choose independent variables $U_1, \cdots, U_d$ and expand the product $\prod_{i=1}^n (1 + \sum_{j=1}^d T_{ij} U_j)$. This yields

$$\prod_{i=1}^{n}(1 + \sum_{j=1}^{d} T_{ij}U_j) = \sum_{j_1,\cdots,j_q} \left(\sum_{i_1 < \cdots < i_q} T_{i_1 j_1} \cdots T_{i_q j_q} \right) U_{j_1} \cdots U_{j_q} = \sum_{\gamma} \sigma_\gamma(T) U^\gamma \tag{1.36}$$

where $U^\gamma = U_1^{\gamma_1} \cdots U_d^{\gamma_d}$. Clearly the coefficients $\sigma_\gamma(T)$ are symmetric. To be more precise, and to state the classical result which goes back to H. Weyl [62], we need some

*generators of the maximal ideal

notations. Assume first $T_1, \cdots, T_d$ are d independent variables, and let us denote for $1 \leq q \leq d$

$$\sigma_q(T_1, \cdots, T_d) = \sum_{1 \leq i_1 < i_2 < \cdots < i_q \leq d} T_{i_1} T_{i_2} \cdots T_{i_q}$$

the q-th *elementary symmetric polynomial* in the $T_i's$. One classically defines the *total polarization* of σ_q as the polynomial $\hat{\sigma}_q \in k[U_{i,j}]_{1 \leq i \leq N; 1 \leq j \leq q}$ given by

$$\hat{\sigma}_q(T_{i,j}) = \sum_{1 \leq i_1, \cdots, i_q \leq d}' U_{i_1,1} U_{i_2,2} \cdots U_{i_q,q} \tag{1.37}$$

where the prime means that we sum over all pairwise distinct indices $i_1, \cdots, i_q \in [1, n]$. The main theorem about the simultaneous invariants of $\mathbf{S}_n$ as stated in [62], see also [11] for some comments:

Theorem 1.18. *With the previous notations, the ring of diagonal invariants $k[T_{i,j}]^{\mathbf{S}_n}$ is generated by the total polarizations of the $\sigma_q's$, i.e by the polynomials*

$$\hat{\sigma}_q(T_{i,\alpha_j}) = \sum_{1 \leq i_1, \cdots, i_q \leq n}' T_{i_1,\alpha_1} T_{i_2,\alpha_2} \cdots T_{i_q,\alpha_q} \tag{1.38}$$

for any choice of $\alpha_1, \cdots, \alpha_q \in [1, d]$ including repeated indices.

In the book [62] the base field k has characteristic zero. Indeed, the result can be false in positive characteristic, unless $d = 1$, see for example [11] for a more complete discussion of this question. As an immediate corollary of this local result, true without restriction on k, the scheme $X^{(n)}$ behaves well if $\dim X = 1$, e.g X is a smooth curve. This is essentially the content of Viete's classical theory revisited (see [45]).

Corollary 1.19. *For a smooth curve X the n-fold symmetric power $X^{(n)}$ is smooth of dimension n. This scheme parameterizes the effective divisors of degree n, i.e. $X^{(n)} = \mathrm{Div}_n(X)$.*

Assuming again X smooth, if $d = \dim X \geq 2$ and $n \geq 2$, the scheme $X^{(n)}$ is always singular. The punctual Hilbert scheme that is the main subject of this set of notes is intended at least partially to rub out this defect.

Proposition 1.20. *Suppose X is a smooth variety of dimension $d \geq 2$. Then for any $n \geq 2$ the singular locus $X^{(n)}_{sing}$ of $X^{(n)}$ is the closure $\overline{X^{(n)}_{2,1^{n-2}}} = \cup_{\lambda \neq (1^n)} X^{(n)}_\lambda$, where the notation 1^n stands for the trivial partition $(1, 1, \cdots, 1)$ on n.*

Proof:
It is easy to see $\cup_{\lambda \neq (1^n)} X^{(n)}_\lambda$ is the closure of the stratum $X^{(n)}_{2,1^{n-2}}$. This strata has codimension $d \geq 2$ in $X^{(n)}$ and is irreducible. The claim is $X^{(n)}_{sing} = \overline{X^{(n)}}_{2,1^{n-2}}$. Since the singular locus is closed, it suffices to check $X^{(n)}_{2,1^{n-2}} \subset X^{(n)}_{sing}$. We can present two arguments for that. First assume the contrary, so $X^{(n)}$ is smooth along $X^{(n)}_{2,1^{n-2}}$. Notice the galois cover induced by π_n over $X^{(n)} - X^{(n)}_{2,1^{n-2}}$ has branch locus $X^{(n)}_{2,1^{n-2}}$, a

subset of codimension $d \geq 2$. The theorem of purity of the branch locus (Prop. 1.14) yields a contradiction.

We can also argue more directly, and more elementary, as follows. Let $x = 2x_1 + x_2 + \cdots + x_{n-1}$ be a point of the stratum $X^{(n)}_{2,1^{n-2}}$. The complete local ring of $X^{(n)}$ at x is described by (1.34) together with Weyl's theorem 1.18. This yields with slightly modified notations

$$\hat{\mathcal{O}}_{X^{(n)},x} = k[[x_1, \cdots, x_d, y_1, \cdots, y_d]]^{\mathbb{Z}/2\mathbb{Z}} \hat{\otimes} k[[z_1, \cdots, z_{d(n-1)}]] \tag{1.39}$$

where the group $\mathbf{S}_2 = \mathbb{Z}/2\mathbb{Z}$ acts through the involution $\sigma(x_i) = y_i \, (1 \leq i \leq d)$. Let us choose the more convenient indeterminates (assuming the characteristic $\neq 2$)

$$u_i = \frac{x_i + y_i}{2}, \;\; v_i = \frac{x_i - y_i}{2}$$

Then $\sigma(u_i) = u_i, \sigma(v_i) = -v_i$. The ring of invariants is now easy to describe:

$$k[[x_1, \cdots, x_d, y_1, \cdots, y_d]]^{\mathbb{Z}/2\mathbb{Z}} = k[[u_1, \cdots, u_d, \{v_i v_j\}_{i \leq j}]] \tag{1.40}$$

If we set $v_{ij} = v_i v_j (= v_{ji})$ then the ideal of relations between these $\frac{d(d+3)}{2}$ generators is spanned by the quadratic relations

$$v_{ij} v_{kl} = v_{ik} v_{jl} \quad (\forall i, j, k, l \in [1, d])$$

It is now readily seen, denoting $\mathcal{M}$ the maximal ideal of the local ring $\hat{\mathcal{O}}_{X^{(n)},x}$, that

$$\dim \mathcal{M}/\mathcal{M}^2 = \frac{d(d+3)}{2} \tag{1.41}$$

showing the local ring $\hat{\mathcal{O}}_{X^{(n)},x}$ is regular iff $d = 1$.

□

We refer to the book ([11], Chapter 7) for a more thorough discussion of the n-fold symmetric products.

Example 1.3.

As an example we are going to describe the local ring of $\mathbb{A}^{(2)}$ at the point $2[0]$ ($0 = (0,0) \in \mathbb{A}^2$), and then to show the blow-up of the singular locus desingularizes $(\mathbb{A}^2)^{(2)}$ (char $k \neq 2$). This example will be useful in the sequel of these notes.
As shown in the proof of Proposition 1.20, it is convenient to work with new coordinates u_1, u_2, v_1, v_2 such that the $\mathbf{S}_2$ action reads

$$(u_1, u_2, v_1, v_2) \mapsto (u_1, u_2, -v_1, -v_2)$$

Then the ring of invariants is $k[u_1, u_2, v_1^2, v_2^2, v_1 v_2] = k[u_1, u_2, x, y, z]/(xz - y^2)$ with $x = v_1^2, y = v_1 v_2, z = v_2^2$. Thus $(\mathbb{A}^2)^{(2)}$ is simply the product $\mathbb{A}^2 \times Q$ where Q denotes the quadric cone $\{(x, y, z) \in k^3, xz = y^2\}$, in other words the A_1-singularity (see section 5). The singular locus is $\mathbb{A}^2 \times (0,0,0)$. For a general description of the blow-up of a point, more generally a closed subscheme, we refer to the book [34]. In our example the description is as follows. The blow-up plane is covered by two coordinate patches $U, V \cong \mathbb{A}^2$, where

$$U = \operatorname{Spec} k[u_1, u_2, x, \frac{y}{x}], \;\; V = \operatorname{Spec} k[u_1, u_2, y, \frac{x}{y}]$$

and $U = V = \mathbb{A}^2$. The corresponding morphisms $U \to (\mathbb{A}^2)^{(2)}$ (resp. $V \to (\mathbb{A}^2)^{(2)}$) are given in term of these coordinates by the obvious morphisms

$$k[u_1, u_2, x, y, z]/(xz - y^2) \longrightarrow k[u_1, u_2, x, \frac{y}{x}]$$

(resp. $k[u_1, u_2, x, y, z]/(xz-y^2) \longrightarrow k[u_1, u_2, \frac{x}{y}, y]$). The exceptional locus $\mathcal{E}$, a divisor, is given in U by $\{x = 0\}$, and in V by $\{y = 0\}$.

Exercise 1.2. *Show* $(\mathbb{P}^1)^{(n)} \cong \mathbb{P}^n$.

Exercise 1.3. *Let E be an elliptic curve over $k = \overline{k}$ with char$k$= 0. (a smooth complete curve of genus one, together with a distinguished point $O \in E$ [34]). Let $n \geq 2$.*

- *i) Use the abelian group law on E with neutral element O to check that $E^n \xrightarrow{\sim} E \times W$, where $W = \{(x_1, \cdots, x_n) \in E^n, \sum_i x_i = 0\}$.*
- *ii) Show $W/\mathbf{S}_n \xrightarrow{\sim} \mathbb{P}^{n-1}$ (hint: use Abel's theorem [34] to identify $W/\mathbf{S}_n$ with the linear system $|nO|$). Then show $E^{(n)} \xrightarrow{\sim} E \times \mathbb{P}^{n-1}$.*

Exercise 1.4. *Let as before X be a quasi-projective scheme, and let $n_1, \cdots, n_r \geq 1$. If we set $n = n_1 + \cdots + n_r$, show there is a sum morphism $\prod_{i=1}^r X^{(n_i)} \longrightarrow X^{(n)}$.*

Exercise 1.5. *Let $X = X_1 \sqcup X_2$ be a disjoint sum of two schemes. Prove that $X^{(n)} = \sqcup_{p+q=n} X_1^{(p)} \times X_2^{(q)}$.*

1.4. **Grassmann blow-up.** The Chevalley-Shephard-Todd theorem 1.15 emphazises the flatness property of X over X/G. If this condition is not fulfilled, it is of interest to explain how to recover it universally by mean of a suitable birational modification of X/G. This will be used in the construction of the equivariant Hilbert scheme. The set-up is as follows. Let X be a scheme, and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. We are going to figure out how to make $\mathcal{F}$ flat i.e locally free by a suitable modification of X and $\mathcal{F}$. This problem has been studied in a very general setting by Raynaud [51]. Our concern here is much more modest. We need some additional assumptions:

(1) Assume there is an open subset U of X such that $\mathcal{F}$ is locally free of rank $d \geq 1$ on U,
(2) U is schematically dense*.

In our examples X will be integral and hence (1) and (2) simply mean $\mathcal{F}$ is generically of rank d. Let us introduce a general definition. Let $f : X' \to X$ be a birational morphism, precisely it will be assumed that

- $f : U' = f^{-1}(U) \to U$ is an isomorphism,
- 1) holds for U', i.e $\mathrm{Ass}(X') \subset U'$.

Denote $\imath : U' \hookrightarrow X'$ the canonical injection. There is a natural map coming from the adjonction property of $\imath_*$ and $\imath^*$

$$f^*(\mathcal{F}) \longrightarrow \imath_* \imath^*(f^*(\mathcal{F})) \tag{1.42}$$

Call the image of this map the *strict transform* of $\mathcal{F}$ under f, and denote it as $f^\natural(\mathcal{F})$. Under our hypothesis $f^\natural(\mathcal{F})$ is coherent, indeed it is the quotient of $f^*(\mathcal{F})$ by the

*This means the morphism $\mathcal{O}_X \to \imath_*(\mathcal{O}_U)$ is injective, where $\imath : U \hookrightarrow X$ denotes the canonical injection sub-sheaf $\mathcal{T}$ whose sections are the sections of $f^*(\mathcal{F})$ with support in $X' - U'$. It is not difficult to check the definition is independent of the choice of U, that is, if the strict transform is defined relatively to an open set V with $V \subset U$, then this yields the same result. Before going further we must recall some basic facts about the Fitting ideals ([16], 20.2). Suppose first $X = \operatorname{Spec} A$, the spectrum of a noetherian ring, then $\mathcal{F}$ is identified with an A-module of finite type M. Let us choose a presentation of M

$$A^m \xrightarrow{\varphi} A^n \to M \to 0 \tag{1.43}$$

Then it is well known that the ideal spanned by the $n-k$-minors of the matrix φ is independent of the choice of the presentation. This ideal $F_k(M)$ is the k-th Fitting ideal of M, or of $\mathcal{F}$. We have $F_k(M) \subset F_l(M)$ if $k \leq l$, and $F_k(M) = A$ when $k \geq n$. It is likely clear that for a general X we can glue together the local Fitting ideals and thus speak of the ideal $F_k(\mathcal{F})$. If $\mathcal{F}$ is locally free of rank $r \geq 1$, then $F_k(\mathcal{F}) = 0$ for $k < r$, and $F_r(\mathcal{F}) = \mathcal{O}_X$. It is worth noting that the closed subset $S_k(\mathcal{F})$ defined by $F_k(\mathcal{F})$ is

$$S_k(\mathcal{F}) = \{x \in X,\ \dim \mathcal{F}_x \otimes k(x) > k\} \tag{1.44}$$

For instance Z_0 is the support of $\mathcal{F}$. Assuming always that $\mathcal{F}$ is locally free of rank $r \geq 1$, the situation we are interersted in is when there is a surjective map $\mathcal{F} \to \mathcal{L}$ onto a locally free sheaf of rank r.

Lemma 1.21. *Let as before $\mathcal{F}$ be a sheaf, locally free of rank r on a schematically dense open subset $U \subset X$. Assume there is a quotient $\mathcal{F}/\mathcal{T}$ locally free of rank $r \geq 1$. Then $\mathcal{T}$ is the subsheaf whose sections are the sections of $\mathcal{F}$ annihilated by $F_r(\mathcal{F})$. We can take for U the open subset $X - Z$ where Z is the support of $\mathcal{O}_X/F_r(\mathcal{F})$.*

Proof:
The problem is local so we can assume* $X = \operatorname{Spec} A$, and $\mathcal{F} = \tilde{M}$, $\mathcal{L} = \tilde{L}$. In that case if $L = M/N$, then $M = L \oplus N$, which in turn yields, see exercice 1.7 below.

$$F_r(M) = F_r(L \oplus N) = F_0(N) \tag{1.45}$$

It follows that the support of N is the closed subset $V(F_r(M))$. Then the restriction of M on the open set $\operatorname{Spec} A - V(F_r(M))$ is locally free of rank r, and this open set contains U. □

The following lemma ensures that under certain conditions the previous hypothesis holds.

Lemma 1.22. *Let M be a finitely generated module over A. Assume $F_r(M)$ is a principal ideal, and condition (1) above holds for $U = \operatorname{Spec} A - V(F_r(M))$. That is U is schematically dense, and the restriction of $\tilde{M}$ on U is locally free of rank r. Then if $T = \{m \in M,\ F_r(M)m = 0\}$, the quotient M/T is locally free of rank r.*

Proof:
Let us choose a presentation of M as (1.43). Let $|a_{ij}|$ denotes the matrix of φ. On a suitable affine open subset, and after a suitable permutation, we can assume that

*As usual $\tilde{M}$ denotes the quasi-coherent sheaf associated to M.

$F_r(A) = A\delta$ where δ the $n-r$-minor of $\delta = \det|a_{ij}|_{i,j\leq n-r}$. All other minors of order $n-r$ are multiples of δ. Denote $(e_i)_{1\leq i\leq n}$ the images in M of the canonical basis of A^n. The Cramer rule yields

$$\delta(e_i - \sum_{j=n-r+1}^{n} b_{ij}e_j) = 0 \quad i = 1, \cdots, n-r \tag{1.46}$$

for some $b_{ij} \in A$, showing $e_i - \sum_{j=n-r+1}^{n} b_{ij}e_j \in N$ the submodule killed by $F_r(M)$. Thus M/N is generated by the r elements $e_{n-r+1}, \cdots, e_n$. We can find a presentation

$$0 \to Q \to A^r \to M/N \tag{1.47}$$

Since M is locally free of rank r on $\operatorname{Spec} A - V(\delta)$, the support of Q is a subset of $V(\delta)$, which in turn says that Q is killed by a power of δ. But we know that δ is a not a zero divisor which in turn implies $Q = 0$. $\square$

The Grassmann blow-up can be described as follow:

Proposition 1.23. *There is a projective morphism $p : X' \to X$ such that $p : U' = p^{-1}(U) \xrightarrow{\sim} U$ is an isomorphism, and*

i) *U' is schematically dense into X'*
ii) *The strict transform $p^\natural(\mathcal{F})$ is locally free of rank d.*
iii) *(X', p) is universal with respect to i) end ii).*

Proof:
The construction of X' goes as follows. First let $g : \mathbf{G}_r(\mathcal{F}) \to X$ be the Grassmann scheme associated to $\mathcal{F}$ (proposition 1.2). The X-points of $\mathbf{G}_r(\mathcal{F})$ correspond to the locally free quotients of $\mathcal{F}$ of rank r. Over U, the sheaf $\mathcal{F}$ is locally free of rank r, thus providing a section of g over U:

$$\begin{tikzcd} \mathbf{G}_r(\mathcal{F}) \arrow[r, "g"] & X \\ & U \arrow[ul, "s"] \arrow[u, hook] \end{tikzcd}$$

Then we define X' as the schematic image of s in $\mathbf{G}_r(\mathcal{F})$. Let π denote the restriction of g to X'. Clearly π induces an isomorphism $U' = \pi^{-1}(U) \xrightarrow{\sim} U$. Furthermore U' is schematically dense in X'. The restriction of the universal quotient to X' yields a canonical surjection

$$\pi^*(\mathcal{F}) \longrightarrow \mathcal{L} \tag{1.48}$$

which is an isomorphism on U'. Due to i), the kernel is precisely the subsheaf of sections with support on $X' - U'$, thus $\mathcal{L} = \pi'^\natural(\mathcal{F})$. We now are going to check that (X', π) satisfies the universal property iii). Suppose a morphism $f : Y \to X$ is given and i) and ii) holds. The quotient $f^*(\mathcal{F}) \to f^\natural(\mathcal{F})$ gives us a section f' of g over Y, that is

$$\begin{tikzcd} X' \subset \mathbf{G}_r(\mathcal{F}) \arrow[r, "g"] & X \\ & Y \arrow[u, "f'"] \arrow[ul] \end{tikzcd} \tag{1.49}$$

Clearly f' factors through $\mathbf{G}_r(\mathcal{F})$ and taking into account i) f even factors through X'.

□

Lemma 1.22 shows the blow up of X with center the ideal $F_r(\mathcal{F})$ must factors through the Grassmann blow-up X', but it is not necessarily isomorphic to X'.

Example 1.4.

Let R be a finitely generated integral k-algebra with fraction field K. Let M be the module given by the presentation

$$R \xrightarrow{\varphi} R^{r+1} \to M \to 0 \tag{1.50}$$

where $\varphi(1) = (a_0, \cdots, a_r)$. We set $J = (a_0, \cdots, a_r) \subset A$, and denoting $J^{-1} = \{x \in K,\, xJ \subset R\}$, assume that $J^{-1} = R$. The module M is torsion free of rank r. To describe the scheme $\mathbf{G}_r(M)$, let us consider the relative projective space $\mathbb{P}^r_R$. A point of $\mathbb{P}^r_R$ is locally given by a matrix $\lambda = |\lambda_{\alpha,\beta}|$ of size $r \times (r+1)$ with entries in a R-algebra A, such that if z_j denotes $(-1)^j$ times the minor obtained by omitting the jth column, then $\sum_{j=0}^r A z_j = A$. We use the $z_j's$ as coordinates of $\mathbb{P}^r_R$. Then $\mathbf{G}_r(M)$ is the closed subscheme given by the equations

$$a_i z_j = a_j z_i \ (1 \leq i, j \leq r) \tag{1.51}$$

On the open subset $U = \operatorname{Spec} R - V(J)$ there is a canonical point $z_j = a_j$. The closure of this point is the Grassmann blow-up X'. One can ask if in any case the result is

$$X' = \mathbf{G}_r(M)$$

Over the affine open set $z_i \neq 0$, the subscheme $\mathbf{G}_r(M)$ is given by the set of equations (1.51), which reduce to $a_i \frac{z_j}{z_i} - a_j = 0$. It is an elementary fact that the ideal $(aX + b)$ in $A[X]$ is prime if A is integral and if (a, b) is a regular sequence. Thus if for any $i \neq j$, (a_i, a_j) is a regular sequence, then $\mathbf{G}_r(M)$ is integral, which in turn yields $X' = \mathbf{G}_r(M)$. If furthermore the whole sequence $(a_0, \cdots, a_r)$ is regular, then it is known that $X' = \mathbf{G}_r(M)$ is the blow-up of $\operatorname{Spec} R$ along the center $V(J)$ ([17], exercise IV-26). In any way the fiber $\pi^{-1}(x)$ over a point $x \in V(J)$ is a projective space $\mathbb{P}^r$.

Example 1.5.

Let X be an integral scheme, with function field $k(X)$, i.e. $k(X) = \mathcal{O}_{X,\xi}$, ξ being the generic point. Recall that a $\mathcal{O}_X$ coherent sheaf $\mathcal{F}$ is torsion free if the canonical map $\mathcal{F} \to \mathcal{F} \otimes k(X) = \mathcal{F}_\xi$ in injective. It is torsion free of rank $r \geq 1$, if furthermore $\dim_{k(X)} \mathcal{F} \otimes k(X) = r$.

It will be convenient for the sequel to say $x \in X$ is a *singular point* of $\mathcal{F}$ if the fiber $\mathcal{J}_x$ is not a free $\mathcal{O}_{X,x}$-module. Since for a coherent module freeness is an open condition, we see that the singular locus of $\mathcal{J}$ is closed. For a general $\mathcal{F}$, the torsion subsheaf is $\mathcal{F}_{tors} = \ker(\mathcal{F} \to \mathcal{F} \otimes K)$.

Suppose now $\mathcal{F}$ is a torsion free sheaf of rank one. Let $\mathbf{Sym}^\bullet(\mathcal{F})$ be the symmetric algebra of the module $\mathcal{F}$, namely

$$\mathbf{Sym}^\bullet(\mathcal{F}) = \bigoplus_{\mathbf{d} \geq \mathbf{0}} \mathcal{F}^{\otimes \mathbf{d}} / \langle (\mathbf{x} \otimes \mathbf{y} - \mathbf{y} \otimes \mathbf{x}) \rangle \tag{1.52}$$

the quotient of the tensor algebra of $\mathcal{F}$ by the two-sided ideal spanned by the commutators $x \otimes y - y \otimes x$. This graded algebra need not be integral. For this reason we replace it by its image $\mathcal{S}$ in $\mathbf{Sym}^\bullet(\mathcal{F}) \otimes_A K$. Therefore $\mathcal{S}$ is an integral graded A-algebra generated by its elements of degree one. We set $\mathbf{P}_{\mathcal{F}} := \mathrm{Proj}(\mathcal{S})$ [34]. This scheme equipped with a canonical (projective) morphism $\pi : \mathbf{P}_{\mathcal{F}} \to X$ is exactly the Grassmann blow-up associated to $\mathcal{F}$. Notice there is a canonical line bundle $\mathcal{O}(1)$ on $\mathbf{P}_{\mathcal{F}}$. Let us record the basic features of this construction.

i) The sheaf $\pi^*(\mathcal{F})/(tors)$ is locally free of rank one, indeed $\pi^*(\mathcal{F})/(tors) = \mathcal{O}(1)$,

ii) Universal property: if $f : Y \to X$ is a dominant morphism, with Y integral, such that $f^*(\mathcal{F})/(tors)$ is locally free of rank one, then f factors uniquely through $\mathbf{P}_{\mathcal{F}}$,

iii) $\mathbf{P}_{\mathcal{F}}$ is an integral scheme, and π is an isomorphism over the regular locus of $\mathcal{J}$.

Property ii) is better explained by a commutative diagram

$$\begin{tikzcd}
\mathbf{P}_{cJ} \arrow[r, "\pi"] & X \\
\mathbf{P}_{f^*(\mathcal{J})/(tors)} \arrow[u, "F"] \arrow[r, "\sim"] & Y \arrow[u, "f"']
\end{tikzcd}$$

where F is the morphism induced by f.

Exercise 1.6. *Let M be a finitely generated module over a noetherian ring A. Prove that M is locally generated by r elements if and only if $F_s(M) = A$ for all $s \geq r$. If furthermore $F_k(M) = 0$ when $k < r$, then M is locally free of rank r.*

Exercise 1.7. *Let $M = P \oplus Q$. Prove that $F_k(M) = \sum_{j=0}^{k} F_j(P) F_{k-j}(Q)$.*

2. Welcome to the punctual Hilbert scheme

In this section the punctual Hilbert functor is defined, and shown to be representable.

2.1. **The punctual Hilbert scheme: definition and construction.**

2.1.1. *The definition.* Let X be an arbitrary scheme, and let us fix an integer $n \geq 1$. The punctual Hilbert scheme is defined by means of its functor of points:

Definition 2.1. *The functor $\mathcal{H}_{X,n} : \mathbf{Sch} \to \mathbf{Ens}$, called the punctual Hilbert functor of degree n of X, is the contravariant functor such that*

$$\mathcal{H}_{X,n}(S) = \{Z \subset X \times S,\ Z \text{ is finite flat and surjective of degree } n \text{ over } S\} \tag{2.1}$$

If $f : T \to S$ is a morphism, the map $\mathcal{H}_{X,n}(S) \to \mathcal{H}_{X,n}(T)$ is the pull-back i.e. the fiber product $Z \mapsto (1 \times f)^{-1}(Z) = Z \times_S T$.

Let $p : X \times S \to S$ denote the projection. A closed subscheme $Z \subset X \times S$ is flat (resp. finite) over S if the restriction $p_Z : Z \to S$ is a flat (resp. finite) morphism. Thus under the assumptions of Definition 2.1 the morphism $p : Z \to S$ is finite flat surjective, equivalently, the $\mathcal{O}_S$ coherent sheaf $p_*(\mathcal{O}_Z)$ is locally free of constant rank n. The definition makes sense since both properties, finiteness and flatness, are preserved by base change. Notice that a finite morphism is affine [34], so if $S = \operatorname{Spec} A$, then $Z = \operatorname{Spec} B$, where B is a finitely generated projective module of constant rank n.

The main result of this section is:

Theorem 2.2. *The Hilbert functor $\mathcal{H}_{X,n}$ is a scheme $\mathbf{H}_{X,n}$, the degree n punctual Hilbert scheme.*

The proof will be given below. Assuming Theorem 2.2, the identity map $1_{\mathbf{H}_{X,n}}$ corresponds to a subscheme $\mathcal{Z} \subset X \times \mathbf{H}_{X,n}$ finite and flat over $\mathbf{H}_{X,n}$, the so-called *universal subscheme*, i.e.

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[dr] & X \times \mathbf{H}_{X,n} \arrow[d, "p"] \\ & \mathbf{H}_{X,n} \end{tikzcd} \tag{2.2}$$

As explained in the previous section, this means that any $Z \in \mathcal{H}_{X,n}(S)$ comes from $\mathcal{Z}$ by pullback: $Z = (1 \times f)^*(\mathcal{Z})$ for a unique morphism $f : S \to \mathbf{H}_{X,n}$. We may call f the classifying map of the subscheme Z. Assuming $k = \overline{k}$, there is a natural bijection between the closed points of $\mathbf{H}_{X,n}$ and the finite subschemes $Z \subset X$ with $\dim \mathcal{O}_Z = n$. The bijection is

$$q \in \mathbf{H}_{X,n} \mapsto Z_q = \mathcal{Z} \cap (X \times \{q\}) \tag{2.3}$$

Such a subscheme can be non-reduced, and clearly its reduced subscheme Z_{red} has no more than n distinct points, strictly less than n if non-reduced. It will be convenient to call a finite subscheme of degree n of X a *cluster* of degree n of X, or in short an n-cluster. Let $|Z| = \{x_1, \cdots, x_d\}$ the support of the subscheme Z. Since Z is the spectrum of a finite dimensional k-algebra we have $\Gamma(Z, \mathcal{O}_Z) = \oplus_{i=1}^d \mathcal{O}_{Z,x_i}$, and $n = \dim_k \Gamma(Z, \mathcal{O}_Z) = \sum_i \dim_k \mathcal{O}_{Z,x_i}$. Call the dimension $\ell_{Z,x_i} = \dim_k \mathcal{O}_{Z,x_i}$ the length of Z at x_i.

Definition 2.3. *The 0-cycle associated to the n-cluster Z is*

$$[Z] = \sum_{x \in |Z|} \ell_{Z,x} x \tag{2.4}$$

Assume $k = \overline{k}$, and X quasi-projective. The most natural examples of n-clusters, are obviously the reduced subschemes of degree n, that is, the collections of n unordered distinct points. If $x_1, \cdots, x_n$ are n distinct points, the associated cluster will be denoted $Z = \sum_{i=1}^{n} x_i$. In this case

$$\mathcal{O}_Z = \prod_{i=1}^{n} k(x_i) = k^n$$

is a reduced algebra. It is not difficult to parameterize this set of reduced n-clusters. Indeed let X_*^n the open subset of X^n locus of points $x = (x_1, \cdots, x_n)$ with $x_i \neq x_j$ if $i \neq j$ i.e. the open stratum of the symmetric product. Then $X_*^n/\mathbf{S}_n$ parameterizes the reduced n-clusters of X. The corresponding universal object $\mathcal{Z}^* \subset X \times X_*^n/\mathbf{S}_n$ as previously explained is obtained as follows. Let $\Delta \subset X \times X^n$ the closed subscheme whose closed points are $(x, x_1, \cdots, x_n)$, such that for some i, $x = x_i$. Denote π the projection on X^n. Clearly the group $\mathbf{S}_n$ acts on Δ, then we set $\mathcal{Z}^* = \Delta/\mathbf{S}_n$. This is a closed subscheme of $X_*^n/\mathbf{S}_n$. The morphism π induces a morphism $p^* : \mathcal{Z}^* \to X_*^n$. We have a commutative diagram

$$\begin{tikzcd} \Delta \arrow[r, "\pi"] \arrow[d] & X_*^n \arrow[d] \\ \mathcal{Z}^* \arrow[r, "p^*"] & X_*^n/\mathbf{S}_n \end{tikzcd}$$

with vertical arrows being the quotient morphisms. Since $\mathbf{S}_n$ acts freely on both sides, the vertical maps are etale. The morphism π is certainly etale surjective, so p^* is etale surjective.

Proposition 2.4. *The scheme $X_*^{(n)} := X_*^n/\mathbf{S}_n$ (the open stratum of $X^{(n)}$) parameterizes the reduced n-clusters. In other words, for any $Z \subset X \times S$, such that $p : Z \to S$ is etale surjective of degree n, there exists a unique morphism $S \to X_*^{(n)}$ with $Z = (1 \times f)^*(\mathcal{Z}^*)$.*

Proof:
Let $W \hookrightarrow X \times S \to S$ a reduced n-cluster over S. Then W is finite etale of degree n onto S. Assume first the covering $W \to S$ is trivial, that is, one can find n disjoint sections $P_i : S \mapsto W$. Define a map $f : S \to X_*^{(n)}$ as $f = \sum_{i=1}^{n} P_i$. Notice this map is independent of the labelling of the $P_i's$. The claim is $W = f^*(\mathcal{Z}^*)$. Denote by $g : W \to X$ the projection on the first factor. Then we have a commutative diagram

$$\begin{tikzcd} \mathcal{Z}^* \arrow[r, hook] & X \times X_*^{(n)} \\ W \arrow[u, "(g,f)"] \arrow[r, hook] & X \times S \arrow[u, "(1,f)"'] \end{tikzcd}$$

It is easy to see this diagram is cartesian. Indeed it defines a morphism $W \longrightarrow (1, f)^*(\mathcal{Z}^*)$ which is both etale and a closed immersion, so an isomorphism. Now in the general case we can choose a finite etale morphism $\varphi : S^* \to S$ such that pulling back to S^*, the covering $W \to S$ is trivialized. This gives us a well-defined classifying morphism $f^* : S^* \to X_*^{(n)}$. It suffices to check that f^* descents to S. This is a typical question of etale descent[*] for which we refer to [58]. One must prove the following fact. Let $h : \tilde{S} \to S^*$ be an etale surjective morphism, then if $\tilde{f}$ denote the classifying map associated to φh, then $\tilde{f} = f^* h$. But this follows from the uniqueness of the classifying map. □

The property to be *reduced* is open, so $X_*^{(n)}$ embeds as an open subset $\mathbf{H}^0_{X,n}$ in $\mathbf{H}_{X,n}$. This is the *easy part* of $\mathbf{H}_{X,n}$.

2.1.2. *Construction: reduction to the affine case.* Let us start the construction of the punctual Hilbert scheme. The construction splits in two parts. We first reduce the problem to the affine case, and then contruct by hand the punctual Hilbert scheme of an affine scheme. As we shall see in some special cases, e.g. the affine plane, there are nice relationship between the punctual Hilbert scheme and some quiver moduli varieties as introduced in Brion's lectures [11].

Proposition 1.1 tells us that $\mathcal{H}_{X,n}$ is representable iff we can find a covering of this functor by a family of representable open subfunctors. Let $U \subset X$ be an open subset. For any $Z \in \mathcal{H}_{n,U}$, since Z is finite over S, hence proper, it follows that the immersion $Z \hookrightarrow X \times S$ is proper and hence closed ([34], corollary 4.8).

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[dr] & U \times S \subset X \times S \arrow[d, "p"] \\ & S \end{tikzcd}$$

showing $Z \in \mathcal{H}_{n,X}$. This defines a morphism of functors

$$\mathcal{H}_{n,U} \hookrightarrow \mathcal{H}_{n,X} \tag{2.5}$$

Lemma 2.5. *The functorial morphism (2.5) is an open immersion.*

Proof:
Let $F = X - U$. Let Z be an S-point of $\mathcal{H}_{n,X}$. Since $p : Z \to S$ is finite, the subset $B = p(Z \cap F \times S) \subset S$ is closed. Let $f : S' \to S$ be a morphism. Then the pull-back $f^*(Z)$ is a subscheme of $U \times S'$ if and only if $f(S')$ is disjoint from the closed set B, that is, f factors through the open subset $S - B$.

□

Let $(U_i)_{i \in I}$ be the family of affine open subsets of X.

Lemma 2.6. *The functors $\mathcal{H}_{n,U_i} \hookrightarrow \mathcal{H}_{n,X}$ define a covering of $\mathcal{H}_{n,X}$, i.e. the representable morphism $\coprod_{i \in I} \mathcal{H}_{n,U_i} \longrightarrow \mathcal{H}_{n,X}$ is surjective.*

Proof:
This amounts to check that any $Z \in \mathcal{H}_{n,X}(S)$ comes from some $\mathcal{H}_{n,U_i}(S)$ locally on S. Let $s \in S$ be an arbitrary point. The quasi-projectivity of X tells us there is an affine open subset U_i such that the fiber Z_s of $p : Z \to S$ at s is included in $U_i \times S$,

[*] Alternatively, one can choose the etale cover to be galois, say with group Γ, and then factors out by Γ

i.e is a subscheme of $U_i \times S$. The projection $p(Z \cap ((X - U_i) \times S))$ is a closed subset of S. Denoting V the complementary open subset, it is readily seen that

$$Z \cap (X \times V) \subset U_i \times V$$

The lemma is proved. □

Exercise 2.1. *Using the fact that X is quasi-projective, use noetherian induction to show that we can find a covering of X by finitely many affine open subsets U_i, such that any n-uple of points of X is contained in one of the $U_i's$.*

2.1.3. *The affine case.* In this subsection we prove the main theorem in the affine case. In the remaining part of this subsection the hypotheses are as follows. Let A be a base ring, not necessarily an algebra of finite type over a field, even not noetherian. Let R be a commutative algebra over A with unit 1. It is convenient to assume R is a free A-module (with arbitrary rank, finite or not), a mild restriction. With more care it is possible to drop the freeness assumption [28]. The (local) Hilbert functor $\mathcal{H}_{n,R/A}$ is the following. Let $\mathbf{A} - Alg$ be the category of (commutative) A-algebras.

Definition 2.7. *The functor $\mathcal{H}_{n,R/A}$ is the covariant functor on the category $\mathbf{A} - Alg$ such that*

$$\mathcal{H}_{n,R/A}(B) = \{\alpha : R \otimes_A B \to E\}/\cong \tag{2.6}$$

that is the set of isomorphism classes of surjective B-algebra morphisms from $R \otimes_A B$ to an algebra E which as a B-module is locally free of rank n.

In the remaining of this subsection we shall fix a basis $((\nu_\mu)_{\mu \in L})$ of the A-module R. We shall assume the unit 1 is a distinguished element ν_1 of the basis. The proof uses once more the criterion 1.1. What we need to do amounts to find a covering of $\mathcal{H}_{n,R/A}$ by representable open subfunctors. If we remove the algebra structure on E then we obtain the functor $\mathcal{Q}_{n,R/A}$ of Grothendieck classifying the locally free quotient A-modules of R of rank n, a kind of Grassmann functor. The Hilbert functor will appear as a closed subfunctor of $\mathcal{Q}_{n,R/A}$. This suggests that to find a cover by affine open subschemes one has to fix a linear map of A-modules $\beta : F = A^n \to R$ and to consider the subfunctor $\mathcal{H}^\beta_{n,R/A}$ parameterizing the quotient A-algebras $\alpha : R \to E$ such that $\ker \alpha \oplus \mathrm{Im}\beta = R$. More precisely, let $F = A^n$ be a free module of rank n, with fixed basis $(e_i)_{1 \le i \le n}$. For any A-linear map $\beta : F \to R$, with $\beta(e_1) = 1$, let us define a subset of $\mathcal{H}_{n,R/A}(B)$

$$\mathcal{H}^\beta_{n,R/A}(B) = \{\alpha : R \otimes_A B \to E,\ \alpha(\beta \otimes 1) = \text{i}somorphism\}/\cong \tag{2.7}$$

that is

$$\begin{tikzcd} F \otimes_A B \arrow[r, "\beta\otimes 1"] \arrow[rr, bend right, "\cong"'] & R \otimes_A B \arrow[r, "\alpha"] & E \end{tikzcd}$$

If $[R \otimes_A B \xrightarrow{\alpha} E]$ represents an object of $\mathcal{H}^\beta_{n,R/A}$, then E must be free. Notice we get a functor isomorphic to the previous one by requiring $E = B^n$ and $\alpha(\beta \otimes 1) = 1$. In the remaining of this subsection this restriction will be assumed.

The main result is:

Theorem 2.8. *The subfunctor $\mathcal{H}^{\beta}_{n,R/A}$ is open in $\mathcal{H}_{n,R/A}$, and representable by an affine scheme. If β runs over the linear maps $F = A^n \to R$ with $\beta(e_1) = 1$, the $(\mathcal{H}^{\beta}_{n,R/A})_{\beta:F\to R}$ define an open cover of $\mathcal{H}_{n,R/A}$. Finally $\mathcal{H}_{n,R/A}$ is representable.*

Proof:
Let the linear map β be defined by the matrix $(a_{i\mu})$ with entries in A such that

$$\beta(e_i) = \sum_{\mu} a_{i\mu}\nu_{\mu}, \quad (\beta(e_1) = 1 = \nu_1)$$

To define the map α amounts to defining a matrix $(\nu_{\mu i})$ with entries in B such that

$$\alpha(\nu_{\mu}\otimes 1) = \sum_{i} \nu_{\mu i} e_i, \quad \nu_{1i} = \delta_{1,i}$$

The equality $\alpha(\beta\otimes 1) = 1$ then translates as

$$\sum_{\mu} a_{i\mu}\nu_{\mu j} = \delta_{i,j} \tag{2.8}$$

This shows the data $(\nu_{\mu i})$ defines a B- valued point of the affine scheme

$$\operatorname{Spec} A[T_{\mu i}]/(\mathcal{J}) \tag{2.9}$$

where $\mathcal{J}$ denotes the ideal generated by the equations 2.8. The last condition that we must implement is that $\ker\alpha$ is an ideal of $R\otimes_A B$. It is readily seen that the elements $\nu_{\mu} - (\beta\otimes 1)\alpha(\nu_{\mu})$ generate the B-module $\ker\alpha$. As a consequence $\ker\alpha$ is an ideal if and only if for all $\lambda\in L$ one has

$$\alpha\left(\nu_{\lambda}(\nu_{\mu} - (\beta\otimes 1)\alpha(\nu_{\mu}))\right) = 0 \tag{2.10}$$

This will translate into a system of equations between the coordinates $\nu_{\mu i}$, for this we need the structure constants of the algebra structure of R. We set

$$\nu_{\lambda}\nu_{\mu} = \sum_{\delta\in L} b^{\delta}_{\lambda\mu}\nu_{\delta} \ (b^{\delta}_{\lambda\mu}\in A)$$

Then the equations (2.10) are equivalent to the system of quadratic equations

$$(\forall j\in[1,n]) \ \sum_{\delta} c^{\delta}_{\lambda\mu}x_{\delta j} - \sum_{i,j,\gamma} a_{i\gamma}c^{\delta}_{\lambda\gamma}x_{\mu i}x_{\delta j} = 0 \tag{2.11}$$

We then see that $\mathcal{H}^{\beta}_{n,R/A}$ is represented by the closed subscheme $H^{\beta}_{n,R/A}\subset \operatorname{Spec} A[\{T_{\mu i}\}]$ defined by the equations (2.8, 2.11). This conclude the proof of representability of the punctual Hilbert functor.

□

The exercises below suggest some variant of the punctual Hilbert scheme*.

Exercise 2.2. *(**The punctual Quot scheme**). With the same hypothesis as before, let X be a quasi-projective scheme. Prove there is a scheme $\mathrm{Quot}_{n,d,X}$ whose closed points are the quotients (up to isomorphism) $\mathcal{O}^d_X \to F$, where F is a coherent scheme of finite length n.*

*For a non-commutative version, see the paper by Vaccarino, at this school [60]

Exercise 2.3. *(***The moduli space of based commutative algebras***) Let $n \in \mathbb{N}$. Define a functor $\mathbf{A}l_n : \mathbf{Sch} \to \mathbf{Ens}$ as follows: if $S \in \mathbf{Sch}$, an element of $\mathbf{A}l_n(S)$ is an isomorphism class of pairs $(\mathcal{A}, \varphi)$ where $\mathcal{A}$ is an $\mathcal{O}_S$-algebra (commutative with unit), and $\varphi : \mathcal{O}_S^n \xrightarrow{\sim} \mathcal{A}$ is an $\mathcal{O}_S$-module isomophism. Two pairs $(\mathcal{A}, \varphi)$ and $(\mathcal{A}', \varphi')$ are isomorphic if there is an algebra isomorphism $\alpha : \mathcal{A} \xrightarrow{\sim} \mathcal{A}'$ such that $\alpha\varphi = \varphi'$. At the level of morphisms, the functor $\mathbf{A}l_n$ is the pull-back. Then show $\mathbf{A}l_n$ is representable by an affine scheme of finite type over $\mathbb{Z}$.*
Derive in a different way this result by finding a relationship with a suitable punctual Hilbert scheme (for details see Poonen [49]).

We can draw two immediate corollaries from the previous proof.

Corollary 2.9. *Let $Y \subset X$ be a closed (resp. open) subscheme, then $\mathbf{H}_{n,Y} \subset \mathbf{H}_{n,X}$ is a closed (resp. open) subscheme.*

Proof:
For an open subscheme this was part on the previous proof. Let us assume Y is closed in X. Since an open cover of X yields an open cover of $\mathbf{H}_{n,X}$, we may assume without loss of generality that X is affine. Indeed with the notations of step 2 (2.1.3), if $J \subset R$ is an ideal such that $\overline{R} := R/J$ is free over A, then $\mathbf{H}_{n,\overline{R}/A}$ is a closed subscheme of $\mathbf{H}_{n,R/A}$. Using the affine cover $\mathbf{H}^{\beta}_{n,R/A}$ (2.1.3), this amounts to check that $\mathbf{H}_{n,\overline{R}/A} \cap \mathbf{H}^{\beta}_{n,R/A}$ is a closed subscheme of $\mathbf{H}^{\beta}_{n,R/A}$. Keeping the same notations as in section (2.1.3), it is readily seen that if $\alpha : R \otimes_A B \to B^n$ yields a point of $\mathbf{H}_{n,\overline{R}/A}$ if and only if $\alpha(J) = 0$. Taking a system of generators $(f_k = \sum_\lambda y_{k,\lambda}\nu_\lambda)_k$ of J, this condition can be translated in a system of linear equations

$$\sum_{\lambda} f_{k,\lambda}\nu_{\lambda,i} = 0 \quad (\forall k, i) \tag{2.12}$$

The conclusion is clear.

□

It is useful to extend somewhat the basic construction, and try to classify the pairs (Z_1, Z_2) of clusters such that $Z_2 \subset Z_1$, i.e Z_2 is a subscheme of Z_1.

Proposition 2.10. *Let $n_1, n_2 \geq 1$. The subset of points $(Z_1, Z_2) \in \mathbf{H}_{n_1,X} \times \mathbf{H}_{n_2,X}$ such that Z_2 is a closed subscheme of Z_1, is a closed subscheme $\mathbf{H}_{n_1,n_2,X}$, the so-called incidence subscheme.*

Proof:
As in our construction of $\mathbf{H}_{n,X}$, we may reduce to X being affine. Then the description of $\mathbf{H}_{n,X}$ given before show we can reduce further to the open affine pieces $\mathcal{H}^{\beta}_{n,R/A}$. The notations being the same as in (2.7), the conditions that Z_2 is a subscheme of Z_1 then reads $\alpha_2(\ker(\alpha_1)) = 0$. With the coordinates introduced in the proof of Theorem 2.8, for both α_1 and α_2, we easily see that this last condition yields a system of equations between these coordinates.

□

We can give a slightly different proof using the exercice below.

Exercise 2.4. *Let $p : Z \to Y$ be a finite flat morphism of degree n_1. If $n_2 < n_1$, show there is a Y-scheme $H_{n_2,Z}$ which parameterises the closed subschemes of Z which are flat of degree n_2 over Y.*

The exercise below shows $\mathbf{H}_{n,X}$ is separated, so really a scheme.

Exercise 2.5. *Show that the valuative criterion of separatedness holds for $\mathcal{H}_{n,X}$.*

2.1.4. *The affine plane $\mathbb{A}^2$.* We now want to apply the general method previously explained to describe charts on the punctual Hilbert scheme, and then explicit coordinates, in a non trivial example. The choice of $R = k[X, Y]$ the polynomial algebra in two indeterminates, is very important both for the applications, as we shall see, but also as a toy model. To start with, the natural choice of a k-basis of R is the set of monomials X^pY^q, $(p, q) \in \mathbb{N}^2$. In order to simplify the notations, let us denote for a moment $\mathbf{H}_n$ what is called $\mathbf{H}_{n,k[X,Y]/k}$. We see $\mathbf{H}_n$ either as the set of ideals of codimension n, or as the set of subschemes of length n. The previous construction gives a method to get a covering of $\mathbf{H}_n$ by affine open subsets. Let $M \subset \mathbb{N}^2$ be a subset with cardinal n. Then set

$$U_M = \{I \subset k[X, Y],\ \oplus_{(p,q)\in M} kX^pY^q \xrightarrow{\sim} k[X, Y]/I\} \tag{2.13}$$

Denote x^py^q the image of X^pY^q in $k[X, Y]/I$. Then for all $(r, s) \notin M$ and $(p, q) \in M$, we have a set of well-defined constants $c^{rs}_{pq} \in k$, such that

$$X^rY^s = \sum_{(p,q)\in M} c^{r,s}_{p,q} X^pY^q \pmod I \tag{2.14}$$

It is convenient to assume that $c^{r,s}_{p,q}$ exists for all (r, s), but if $(r, s) \in M$ then $c^{r,s}_{p,q} = 0$ if $(p, q) \neq (r, s)$ and $c^{r,s}_{r,s} = 1$. The fact that I must be an ideal amounts to the two conditions $XI \subset I$, and $YI \subset I$. Indeed for any $(\alpha, \beta) \in \mathbb{N}^2$ making the product of both members of 2.14 by $X^\alpha Y^\beta$ yields first the relation

$$X^{r+\alpha}Y^{s+\beta} = \sum_{(k,l)} c^{r,s}_{p,q} c^{p+\alpha,q+\beta}_{k,l} X^kY^l \pmod I \tag{2.15}$$

and then expanding the left-hand side, we get the system of quadratic equations

$$(\forall (r, s), (\alpha, \beta) \in \mathbb{N}^2) \qquad \sum_{(p,q)\in M} c^{r,s}_{p,q} c^{p+\alpha,q+\beta}_{k,l} = c^{r+\alpha,s+\beta}_{k,l} \tag{2.16}$$

Specializing (α, β) to $(1, 0)$, or $(0, 1)$, 2.16 becomes equivalent to

$$\left\{ \begin{array}{l} c^{r+1,s}_{k,l} = \sum_{(p,q)\in M} c^{r,s}_{p,q} c^{p+1,q}_{k,l} \\ c^{r,s+1}_{k,l} = \sum_{(p,q)\in M} c^{r,s}_{p,q} c^{p,q+1}_{k,l} \end{array} \right. \tag{2.17}$$

Conversely it is easily seen that the equations (2.17) ensure that the vector space I spanned by the elements $X^rY^s - \sum_{(p,q)\in M} c^{rs}_{pq} X^pY^s$ is an ideal. Thus we get a very explicit affine open covering of $\mathbf{H}_{n,\mathbb{A}^2}$, which will be used later.

2.1.5. *Local structure of the Hilbert scheme.* Once the scheme $\mathbf{H}_{n,X}$ constructed, it is natural to ask about its structure, either in global terms, or local terms. The most natural question would be to describe the local ring $\mathcal{O}_{\mathbf{H}_{n,X},Z}$ at the point Z. This is not easy. We are able to answer a weaker question, viz. describe the tangent space at a point Z. I want to remind you that the tangent space of a scheme $X \in Sch$ at a point $x \in X$ with residue field $k(x)$, not necessarily k-rational, is

$$\mathbf{T}_{X,x} = \mathrm{Hom}_{k(x)}(\mathcal{M}_x/\mathcal{M}_x^2)^* \tag{2.18}$$

Let A be a ring. We set $A[\epsilon] = A[X]/(X^2)$ ($\epsilon = \overline{X}$) the A-algebra of dual numbers. Thus $A[\epsilon] = A \oplus A\epsilon$ since $\epsilon^2 = 0$. Assuming $k(x) = k$, e.g. x is rational, the tangent space admits an alternative description

$$\mathbf{T}_{X,x} = \mathrm{Hom}_{k-alg}(\mathcal{O}_x, k[\epsilon]) = X_x(k[\epsilon]) \tag{2.19}$$

the set of $k[\epsilon]$-points of X over x. Now $X = \mathbf{H}_{n,X}$ and $Z \subset X$ is an n-cluster. We denote by $\mathcal{I}_Z$ the ideal sheaf of the closed subscheme Z, that is $\mathcal{O}_Z = \mathcal{O}_X/\mathcal{I}_Z$.

Proposition 2.11. *The tangent space $\mathbf{T}_Z$ to $\mathbf{H}_{x,X}$ at Z is*

$$\mathbf{T}_Z = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{I}_Z, \mathcal{O}_Z) \tag{2.20}$$

Proof:
We saw the Hilbert scheme $\mathbf{H}_{n,X}$ can be covered by open subsets $\mathbf{H}_{n,U}$ with $U \subset X$ open, then if $Z \in \mathbf{H}_{n,U}$ we can restrict ourselves to X affine, i.e. to the local setting R/A of section (2.1.3). The problem translates more generally as follows: to describe the clusters $\mathcal{Z} \subset R[\epsilon]$ which reduce modulo ϵ to $Z \subset \mathrm{Spec}\, R$. Let Z be given by the ideal $I \subset R$. Recall R/I is flat over A, this implies that I is flat.

Proposition 2.12. *There is a one-to-one correspondance between the set of liftings to Z in $\mathbf{H}_{n,R/A}(A[\epsilon])$, and* $\mathrm{Hom}_R(I, R/I)$.

Proof:
We have to describe all ideals $\mathcal{I} \subset R[\epsilon]$ such that i) $\mathcal{I}$ is flat over $A[\epsilon]$ and ii) $\mathcal{I} + \epsilon R[\epsilon]/\epsilon R[\epsilon] = I$.

Notice if i) holds then it is easy to see that

$$\mathcal{I} \cap \epsilon R[\epsilon] = \epsilon\mathcal{I} \cong \epsilon R[\epsilon] \otimes_{A[\epsilon]} \mathcal{I}, \tag{2.21}$$

in this way ii) translates as $\mathcal{I}/\epsilon\mathcal{I} \cong I$. It is an interesting fact, that conversely (2.21) implies the flatness of $\mathcal{I}$. This is the content of (a very particular case) of the local criterion of flatness* ([16], Theorem 6.8, cor 6.9). In our setting the criterion is as follows:

- $\mathcal{I}$ is flat over $A[\epsilon] \iff \mathcal{I}/\epsilon\mathcal{I}$ is flat over A, and the canonical surjective map

$$\epsilon A[\epsilon] \otimes_{A[\epsilon]} \mathcal{I} \to \epsilon\mathcal{I}$$

is bijective, i.e injective. Coming back to our problem, given $\mathcal{I}$ we define a map

$$\varphi : I \longrightarrow R/I \tag{2.22}$$

*In Eisenbud's book the local criterion of flatness is stated in the local case, but the same proof, even simpler, shows the result holds true with a nilpotent ideal instead of the maximal ideal.

as follows. If $a \in I$, we can lift a to $x = a + b\epsilon \in \mathcal{I}$. Then we set $\varphi(a) = \bar{b} \in R/I$. This is well-defined because $a = 0 \Longrightarrow b \in I$, so $\bar{b} = 0$. It is readily seen that φ is R-linear. We can reconstruct $\mathcal{I}$ from φ as follows

$$\mathcal{I} = \{x = a + b\epsilon,\ a \in I, \bar{b} = \varphi(a)\} \tag{2.23}$$

2.1.6. *Global structure.* A punctual Hilbert scheme is in general a dramatically complicated object. It is even in simple examples, highly singular, even not irreducible, nor equidimensional. Such example has been provided by A. Iarrobino ([36]). However they share some basic global properties. A feedback of our construction of $\mathbf{H}_{n,X}$ is the projectivity property. Recall [34] a scheme is projective if it is a closed subscheme of some projective space.

Proposition 2.13. *Let us assume X is projective, then $\mathbf{H}_{n,X}$ is projective.*

Proof:
If $X \subset \mathbf{P}^N$ then $\mathbf{H}_{n,X}$ is a closed subscheme of $\mathbf{H}_{n,\mathbf{P}^N}$, thus we may assume $X = \mathbf{P}^N$. Let $\mathcal{O}(1)$ be the tautological line bundle on $\mathbf{P}^N$ with global sections of $\mathcal{O}(k) = \mathcal{O}(1)^{\otimes k}$ identified to $\Gamma(\mathbf{P}^N, \mathcal{O}(k)) = k[X_0, \cdots, X_N]_k$, the vector space of homogeneous polynomials of degree k [34]. Let $Z \subset \mathbf{P}^N$ be a cluster of degree n.

Lemma 2.14. *There is an integer d depending only on n, N such that for all n-clusters Z, the restriction map*

$$\Gamma(\mathbf{P}^N, \mathcal{O}(n)) \longrightarrow \Gamma(Z, \mathcal{O}_Z(n)) = \Gamma(Z, \mathcal{O}_Z) \tag{2.24}$$

is onto.

Proof:
We may obviously assume that the support of Z lies in the open affine subset $X_0 \neq 0$. The above map amounts to

$$R = k[x_1, \cdots, x_N]_{\leq d} \longrightarrow \mathcal{O}_{Z,0} = R/I$$

where the left hand side means the vector space of polynomials of degree least or equal to d, and $I = I_Z$. We proceed by induction on n and N. There is no loss of generality to assume that $Z \cap \{X_N = 0\} \neq \emptyset$. We have an exact sequence

$$0 \to \frac{I + (X_N)}{I} \to \frac{R}{I} \to \frac{R}{I + (X_N)} \to 0 \tag{2.25}$$

There is an ideal I' such that

$$\frac{R}{I + (X_N)} = \frac{k[X_1, \cdots, X_{N-1}]}{I'}$$

then $1 \leq \dim \frac{R}{I'} < n = \dim \frac{R}{I}$. Likewise

$$\dim \frac{I + (X_N)}{I} = \dim \frac{R}{X_N^{-1}(I \cap (X_N))} < n$$

The conclusion follows from our inductive assuption.

□

As a consequence any n-cluster $Z \subset \mathbf{P}^N$ yields a well-defined surjective map $V = \Gamma(\mathbf{P}^N, \mathcal{O}(n)) \longrightarrow \Gamma(Z, \mathcal{O}_Z)$, that is a point of the Grassmannian $\mathbf{G}_{d,n}$ where $d = \dim V$. It not difficult to check that this defines a closed embedding of $\mathbf{H}_{n,X}$ into the Grassmannian. Since the Grassmannian is naturally embedded in a projective space, via the Plücker embedding, this shows $\mathbf{H}_{n,X}$ is a projective scheme.

$\square$

One problem concerning clusters is the lack of a well-defined addition, contrary to 0-cycles. Indeed if $n = n_1 + n_2$ $(n_i \geq 1)$, there is an obvious sum morphism

$$X^{(n_1} \times X^{(n_2)} \to X^{(n)}, \; (\xi_1, \xi_2) \mapsto \xi_1 + \xi_2$$

Such an operation only exists at the Hilbert scheme level under a strong restriction. Namely, let $(Z_1, Z_2) \in \mathbf{H}_{n_1,X} \times \mathbf{H}_{n_2,X}$. We may assume both clusters living in an open affine subset $U = \operatorname{Spec} R \subset X$. Let I_i $(i = 1, 2)$ be the ideal of Z_i. The natural counterpart of the sum of 0-cycles, should be either $I_1.I_2$ or $I_1 \cap I_2$. The difficulty is even if any of these two ideals leads to a subscheme with support $|Z_1| \cup |Z_2|$, the length may be wrong. If one imposes that Z_1 and Z_2 are disjoint, i.e. $I_1 + I_2 = R$, then $I_1.I_2 = I_1 \cap I_2$ defines the subscheme $Z_1 + Z_2$, and the difficulty disappears. The reason is the Chinese Remainder Theorem which yields the isomorphism

$$R/I_1 \cap I_2 \xrightarrow{\sim} R/I_1 \oplus R/I_2$$

The result of this construction is the partially defined sum morphism

$$+ : (\mathbf{H}_{n_1,X} \times \mathbf{H}_{n_2,X})_0 \longrightarrow \mathbf{H}_{n,X}, \quad (Z_1, Z_2) \mapsto Z_1 + Z_2 \tag{2.26}$$

where the subscript means the domain of the morphism is the subset of pairs (Z_1, Z_2) with $|Z_1| \cap |Z_2| = \emptyset$. This morphism plays an important role in the work of Nakajima about the cohomology of the punctual Hilbert scheme.

Let $\mathcal{Z} \subset X \times \mathbf{H}_{n,X}$ be the universal n-cluster. It provides us with a rank-n vector bundle (locally free sheaf of rank n) on $\mathbf{H}_{n,X}$, namely the direct image

$$\mathbb{E} := p_*(\mathcal{O}_{\mathcal{Z}}) \tag{2.27}$$

as $\mathcal{O}_{\mathbf{H}_{n,X}}$-module.

Definition 2.15. *We will refer to the bundle $\mathbb{E}$ as the universal bundle on $\mathbf{H}_{n,X}$.*

Exercise 2.6. *Show independently of Proposition 2.13 that if X is projective, then the valuative criterion of properness holds true for $\mathbf{H}_{n,X}$.*

2.2. **The Hilbert-Chow morphism.** We keep the same hypothesis as in the previous section, in particular $k = \overline{k}$. We saw (proposition 2.4) that the reduced clusters of degree n of a quasi-projective scheme are parameterized by the open stratum of the symmetric product $X^{(n)}$. It is of fundamental importance to be able to extend this correspondence to a full morphism $\varphi_{n,X} : \mathbf{H}_{n,X} \to X^{(n)}$, the so called *Hilbert-Chow* morphism.

The construction is somewhat intricate due to the difficulty to give a simple definition of the functor of points of the symmetric product ([23],[46]), namely to define flat families of 0-cycles. It should be noted that the restriction on the base field is unnecessary to construct the Hilbert-Chow morphism. Recall that the cycle of a cluster

Z with support $|Z| = \{x_1, \cdots, x_r\}$ is $[Z] = \sum_{i=1}^r n_i x_i$, where n_i is the length of $\mathcal{O}_Z$ at x_i, i.e. $n_i = \dim \mathcal{O}_{Z,x_i}$. The result we are going to prove is:

Theorem 2.16. *Let X be a quasi-projective scheme. Then there is a morphism*

$$\varphi_{n,X} : \mathbf{H}_{n,X} \longrightarrow X^{(n)} \tag{2.28}$$

such that set-theoretically $\varphi_{n,X}(Z) = [Z]$ (the cycle of Z). The morphism φ_X is an isomorphism over the open subset of reduced clusters.

Proof:
To simplify our notations, we can drop the subscripts n, X. The construction of the Hilbert-Chow morphism amounts to build from a cluster $Z \subset X \times S$ a 0-dimensional relative cycle [*] $[Z]$ functorially in S. Let us sketch an idea which goes back to Mumford [46], and further developed by Fogarty in [23], and makes use of a lot of homological algebra. Let us assume to start with that $X = \mathbb{P}^N$ with homogeneous coordinates $(x_0, \cdots, x_N)$. The space of hyperplanes of $\mathbb{P}^N$, the so-called dual projective space is denoted $\check{\mathbb{P}}^N$, with homogeneous coordinates $(\lambda_0, \cdots, \lambda_N)$. To $\lambda \in \check{\mathbb{P}}^n$ is associated the hyperplane $\sum_{i=0}^N \lambda_i x_i = 0$. The incidence correspondance

$$\Sigma = \{(x, H) \in \mathbb{P}^N \times \check{\mathbb{P}}^N,\ x \in H\} \tag{2.29}$$

is the subvariety defined by the equation $\sum_{i=0}^N \lambda_i x_i = 0$. Note the diagram

$$\begin{tikzcd} & \Sigma \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{P}^N & & \check{\mathbb{P}}^N \end{tikzcd} \tag{2.30}$$

where p and q are the projectors on the two factors. If $x \in \mathbb{P}^N$ then $H_x := qp^{-1}(x) \subset \check{\mathbb{P}}^N$ is the hyperplane with equation $\sum_i \lambda_i x_i = 0$. We can extend this and associate to a cluster $Z \subset \mathbb{P}^N$ a hypersurface H_Z (of a very special type). Let us consider the coherent sheaf $\mathcal{F}_Z = q_* p^*(\mathcal{O}_Z)$ clearly supported in codimension one. Namely if $\{p_1, \cdots, p_r\}$ is the support of Z, then the support of $\mathcal{F}_Z$ is $\bigcup_{j=1}^r H_{p_j}$. Therefore, using a finite resolution of $\mathcal{F}_Z$ by locally free sheaves of finite rank we may associate a divisor $H_Z := \mathrm{Div}(\mathcal{F}_Z) \subset \check{\mathbb{P}}^N$ ([46]). We don't give the precise definition of the divisor $\mathrm{Div}(\mathcal{F})$ attached to a torsion coherent sheaf $\mathcal{F}$, details are in [46], [23], but just explain the plausability of this construction. Since the restriction of q to the support of $p^*(\mathcal{O}_\mathcal{Z})$ is finite, then $\mathbf{R}^i q_*(p^*(\mathcal{O}_\mathcal{Z}) = 0$ if $i > 0$. This is the reason why we work with the sheaf $q_* p^*(\mathcal{O}_Z)$ instead of the complex of sheaves $\mathbf{R}^\bullet(p^* \mathcal{O}_Z)$. Furthermore this construction works well in a family $Z \subset \mathbf{P}^N \times S$ over $S \in \mathbf{Sch}$. Assuming $S = \mathrm{Spec}\, k$, if Z is reduced with support at the points $p_1 = (x_0^1, \cdots, x_N^1), \cdots, p_r = (x_0^r, \cdots, x_N^r)$, so that $\mathcal{O}_Z = \oplus_{j=1}^r k(p_j)$, then our previous remark yields

$$\mathrm{Div}(\mathcal{F}_Z) := \{\prod_{j=1}^r \sum_{i=0}^N x_i^j \lambda_i = 0\} \tag{2.31}$$

[*] This is a vaguely defined concept

In the general case, we can find a filtration $\mathcal{H}_0 = 0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_s = \mathcal{O}_Z$ such that for any $k > 0$

$$\mathcal{H}_k/\mathcal{H}_{k-1} \cong \bigoplus_{j=1}^{r} k(p_j)^{e_j} \quad (e_j = 0, 1)$$

The functor Div being additive on exact sequences (loc.cit) it follows

$$\mathrm{Div}\,(\mathcal{F}_Z) = \{\prod_{j=1}^{r}(\sum_{i=0}^{N} x_i^j \lambda_i)^{n_j} = 0\} \tag{2.32}$$

where n_j is the length of Z at p_j, namely $n_j = \dim_k(\mathcal{O}_Z)_{p_j}$. In a sense previously explained, this divisor is attached to the 0-cycle $[Z] = \sum_{j=1}^{r} n_j p_j$ ($\sum_j n_j = n$). The form of the equation (2.32) makes plausible that the result of this construction goes into the n- fold symmetric product $(\mathbf{P}^N)^{(n)}$. As a consequence we get the Hilbert-Chow morphism, but defined only on the reduced Hilbert scheme

$$\varphi : (H_{n,\mathbf{P}^N})_{red} \longrightarrow (\mathbf{P}^N)^{(n)}$$

We are to use now a different and more elementary path to validate the existence of the Hilbert-Chow map φ. It is worth to note this construction is valid in any characteristic. The basic idea is as follows. Let $Z \subset X \times S$ a cluster of degree n over S. The *relative n-fold symmetric product*

$$(Z/S)^{(n)} := \overbrace{Z \times_S Z \times_S \cdots \times_S Z}^{n} / \mathbf{S}_n \tag{2.33}$$

makes sense. If we can build a canonical, i.e. functorial S-point $\imath_Z : S \to (Z/S)^{(n)}$, then this will yield a morphism $S \longrightarrow X^{(n)}$, namely

$$S \xrightarrow{\imath_Z} (Z/S)^{(n)} \to (X \times S)^{(n)} = X^{(n)} \times S \xrightarrow{pr} S \tag{2.34}$$

Applying this to the universal cluster $\mathcal{Z} \subset X \times \mathbf{H}_{n,X}$, we will get a morphism $H_{n,X} \to X^{(n)}$. This morphism is precisely the Hilbert-Chow morphism. Notice this construction is implicit in [27].

Our final task is to prove the existence of the alluded canonical point, and clearly it suffices to check this in the affine case. Indeed, if we cover S by affine open subsets S_i, and if we denote $\pi : Z \to S$ the projection onto S, then $Z_i = \pi^{-1}(S_i)$ is affine. Let denote $\varphi_i : S_i \to (Z_i/S_i)^{(n)}$ the expected canonical point. Then φ_i and φ_j both restrict to the canonical point on $S_i \cap S_j$, thus are equal on $S_i \cap S_j$. Glueing them we get $\varphi : S \to (Z/S)^{n)}$. Therefore we may assume S, and hence Z, affine. We set $Z = \mathrm{Spec}\,R$, $S = \mathrm{Spec}\,A$. Localizing further we may assume R is a free A-algebra of rank n. Let $h_x : R \to R$ be the map $y \mapsto xy$. This yields an A-algebra morphism

$$h : R \to \mathrm{End}_A(R) \cong \mathbf{M}_n(A)$$

As is well known $\det(h_x)$ is called the *Norm* of x over A, and denoted $\mathrm{N}m_{R/A}(x)$. The key technical tool we are using is the *linearized norm*, a by-product of the *linearized determinant* [41].

Proposition 2.17. *Let A be a commutative ring, and let $n \geq 1$ be an integer. The symmetric group acts naturally on the tensor product algebra $M_n(A)^{\otimes n}$ by permutation of the factors, namely*

$$\sigma(M_1 \otimes \cdots \otimes M_n) = M_{\sigma^{-1}(1)} \otimes \cdots \otimes M_{\sigma^{-1}(n)} \tag{2.35}$$

Let $(M_n(A)^{\otimes n})^{\mathbf{S}_n}$ be the subring of symmetric elements. Then there exists a unique A-algebra morphism

$$\mathrm{L}det : (M_n(A)^{\otimes n})^{\mathbf{S}_n} \longrightarrow A \tag{2.36}$$

such that $\mathrm{L}det(M \otimes \cdots \otimes M) = \det M$. This morphism (the "the linearized determinant") commutes with an arbitrary base change.

Proof:
The proof is easy if we assume $n!$ invertible in A. In that case we have the standard projector onto the invariant subring $(M_n(A)^{\otimes n})^{\mathbf{S}_n}$, viz.

$$\mathcal{S} : M_n(A)^{\otimes n} \to (M_n(A)^{\otimes n})^{\mathbf{S}_n},\ \mathcal{S}(a_1 \otimes \cdots \otimes a_n) = \frac{1}{n!}\sum_{\sigma \in \mathbf{S}_n} a_{\sigma(1)} \otimes \cdots a_{\sigma(n)}$$

Let us denote M_i^j the j-th column of $M_i \in M_n(A)$. Define a linear map $M_n(A)^{\otimes n} \longrightarrow A$ by

$$M_1 \otimes \cdots \otimes M_n \mapsto \det(M_1^1, \cdots, M_n^n)$$

Call $\mathrm{L}det$ its restriction to $(M_n(A)^{\otimes n})^{\mathbf{S}_n}$. Namely

$$\mathrm{L}det(\mathcal{S}(M_1 \otimes \cdots \otimes M_n)) = \frac{1}{n!}\sum_{\sigma \in \mathbf{S}_n} \det\left(M_{\sigma(1)}^1, \cdots, M_{\sigma(n)}^n\right) \tag{2.37}$$

Clearly $\mathrm{L}det(M \otimes \cdots \otimes M) = \det M$. We only need to check $\mathrm{L}det$ is a ring homomorphism. This amounts to prove the identity

$$\mathrm{L}det(\mathcal{S}(a_1 \otimes \cdots a_n).\mathcal{S}(b_1 \otimes \cdots \otimes b_n)) = \mathrm{L}det(\mathcal{S}(a_1 \otimes \cdots a_n))\mathrm{L}det(\mathcal{S}(b_1 \otimes \cdots \otimes b_n))$$

We can check this by direct computation. The resulting morphism $\mathrm{L}det$ is uniquely characterized by $\mathrm{L}det(M^{\otimes n}) = \det M$. Indeed, we know the pure tensors $M^{\otimes n}$ generate the A-module $(M_n(A)^{\otimes n})^{\mathbf{S}_n}$. For example, we can deduce this from the general identity making sense in any associative algebra

$$\sum_{\sigma \in \mathbf{S}_n} a_{\sigma(1)} \cdots a_{\sigma(n)} = \sum_{I \subset [1,n]} (-1)^{n-|I|} \left(\sum_{i \in I} a_i\right)^n \tag{2.38}$$

It should be instructive to give a proof valid in any characteristic. We refer to exercises at the end of this subsection for such a proof, see also [41].

□

A direct corollary of the existence and uniqueness of $\mathrm{L}det$ is what we may call the linearized norm:

Corollary 2.18. *Let R be a free A-algebra of finite rank n. There is a uniquely defined ring morphism $\mathrm{L}Nm : (\overbrace{R \otimes_A \cdots \otimes_A R}^{n})^{\mathbf{S}_n} \longrightarrow A$ such that $\mathrm{L}Nm(\overbrace{a \otimes \cdots \otimes a}^{n}) = \mathrm{N}m_{R/A}(a)$.*

Proof:
Let us fix a basis of the A-module R. The morphism $h : R \to M_n(A) = \mathrm{End}_A(R)$ extends to $h^{\otimes n} : R^{\otimes n} \to M_n(A)^{\otimes n}$. This morphism commutes with the action of $\mathbf{S}_n$ and hence maps $(R^{\otimes})^{\mathbf{S}_n}$ into $(M_n(A)^{\otimes n})^{\mathbf{S}_n}$. Define L$Nm$ as the product

$$\mathrm{L}Nm : (R^{\otimes n})^{\mathbf{S}_n} \xrightarrow{h^{\otimes n}} (M_n(A)^{\otimes n})^{\mathbf{S}_n} \xrightarrow{\mathrm{L}det} A \tag{2.39}$$

□

As a consequence, the previous construction together with an obvious gleing process yields for any finite and flat morphism $Z \to S$ of constant rank n, a canonical section L$Nm : S \to Z^{(n)}$ of

$$Z^{(n)} = \overbrace{Z \times_S \cdots \times_S Z}^{n} \longrightarrow S \tag{2.40}$$

Finally, the Hilbert-Chow morphism is obtained as follows. Let $\mathcal{Z} \subset X \times \mathbf{H}_{n,X}$. The morphism $\mathcal{Z} \hookrightarrow X \times \mathbf{H}_{n,X} \xrightarrow{p_2} \mathbf{H}_{n,X}$ is locally free of rank n. Let L$Nm : \mathbf{H}_{n,X} \to \mathcal{Z}^{(n)}$ be the corresponding canonical section. Then define φ_X as the product

$$\varphi_X : \mathbf{H}_{n,X} \xrightarrow{\mathrm{L}Nm} \mathcal{Z}^{(n)} \longrightarrow X^{(n)} \times \mathbf{H}_{n,X} \xrightarrow{pr_1} X^{(n)} \tag{2.41}$$

Notice the relative symmetric product $(X \times \mathbf{H}_{n,X}/\mathbf{H}_{n,X})^{(n)}$ is nothing but $X^{(n)} \times \mathbf{H}_{n,X}$. To complete the proof, we need to check that for any $Z \in \mathbf{H}_{n,X}$ we have $\varphi(Z) = [Z]$ the cycle defined by Z. But the morphism φ commutes with an arbitrary base change, so making the base change $Z : \mathrm{Spec}\, k \to \mathbf{H}_{n,X}$ we see the result amounts to the fact that φ evaluated at Z is the same as the point

$$\mathrm{Spec}\, k \xrightarrow{\mathrm{L}Nm} Z^{(n)} \longrightarrow X^{n)}$$

If the support of Z is reduced to one point x, this is obvious since the image is the cycle nx, the only n-cycle with support x. In the general case, writing $Z = \sqcup_{i=1}^r Z_i$ with pairwise disjoint $Z_i's$, the conclusion will follows from the fact that our construction of $\varphi_X(Z)$ commutes in an obvious sense with the sum morphism (2.26), i.e.

$$\varphi(Z) = \sum_i \varphi(Z_i)$$

To prove this fact it suffices to treat the case $r = 2$. Moreover, one may assume $X = \mathrm{Spec}\, R$ affine. The disjointness of Z_1 and Z_2 yields for $B = \mathcal{O}_Z$, and the associated algebras $B_i = R/I_i$, that $B = B_1 \times B_2$. Then it is readily seen (Exercise 2.10) that the linearized norm of B factors through the product of the linearized norm of B_1 and B_2, viz.

$$\begin{tikzcd}
(B^{\otimes n})^{\mathbf{S}_n} \arrow[rr, "\mathrm{L}Nm_n"] \arrow[dr] & & A \\
& (B_1^{\otimes n_1})^{\mathbf{S}_{n_1}} \otimes (B_2^{\otimes n_2})^{\mathbf{S}_{n_2}} \arrow[ur, "\mathrm{L}Nm_{n_1} \otimes \mathrm{L}Nm_{n_2}"'] &
\end{tikzcd}$$

As a consequence of the construction, we see that the Hilbert-Chow morphism commutes with the sum of two disjoint clusters, thus making the diagram below commutative

$$\begin{tikzcd}
(\mathbf{H}_{n_1,X} \times \mathbf{H}_{n_2,X})_0 \arrow[r, "+"] \arrow[d, "\varphi_{n_1}\times\varphi_{n_2}"] & \mathbf{H}_{n,X} \arrow[d, "\varphi_n"] \\
X^{(n_1)} \times X^{(n_2)} \arrow[r, "+"] & X^{(n)}
\end{tikzcd}$$

Remark 2.19.

Let $Y \subset X$ be a closed (resp. open) subscheme. Then the following diagram, the vertical arrows being induced from $Y \hookrightarrow X$, is commutative, even cartesian for an open immersion $Y \subset X$.

$$\begin{tikzcd}
\mathbf{H}_{n,X} \arrow[r, "\varphi_X"] & X^{(n)} \\
\mathbf{H}_{n,Y} \arrow[r, "\varphi_Y"] \arrow[u, "i"] & Y^{(n)} \arrow[u, "j"']
\end{tikzcd}$$

As a consequence one can show the Hilbert-Chow morphism is projective, in particular proper. To check this, we embed X as a locally closed subscheme of a projective scheme, and using the previous diagram we see that it suffices to check the result for the closure $\overline{X}$, that is, in the case where X a projective scheme. In that case since $H_{n,X}$ is a projective scheme, then (2.41) shows φ_X is projective. More specific properties of φ_X will be shown if X is a smooth surface.

Exercise 2.7. *Let X be a non singular variety (char. $k \neq 2$). Show the Hilbert-Chow morphism identifies $\mathbf{H}_{2,X}$ with the blow-up of $X^{(2)}$ along the singular locus (see exercice 1.3), or [10], p. 224).*

Finally using similar ideas, we can extend without extra efforts the Hilbert-Chow morphism to the Incidence scheme $\mathbf{H}_{n_1,n_2,X}$ (see proposition 2.10).

Proposition 2.20. *There is an extended Hilbert-Chow morphism*

$$\varphi_{n_1,n_2} : \mathbf{H}_{n_1,n_2,X} \longrightarrow X^{(n_1-n_2)} \tag{2.42}$$

which maps the point (Z_1, Z_2) with $Z_2 \subset Z_1$ to the cycle $[Z_1] - [Z_2]$.

Proof:
We proceed as in the proof of Theorem 2.16, so the proof will be more sketchy. Let (Z_1, Z_2) be a closed point of $\mathbf{H}_{n_1,n_2,X}$. Then since Z_2 is a subscheme of Z_1, we have an exact sequence

$$0 \to I \longrightarrow \mathcal{O}_{Z_1} \longrightarrow \mathcal{O}_{Z_2} \longrightarrow 0$$

for some ideal I of $\mathcal{O}_{Z_1}$. For $a \in \mathcal{O}_1$, let h_a be the endomorphism $x \mapsto ax$ of I. We get in this way an algebra morphism $\mathcal{O}_{Z_1} \to \mathrm{End}_k(I)$, then $\mathcal{O}_{Z_1}^{\otimes n_1-n_2} \to I^{\otimes n_1-n_2}$, the tensor product being taken over k. As before we can take the "linearized norm" $\mathrm{L}Nm : (\mathrm{End}_k(I)^{\otimes n_1-n_2})^{\mathbf{S}_{n_1-n_2}} \xrightarrow{Ldet} k$, and then the product morphism

$$\left(\mathcal{O}_{Z_1}^{\otimes(n_1-n_2)}\right)^{\mathbf{S}_{n_1-n_2}} \longrightarrow \left(\mathrm{End}_k(I)^{\otimes(n_1-n_2)}\right)^{\mathbf{S}_{n_1-n_2}} \xrightarrow{Ldet} k \tag{2.43}$$

In other words, we get a canonical point of $Z_1^{(n_1-n_2)}$. Now we must see that the image of this point in $X^{(n_1-n_2)}$ is the cycle $[Z_1][Z_2]$. It is easy using the result of exercice (2.10,2) to reduce this check to the case where Z_1 has support consisting of one point x. In that case, the construction above yields a point of $X^{(n_1-n_2)}$ of the form

$$\mathrm{Spec}\, k \to Z_1^{(n_1-n_2)} \to X^{(n_1-n_2)}$$

Thus this point must necessarily be the cycle $(n_1 - n_2)x$. To complete the proof, notice the previous construction works well in families, then working applying this to the universal family $\mathcal{Z}_{Z_1}$, with base $H_{n_1,n_2,X}$, this yields the expected morphism φ_{n_1,n_2}.

□

A very useful particular case is $n_1 = n + 1$, $n_2 = n$. Then the Hilbert-Chow morphism is a morphism

$$\varphi : \mathbf{H}_{n+1,n,X} \longrightarrow X \tag{2.44}$$

The induction scheme $\mathbf{H}_{n+1,n,X}$ fits into a diagram

$$\begin{tikzcd} \mathbf{H}_{n,X} & \mathbf{H}_{n+1,n,X} \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, "\varphi"] & \mathbf{H}_{n+1} \\ & X & \end{tikzcd} \tag{2.45}$$

Let (Z_1, Z_2) be a closed point of $\mathbf{H}_{n+1,n,X}$. If $x \notin |Z_2|$, then Z_1 is just the sum $Z_1 = Z_2 + x$ (2.26), showing the morphism $(p, \varphi) : \mathbf{H}_{n+1,n,X} \longrightarrow \mathbf{H}_{n,X} \times X$ is an isomorphism above the open subset $\mathbf{H}_{n,X} \times X - Z_2$. To recover Z_1 from the pair (Z_2, x) we must find the ideal I_{Z_1} inside I_{Z_2} such that $I_{Z_1}/I_{Z_2} \cong k(x) = k$, the residue field at x. In other words we must select a closed point of the projectivized scheme

$$\mathbb{P}(I_{Z_2}) \longrightarrow \mathbf{H}_{n,X} \times X \tag{2.46}$$

This gives us the result:

Proposition 2.21. *The morphism $(p, \varphi) : \mathbf{H}_{n+1,n,X} \longrightarrow \mathbf{H}_{n,X} \times X$ identifies $\mathbf{H}_{n+1,n,X}$ with the projectivized scheme $\mathbb{P}(I_{Z_2}) \longrightarrow \mathbf{H}_{n,X} \times X$.*

□

It is not difficult to prove by induction, thanks to proposition 2.21, that for any quasi-projective scheme X, the Hilbert scheme $\mathbf{H}_{n,X}$ is connected. We leave this as an exercise.

Exercise 2.8. *Let X be a connected quasi-projective scheme. Prove by induction on n, with the help of the incidence scheme that $\mathbf{H}_{n,X}$ is connected.*

Exercise 2.9. *(***The linearized determinant***) Let A be a commutative ring. In the following $e_{r,s}$ stands for the elementary matrix with all entries equal to zero, excepted the (r,s) entry equal to one. Let us denote T_n the set of maps $[1,n] \to [1,n]$. If $f \in T_n$, then $\epsilon(f)$ will mean the signature of f, i.e. zero if f is not bijective, otherwise its signature in the usual sense. For any $f, g \in T_n$ define $E_{f,g} := e_{f(1),g(1)} \otimes \cdots \otimes e_{f(n),g(n)}$. Recall that the symmetric group $\mathbf{S}_n$ acts on $M_n(A)^{\otimes n}$ according to the rule $\sigma E_{f,g} = E_{f\sigma^{-1},g\sigma^{-1}}$.*

(1) *i) Show the invariant subring is given by $(M_n(A)^{\otimes n})^{\mathbf{S}_n} = \oplus_{(f,g)\in T_n^2/\mathbf{S}_n} A\Box E_{f,g}$, where $\Box E_{f,g} := \sum_{(f',g')\in \mathbf{S}_n(f,g)} E_{f',g'}$.*
(2) *ii) Show there is a unique ring morphism $\mathrm{L}det : (M_n(A)^{\otimes n})^{\mathbf{S}_n} \longrightarrow A$ such that $\mathrm{L}det(\Box E_{f,g}) = \epsilon(f)\epsilon(g)$. Then show $\mathrm{L}det(M \otimes \cdots \otimes M) = \det M$.*

Exercise 2.10. *(***The linearized norm** *) Let B be a free A-algebra. The regular representation yields an algebra morphism $h : B \to \mathrm{End}_A(B)$.*

(1) *Deduce from this, and from the previous exercise, that there is a unique ring morphism $\mathrm{L}Nm_n : (B^{\otimes n})^{\mathbf{S}_n} \to A$ such that $\mathrm{L}Nm_n(b\otimes\cdots\otimes b) = \mathrm{L}det(h(b) := Nm_{B/A}(b)$.*
(2) *Suppose $B = B_1 \times B_2$ is a product, where B_i is free as module of rank n_i ($i = 1, 2$). Then show the morphism $\mathrm{L}Nm_n$ factorizes through the factor $(B_1^{\otimes n_1})^{\mathbf{S}_{n_1}} \otimes (B_2^{\otimes n_2})^{\mathbf{S}_{n_2}}$ as follows*

$$\begin{tikzcd}
(B^{\otimes n})^{\mathbf{S}_n} \arrow[r, "\mathrm{L}Nm_n"] \arrow[dr] & A \\
& (B_1^{\otimes n_1})^{\mathbf{S}_{n_1}} \otimes (B_2^{\otimes n_2})^{\mathbf{S}_{n_2}} \arrow[u, "\mathrm{L}Nm_{n_1}\otimes \mathrm{L}Nm_{n_2}"']
\end{tikzcd}$$

Exercise 2.11. *Check that the subset $(\mathbf{H}_{n_1,X} \times \mathbf{H}_{n_2,X})_0 \subset \mathbf{H}_{n_1,X} \times H_{n_2,X}$ is open.*

2.3. **The local Punctual Hilbert scheme.**

2.3.1. *The local punctual Hilbert scheme.* To get insight about the fibers of the Hilbert-Chow morphism, we need further properties of totally degenerated clusters, i.e. those supported at one point. Indeed, the fiber of $\varphi_{n,X}$ at the point $\sum_{i=1}^r n_i x_i$ is, at least set-theoretically, the locus of n-clusters $Z \subset X$ such that $Z = \sqcup_{i=1}^r Z_i$ where Z_i is supported at x_i. Let us denote $\mathbf{H}_{n,X,x}$ this fiber, i.e. the locus of n-clusters with support x. To say something about $\mathbf{H}_{n,X,x}$ there is no loss of generality to assume $X = \mathrm{Spec}\, R$ and $x = \mathcal{M} \subset R$ a maximal ideal. Indeed if $Z = \mathrm{Spec}\, R/I$ is concentrated at x then R/I is local with only prime ideal $\mathcal{M}/I$, in particular $R/I = \mathcal{O}_{X,x}/I\mathcal{O}_{X,x}$, and conversely. This shows the fiber coincides set-theoretically with the *local* Hilbert scheme $H_{n,\mathcal{O}_{X,x}}$.

By a local punctual Hilbert scheme we mean the punctual Hilbert scheme $\mathbf{H}_{n,R/k}$ of a local artinian k-algebra. To see this terminology is the correct one, note the following fact: Let R be a local noetherian k-algebra with $\mathcal{M}$ its maximal ideal.

Lemma 2.22. *If $I \in \mathbf{H}_{n,R/K}$ then $\mathcal{M}^n \subset I$. In particular $\mathbf{H}_{n,R/k} = \mathbf{H}_{n,R_n/k}$ where $R_n = R/\mathcal{M}^n$.*

Proof:
Indeed, the algebra R/I is a finitely generated commutative local k-algebra, thus with

$\mathcal{M}/I$ as its unique prime ideal. Therefore this ideal is nilpotent, and being finitely generated, some power must be zero. To check that $k \leq n$, let $\overline{\mathcal{M}}$ be the image of $\mathcal{M}$ in R/I. If for $j > 0$ we have $\overline{\mathcal{M}}^{j+1} = \overline{\mathcal{M}}^{j}$, then Nakayama's lemma shows $\overline{\mathcal{M}}^{j} = 0$. Thus if k is the least integer such that $\overline{\mathcal{M}}^{k} = 0$, then for $1 \leq j < k$ we must have $\overline{\mathcal{M}}^{j+1} \not\subseteq \overline{\mathcal{M}}^{j}$. Dimension counting yields $k \leq n$. □

This lemma justifies why one may be interested by the Hilbert scheme $\mathbf{H}_{n,R/k}$ when R is a finitely dimensional local k-algebra. This *local* Hilbert scheme describes the fibers of the Hilbert-Chow morphism over totally degenerated points. It should be noted the local Hilbert scheme at $x \in X$ is really a local-etale invariant, depending only on the local ring $\mathcal{O}_x$, even on the completed local ring $\hat{\mathcal{O}}_x$.

In the local setting, there are at least two main questions, the first, how to build n-clusters, and then how to distinguish them ? Suppose $\mathcal{O}$ is a local noetherian ring with maximal ideal $\mathcal{M}$. Then $\mathbf{H}_{n,\mathrm{Spec}\,\mathcal{O}/k} = \mathbf{H}_{n,\mathcal{O}_n/k}$ where $\mathcal{O}_n = \mathcal{O}/\mathcal{M}^n$. Let Z be some point of $\mathbf{H}_{n,\mathrm{Spec}\,\mathcal{O}/k}$ with defining ideal $I \subset \mathcal{O}$. The quotient algebra $\mathcal{O}/I$ has a natural filtration

$$0 = (\mathcal{M}^n + I)/I \subset \cdots \subset (\mathcal{M}^{k+1} + I)/I \subset (\mathcal{M}^k + I)/I \subset \cdots \subset \mathcal{O}/I \tag{2.47}$$

If k is the least integer with$(\mathcal{M}^k + I)/I \neq 0$, e.g $\mathcal{M}^k \not\subseteq I$, then for $k \leq j \leq 0$, the inclusion $(\mathcal{M}^{j+1} + I)/I \subset (\mathcal{M}^j + I)/I$ is strict. Furthermore we have

$$\sum_j \dim(\mathcal{M}^j + I)/(\mathcal{M}^{j+1} + I) = n \tag{2.48}$$

The filtration (2.47) exhibits some properties of the cluster Z, as we will see below.

The rest of this section is devoted to the study of some examples. The first is a trivial one, i.e. $R = k[X]$. Clearly $\mathbf{H}_{n,k[X]/k} = \mathrm{Spec}\,k$, i.e one reduced point. A non-trivial example, where φ_X is not an isomorphism, occurs when X is a singular curve.

Example 2.1. *The punctual Hilbert scheme of a nodal algebra* [50]

Recall a node or an ordinary double point of a curve (a reduced one dimensional scheme) is a point $p \in X$ such that formally X looks near p like to the plane curve $\{xy = 0\} \subset \mathbb{A}^2$ at $(0,0)$. Equivalently, the complete local ring $\hat{\mathcal{O}}_{X,p}$ is isomorphic to

$$\hat{\mathcal{O}}_{X,p} \cong k[[x,y]]/(xy) \tag{2.49}$$

We are interested in the fiber of $\varphi_X : \mathbf{H}_{n,X} \to X^{(n)}$ at the 0-cycle $n[0]$. This fiber is the locus of ideals $I \subset \mathcal{O}_{X,p}$ of colength n, i.e $\dim \mathcal{O}_{X,p}/I = n$. Let $\mathcal{M}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Notice $\mathbf{H}_{n,\mathcal{O}_{X,p}} = \mathbf{H}_{n,\hat{\mathcal{O}}_{X,p}}$, so we may assume $R = k[[X,Y]]/(XY)$.

It is not too difficult to describe explicitly the scheme $\mathbf{H}_{n,R/k}$. Let x (resp. y) be the residue classes of X (resp. Y). An element of $\xi \in R$ can be reduced to a (unique) normal form

$$\xi = a + f(x) + g(y), \quad (a \in k,\ f(x) \in xk[[x]],\ g(y) \in yk[[y]])$$

The maximal ideal is $\mathcal{M} = (x,y) = \{f(x) + g(y),\ f(0) = g(0) = 0\}$. The notation $val(f)$ stands for the valuation of the formal power series f. The result is as follows:

Proposition 2.23. *An ideal $I \subset R$ of colength $n \geq 2$ falls into one of the two families*

i) $I = (x^a, y^b)$ $(a + b = n + 1)$

ii) $I = (x^a + \alpha y^b)$, $(\alpha \in k^*,\ a + b = n,\ a, b \geq 1)$

In particular $H_{n,R/k} = \varphi_X^{-1}(n[0])$ is a string of $n-1$ copies of $\mathbb{P}^1$. The i-th strand minus the points 0 and ∞ is the locus of points of type ii) where $a = i$. The 0 point is given by (x^i, y^{n-i+1}), and the ∞ point is given by (x^{i+1}, y^{n-i}).

Proof:
Let $(f(x), g(y))$ be an element of $\mathcal{M}$. Notice the following equalities in R

$$x(f(x), g(y)) = xf(x),\ \ y(f(x), g(y)) = yg(y)$$

Let $I \subset \mathcal{M}$ be an ideal of colength n. Define a as the smallest $i \geq 1$ such that there exists $(f(x), g(y)) \in I$ with $val(f) = i$, and define similarly b. This means $\exists(f, g)$ with $(x^a + g(y)) \in I$, $(f(x), y^b) \in I$. If $g = 0$, then $x^a \in I$. But $val(f) \geq a$, so $y^b \in I$. In this case $I = (x^a, y^b)$ and clearly $\dim R/I = a + b - 1$.

Assume now $g \neq 0$. The claim is the ideal I is principal with generator $x^a + \alpha y^b$ for some $\alpha \neq 0$. Notice $x^{a+1} \in I, y^{b+1} \in I$. Thus we may truncate $g(y)$ and assume that $g(y) = \alpha y^b$. Let us take $\xi = p(x) + q(y) \in I$. If either $val(p) \geq a + 1$ or $val(q) \geq b + 1$ then $val(p) \geq a + 1$ and $val(q) \geq b + 1$, thus $\xi \in R(x^a + \alpha y^b)$.

The last case we need to consider is $val(p) = a$ and $val(q) = b$. Then if $\beta(x) \in k[[x]]$is an invertible element is such that $\beta f = x^a$, and likewise $\gamma g = y^b$, then

$$\gamma\beta\xi = (x^a + \alpha^* y^b) \in I$$

This forces the equality $\alpha = \alpha^*$. Finally in this case I is generated by ξ. Obviously $\dim R/(x^a + \alpha y^b)R = a + b$.

Thus for any pair $(a, b) \in (\mathbb{N}^*)^2$ with $a + b = n$ we found a one parameter family $\mathbb{A}^1 = C_{a,b} \subset \mathbf{H}_{n,R/k}$, viz.

$$\alpha \in k^* \mapsto I_{a,b}(\alpha) = (x^a + \alpha y^b) \tag{2.50}$$

To check this yields an open embedding into $\mathbf{H}_{n,R/k}$, one has to compute the tangent space of $H_{n,R/k}$ at the corresponding point. An easy calculation shows the tangent space is one dimensional and the differential of (2.50) is bijective.

Next, one can also show that the limit $\lim_{\alpha \to 0} I_{a,b}(\alpha)$ exists in $\mathbf{H}_{n,R/k}$, indeed this limit is the non-principal ideal (x^a, y^{b+1}). Similarly

$$\lim_{\alpha \to \infty} I_{a,b}(\alpha) = (x^{a+1}, y^b)$$

The last assertion follows from these facts. $\square$

The special fiber of the Hilbert scheme of a nodal algebra

2.3.2. *Curvilinear clusters.* Let X be a smooth quasi-projective scheme. Let $Y \subset X$ be a closed subscheme. Recall $\mathbf{H}_{n,Y} \subset \mathbf{H}_{n,X}$ is a closed subscheme (corollary 2.9). Since the Hilbert scheme $\mathbf{H}_{n,Y}$ is easy to describe, if Y is a smooth curve, it is natural to ask how these n-clusters fit into the whole $H_{n,X}$. The clusters that comes from smooth curves germs will be called **curvilinear**. The notation $H_{n,X}^{curv}$ stands for the subset of curvilinear clusters. Clearly to test if Z is curvilinear, we may restrict to the case $Z \in \mathbf{H}_{n,X}$ is supported at one point $x \in X$. If Z is given by the ideal $I \subset \mathcal{O}_x$ with $\dim \mathcal{O}_x/I = n$, then

$$\mathcal{O}_Z = \mathcal{O}_x/I \cong k[t]/(t^n)$$

With respect to the filtration (2.47) we see its length is n, and the corresponding sequence is $(\overbrace{1, 1, \cdots, 1}^{n})$

Throughout the rest of this section it will be assumed that X is a smooth surface. In this case a smooth curve is locally defined by one equation. More precisely:

Definition 2.24. *By a germ of smooth curve $(C, x) \subset X$, we mean a subscheme $\mathrm{Spec}\,\mathcal{O}_x/(f) \subset \mathrm{Spec}\,\mathcal{O}_x$ where $f \in \mathcal{M}_x$, $f \notin \mathcal{M}_x^2$. We say that the cluster Z given by the ideal $I \subset \mathcal{O}_x$ lies on the curve (C, x) if $f \in I$. A n-cluster is called curvilinear if it lies on a smooth curve germ through x.*

Here is another way to think about a curvilinear cluster:

Proposition 2.25. *A cluster is curvilinear if and only if the algebra $\mathcal{O}_Z$ can be generated by one element, i.e. $\mathcal{O}_Z = k[f]$ for some $f \in \mathcal{O}_Z$. Moreover $\mathbf{H}_{n,X}^{\mathrm{curv}}$ is open in $\mathbf{H}_{n,X}$.*

Proof:
Notice the notation $\mathcal{O}_Z = k[f]$ makes sense since Z is affine, i.e. $Z = \mathrm{Spec}\,\mathcal{O}_Z$. In that case $(1, f, \cdots, f^{n-1})$ must be a k-basis of $\mathcal{O}_Z$, which in turn means there is a monic polynomial $P(T)$ of degree n such that

$$\mathcal{O}_Z = k[T]/(P(T)) \tag{2.51}$$

This definition should be compared with the definition of a cyclic endomorphism in linear algebra. The only thing to prove is if $\mathcal{O}_Z$ is of the form $k[f]$, then Z is curvilinear, and for this we may assume that $|Z| = p$. Then $\mathcal{O}_Z = \mathcal{O}_{X,p}/I$ is local, and there is no loss of generality to assume that $f \in \mathcal{M}_{X,p}/I$. Since $\mathcal{M}_{X,p}^n \subset I$, necessarily $f \notin (I + \mathcal{M}_{X,p}^2)/I$. Thus $\dim(I + \mathcal{M}_{X,p}^2)/\mathcal{M}_{X,p}^2 = 1$. This means we can choose $g \in I$ such that $(I + \mathcal{M}_{X,p}^2)/\mathcal{M}_{X,p}^2 = k\overline{g}$. Then Z is drawn on the smooth germ $\{g = 0\}$.

To prove the openness of $H_{n,X}^{\mathrm{curv}}$, we may restrict to the case of an affine surface $\mathrm{Spec}\,R$. Then it suffices to note this subset is the union of all open charts of $H_{n,R/k}$ given by the choices $(e_1 = 1, e_2 = f, \cdots, e_n = f^{n-1})$ (see section 2.1.3). See also Exercise 2.12.

$\square$

To give a more concrete description of the subset $\mathbf{H}_{n,X}^{\mathrm{curv}}$, let us denote $\mathbf{H}_{n,X,x}^{\mathrm{curv}}$ the set of curvilinear n-clusters with support x. To describe this set there is no loss of generality to work with the artinian ring $\mathcal{O}_x/\mathcal{M}_x^n$ (Lemma 2.22). Let C be a germ of smooth curve drawn on X localized at some point $p \in X$. Let (x,y) be a system of parameters at p, i.e $\mathcal{M}_p = (x,y)$. We can write

$$f(x,y) = ax + by + \sum_{2 \le i+j \le n-1} a_{ij} x^i y^j \pmod{\mathcal{M}_x^n} \tag{2.52}$$

The smoothness condition means $ax + by \neq 0$. The line $ax + by = 0$ yields the tangential direction of the curve. Assume for example that $b \neq 0$. Changing y to αy for some unit $\alpha \in \mathcal{O}_p^*$, we may reduce f to the normal form $f(x,y) = y + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \pmod{\mathcal{M}_p^n}$ and

$$\mathcal{O}_Z \xrightarrow{\sim} k[x,y]/(x^n, y + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}) = k[x]/(x^n) \tag{2.53}$$

It is not difficult for a fixed system of coordinates (x,y), to check the $a_i's$ in (2.53) are unique. Thus what we get for a fixed tangential direction, is a $(n-1)$-parameter family of n-clusters with support p, viz.

$$\begin{aligned} k[x,y,a_2,\cdots,a_{n-1}] \to \mathcal{O}_{\mathcal{Z}} &= k[x,y,a_2,\cdots,a_{n-1}]/(x^n, y + a_1 x + a_2 x^2 + \cdots + a_{n-1}x^{n-1}) \\ &\xrightarrow{\sim} k[x,a_2,\cdots,a_{n-1}]/(x^n) \end{aligned} \tag{2.54}$$

This yields a morphism $\mathbb{A}^{n-2} \longrightarrow H_{n,X,p}$. Obviously the previous family depends on how C fits into X, for example it is sensitive to the tangential direction of C at p. Furthermore $H_{n,X,p}^{curv}$ can be made into a smooth algebraic variety of dimension $n-1$, showing that $\dim \varphi_X^{-1}(np) \ge n-1$. Below we shall prove the equality.

Remark 2.26.

If Z is a cluster with irreducible components $Z_1, \cdots, Z_r$, then it is curvilinear if and only if any component Z_i is curvilinear. This show that any reduced cluster is curvilinear. More generally if the length of Z_i is one for $i = 2, \cdots, r$, but the length of Z_1 is equal to 2, then Z is curvilinear. Indeed it suffices to check this when $n = 2$. In this case this can be seen by direct inspection.

Exercise 2.12. *Let R be a free A-algebra of rank n. Show the set of $P \in \operatorname{Spec} A$ such that the $k(P)$-algebra $R \otimes_A k(P)$ is cyclic (generated by one element) is open.*

Exercise 2.13. *Let $\mathbb{P}(T_{X,p})$ be the projectivized tangent space of X at p. Show there is a morphism*

$$H_{n,X,p}^{curv} \longrightarrow \mathbb{P}(T_{X,p})$$

which is a locally trivial fibration with fibers $\mathbb{A}^{n-1}$. In particular, $H_{n,X,p}^{curv}$ is smooth.

2.3.3. *Connectedness theorem.* We have already observed that if X is quasi-projective and connected, then $\mathbf{H}_{n,X}$ is connected (exercise 2.27). The result below which shows that a local Hilbert scheme is always connected is due to Fogarty [21]

Proposition 2.27. *Let R/k be a local commutative k-algebra of finite dimension. The Hilbert scheme $\mathbf{H}_{n,R/k}$ is connected.*

Proof:
For a different proof see ([21], prop. 2.2), or [10]. We proceed by induction on both n and $d = \dim_k R$. The case $d = \dim R = 1$ being trivial, we assume $d = \dim R \geq 2$. Let $\mathcal{M}$ be the maximal ideal of R. There is an integer $m \geq 1$ such that $\mathcal{M}^m = 0$ and $\mathcal{M}^{m-1} \neq 0$ (2.22). The case $m = 2$ is very easy. Indeed any vector subspace $I \subset \mathcal{M}$ is an ideal. This identifies $\mathbf{H}_{n,R/k}$ with $\mathbf{G}_n(R)$ the Grassmann variety of n-dimensional subspaces of R. The result in this case is stronger since we know $\mathbf{G}_n(R)$ is a smooth irreducible variety of dimension $n(d-n)$.

Assume now $m \geq 3$. Choose $0 \neq t \in \mathcal{M}^{m-1}$. Then $\mathcal{M}t = 0$. We set $\overline{R} = R/tR$. We know $\mathbf{H}_{n,\overline{R}} \subset \mathbf{H}_{n,R}$ is a closed subscheme. The proof amounts to understand the structure of the open subscheme $\Omega = \mathbf{H}_{n,R} - \mathbf{H}_{n,\overline{R}}$. Let I be a point of Ω. This means that $t \notin I$. Then $I + tR = I \oplus tR$ is readily seen to be a point of $H_{n-1,R}$. This suggests we can define a morphism

$$\pi : \Omega \longrightarrow \mathbf{H}_{n-1,R},\ I \mapsto I + tR \tag{2.55}$$

To properly check this claim, we need to validate the construction over a base $S = \operatorname{Spec} A$. Namely if $I \subset R \otimes_k A$ is a point of Ω, meaning $t \notin I \otimes k(s)$ for any (closed) point $s \in S$, we must check $I + (t) = I \oplus (t)$ defines an S-point of $H_{n-1,R}$. For this purpose, let

$$I \oplus tR \otimes_k A \longrightarrow R \otimes_k A$$

be the canonical morphism, i.e. the sum morphism. Notice the source and the target of this map are flat modules. Our hypothesis says that this morphism is fibrewise injective. Then it is necessarily injective with cokernel flat over A as follows from the local flatness lemma [16].

Now to check our claim about Ω we must study the fibers of π. Namely let us fix $I \in \Omega$. The points belonging to the fiber $\pi^{-1}\pi(I)$ correspond to the ideals $J \subset I \oplus (t)$ with $I \oplus (t) = J \oplus (t)$, $t \notin J$. This equality yields $\mathcal{M}I = \mathcal{M}J$. Arguing as at the beginning of the proof, we see that such ideals are in $1:1$ correspondence with the hyperplanes of the vector space $I/\mathcal{M}I \oplus (t)$ not containing the line (t). The fiber is then an affine space of dimension $\dim I/\mathcal{M}I$. This shows the fibers of the morphism (2.55) are connected and not closed, since $H_{n,R}$ is known to be a projective scheme. To complete the proof, let us assume $H_{n,R}$ is the disjoint union of the open subsets $U_i\,(i = 1, 2)$. Since $\mathbf{H}_{n,\overline{R}}$ is connected, due to the induction hypothesis, we can assume $\mathbf{H}_{n,\overline{R}} \subset U_1$. Let F be a fiber of the morphism (2.55). If $F \cap U_1 = \emptyset$ then $F = F \cap U_2$ would be closed, contrary to our previous remark. This contradiction completes the proof.

□

Let us describe $\mathbf{H}_{3,R/k}$ where $R = k[[x_1, \cdots, x_r]]\ (r \geq 2)$ is the ring of formal power series in $r \geq 2$ formal variables. Let $\mathcal{M} = (x_1, \cdots, x_r)$ be the maximal ideal. Then $\mathcal{M}^3 \subset I$ for any $I \in H_{3,R/k}$. We need to distinguish two cases:

i) $I \subset \mathcal{M}^2$. As before there is a $1:1$ correspondance between these ideals and the vector subspaces $V \subset \mathcal{M}^2/\mathcal{M}^3$, namely $V = I/\mathcal{M}^3$. If $k = \dim V$, then

$$\dim R/I = n = \dim R/\mathcal{M}^3 - k = r + 1 + \frac{r(r+1)}{2} - k$$

ii) $I \nsubseteq \mathcal{M}^2$. Then $W = (I + \mathcal{M}^2)/\mathcal{M}^2 \subset \mathcal{M}/\mathcal{M}^2$ is a vector subspace, say of dimension l, the space of initial linear forms of elements of I. Writing a power series $f = f_1 + f_2 + \cdots + f_d + \cdots$ where f_d is a homogeneous polynomial de degree d, then W is the space of initial terms f_1. Notice $W.\mathcal{M}/\mathcal{M}^2 = \{PQ,\ P \in W, Q \in \mathcal{M}/\mathcal{M}^2\} \subset I$. This shows to catch I it suffices to know the following data. Together with W we need to know $I \cap \mathcal{M}^2/\mathcal{M}^3$ a subspace of $\mathcal{M}^2/\mathcal{M}^3$ containing $(\mathcal{M}W + \mathcal{M}^3)/\mathcal{M}^3$. Equivalently we need to specify the subspace $V = (I \cap \mathcal{M}^2 + \mathcal{M}^3)/\mathcal{M}^3$. To recover I starting from this data, notice if $f = f_1 + f_2 \pmod{\mathcal{M}^3} \in I$ (f_i homogeneous of degree i), then the residue class $\overline{f}_2 \in (\mathcal{M}^2/\mathcal{M}^3)/V$ depends only of $f_1 \in V$, yielding a linear map

$$\varphi : W \longrightarrow (\mathcal{M}^2/\mathcal{M}^3)/V \tag{2.56}$$

The ideal I is completely defined by the data (W, V, φ). Indeed

$$f = f_1 + f_2 + \cdots \in I \Longleftrightarrow f_2 \in \varphi(f_1) + V \tag{2.57}$$

It is readily seen that I as defined by (2.57) is an ideal. It is also not difficult to compute $\dim R/I$ in terms of $\dim V, \dim W$. We get

$$n = \dim R/I = 1 + r + \frac{r(r+1)}{2} - \dim V - \dim W$$

Let now $r = 2$. In this case we see $\dim W = 1$, $\dim V = 2$, indeed $V = \mathcal{M}.W + \mathcal{M}^3/\mathcal{M}^3$. Suppose W is the line generated by $ax + by \neq 0$, and $\varphi(ax + by) = \phi$ a degree two form. Then $\phi, x(ax + by), y(ax + by)$ must be a basis of $\mathcal{M}^2/\mathcal{M}^3$. Notice the data (a, b, ϕ) and (a', b', ϕ') define the same ideal if and only if $\exists \lambda \in k^*$ such that

$$a' = \lambda a, b' = \lambda b, \phi' = \lambda \phi \pmod{x(ax + by), y(ax + by)} \tag{2.58}$$

Inserting a parameter i.e. $\phi \mapsto t\phi\, (t \in k^*)$, we get a one-parameter family of ideals $I_t \in \mathbf{H}_{3,R/k}$. Now if $t \to \infty$, we see

$$I_t \to \mathcal{M}^2/\mathcal{M}^3$$

This show $\mathbf{H}_{3,R/k}$ (if $r = 2$) is a two-dimensional smooth projective cone with vertex $\mathcal{M}^2$.

$\square$

Exercise 2.14. *In the previous example ($r = 2$) find a description of Ω by two charts $\Omega_1 = \{(y^3, a_1x + a_2x^2)\}$ and $\Omega_2 = \{(x, x + b_1y + b_2y^2)\}$, and give the glueing relations between these coordinates.*

3. Case of a smooth surface

From now on X is a smooth surface over $k = \overline{k}$. It is known that X is quasi-projective ([34], Chap II, Remark 4.10.2). In this setting the punctual Hilbert scheme has a very interesting behavior, either local or global, as we are going to explain. The first main result is Fogarty's theorem below. During the last decade important results have been obtained about the geometry of punctual Hilbert scheme of a smooth surface, especially on its cohomology ring. The most prominent example is the affine plane $X = \mathbb{A}^2$. The Hilbert scheme $\mathbf{H}_n := H_{n,\mathbb{A}^2}$ has a particularly rich structure, sometimes unexpected. This explain why $\mathbf{H}_n$ is a ubiquitous object.

3.1. The theorems of Briançon and Fogarty.

3.1.1. *Fogarty's theorem.* The main result of this section shows the case of smooth surfaces is somewhat exceptional [21] :

Theorem 3.1. *Let X be a smooth connected surface. Then the Hilbert scheme $\mathbf{H}_{n,X}$ is connected and smooth of dimension $2n$.*

Proof:
The first point will be proven if we show that the fibers of the Hilbert-Chow morphism $\varphi : H_{n,X} \to X^{(n)}$ over the closed points are connected. Indeed if $U \subset H_{n,X}$ is open and closed, then for a fiber $\varphi^{-1}(x)$ either $\varphi^{-1}(x) \subset U$ or $\varphi^{-1}(x) \cap U = \emptyset$. This shows $U = \varphi^{-1}(\varphi(U))$. Now φ is projective (see remark 2.19), thus a closed morphism. This yields that $\varphi(U)$ is closed. For the same reason the complementary subset $X^{(n)} - \varphi(U)$ is closed. The claim follows from the connectedness of $X^{(n)}$.

Let $z = \sum_{i=1}^r n_i x_i$ be a point of $X^{(n)}$ with the $x_i \in X$ distincts, and $\sum_i n_i = n$. Let $\mathcal{O}_{x_i}$ the local ring of X at x_i with maximal ideal $\mathcal{M}_i$. At least set theoretically the fiber $\varphi^{-1}(z)$ is the product (see remark p 33)

$$\prod_{i=1}^{r} \mathbf{H}_{n_i,\mathcal{O}_{x_i}} = \prod_{i=1}^{r} \mathbf{H}_{n_i,\mathcal{O}_{x_i}/\mathcal{M}_i^{n_i}} \tag{3.1}$$

The connectedness of this fiber follows from proposition 2.27.

We now are going to prove the smoothness of $\mathbf{H}_{n,X}$. Notice since $k = \overline{k}$, smooth is synonym to regular. This is a local problem around a given point $Z \in \mathbf{H}_{n,X}$. Our construction of $H_{n,X}$ gives an open cover of this scheme, and each open piece is given explicitly as a subscheme of an affine space by a set of relations. But these relations are intractable, to check the smoothness by means of the jacobian criterion. We need a different strategy. The proof that follows is Fogarty's proof. It uses some homological algebra of regular local rings. For background about regular local rings see ([16], Ch 19). A second proof will be given in the next subsection in the special but sufficient case $X = \mathbb{A}^2$. Then we will identify $\mathbf{H}_{n,\mathbb{A}^2}$ with a suitable quiver variety, yielding a nice relationship with varieties of representations constructed in Brion's lectures [10], [24].

To help the reader we list below the necessary facts we are using. Let A be a local noetherian ring with maximal ideal $\mathcal{M}$. The (Krull) dimension of A is the least $d \in \mathbb{N}$ such that we can find $(x_1, \cdots, x_d) \subset \mathcal{M}$, with $A/(x_1, \cdots, x_d)$ of finite length, i.e. artinian. Then $d \leq \dim \mathcal{M}/\mathcal{M}^2$. The ring A is regular if and only if this inequality is an equality. A regular local ring is a domain.

There is an important homological characterization of regular local ring. Let M be a finitely generated A-module. Recall a finite free resolution of M is an exact sequence

$$0 \to L_m \to L_{m-1} \to \cdots \to L_0 \to M \to 0 \tag{3.2}$$

with L_i free of finite rank. In this case we say M is of finite homological dimension. Indeed the homological dimension $\mathrm{d}h_A(M)$ is the infimum of the length of the finite free resolutions (3.2). Then ([16], thm 19.12)

Theorem 3.2. *A noetherian local ring is regular if and only if for any finitely generated A-module M then $\mathrm{d}h_A(M) < \infty$. Then $\dim A = \max \mathrm{d}h_A(M)$. If this is the case then for $k = A/\mathcal{M}$, $dh_A(k) = \dim A$.*

We can compute $\mathrm{d}h_A(M)$ (finite or not) with the help of the $\mathrm{Ext}^k(-,-)$ modules ([16], Appendix 3). Namely $\mathrm{d}h_A(M) = k \iff \mathrm{Ext}^j(M,N) = 0$ for all $j > k$, and for some N, $\mathrm{Ext}^k(M,N) \neq 0$. If A is regular with $\dim A = 2$, then $\mathrm{Ext}^3(-,-) = 0$. We need one more fact about regular local rings. It says the dimension d of A is also the greatest length of the regular sequences included in $\mathcal{M}$. From a homological point of view this says $\mathrm{Ext}^j(M,A) = 0$ when $j < d = \dim A$, and when M is a finitely generated module of finite length, i.e annihilated by a power of $\mathcal{M}$.

Let us go back to the theorem. The idea is to estimate the dimension of the tangent space T_Z at $Z \in \mathbf{H}_{n,X}$, specifically

$$\dim T_Z \leq 2n \tag{3.3}$$

Assuming this claim let us see how to get the result. We know the points of $\mathbf{H}_{n,X}$ defined by the sum of n distinct points of X are smooth points, then regular. If Z is such a point then the dimension of the local ring of $\mathbf{H}_{n,X}$ at Z is $2n$. This follows from the fact that the Hilbert-Chow morphism is an isomorphism at this point. Furthermore the locus Ω of these points is irreducible, and so is the closure $W = \overline{\Omega}$. Since $\dim W = 2n$, for any $Z \in W$ we have $\dim \mathcal{O}_Z = 2n$ but (3.3) implies $\mathcal{O}_Z$ is regular. Then any point of the irreducible component W is smooth. Let W' be another irreducible component. Then $W' \cap W = \emptyset$ otherwise a point $Z \in W \cap W'$ would be singular. The connectedness of $\mathbf{H}_{n,X}$ then shows $W = \mathbf{H}_{n,X}$, and the theorem is proved.

Proof of (3.3). This amounts to check for any 2-dimensional regular local ring A, and any $I \subset A$ with $\ell(A/I) = n$ that $\ell(\mathrm{Hom}_A(I, A/I)) \leq 2n$. Here $\ell(-)$ means the length, that is the dimension over k if $k \subset A$ and $A/\mathcal{M} = k$. Notice the homological dimension of A/I must be 2. Otherwise from the exact sequence

$$0 \to I \to A \to A/I \to 0 \tag{3.4}$$

we would get $\mathrm{d}hI = 0$, forcing I to be a principal ideal. But then $\dim A/I = 1$, and A/I would not be artinian, i.e 0-dimensional. Thus $\mathrm{d}h\, A/I = 2$ and hence $\mathrm{d}h\, I = 1$. We can take a free resolution of the form

$$0 \to A^r \longrightarrow A^{r+1} \longrightarrow I \to 0 \tag{3.5}$$

To see the exponents are the correct ones one can tensor with K, the fraction field of A, and notice I is of rank one. Now we have the Ext exact sequence which we derive from the short exact sequence (3.5), taking into account the vanishing property $\mathrm{Ext}^1(A^{r+1}, A/I) = 0$

$$0 \to \mathrm{Hom}(I, A/I) \to \mathrm{Hom}(A^{r+1}, A/I) \to \mathrm{Hom}(A^r, A/I) \to \mathrm{Ext}^1(I, A/I) \to 0$$

This yields $\ell\,(\mathrm{Hom}\,(I, A/I)) = n + \ell\,(\mathrm{Ext}^1(I, A/I))$. It suffices to check

$$\ell\,(\mathrm{Ext}^1(I, A/I)) \leq n$$

The exact sequence (3.4) yields on one hand

$$\mathrm{Ext}^1(I, A/I) \xrightarrow{\sim} \mathrm{Ext}^2(A/I, A/I)$$

and on the other hand a surjective map $\mathrm{Ext}^2(A/I, A) \longrightarrow \mathrm{Ext}^2(A/I, A/I) \to 0$. The conclusion will follows from the fact that

$$\ell\,\left(\mathrm{Ext}^2(A/I, A)\right) = \ell\,(A/I) = n \tag{3.6}$$

Since the support of the module A/I is the closed point, we know $\mathrm{Ext}^1(A/I, A) = 0$, more generally $\mathrm{Ext}^1(M, A) = 0$ for any M with $\ell(M) < \infty$. Since A is a two dimensional regular ring, $\mathrm{Ext}^3(-,-) = 0$. Therefore the functor $M \to \mathrm{Ext}^2(M, A)$ is exact on the category of finite length modules. Then by an obvious devissage, it suffices to check that $\ell(\mathrm{Ext}^2(k, A)) = 1$. But in this case we can use a Koszul type resolution to compute this module. Let $\mathcal{M} = (x, y)$ where (x, y) are a regular system of parameters [16]. This yields an exact sequence

$$0 \to A = \wedge^2 A^2 \xrightarrow{\partial} A \xrightarrow{\varphi} A \to k \to 0$$

where $\partial(1) = (y, -x)$ and $\varphi(a, b) = ax + yb$. Dualizing yields

$$\mathrm{Ext}^2(k, A) = \mathrm{Coker}(u, v) \in A^2 \mapsto uy - vx \in A = A/\mathcal{M} = k$$

The proof is now complete.

□

A different proof will be obtained below as a corollary of Haiman's very explicit description of the punctual Hilbert scheme of the affine plane.

We close this section with a remark about the sum morphism (2.26).

Proposition 3.3. *The sum morphism* $+ : (H_{n_1,X} \times H_{n_2,X})_0 \longrightarrow H_{n,X}$ *is etale.*

Proof:
Let me remind you the definition, in a special case, of an etale morphism. Let $f : X \to Y$ be a morphism between schemes. Then f is etale at the closed point $x \in X$ if the induced ring morphism $f_x^* : \hat{\mathcal{O}}_{Y,f(x)} \to \hat{\mathcal{O}}_{X,x}$ between complete local rings is an isomorphism [34]. If moreover both schemes are smooth, then to check f is etale at x, it is sufficient to prove the differential $df_x : T_{X,x} \to T_{Y,f(x)}$ is bijective. Finally an etale morphism is quasi-finite, i.e. with finite fibers.

Then to check the sim morphism is etale at a pair (Z_1, Z_2) of two disjoint clusters, it suffices to compare both tangent spaces. We may assume $X = \mathrm{Spec}\,R$. The tangent space of $\mathbf{H}_{n,X}$ at $Z_1 + Z_2$ is $\mathrm{Hom}\,(I_1 \cap I_2, R/I_1 \cap I_2)$. Since Z_1 and Z_2 are disjoint, the Chinese Remainder Theorem yields $R/I_1 \cap I_2 = R/I_1 \oplus R/I_2$, therefore by elementary algebra

$$\mathrm{Hom}\,(I_1 \cap I_2, R/I_1 \cap I_2) = \mathrm{Hom}\,(I_1 \cap I_2, R/I_1) \oplus \mathrm{Hom}\,(I_1 \cap I_2, R/I_2)$$

The hypothesis $I_1 + I_2 = (1)$ ensures the result is the same as $\mathrm{Hom}\,(I_1, R/I_1) \oplus \mathrm{Hom}\,(I_2, R/I_2)$, that is the tangent space at (Z_1, Z_2). This is precisely what we want.

□

3.1.2. *The fibers of the Hilbert-Chow morphism: Briançon 's theorem.* The example of the previous section is a particular case of an important result governing the structure of the fibers of the Hilbert-Chow morphism (Briançon [8], see also [18], [37]). We shall not give the proof of this result, refering to papers cited above, or to the survey by Lehn [43] for a geometric proof based on the incidence scheme 2.10 plus an inductive argument.

Theorem 3.4. *Let X be a smooth surface, and let $x \in X$. The fiber $H_{n,X,x} = \varphi_X^{-1}(nx)$ (at a totally degenerated cycle) is irreducible of dimension $n-1$.*

□

The theorem gives a good deal of information about the fibers of the morphism $\varphi_n : \mathbf{H}_{n,X} \to X^{(n)}$, namely:

Corollary 3.5. *Let X be a smooth surface. For any partition λ of weight n, length ℓ, and for any cycle $\xi \in X_\lambda^{(n)}$, the fiber $\varphi_n^{-1}(\xi)$ is irreducible of dimension $n-\ell$.*

Proof:
Write $\xi = \sum_{i=1}^{\ell} n_i x_i$ with pairwise distinct points x_i. Any cluster in the fiber $\varphi_X^{-1}(\xi)$ is a disjoint sum $Z = \sum_{i=1}^{r} Z_i$ for a r-tuple $(Z_1, \cdots, Z_r) \in \prod_{i=1}^{r} \mathbf{H}_{n_i,X}$. The sum map being etale, thus with finite fibers, we see the induced morphism

$$\varphi_{n_1}^{-1}(\xi_1) \times \cdots \varphi_{n_r}^{-1}(\xi_r) \longrightarrow \varphi_n^{-1}(\xi)$$

is surjective with finite fibers. In turn this yields

$$\dim \varphi_n^{-1}(\xi) = \sum_{i=1}^{r} \dim \varphi_{n_i}^{-1}(\xi_i) = \sum_{i=1}^{r} (n_i - 1) = n - \ell$$

□

We saw the symmetric product $X^{(n)}$ has singularities if $n \geq 2$. The punctual Hilbert scheme $\mathbf{H}_{n,X}$ solves this defect. We first record some definitions, to state properly the next theorem.

Let X be a smooth variety of dimension n with function field $k(X)$. Let me recall that a rational differential n-form, ω, is an object that can be written in term of a system of parameters[*] $(U, x_1, \cdots, x_n)$ of X as

$$\omega_U = \varphi_U dx_1 \wedge \cdots \wedge dx_n \tag{3.7}$$

If $(V, y_1, \cdots, y_n)$ is another system of local coordinates, then the condition $\omega_U = \omega_V$ on $U \cap V$ yields the following transformation rule

$$\varphi_U = \varphi_V \frac{\partial(y_1, \cdots, y_n)}{\partial(x_1, \cdots, x_n)} \quad (\varphi_U \in \mathcal{O}_X(U)) \tag{3.8}$$

A rational n-form defines a divisor $\mathrm{Div}(\omega)$ of X, where locally on $(U, x_1, \cdots, x_n)$ we put $\mathrm{Div}(\omega)_{|U} = \mathrm{Div}(\varphi_U)$, the divisor of the rational function φ_U. A *canonical divisor* of X is a divisor of the form $\mathrm{Div}(\omega)$, usually denoted K_X. Clearly two canonical divisors differ by a principal divisor (the divisor of a rational function). Likewise the

[*] $\Omega^1_{U/k} = \oplus_{i=1}^{n} \mathcal{O}_U dx_i$

definition applies if X is only non singular in codimension one, for example X normal. Then a canonical divisor (a Weil divisor) is the closure of a canonical divisor of the smooth part.

Let Y be a normal variety of dimension n, a with smooth part Y_{sm}. Then codim$(Y - Y_{sm}) \geq 2$. Let $\pi : X \to Y$ be a resolution of singularities of Y. This means two things:

(1) X is a smooth variety with π a proper map which is an isomorphism over Y_{sm},
(2) the exceptional locus $\pi^{-1}(Y_{sg})$, the preimage of the singular part of Y, is purely of codimension one.

Since $k(X) = k(Y)$, the vector spaces of rational differential n-forms for X and Y are identical. If ω is such an n-form, we have two canonical divisors, namely $\mathrm{Div}_Y(\omega)$ and $\mathrm{Div}_X(\omega)$. Then pulling back $\mathrm{Div}_Y(\omega)$ to X, yields a divisor

$$\mathrm{Div}_X(\omega) - \pi^*(\mathrm{Div}_Y(\omega)) \tag{3.9}$$

which is readily seen to be independent of the choice of ω. It is not difficult to see this divisor must be a linear combination of exceptional divisors, i.e. of those prime divisors $E \subset X$ such that codim$\pi(E) \geq 2$. We say the resolution is *crepant* if $\mathrm{Div}_X(\omega) = \pi^*(\mathrm{Div}_Y(\omega))$, equivalently $K_X \sim \pi^*(K_Y)$ where $\sim$ stands for the linear equivalence. A stronger assumption will be $K_X \sim 0$ and $K_Y \sim 0$.

Theorem 3.6. *Let X be a smooth surface.*

- *i) The Hilbert-Chow morphism defines a crepant resolution of singularities of the symmetric product $X^{(n)}$.*
- *ii) The exceptional divisor $\mathcal{E} \subset \mathbf{H}_{n,X}$ is irreducible, precisely $\mathcal{E}$ is the closure*

$$\mathcal{E} = \overline{\varphi_{n,X}^{-1}(X^{(n)}_{(2,1^{n-2})})} \tag{3.10}$$

Proof:
The first part of i) is a just a recollection of known facts. First $\mathbf{H}_n$ is a smooth variety, and secondly, the Hilbert-Chow morphism is birational. The adjective semi-small translates the fact that the fibers have small dimensions (corollary 3.5). The only serious part of i) is that $\varphi_{n,X}$ is crepant, i.e $K_{\mathbf{H}_{n,X}} \sim \varphi_{n,X}(K_{X^{(n)}})$. We don't give a complete proof of this fact here, for details see ([11], Chap 7). However a proof for $X = \mathbb{A}^2$ will be given below.

ii) Recall we defined for any partition λ of weight n a stratum $X^{(n)}_\lambda \subset X^{(n)}$, with dimension 2ℓ where $\ell = \ell(\lambda)$ is the length. Since $X^{(n)}_\lambda$ is irreducible, and the fibers of the Hilbert-Chow morphism over the points of $X^{(n)}_\lambda$ are irreducible of fixed dimension $n - \ell$ (Theorem 3.4), then a known elementary result ([55], Theorem 8, Chap 1) ensures the preimage $\mathbf{H}_{n,X}(\lambda) := \varphi_{n,X}^{-1}(X^{(n)}_\lambda)$ is also irreducible of dimension $n + \ell$. The result amounts to check that $\mathcal{E}$ is the closure of the cell $\mathbf{H}_{n,X}(2, 1^{n-2})$. In other words

$$\overline{\mathbf{H}_{n,X}(2, 1^{n-2})} = \bigcup_{\lambda, \ell(\lambda) < n} \mathbf{H}_{n,X}(\lambda)$$

Let $\lambda = (n_1 \geq \cdots, n_\ell)$, with $n_1 \geq 2$. We proceed by induction on n, assuming the result true for lower dimensions. Let us define the open cell

$$\Omega = \left(\mathbf{H}_{n_1,X}(2, 1^{n_1-2}) \times \mathbf{H}_{n_1,X}(1^{n_2}) \times \cdots \times \mathbf{H}_{n_\ell,X}(1^{n_\ell})\right)_0$$

where the subscript means the subset of $(Z_1, \cdots, Z_\ell)$ with pairwise disjoint supports. The sum maps Ω into $\mathbf{H}_{n,X}(2, 1^{n-2})$. The closure of Ω is $\overline{\Omega} = \mathcal{E}_{n_1} \times \mathbf{H}_{n_2,X} \times \cdots \times \mathbf{H}_{n_\ell,X}$, in particular is irreducible. Here we used the induction hypothesis. Since our initial point Z lies in the image of $\overline{\Omega}_0$ by the sum map, the conclusion follows.

□

It is known that for singular varieties of dimension ≥ 3, there is in general no canonical way to select a desingularization. Thus for $n \geq 2$, the symmetric product $X^{(n)}$ looks very particular, since $\mathbf{H}_n$ appears as a preferred desingularization. Indeed, it is known that $\mathbf{H}_{n,X}$ (X a smooth surface) is the only crepant desingularization of the symmetric product $X^{(n)}$, even if $char k = p > 0$ (see [11], and references therein). We will see in section 5 that the G-equivariant Hilbert scheme enjoys a similar property, at least in some important cases. This is more or less the content of the McKay correspondence ([25]).

It is very natural to ask if a given geometric property of the smooth surface X transfers to $\mathbf{H}_{n,X}$. There is an interesting result in this direction due to Beauville and Fujiki (see [47], and the references therein). Recall that a *symplectic structure* on Z, on a smooth variety Z of even dimension $2n$, is given by a regular 2-form ω which is non degenerate at any point, in other words $\omega^n = \omega \wedge \cdots \wedge \omega$ is a nowhere vanishing section of the canonical line bundle.

Theorem 3.7. *Suppose the smooth surface X has a symplectic structure, then likewise $\mathbf{H}_{n,X}$ can be endowed with a sympletic structure.*

Proof:
We only give the ideas, and omit some details. A detailed proof will follow for $X = \mathbb{A}^2$. The idea is to work over an open subset $\Omega \subset \mathbf{H}_{n,X}$ with codimension at least two. Indeed if we have such regular 2-form on Ω, then it is standard to extend it to the whole of $\mathbf{H}_{n,X}$. The choice of Ω cannot be $\mathbf{H}_{n,X}(1^n)$ since this cell is of codimension one. Thus we must take

$$\Omega = \mathbf{H}_{n,X}(1^n) \cup \mathbf{H}_{n,X}(2, 1^{n-2})$$

i.e. the locus of clusters supported at no less than $n-1$ distinct points. Likewise we define the open subsets $U \subset X^n$ and $V \subset X^{(n)}$. Notice that $U/\mathbf{S}_n = V$, and that $\varphi_{n,X}(\Omega) = V$. The ramification locus of the $\mathbf{S}_n$-action on U is the disjoint union of the diagonals $H_{i,j} = \{x,\, x_i = x_j\}$. Let W be the blow-up of U along these disjoint diagonals. Then the action of $\mathbf{S}_n$ lifts to W, and $W/\mathbf{S}_n = \Omega$. This yields a commutative square

$$\begin{tikzcd} W \arrow[r, "\psi"] \arrow[d, "q"'] & U \arrow[d, "\pi"] \\ \Omega \arrow[r, "\varphi"] & V \end{tikzcd}$$

where q is the quotient map by the $\mathbf{S}_n$ action. This very easy to see. Indeed everything reduces to a local computation around a diagonal, that is to the case $\mathbb{A}^2$. Let ω be the regular 2-form giving the symplectic structure of X. Let $p_i : X^n \to X$ be the projection on the i-th factor. Then $\alpha := \sum_{i=1}^n p_i^*(\omega)$ is a regular 2-form on X^n, which yields a symplectic structure. Notice this form is $\mathbf{S}_n$-invariant, thus $\psi^*(\alpha)$ is also $\mathbf{S}_n$-invariant. Therefore we can find a regular 2-form η on Ω such that $q^*(\eta) = \psi^*(\alpha)$. The conclusion will follows if we can check that η is non degenerate on Ω, i.e. $\wedge^n\eta$ is non zero anywhere on Ω. It suffices to check this claim for $q^*(\wedge^n\eta)$ on W. But an easy and classical computation yields, if we denote by $E \subset W$ the ramification divisor, which is the same as the exceptional divisor

$$q^*(\mathrm{Div}\,(\wedge^n\eta) + E = \mathrm{Div}\,(q^*(\wedge^n\eta) = \psi^*(\mathrm{Div}\,(\wedge^n(\alpha)) + E = R$$

Therefore $\mathrm{Div}\,(\wedge^n(\eta) = 0$ as expected.

□

3.2. **The affine plane.** We return to our toy model, the affine plane $\mathbb{A}^2$, see example 2.1.4. As shown by M. Haiman, it is possible to extract from a very explicit description of $\mathbf{H}_{n,\mathbb{A}^2}$ (2.1.4), especially 2.16, important informations about this punctual Hilbert scheme. The first result is a direct proof of the smoothness of $\mathbf{H}_{n,\mathbb{A}^2}$, which in turn gives also a different proof of Fogarty's theorem (Theorem 3.1), i.e. without homological algebra. The second result, which is the main goal of this section, is a very nice interpretation of the Hilbert-Chow morphism as a blow-up of $(\mathbb{A}^2)^{(n)}$ along a suitable closed subscheme. As a consequence of this precise understanding of the geometry of $\mathbf{H}_{n,\mathbb{A}^2}$, and related schemes, Haiman was able to prove difficult combinatorial results, see the notes of I. Gordon at this school [26] for a complete discussion and references.

3.2.1. *Haiman's local coordinates.* The key point that distinguishes $\mathbb{A}^2$ among arbitrary surfaces, is the fact that $\mathbf{T}^2 = (k^*)^2$ acts* on the plane $\mathbb{A}^2$, viz. $(\lambda,\mu).(x,y) = (\lambda x, \mu y)$. This translates algebraically into the fact that $k[X,Y]$ is a bigraded algebra

$$k[X,Y] = \oplus_{(p,q)\in\mathbb{N}^2} kX^pY^q \tag{3.11}$$

It is readily seen that $(0,0)$ is the only fixed point, but more specifically, for any $(x,y) \in \mathbb{A}^2$, the closure $\overline{\mathbf{T}^2(x,y)}$ contains $(0,0)$. Clearly there is an induced an action of $\mathbf{T}^2$ on $(\mathbb{A}^2)^{(n)}$ and respectively H_n, making the Hilbert-Chow morphism $\mathbf{T}^2$-equivariant. This action is rather explicit. Using the notation of example 2.1.4, we see that each open set U_M is invariant, and the induced action on this affine open set reads

$$(\lambda,\mu).c_{p,q}^{r,s} = \lambda^{p-r}\mu^{q-s}c_{p,q}^{r,s} \tag{3.12}$$

in other words

$$c_{p,q}^{r,s}((\lambda,\mu).I) = \lambda^{p-r}\mu^{q-s}c_{p,q}^{r,s} \tag{3.13}$$

Particularly interesting are the fixed points of $\mathbf{T}^2$ on $\mathbf{H}_n$.

*A toric surface has a similar action

Lemma 3.8. *The fixed points of $\mathbf{T}^2$ on $\mathbf{H}_n$ are the homogeneous ideals of colength n, that is the ideals spanned by subsets of the set of monomials (X^pY^q).*

Proof:
Clearly the $\mathbf{T}^2$-stable ideals are exactly the homogeneous ideals, making the lemma clear.

□

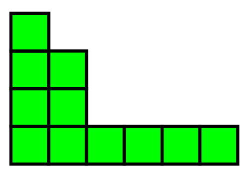
It has been pointed out in example 2.1.4 that $\mathbf{H}_n$ can be covered by affine pieces U_M labelled by the subset $M \subset \mathbb{N}^2$ of cardinal n. Among these subsets, those looking like stairs will play a prominent role. The definition is as follows:

Definition 3.9. *Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0)$ be a partition of length l, and of weight $|\mu| := \sum_i \mu_i = n$. We set*

$$M_\mu = \{(p,q) \in \mathbb{N}^2,\ 0 \leq q < \mu_{p+1}\} \tag{3.14}$$

(the diagram of μ).

For example if $\mu = (6,2,2,1)$ with weight 11, and length 4, the diagram M_μ is



The usefulness of these diagrams is justified by the Proposition below:

Proposition 3.10. i) *The fixed points of $\mathbf{T}^2$ on $\mathbf{H}_n$ are labelled by the partitions of weight n, namely $I_\mu = \bigoplus_{(p,q)\notin\delta_\mu} k\, X^pY^q$. Moreover I_μ is the only fixed point contained in $U_\mu := U_{M_\mu}$.*

ii) *For any $I \in \mathbf{H}_n$ there is a partition μ such that $I_\mu \in \overline{\mathbf{T}^2 I}$.*

iii) *$\mathbf{H}_n = \cup_{\mu,\,|\mu|=n} U_\mu$.*

Proof:
i) The first point is easy. Indeed, due to the fact that I is homogeneous, i.e. generated by the monomials X^pY^q belonging to I, if we set $\Delta = \{(p,q),\ X^pY^q \in I\}$, then Δ enjoys the property

$$\Delta + \mathbb{N}^2 = \Delta \tag{3.15}$$

from which follows easily the fact that $\mathbb{N}^2 - \Delta = M_\mu$ for a well-defined partition μ of weight $n = \dim k[X,Y]/I$. Call $I = I_\mu$. Clearly the $I_\mu' s$ exhaust all fixed points of $\mathbf{H}_n$.

ii) We can either use either the properness of the Hilbert-Chow morphism, or to proceed by a direct reasoning. The only fixed point of $\mathbf{T^2}$ on $(\mathbb{A}^2)^{(n)}$ is $n(0,0)$ and for any cycle $\xi \in (\mathbb{A}^2)^{(n)}$, one has $n(0,0) \in \overline{\mathbf{T}^2.\xi}$. Since $\varphi : \mathbf{H}_n \to (\mathbb{A}^2)^{(n)}$ is closed, one has for any $I \in \mathbf{H}_n$, $\varphi(\overline{\mathbf{T}^2.I}) = \overline{\varphi(\mathbf{T}^2.I)}$. Thus

$$\overline{\mathbf{T}^2.I} \cap \varphi^{-1}(n(0,0)) \neq \emptyset$$

To show this closure contains a fixed point we can assume that $I \in \varphi^{-1}(n(0,0))$. But then this closure is contained entirely in $\varphi^{-1}(n(0,0))$ which is complete, thus the only closed orbit contained in this fiber is a fixed point.This proves that the closure of every orbit contains a fixed point, not unique unless I is already a fixed point.

Once the fact that the closure of any $\mathbf{T}^2$ orbit contains a fixed point is proven, assertion iii) readily follows. Indeed, let F be the closed subset $\mathbf{H}_n - \cup_{\mu,|\mu|}U_\mu$. If $F \neq \emptyset$, then taking $I \in F = \overline{\mathbf{T}^2 I}$, we see F must contains a fixed point, showing necessarily $F = \emptyset$.

A direct and more illuminating proof of ii) goes as follows. It is convenient to use a lexicographic order on the monomials, i.e on $\mathbb{N}^2$. Our choice is

$$X^pY^q \leq X^rY^s \iff q < s \text{ or } q = s, p < r \tag{3.16}$$

For any polynomials, define $in(P)$ as the greatest monomial which appears with a non zero coefficient in P. Now for any ideal $I \subset k[X,Y]$ we put

$$in(I) := \bigoplus_{P \in I} k\, in(P) \tag{3.17}$$

It is easy to check $in(I)$ is indeed an ideal. Let M^* be the set of monomials (a monomial is identified with its exponents) not belonging to $in(I)$. The claim is that the set of monomials $X^rY^s \in M^*$ forms a basis of $k[X,Y]/I$. First these elements are independent. In the contrary, if $\sum_{(r,s)\in M^*} b_{rs}X^rY^s \in I$ is a non trivial linear relation, then the initial form of this polynomial yields a contradiction. Now set $V = \oplus_{(r,s)\in M^*} kX^rY^s + I$. Suppose $V \neq k[X,Y]$. Choose a monomial $X^pY^q \notin V$ which is minimal for the lexicographic order for this property. Then $X^pY^q \notin M^*$, thus this monomial is the initial form of an element of I. An obvious induction hypothesis yields a contradiction. As a consequence we have $|M^*| = n$, so $M^* = M_\mu$ with respect to some partition μ. Furthermore $I \in M_\mu$.

To complete the proof of ii) we need to compute the limit

$$\lim_{\lambda\to\infty} \lim_{\mu\to\infty} (\lambda,\mu).I = in(I) \tag{3.18}$$

Let $X^rY^s \notin M_\mu$. We have a relation

$$X^rY^s \equiv \sum_{(p,q)\in M_\mu} c^{r,s}_{p,q} X^pY^q \pmod{I}$$

thus the leading term of this element of I must be X^rY^s. Therefore invoking (3.13), the action of $(\lambda,\mu) \in \mathbf{T}^2$ reads $c^{r,s}_{p,q}((\lambda,\mu).I) = \lambda^{p-r}\mu^{q-s}c^{r,s}_{p,q}(I)$. In turn, if $c^{r,s}_{p,q}(I) \neq 0$, noticing that either $q < s$, or $q = s$ and $p < r$, we see that

$$\lim_{\lambda\to\infty} \lim_{\mu\to\infty} c^{r,s}_{p,q}((\lambda,\mu).I) \to X^rY^s$$

This precisely means that $\lim_{\lambda\to\infty} \lim_{\mu\to\infty}(\lambda,\mu).I$ exists, and this limit must be $I_\mu = in(I)$.

□

Proposition 3.10 opens the way to a completely elementary treatment of $\mathbf{H}_n$. Namely, we have a distinguished open affine covering $\mathbf{H}_n = \cup_{\mu,|\mu|=n}U_\mu$, and for each open piece the description

$$U_\mu = \mathrm{Spec}\, k[(c^{r,s}_{p,q})_{(p,q)\in M_\mu}]/(\text{equations } 2.17) \tag{3.19}$$

To see the efficiency of this description, we can now check the smoothness of $\mathbf{H}_n$ by a direct reasoning, which in turn yields a new proof of Fogarty's smoothness theorem, compare with Nakajima gauge theoretic proof [47], [24].

Proposition 3.11. *The punctual Hilbert scheme $\mathbf{H}_n$ is smooth.*

Proof:
It suffices to check $\mathbf{H}_n$ is smooth at each fixed point I_μ. Indeed, the singular locus being closed and invariant under the action of $\mathbf{T}^2$, if non empty, must contains a fixed point (Proposition 3.10). We now focus on the fixed point I_μ, which in the presentation (3.19) is identified to the origin $c^{r,s}_{p,q} = 0$. Let $\mathcal{M} = (c^{r,s}_{p,q})_{(p,q)\in M_\mu}$ be its ideal. The claim will follows if we can prove that $\dim \mathcal{M}/\mathcal{M}^2 = 2n$. From the equations (2.17) we can see two useful things. Let $(h,k) \in M_\mu$, $(r,s) \notin M_\mu$, then

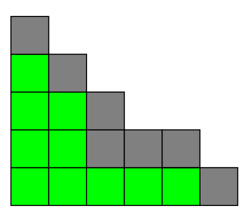
(1) If for all $(p,q) \in M_\mu$ we have $(p+1,q) \neq (h,k)$ then $c^{p+1,q}_{h,k} \in \mathcal{M}^2$, likewise,

(2) If for all $(p,q) \in M_\mu$, $(p,q+1) \neq (h,k) \Longrightarrow c^{r,s+1}_{h,k} \in \mathcal{M}^2$.

This in turn yields the congruences

$$c^{r+1,s}_{h,k} \equiv c^{r,s}_{h-1,k}\,(\equiv 0 \text{ if } h=0) \mod \mathcal{M}^2, \quad c^{r,s+1}_{h,k} \equiv c^{r,s}_{h,k-1}\,(\equiv 0 \text{ if } k=0) \mod \mathcal{M}^2 \tag{3.20}$$

To be able to exhibit a basis of $\mathcal{M}/\mathcal{M}^2$, let $\mathcal{B}$ be the set of points $(f,g) \in \mathbb{N}^2 - M_\mu$ for which the distance of (f,g) to M_μ is one, as pictured below



Here the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ is the standard one, i.e. the unit length is one. For each $(h,k) \in M_\mu$ (the index k is on the horizontal axis), one can define two points of $\mathcal{B}$, one (f,k) on the vertical line $(-,q)$, the other (p,g) on the horizontal line $(p,-)$. In turn, this define a set with $2n$ elements viz. $\{((h,k),(f,k)),\ ((h,k),(h,g)),\ (h,k) \in M_\mu\}$.

Taking into account conguences 1) and 2) leads to the following result:

Lemma 3.12. *For any $(h,k) \in M_\mu$ and $(p,q) \in \mathbb{N}^2 - M_\mu$, we have either $c^{r,s}_{h,k} \in \mathcal{M}^2$ or one of the congruence $c^{r,s}_{h,g} \equiv c^{f,k}_{h,k}$, or $c^{r,s}_{h,g} \equiv c^{h,g}_{h,k}$.*

This lemma shows $\dim \mathcal{M}/\mathcal{M}^2 \leq 2n$, and we know this is sufficient to prove the smoothness, at least for $X = \mathbb{A}^2$. But as seen before this also yields the smoothness for any smooth surface.

$\square$

With minor extra efforts we can show directly this set of $2n$ elements is a basis of $\mathcal{M}/\mathcal{M}^2$ (see exercice below) without appealing to the connectedness of H_n.

3.2.2. *Curvilinear clusters revisited.* Let us return to the open subset $\mathbf{H}_n^{curv}$ of curvilinear clusters, (see Proposition 2.25). Recall we have previously shown that

$$Z \in \mathbf{H}_n^{curv} \iff \exists f,\ \mathcal{O}_Z \xrightarrow{\sim} k[f]$$

When the surface is the affine plane, this can be made more precise. Suppose $|Z| = \{p_1, \cdots, p_r\}$ is the support of Z. Recall that locally at p_i the cluster Z is drawn on the germ of a smooth curve C_i. Let us denote τ_i the tangential direction of C_i at p_i. If the linear form $\ell(x,y) = ax + by$ is sufficiently general, then clearly we may assume that first, the values $\ell(p_i)$ are pairwise distinct, and also that $\ell \notin \{\tau_1, \cdots, \tau_r\}$. Let $\pi : \mathbb{A}^2 \to \mathbb{A}^1$ be the linear projection $\pi(x,y) = ax+by$, then π induces an isomorphism

$$Z \hookrightarrow \mathbb{A}^2 \xrightarrow{\pi} \{\prod_{i=1}^{r}(T - \ell(p_i))^{n_i} = 0\} \subset \mathbb{A}^1 \tag{3.21}$$

is an isomorphism. In particular if $\mathcal{O}_Z = k[X,Y]/I$, then $\prod_{i=1}^{r}(\ell - \ell(p_i))^{n_i} \in I$. Let η be a line transversal to ℓ, i.e. $k[X,Y] = k[\eta, \ell]$. Then we can write in $\mathcal{O}_Z$, $\overline{\eta} = \sum_{i=0}^{n-1} a_i \ell^i$. Finally this yields

$$I = \left(\eta - (\sum_{i=0}^{n-1} a_i \ell^i)\,,\ \prod_{j=1}^{r}(\ell - \ell(p_i))^{n_i} = \ell^n + b_1\ell^{n-1} + \cdots + b_n\right) \tag{3.22}$$

Let us denote $U_{\ell,\eta}$ the subset of $\mathbf{H}_n^{curv}$ described by (3.22). Viewing the $a_i's$ and the $b_j's$ as coordinates on $U_{\ell,\eta}$, then $U_{\ell,\eta} \cong \mathbb{A}^{2n}$, and $\mathbf{H}_n^{curv} = \bigcup_{\ell,\eta} U_{\ell,\eta}$.

We can use these coordinates to refine Theorem 3.6.

Proposition 3.13. *The Hilbert scheme $\mathbf{H}_n$ defines a crepant resolution of singularities of $(\mathbb{A}^2)^{(n)}$.*

Proof:
Indeed we are going to prove a stronger property, that is

$$K_{\mathbf{H}_n} \sim 0 \tag{3.23}$$

The reason for this enhancement is that $(\mathbb{A}^2)^{(n)}$ has a trivial dualizing sheaf, i.e. it is Gorenstein. To check that (3.23) it is tempting to use once more the locus of curvilinear clusters. Let $U_{x,y} = \mathbb{A}^{2n}$ be the open set as defined above. On this open set we have the coordinates $(a,b) = (a_1, \cdots, a_n, b_1, \cdots, b_n)$, and the universal cluster reads (3.22)

$$\mathcal{I} = \left(y - (\sum_{i=0}^{n-1} a_i x^i)\,, P(x) = x^n + b_1 x^{n-1} + \cdots + b_n\right)$$

Using the Viete morphism we can see $U_{x,y}$ as the quotient $\mathbb{A}^n \times (\mathbb{A}^n/\mathbf{S}_n)$, with quotient morphism $(a, x_1, \cdots, x_n) \mapsto (a, b_1, \cdots, b_n)$, the $b_j's$ being the elementary symmetric functions in the $x_i's$. An elementary computation shows the Jacobian of this morphism is the Vandermonde determinant $\Delta(x) = \prod_{i<j}(x_i - x_j)$. If we perform the wedge product of the relations

$$P'(x_i)dx_i + db_1 x_i^{n-1} + \cdots + db_n = 0$$

we obtain $db_1 \wedge \cdots \wedge db_n = \Delta(x) dx_1 \wedge \cdots \wedge dx_n$. On the other hand, if we change the coordinates $a_i's$ to $y_i's$ with $y_i = \sum_{j=0}^{n-1} a_j x_i^j$, then we see readily that

$$dy_1 \wedge \cdots \wedge dy_n = \Delta(x) da_0 \wedge \cdots \wedge da_{n-1} + \bigstar$$

where $\bigstar$ means a sum of terms with at least one dx_i inside. Putting together these relations yields

$$da_0 \wedge \cdots \wedge da_{n-1} \wedge db_1 \wedge \cdots \wedge db_n = dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \tag{3.24}$$

Let T be a matrix in $\mathrm{SL}_2(k)$. Working with the frame (ℓ, η) deduced from (x, y) by means of T, we see the $2n$-form $dx \wedge dy$ is unchanged. This show the forms $da_0 \wedge \cdots \wedge da_{n-1} \wedge db_1 \wedge \cdots \wedge db_n$ glue together and in consequence we get a regular $2n$-form without zero on the open set $\mathbf{H}_n^{curv} = \cup_{\ell,\eta} U_{\ell,\eta}$. Since this open set has a codimension greater than or equal to 2, this shows $K_{\mathbf{H}_n} \sim 0$ as expected. □

Remark 3.14.

Symplectic structure on $\mathbf{H}_n$*:* The previous proof gives us more. Namely we can exhibit explicitly the canonical symplectic structure on $\mathbf{H}_n$, at least on the curvilinear locus, i.e. on the open subset $U_{x,y}$. What is shown by our previous computation is that

$$\eta = \sum_{j,k} \left(\sum_i \frac{x_i^{n+j-k}}{P'(x_i)} \right) da_j \wedge db_k \tag{3.25}$$

is expected to give this symplectic structure (see Theorem 3.7). It is interesting however to check η is regular on $U_{x,y}$. This amounts to check that the functions $\sum_i \frac{x_i^{n+j-k}}{P'(x_i)}$ belong to $k[b_1, \cdots, b_n]$. This is a well-known exercise. Namely (Euler's formula)

$$\sum_i \frac{x_i^{n+j-k}}{P'(x_i)} = \begin{cases} 0 & \text{if } k > j+1 \\ 1 & \text{if } k = j+1 \end{cases} \tag{3.26}$$

It is therefore clear that $\wedge^n \eta = da_1 \wedge \cdots \wedge da_n \wedge db_1 \wedge \cdots \wedge db_n$.

Exercise 3.1. *Let* $(\ell, \eta) = (x, y)$*. The universal cluster over* $U_{x,y} = \mathbb{A}^2$ *is*

$$\mathcal{Z} = \operatorname{Spec} k[a_0, \cdots, a_{n-1}, b_1, \cdots, b_n]/(y - \sum_{i=0}^{n-1} a_i x^i \, , \, x_n + \sum_{j=0}^{n-1} b_i x^{n-i}) \tag{3.27}$$

Identify the polynomial $\sum_{i=0}^{n-1} a_i x^i$ *evaluated at* $Z \in U_{x,y}$ *with a suitable Lagrange interpolation polynomial.*

3.2.3. *More on the Geometry of* $\mathbf{H}_n$. We are going to prove a result due to Haiman [29], which provides a striking explicit form of the Hilbert-Chow morphism for the affine plane. We need some algebra. Throughout $k = \overline{k}$ is a field of characateristic zero. Recall we have a diagonal action of $\mathbf{S}_n$ on the polynomial ring $R = k[x_1, \cdots, x_n, y_1, \cdots, y_n]$, coming from the diagonal action on the affine space $\mathbb{A}^{2n}$, i.e.

$$\sigma(x_i) = x_{\sigma^{-1}(i)},\ \sigma(y_i) = y_{\sigma^{-1}(i)}$$

This is a *symplectic action*, in the sense that the 2-form $\sum_{i=1}^n dx_i \wedge dy_i$ is preserved. The preimage in $\mathbb{A}^{2n}$ of the subset $(\mathbb{A}^2)^{(n)}_{2,1^{n-2}}$ is the union $\mathcal{H} = \cup_{i<j} H_{i,j}$ with

$$H_{i,j} = \{(x, y) \in \mathbb{A}^{2n},\ x_i = x_j,\ y_i = y_j\}$$

the fixed point set of the transposition (i, j).

Previously our interest was the algebra of $\mathbf{S}_n$-invariant polynomials. As Haiman has pointed out, the space of alternating polynomials also plays a very interesting role. We set $\mathcal{A} = k[x_1, \cdots, x_n, y_1, \cdots, y_n]^\epsilon$, the space of alternating polynomials, i.e. such that $\sigma(P) = \epsilon(\sigma)P$. Recall the determinants $\Delta_r = \det |x_i^{r_j}|$ where r runs over the set of strictly decreasing sequences of non negative integers $r_1 > r_2 > \cdots > r_n$, form a basis of the space of ordinary alternating polynomials $k[x_1, \cdots, x_n]^\epsilon$. Likewise, if for any subset $M \subset \mathbb{N}^2$, we set

$$\Delta_M(X, Y) := \det |x_i^{p_j} y_i^{q_j}|_{1\le i,j\le n} \tag{3.28}$$

where $M = \{(p_1, q_1), \cdots, (p_n, q_n)\}$, then it is not difficult to extend the previous remark. Notice the definition has a sign ambiguity, in fact a different labeling of the points of M of the elements of M changes Δ_M by a sign. We may assume that to fix this ambiguity, the labeling is fixed.

Lemma 3.15. *The polynomials Δ_M where M runs over the subsets of $\mathbb{N}^2$ with n elements form a basis of the vector space $k[x_1, \cdots, x_n, y_1, \cdots, y_n]^\epsilon$ of alternating polynomials.*

Proof:
Let $\mathcal{A} = \frac{1}{n!} \sum_{\sigma\in\mathbf{S}_n} \epsilon(\sigma)\, \sigma$ be the projector on $k[X_1, \cdots, X_n, Y_1, \cdots, Y_n]^\epsilon$. Then

$$\Delta_M = \mathcal{A}(x_1^{p_1} \cdots x_n^{p_n} y_1^{q_1} \cdots y_n^{q_n})$$

showing these polynomials generates $k[X_1, \cdots, X_n, Y_1, \cdots, Y_n]^\epsilon$. The fact that this yields a basis is readily seen.

□

We set $\mathcal{J} = k[x_1, \cdots, y_n]^\epsilon$. Thus $\mathcal{J}$ is a rank one torsion free module over the coordinate ring $k[x_1, \cdots, y_n]^{\mathbf{S}_n} = k[(\mathbb{A}^2)^{(n)}]$. For any Δ_M it is clear that $\mathcal{J} \xrightarrow{\sim} \Delta_M\mathcal{J} \subset k[x_1, \cdots, y_n]^{\mathbf{S}_n}$, the latter being an ideal of $k[x_1, \cdots, y_n]^{\mathbf{S}_n}$. Since all rings are without zero divisors, $\mathcal{J}$ is a torsion free module of rank one, but not a projective module. It is instructive to make a comparison with the classical case, viz. the action of $\mathbf{S}_n$ over $k[x_1, \cdots, x_n]$. In this case the module of alternating polynomials $k[x_1, \cdots, x_n]^\epsilon$ is a free module of rank one over $k[x_1, \cdots, x_n]$. The quotient

$$\frac{\det |x_i^{p_j}|}{\Delta(x_1, \cdots, x_n)}$$

is the well-known definition of a Schur polynomial.

Before we state the main result, we record some facts about the *flatification process* of $\mathcal{J}$, i.e. the process that makes $\mathcal{J}$ locally free of rank one (section 1.4). The construction works over an arbitrary integral scheme X. Let $\mathbf{Sym}^\bullet(\mathcal{J})$ be the symmetric algebra of the module $\mathcal{J}$. This graded algebra need not be integral. For this reason we replace it by its image $\mathcal{S}$ in $\mathbf{Sym}^\bullet(\mathcal{J}) \otimes_{\mathbf{A}} \mathbf{K}$. Therefore $\mathcal{S} = \oplus_{k=0}^n \mathcal{S}_k$ is an integral graded $\mathcal{O}_{\mathbb{A}^2)^{(n)}}$-algebra generated by its elements of degree one viz. $\mathcal{J}$. The graded part $\mathcal{S}_k \subset k[x_1, \cdots, y_n]$ is the submodule generated by $(k[x_1, \cdots, y_n]^\epsilon)^k$. We set $\mathbf{P}_\mathcal{J} := \mathrm{Proj}(\mathcal{S})$. This scheme is equipped with a canonical (projective) morphism $\pi : \mathbf{P}_\mathcal{J} \to X$, and also with a canonical line bundle $\mathcal{O}(1)$. The scheme $(\mathbf{P}_\mathcal{J}, \pi)$ enjoys a universal property (proposition 1.23).

After this preparation, the expected result is:

Theorem 3.16. *We have* $\mathbf{H}_n = \mathrm{Proj}(\mathcal{S})$*, furthermore there is a canonical identification between the Hilbert-Chow morphism* $\varphi : \mathbf{H}_n \to (\mathbb{A}^2)^{(n)}$ *and the morphism* $\pi : \mathrm{Proj}(\mathcal{S}) \to (\mathbb{A}^2)^{(n)}$*. In other words the Hilbert-Chow morphism identifies* $\mathbf{H}_n$ *with the blow-up of* $(\mathbb{A}^2)^{(n)}$ *with center the subscheme defined by the ideal* $\mathcal{J}^2$*.*

Proof:
Thanks to the the universal property of Proposition 1.4, the proof amounts to first check that $\varphi^*(\mathcal{J})/(tors)$ is a locally free module of rank one on the Hilbert scheme $\mathbf{H}_n$. As a consequence, we will get a canonical factorization of φ through the blow-up

$$\mathrm{Proj}(\mathbf{Sym}^\bullet(\mathcal{J})/(\mathbf{tors})) = \mathrm{Proj}(\mathcal{S})$$

Then we shall prove that the induced morphism $\mathbf{H}_n \to \mathrm{Proj}(\mathcal{S})$ is an isomorphism.

Recall that $\mathcal{J}$ is the module over $\mathcal{O}_{(\mathbb{A}^2)^{(n)}}$ generated by the skew-symmetric polynomials $\Delta_M(X, Y)$ where Δ runs over all subsets of $\mathbb{N}^2$ with cardinal n. The polynomial Δ_M is only a regular function on $(\mathbb{A}^2)^n$, not on $(\mathbb{A}^2)^{(n)}$. However a product $\Delta_{M_1}\Delta_{M_2}$ of two such functions becomes a rational function (resp a regular function) on the symmetric product $(\mathbb{A}^2)^{(n)}$. Likewise a quotient

$$\frac{\Delta_{M_1}}{\Delta_{M_2}} = \frac{\Delta_{M_1}\Delta_{M_2}}{\Delta_{M_2}^2}$$

is a rational function on $(\mathbb{A}^2)^{(n)}$.

To check that $\varphi^*(\mathcal{J})/(tors)$ is locally free of rank one, it suffices to test this on each open subset U_μ of $\mathbf{H}_n$. Recall these special open subsets form a covering of $\mathbf{H}_n$ (Proposition 3.2.1, iii)). Our goal will be achieved if we can prove that for any choice of M then $\varphi^*(\Delta_M) \in \mathcal{O}_{U_\mu}\Delta_\mu$. It is easy to see that the function $\varphi^*(\frac{\Delta_M}{\Delta_\mu})$, which is a rational function on $\mathbf{H}_n$, is regular on the complementary open set $U_\mu^* := U_\mu - \mathcal{E}$. Indeed if $I \in U_\mu^*$ with $\varphi(I) = \sum_{i=1}^n (x_i, y_i)$ is a reduced 0-cycle, then

$$\varphi^*(\frac{\Delta_M}{\Delta_\mu})(I) = \frac{\Delta_M}{\Delta_\mu}(\sum_{i=1}^n (p_i, q_i)) = \frac{\Delta_M\left((x_1, y_1, \cdots, (x_n, y_n)\right)}{\Delta_\mu\left((x_1, y_1), \cdots, (x_n, y_n)\right)}$$

makes sense, i.e. yields a regular function since the denominator is nonzero. It is nonzero due to the hypothesis, i.e. that the monomials $x^{p_j}y^{q_j}$ with

$$\Delta_\mu = \{(h_1, k_1), \cdots, (h_n, k_n)\}$$

form a basis of $k[X, Y]/I = k^n$, therefore

$$\Delta_\mu((x_1, y_1), \cdots, (x_n, y_n)) = \det |x_i^{h_j} y_i^{k_j}| \neq 0$$

The next step is to extend this function as a regular function on the whole of U_μ. Let us record the identities (2.17) making sense in $k[X, Y]/I$, with I as before:

$$x_i^r y_i^s = \sum_{j=1}^{n} c_{h_j,k_j}^{r,s} x_i^{h_j} y_i^{k_j}$$

Making the choice $(r, s) \in M = \{(p_1, q_1), \cdots, (p_n, q_n)\}$, we get the matrix identity

$$|x_i^{p_j} y_i^{q_j}| = |c_{h_i,k_i}^{p_l,q_l}|.|x_i^{h_j} y_i^{k_j}| \tag{3.29}$$

Taking the determinant of both members, we get an identity of rational functions on $\mathbf{H}_n$

$$\varphi^*(\frac{\Delta_M}{\Delta_\mu}) = \det |c_{h_i,k_i}^{p_j,q_j}| \tag{3.30}$$

But the right hand side is regular on the whole of U_μ (see 3.19), thus we have $f_{\mu,M} := \frac{\Delta_M}{\Delta_\mu} \in \mathcal{O}_{U_\mu}$, showing that $\varphi^*(\Delta_M) = f_{M,\mu}\varphi^*(\Delta_\mu)$ in $\varphi^*(\mathcal{J})/(tors)$. The first point is then completely proved.

Let $\overline{\varphi} : \mathbf{H}_n \longrightarrow \mathrm{Proj}(\mathcal{S})$ be the factorization of φ obtained in the previous step. We are going to prove that $\overline{\varphi}$ is a closed immersion. To this end it suffices to check that the restriction $U_\mu \hookrightarrow \mathrm{Spec}\, k[\frac{\Delta_M}{\Delta_\mu}]$ is a closed immersion, equivalently the ring morphism

$$k[\frac{\Delta_M}{\Delta_\mu}] \longrightarrow \mathcal{O}_{U_\mu} = k[c_{h_i,k_i}^{p,q}] \tag{3.31}$$

is surjective (see (3.19)). For any $(p, q) \notin M_\mu$, and $(h_i, k_i) \in M_\mu$, set $M = (M_\mu - (h_i, k_i)) \cup (p, q)$. Then it is easy to see that for this M:

$$\frac{\Delta_M}{\Delta_\mu} = c_{h_i,k_i}^{p,q}$$

therefore the surjectivity is verified. Since both schemes are integral, with a common open subset, viz. the big cell of $\mathbf{H}_n$, the conclusion follows. $\square$

Remark 3.17.

In [31], [30] Haiman proved that the ideal $\mathcal{J}^2 \subset \mathcal{O}_{(\mathbb{A}^2)^{(n)}}$ is precisely the ideal of polynomial functions vanishing on the singular locus, i.e on the complementary of the big cell. This amounts to check that $\mathcal{J}^2 = \sqrt{\mathcal{J}^2}$ is a radical ideal. This fact sounds elementary but needs more work. As a consequence $\mathbf{H}_n$ is the blow-up of $(\mathbb{A}^2)^{(n)}$ with center the singular locus, a codimension-two subset. This is actually true also for all smooth surfaces.

Example 3.1.

This example provides an answer to exercise (1.3). We set $n = 2$. It is not difficult to check that $\mathcal{J}$ is generated by the pair of skew-polynomials ($\xi = X_2 - X_1$, $\eta = Y_2 - Y_1$). It is convenient to work with the new variables $x = X_1 + X_2, y = Y_1, Y_2, z = X_2 - X_1, t = Y_2 - Y_1$. Then $\mathcal{O}_{(\mathbb{A}^2)^{(2)}} = k[x, y, \xi^2, \eta^2, \xi\eta]$. It is also not difficult to find the relations between these generators which in turn gives the presentation

$$\mathcal{J} = \mathcal{O}^2_{(\mathbb{A}^2)^{(2)}}\xi \oplus \mathcal{O}^2_{(\mathbb{A}^2)^{(2)}}\eta / \,(A\xi - B\eta\,,\, C\xi - B\eta)$$

where we set $A = \xi\eta, ,\ B = \xi^2,\ , C = \eta^2$.

Therefore $\mathbf{Sym}^\bullet(\mathcal{J}) = \oplus_{\mathbf{k=0}}^\infty \mathbf{Sym}^\mathbf{k}(\mathcal{J})$ is the quotient algebra

$$\mathcal{O}^2_{(\mathbb{A}^2)^{(2)}}[\xi, \eta]/(A\xi - B\eta, C\xi - A\eta) \tag{3.32}$$

It is not difficult to check that $\mathrm{Proj}\,(\mathcal{S})$ is smooth and irreducible. Indeed this scheme is covered by two charts isomorphic to the affine space $\mathbb{A}^4$. This is not exactly the presentation of $\mathbf{H}_2$ as a blow-up, but $\mathrm{Proj}\,(\oplus_{k=0}^\infty \mathbf{Sym}^\mathbf{2k}(\mathcal{J}))$ is exactly the blow-up of $(\mathbb{A}^2)^{(2)}$ along its singular locus $A = B = C = 0$.

Exercise 3.2. *The identification of* $\mathbf{H}_n$ *with* $\mathrm{Proj}\,(\mathbf{Sym}^\bullet(\mathcal{J})/(\mathbf{tors}))$ *provides a tautological invertible sheaf* $\mathcal{O}(1)$ *on* $\mathbf{H}_n$ *([34]). Show* $\mathcal{O}(1)$ *can be identify with the top exterior power of the rank* n *universal bundle* $\mathbb{E}$ *(see 2.27).*

The punctual Hilbert scheme $\mathbf{H}_n$ of the affine plane enjoys many others interesting geometric properties. For example one can says something about the geometry of the universal subscheme $\mathcal{Z} \subset \mathbb{A}^2 \times \mathbf{H}_n$. We refer to the papers [29], [30] for precise statements.

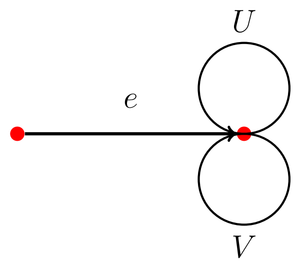
3.2.4. *The Hilbert scheme* $\mathbf{H}_n$ *as a Quiver variety.* It is interesting to point out a natural and fruitful relationship between $\mathbf{H}_n$ and a certain quiver variety, the subject of Brion's lectures ([10], see quivers with relations). The strategy to build the scheme $\mathbf{H}_n$ was to consider the space of finite-dimensional quotients of the vector space $k[X, Y]$, viz. $\varphi : k[X, Y] \longrightarrow V$, i.e. an infinite-dimensional Grassmann variety, and to add constraints that force the quotient vector space V to be a quotient algebra, equivalently the kernel I to be an ideal. These constraints can be summarized as

1. I is stable under the multiplication by X and Y,
2. The vector $e = \varphi(1)$ is *cyclic*, meaning the images $X^pY^q(e)$ form a generating system of V.

Therefore we can see the quotient algebra (together with the quotient morphism) as a collection of a pair of *commuting* linear operators $U, V : k^n \to k^n$, the multiplication by X or Y, and the choice of a cyclic vector $e \in k^n$. In this dictionary, isomorphic algebras correspond to equivalent collections, where

$$(u, v, e) \sim (u', v', e') \iff \exists P \in \mathbf{GL}_n(k),\ u' = PuP^{-1}, v' = PvP^{-1},\ e' = Pe \tag{3.33}$$

We recognize the points of the representation variety of a quiver with two vertices, one arrow and two loops, i.e.the eight figure, and dimension vector $v = (1, n)$. We must add the defining relation $UV - VU = 0$ [10],[24].



Quiver description of $\mathbf{H}_n$

The condition that e is a cyclic vector is a stability condition in the sense explained in [24]. With the choice of $\theta = (-n, 1)$, then $\theta.v = 0$, and the θ-stability of a representation means that e is cyclic vector. Therefore the punctual Hilbert scheme $\mathbf{H}_n$ can be identified with the *quotient* variety $\mathbf{M}/\mathbf{GL}_n(k)$ where

$$\mathbf{M} = \{(U, V, e) \in \mathrm{M}_n(k)^2 \times k^n,\ UV = VU,\ e\ \mathit{cyclic}\} \tag{3.34}$$

where $\mathbf{GL}_n(k)$ acts by simultaneous conjugation $(U, V, e) \mapsto (TUT^{-1}, TVT^{-1}, Te)$. Note the easy observation:

Lemma 3.18. *The action of* $\mathbf{GL}_n(k)$ *on* $\mathbf{M}$ *is free.*

Proof:
Assume $T \in \mathbf{GL}_n(k)$ fixes (u, V, e). Then the kernel of $T - 1$ contains e, and is stable by U and V. Therefore $\ker(T - 1) = k^n$, thus $T = 1$.

□

This is the point of view of Nakajima [47], it relies on ideas originated from gauge theory, see also Ginzburg's notes [24]. One should notice that many structural results about $\mathbf{H}_n$ can be proved within this framework, for example, smoothness. It is an instructive exercise to describe the Hilbert-Chow morphism in term of the simultaneous eigenvalues of the commuting matrices U, V [24]. It is straightforward to extend this description to $\mathbf{H}_n(\mathbb{A}^r)$ $(r \geq 2)$. We set $R = k^n$. To make R a $k[X_1, \cdots, X_r]$-module is the same as to give a linear map

$$\Omega = \oplus_{i=1}^r kX_i \longrightarrow \mathrm{Hom}(R, R)$$

We can identify Ω with a cotangent space of the affine space at the origin, therefore we can understand this linear map as a map

$$\Phi : R \longrightarrow \Omega^* \otimes R = \mathrm{Hom}(\Omega, R) \tag{3.35}$$

If B_i denotes the operator $f \mapsto \Phi(f)(dx_i)$ then we must add to this construction the commutativity condition $[B_i, B_j] = 0$. The map $\Phi^2 = (1 \otimes \Phi)\Phi : R \longrightarrow (\Omega^*)^{\otimes 2} \otimes R$ is a sum of a symmetric piece with an skew-symmetric piece $\Phi \wedge \Phi$. Clearly the commutativity condition amounts to $\Phi \wedge \Phi = 0$. To get a representation space of the Hilbert scheme we must add to this picture a vector $e : k \to R$. As before the condition that e is a cyclic vector is equivalent to a stability condition.

4. The G-Hilbert scheme

In this section our purpose is to revisit the punctual Hilbert scheme of a scheme X, when X has an additional structure. Our main choice is to add a finite group action on X. The first subsection is a recollection of standard facts of representation theory of finite groups. In subsection two, the G-Hilbert scheme will be defined, here G stands for a finite group. In the last subsection we will discuss briefly how the G-Hilbert scheme is a cornerstone of the McKay correspondence.

Throughout this section, schemes are defined over a field $k = \overline{k}$ of characteristic zero*. One may even assume that $k = \mathbb{C}$. Let us fix a finite group G, of order $n = |G|$. If G acts on a scheme X, the action will be denoted $(g, x) \mapsto gx$. The stabilizer of $x \in X$ is $G_x = \{g \in G, gx = x\}$. We shall say the point x, or the G-orbit Gx is free (or regular) if $G_x = 1$. If the action of G is faithful, then the set of free points denoted X_{reg} is open and non empty. Thus X_{reg}/G parameterizes the free orbits.

4.1. Definition and construction of the G-Hilbert scheme.

4.1.1. *Glossary of representation theory of finite groups.* For background on representation theory of finite groups, we refer to [57]. I start by recording some notations and results useful for the treatment of the G-Hilbert scheme.

Let $\mathbf{Irrep}(G)$ be the set of irreducible representations, namely

$$\mathbf{Irrep}(G) = \{V_1 = k, \cdots, V_N\}$$

If V is a representation* of G, i.e. a left $k[G]$-module, χ_V stands for the character of V. The degree of V, is $d_V = \dim V = \chi_V(1)$. The character of V_i is χ_i, and $d_i = \chi_i(1) = \dim V_i$. The characters group, or representations group, is $R(G) = \oplus_{i=1}^N \mathbb{Z}\chi_i$. If we are given two representations V, W, then $\mathrm{Hom}_G(V, W)$ denoted the vector space of intertwinning linear operators, i.e. those commuting with the G-action. We set V^G for the subspace of invariant vectors. We have

$$\dim \mathrm{Hom}_G(V, W) = \frac{1}{|G|} \sum_{g \in G} \chi_V(g)\overline{\chi_W(g)} \tag{4.1}$$

The right-hand side defines the standard hermitian scalar product (χ_V, χ_W). This definition extends to the space $R(G) \otimes \mathbb{C}$. In this way the $\chi_i's$ form an orthonormal basis of $R(G) \otimes \mathbb{C}$.

The notation Reg_G stands for *regular representation.* It is a classical fact that $\mathrm{Reg}_G = \oplus_{i=1}^N V_i^{\oplus d_i}$. Let V be any representation of G. Since G is reductive, V splits as a direct sum of irreducible sub-representations. Another way to express this semi-simplicity property is to write V as the direct sum of its isotypic summands. This can be written in a canonical form

$$V = \bigoplus_{i=1}^N \mathrm{Hom}_G(V_i, V) \otimes V_i \tag{4.2}$$

*One needs the complete reductibility of the group algebra

*of finite dimension

The map from the righthand side to V is $(f_i \otimes y_i)_i \mapsto \sum_{i=1}^N f_i(x_i)$.

We need to extend the isotypic decomposition to a global setting i.e. to locally free G-sheaves. Let the group G acts trivially on X. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ is a G-sheaf if we are given a group morphism $G \to \mathrm{Aut}_{\mathcal{O}_X}(\mathcal{F})$. When $X = \operatorname{Spec} R$, and $\mathcal{F} = \tilde{F}$, then this means that G acts R-linearly on F. The decomposition (4.2) can be extended as follows:

Lemma 4.1. *There is a canonical decomposition of a given G-sheaf $\mathcal{F}$ as direct sum of its isotypic components*

$$\bigoplus_{i=1}^{N} \mathcal{F}_i \otimes V_i \xrightarrow{\sim} \mathcal{F} \tag{4.3}$$

where $\mathcal{F}_i = \mathcal{H}om_{\mathcal{O}_X,G}(V_i, \mathcal{F})$ is the sheaf on G-linear maps $V_i \otimes \mathcal{O}_X \to \mathcal{F}$.

Proof:
The map (4.3) from the right hand side to $\mathcal{F}$ is the same as in (4.2). To check this is an isomorphism it suffices to argue fibrewise, but then this reduces to the previous result.

$\square$

Suppose now that $\mathcal{F}$ is locally free. It is well-known, and easy to check that the rank of the fiber $\mathcal{F}_x \otimes k(x)$ is locally constant, hence constant if X is connected ([45]). We can say more in presence of a G-action. Indeed at every point $x \in X$, we can see the vector space $\mathcal{F}_x \otimes k(x)$ as a representation of G. The type of this representation, i.e. its isomorphism class, is given in the representation ring by (4.3)

$$\mathcal{F}_x \otimes k(x) = \sum_{i=1}^{N} \dim(\mathcal{F}_i \otimes k(x))[V_i] \tag{4.4}$$

This yields the easy result:

Proposition 4.2. *Let $\mathcal{F}$ be a G-sheaf on connected scheme X, where G acts trivially. Let us assume that $\mathcal{F}$ is locally free. Then the representation of G on the fiber $\mathcal{F}_x \otimes k(x)$ is constant, i.e. does not depend of x.*

Proof:
This follows readily from the isotypic decomposition 4.3. Indeed the fact that $\mathcal{F}$ is locally free implies that for any i, the sheaf $\mathcal{F}_i = \mathcal{H}om_{\mathcal{O}_X,G}(V_i, \mathcal{F})$ is alos locally free. This is easy to check. Therefore the rank of $\mathcal{F}_i$ must be constant since X is connected. The result follows. $\square$

For the sequel we will need the following elementary result about induction. Let H and K be two subgroups of the group G, and let W a representation of H. We are interested in the subspace $\mathbf{Ind}_H^G(W)^K$ of K-fixed vectors in $\mathbf{Ind}_H^G(W)$. A very particular case of the Mackey restriction theorem is:

Lemma 4.3. *We have $\mathbf{Ind}_H^G(W)^K = \bigoplus_{s \in H\backslash G/K} W^{H \cap sKs^{-1}}$ where s runs over the double classes modulo (H, K).*

$\square$

4.1.2. *The G-Hilbert scheme.* We keep the previous assumptions, in particular we assume that X is a connected G-scheme, i.e. endowed with a G-action. Let Z be an S-point of $\mathbf{H}_{n,X}$. There is an obvious action of G on $X \times S$, i.e. trivial on the factor S. Then the subscheme $g.Z \subset X \times S$ is clearly a G-cluster. Therefore we get a G-action on the functor of points of the scheme $\mathbf{H}_{n,X}$ which translates into a G-action on $\mathbf{H}_{n,X}$ itself. At the level of closed points, this action is the obvious one $Z \mapsto gZ$. This defines a G-action on the universal cluster $\mathcal{Z} \subset X \times \mathbf{H}_{n,X}$ coming from the product action of G on $X \times \mathbf{H}_{n,X}$. Consider the cartesian square

$$\begin{array}{ccc} \mathcal{Z}_G & \longrightarrow & X \times \mathbf{H}_{n,X} \\ \varphi\downarrow & & \downarrow p_2 \\ \mathbf{H}^G_{n,X} & \longrightarrow & \mathbf{H}_{n,X} \end{array} \tag{4.5}$$

This defines $\mathcal{Z}_G := \mathcal{Z} \times_{X \times \mathbf{H}_{n,X}} \mathbf{H}_{n,X} = \mathcal{Z} \cap (X \times \mathbf{H}^G_{n,X})$ as the universal family of G-invariant clusters. We want to focus on very specific G-invariant clusters, essentially those which are limits of regular G-orbits.

Let $\mathbf{H}^G_{n,X}$ be the fixed point subscheme (1.8). The group G clearly acts on the cluster $\mathcal{Z}_G$ as explained before (4.5), therefore the bundle $\mathbb{E} = p_*(\mathcal{O}_{\mathcal{Z}})$ over $\mathbf{H}^G_{n,X}$ is a G-bundle. Proposition 4.2 says that the representation type of G on the fibers of this bundle is constant on each connected component of $\mathbf{H}^G_{n,X}$. Our preferred representation is the regular representation. From now we drop the subscript n, since it is implicitly understood as $n = |G|$.

Therefore a natural definition of the G-Hilbert scheme is as follows:

Definition 4.4. *The G-Hilbert scheme, denoted $G - \mathbf{H}_X$, is the union of all connected components $\bigcup_\alpha (\mathbf{H}^G_{n,X})_\alpha$ of $\mathbf{H}^G_{n,X}$, such that when Z runs over $(\mathbf{H}^G_{n,X})_\alpha$ then the representation of G in $\mathcal{O}_Z$ is the regular representation. The points of $G - \mathbf{H}_X$ are called G-clusters.*

The choice of the regular representation is natural since the representation that comes from a reduced G-cluster is the regular representation. To get some control about the representation provided by a G-invariant cluster $Z \in G - \mathbf{H}_{n,X}$, notice the elementary fact:

Lemma 4.5. *Let Z be a G-cluster. The support $|Z|$ is a G-orbit.*

Proof:
Clearly $|Z|$ must be a G-invariant subset, therefore a union of G-orbits. This yields a splitting $Z = \sqcup_j Z_j$ into disjoint G-subschemes. For any G-cluster the invariant ring $\mathcal{O}_Z^G$ contains the constants, thus $\dim \mathcal{O}_Z^G \geq 1$. Since $\mathcal{O}_Z$ is the regular representation, then $\dim \mathcal{O}_Z^G = 1$. This forces the support of Z to be a single G-orbit. $\square$

From now on by a G-cluster we mean a G-invariant subscheme such that the representation afforded by $\mathcal{O}_Z$ is the regular representation. The G-Hilbert scheme of X is therefore the scheme which parametrizes the G-clusters. The universal G-cluster

$\mathcal{Z}_G$ is simply deduced by base change from $\mathcal{Z}$, see the diagram (4.5). The picture is

$$\begin{tikzcd} \mathcal{Z}_G \arrow[r, hook] \arrow[dr] & X \times G - \mathbf{H}_X \arrow[d, "p_2"] \\ & G - \mathbf{H}_X \end{tikzcd} \tag{4.6}$$

As defined it is not clear whether the G-Hilbert scheme is smooth and even irreducible, see for example [43]. In some important cases, smooth surfaces for instance, see below, the G-Hilbert scheme will be smooth, but not necessarily irreducible.

Let $x \in |Z|$, with stabilizer $H = G_x$. It is readily seen that not only as representation, but as closed subscheme

$$\mathcal{O}_Z = \mathrm{Ind}_H^G(\mathcal{O}_{Z,x})$$

the induced representation (subscheme) of $\mathcal{O}_{Z,x}$ from H to G. Let $x \in X$ be a point with $G_x = 1$, and G-orbit $Gx = \{x_1 = x, \cdots, x_n\}$. Then a reduced cluster Z with support a free orbit Gx is a G-cluster. We see the reduced G-clusters correspond bijectively with the regular (or free) G-orbits, those of lenght n. Precisely:

Lemma 4.6. *The regular G-clusters are the points of an open subset of $G - \mathbf{H}_X$ naturally isomorphic to X_{reg}/G. In other words the G-cluster $\mathcal{Z}_{reg} := \{(x, z) \in X \times X_{reg}/G,\ x \in z\}$ is the universal family of regular G-clusters.*

Proof:
This is the equivariant counterpart of Lemma 2.4. The proof will not be reproduced. □

Thus among the connected components of $G - \mathbf{H}_X$ there is at least the component containing the regular orbits, i.e. the closure of X_{reg}/G, as a consequence $G - \mathbf{H}_X$ is not empty. The scheme $G - \mathbf{H}_X$ need not be irreducible, even connected. This suggests that a more restrictive definition of the G-Hilbert scheme would be:

Definition 4.7. *The reduced, or dynamical* ⋆ *G-Hilbert scheme $X//G$ is defined as the closure in $G - \mathbf{H}_X$ of the set X_{red}/G of reduced G-clusters (i.e. regular G-orbits). Therefore the scheme $X//G$ is reduced and irreducible.*

The universal property that leads to the G-Hilbert scheme is somewhat lost by $X//G$. However we will see below a different construction of $X//G$ that exhibits some kind of a universal property.

Remark 4.8.

We wish to add one more remark about G-clusters. Assume $Z \subset X$ is a G-cluster with support $|Z| = \{x_1, \cdots, x_r\}$. Denote $H = G_{x_1}$ the stabilizer of x_1, so that $|Z| = Gx_1$. The finite subscheme Z is a sum $Z = \sqcup_{i=1}^r Z_i$, where $|Z_i| = \{x_i\}$. Clearly Z_1 is an H-cluster

Some additional remarks are added as exercises.

Exercise 4.1. *Let $U \subset X$ be an open G-invariant subset. Then show that $G - \mathbf{H}_U$ is an open subset of $G - \mathbf{H}_X$, resp $U//G$ is open in $X//G$.*

⋆ We hope this notation will not be confused with another meaning

Exercise 4.2. *Assume $X = X_1 \times X_2$, where G acts only on X_2, and trivially on X_1. Prove that $G - \mathbf{H}_{X_1 \times X_2} \cong X_1 \times G - \mathbf{H}_{X_2}$.*

Exercise 4.3. *(Induction) Let H be a subgroup of G. Assume H acts on Y. The induced G-scheme is the quotient $\mathbf{Ind}_H^G(Y) = (G \times Y)/H$, where H acts by $h(g, y) = (gh^{-1}, hy)$. Show there is a natural isomorphism $G - \mathbf{H}_{\mathbf{Ind}_H^G(Y)} \cong H - \mathbf{H}_Y$.*

4.1.3. *The equivariant Hilbert-Chow morphism.* Interestingly, the G-Hilbert scheme is related to the quotient X/G in the same way $\mathbf{H}_{n,X}$ is related to the symmetric product $X^{(n)}$ by means of the Hilbert-Chow morphism. Consider the universal G-cluster $\mathcal{Z}_G \subset X \times G - \mathbf{H}_X$ (see (4.6)). If we perform the quotient by G of both members, this yields a closed subscheme $\mathcal{Z}_G/G \subset X/G \times G - \mathbf{H}_X$.

Lemma 4.9. *The projection $G - \mathbf{H}_X$ yields an isomorphism $\mathcal{Z}_G/G \xrightarrow{\sim} G - \mathbf{H}_X$.*

Proof:
The projection factors as $\mathcal{Z}_G \to \mathcal{Z}_G/G \to G - \mathbf{H}_X$. Since the quotient by G commutes with any base change, the fiber of $\mathcal{Z}_G/G \to G - \mathbf{H}_X$ at ξ is the quotient of the G-cluster Z_ξ by G. But as previously observed, due to the fact that $\mathcal{O}_{Z\xi}$ is the regular representation, this quotient is $\mathrm{Spec}(\mathcal{O}_{Z_\xi})^G = \mathrm{Spec}\, k$. The result follows.

□

As a consequence we get a morphism $\varphi_X : G - \mathbf{H}_X \to X/G$ such that

(4.7)
$$\begin{tikzcd} & G - \mathbf{H}_X \arrow[dr, "\varphi_X"] & \\ \mathcal{Z}_G/G \arrow[ur, "\sim"] \arrow[rr, "p_1"] & & X/G \end{tikzcd}$$

commutes. It is not difficult to see the morphism φ_X if restricted to the open subset of regular G-clusters is the previously observed isomorphism onto the subset X_{reg}/G of regular G-orbit. The same remarks holds for the reduced G-Hilbert scheme.

Definition 4.10. *We call the morphism $\varphi_X : G - \mathbf{H}_X \longrightarrow X/G$ the equivariant Hilbert-Chow morphism.*

This morphism is closely related to the ordinary Hilbert-Chow morphism $\mathbf{H}_{n,X} \to X^{(n)}$. Indeed we can define a morphism but in a set-theoretic sense, $X/G \to X^{(n)}$, by $Gx \in X/G \mapsto (\sum_{g\in G} gx)$. We can defined it in a schematic sense in the following way. Consider the morphism $X \to X^n \to X^{(n)}$, where the first morphism is $x \mapsto (gx)_{g\in G}$. This morphism clearly factors through X/G. This leads to a commutative diagram, where the arrows are those previously defined

(4.8)
$$\begin{tikzcd} \mathbf{H}_{n,X} \arrow[r, "\varphi_{n,X}"] & X^{(n)} \\ G - \mathbf{H}_X \arrow[u] \arrow[r, "\varphi_X"] & X/G \arrow[u] \end{tikzcd}$$

Therefore the equivariant Hilbert-Chow morphism is in some sense the "restriction" of the ordinary Hilbert-Chow morphism.

Assume that X is smooth. The construction shows

$$\varphi_X : G - \mathbf{H}_X \longrightarrow X/G \tag{4.9}$$

is an isomorphism above the open subset whose points are the regular G-orbits. This is an open subset of the smooth part of X/G. If by chance $G - \mathbf{H}_X$ is a smooth irreducible scheme, then it defines a resolution of the singularities of X/G. The McKay correspondance speculates about this fact and their consequences. We shall comment on this hope in the section below.

4.1.4. *The action of $\mathbf{S}_n$ on $\mathbb{A}^n$.* The Viete map which is the heart of classical symmetric polynomials, is very instructive. Let us consider the natural action of $\mathbf{S}_n$ on $\mathbb{A}^n = \mathbb{C}^n$ (or any algebraically closed field of charateristic zero). In this case we are going to check that the $\mathbf{S}_n$-Hilbert scheme of $\mathbb{A}^n$ is the same as the quotient $\mathbb{A}^n/\mathbf{S}_n \cong \mathbb{A}^n$. This remark works more generally for G a finite subgroup of $\mathbf{GL}_n(\mathbb{C})$ generated by a set of pseudo-reflections. The key point is the fact that $\mathbb{A}^n/G$ is smooth, see Sheppard-Todd-Chevalley's theorem 1.15.

Theorem 4.11. *Let G be a finite group acting on a smooth (connected) variety X such that the quotient X/G is smooth (we don't assume the action is free). Then the equivariant Hilbert-Chow morphism is an isomorphism, i.e.*

$$G - \mathbf{H}_{\mathbb{A}^n} \xrightarrow{\sim} \mathbb{A}^n/G \tag{4.10}$$

More generally if X/G is smooth then $G - \mathbf{H}_X = X/G$.

Proof:
Let $\pi : X \to Y = X/G$ the quotient morphism. The key point under this smoothness assumption, is the fact that $\pi_*(\mathcal{O}_X)$ is a locally free module over $\mathcal{O}_{X/G}$. This follows from a general flatness property, see for example Eisenbud [16], corollary 18.17, for details: let A be a regular local ring, and let $A \to B$ be a morphism between local rings, with B Cohen-Macaulay, and finite as A-module; then B is A-flat.

With this very strong property in mind, the isotypic factors of the locally free sheaf $\pi_*(\mathcal{O}_X)$ are also locally free (4.3), namely

$$\pi_*(\mathcal{O}_X) = \bigoplus_{V_i \in \mathbf{Irrep}(G)} E_i \otimes V_i \tag{4.11}$$

for E_i a locally free module over $\mathcal{O}_{X/G}$ of rank $n_i = \deg V_i$. Consider the graph of $\pi : X \to X/G$, that is

$$X \hookrightarrow X \times X/G \tag{4.12}$$

Our previous remark yields that X can be seen as a G-cluster over X/G. This is clear since π is flat, and the fiber of π at any point $y \in X/G$ is the spectrum of $\pi_*(\mathcal{O}_X)_y \otimes k$ which affords the regular representation

$$\bigoplus_i ((E_i)_y \otimes k) \otimes V_i$$

Now our claim amounts to check that this G-cluster is universal. Let $Z \subset X \times S \xrightarrow{p_2} S$ be a G-cluster. The morphism $p : Z \to S$ is flat of rank $n = |G|$, let

$\overline{p} = Z/G \to S$ denote the factorization. Then $\overline{p}$ is an isomorphism, see Lemma 4.9. Thus the morphism $Z \to X \to X/G$ induced by the second projection factors through $Z/G = S$. This yields a commutative diagram

$$\begin{tikzcd} Z \arrow[r, "\imath"] \arrow[dr, "p"'] & S \times_{X/G} X \arrow[r] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ & S = Z/G \arrow[r, "f"] & X/G \end{tikzcd}$$

The morphism $\imath : Z \to S \times_{X/G} X$ is a G-equivariant morphism between two G-clusters with a common basis $S = X/G$. It is not difficult to see that $\imath$ is indeed an isomorphism. It suffices to check this fibrewise. But then we have a G-linear map between two copies of the regular representation, i.e. $k[G] \to k[G]$, such that $1 \mapsto 1$. Clearly this must be an isomorphism. This completes the proof. $\square$

4.1.5. *Another view on the G-Hilbert scheme.* What the G-Hilbert scheme really does is to flatify the coherent sheaf $\pi_*(\mathcal{O}_x)$ over $Y = X/G$. As shown in theorem 4.11, if $\pi_*(\mathcal{O}_X)$ is a flat i.e locally free $\mathcal{O}_Y$-module, then $G - \mathbf{H}_X = Y$. In section 1.4 we described a way to make a torsion free coherent sheaf flat by mean of a Grassmann blow-up. Let us denote E_χ the torsion free sheaves defined by the isotypic component of $\pi_*(\mathcal{O}_X)$ (4.3). Let for $\chi \neq 1$, $\rho_\chi : G_\chi \to X/G$ the Grassmann blow-up that makes E_χ locally free (theorem 1.4).

Proposition 4.12. *The Hilbert-Chow morphism $\varphi : G - \mathbf{H}_X \to X/G$ restricted to $X//G$ factors through G_χ. Moreover the induced morphism $X//G \longrightarrow \prod_{X/G} G_\chi$ is an isomorphism (the right-hand side means the fiber product of the G_χ's).*

Proof:
Recall the commutative diagram (4.7)

$$\begin{tikzcd} \mathcal{Z}_G \arrow[r, hook] \arrow[rr, bend left, "q"] \arrow[dr, "p"'] & X//G \times_{X/G} X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "\pi"] \\ & X//G \arrow[r, "\varphi_X"] & X/G \end{tikzcd} \tag{4.13}$$

Since π is affine the natural morphism $\psi : \varphi_X^*(\pi_*(\mathcal{O}_X) \longrightarrow p_{1*}(\mathcal{O}_{X//G\times_{X/G}X})$ is an isomorphism. This isomorphism together with the surjection $\mathcal{O}_{X//G\times_{X/G}X} \to \mathcal{O}_{\mathcal{Z}_G}$ yield a surjective map

$$\varphi^*(\pi_*(\mathcal{O}_X)) \longrightarrow q_*(\mathcal{O}_{\mathcal{Z}_G}) = \bigoplus_\chi (q_*(\mathcal{O}_{\mathcal{Z}_G}))_\chi \tag{4.14}$$

This morphism is clearly G-equivariant, thus for any irreducible character χ we get a surjective morphism between isotypic factors

$$\varphi_X^*(E_\chi) \longrightarrow q_*(\mathcal{O}_{\mathcal{Z}_G}) \tag{4.15}$$

Notice the right-hand side is locally free of rank $\deg\chi$, that makes it the quotient of $\varphi_X^*(E_\chi)$ by its torsion sub-sheaf. Therefore, from the universal property of the Grassmann blow-up, we get a factorization of φ through G_χ. This in turn yields a canonical morphism to the fiber product $\widetilde{X/G}$ of the $G'_\chi s$

$$X//G \longrightarrow \widetilde{X/G} = \prod_{X/G} G_\chi \tag{4.16}$$

To check this morphism is an isomorphism we must construct the inverse morphism. Let $\rho : \widetilde{X/G} \to X/G$ be the canonical morphism. We know the quotient sheaf $\mathcal{A} = \rho^*(\pi_*(\mathcal{O}_X))/(tors)$ is locally free, then a locally free algebra over $\widetilde{X/G}$. Clearly there is an induced G-action on $\mathcal{A}$. Let $\operatorname{Spec}\mathcal{A} \to X\tilde{/}G$ be the corresponding scheme, affine over $\widetilde{X/G}$. It is not difficult to see there is a G-equivariant morphism $f : \operatorname{Spec}\mathcal{A} \to X$ that fits into a commutative diagram

$$\begin{tikzcd} \operatorname{Spec}\mathcal{A} \arrow[r, "f"] \arrow[d, "r"] & X \arrow[d, "\pi"] \\ \widetilde{X/G} \arrow[r, "\rho"] & X/G \end{tikzcd} \tag{4.17}$$

The morphism r is flat, therefore we can see $\operatorname{Spec}\mathcal{A}$ as a G-cluster with base $\widetilde{X/G}$. We get in this way a morphism $\widetilde{X/G} \to G-\mathbf{H}_X$ which clearly factors though $X//G$. Clearly this yields the inverse of ρ. □

4.1.6. *The punctual Hilbert scheme of $\mathbb{A}^2$ revisited.* The Hilbert scheme $\mathbf{H}_n$ classifies the finite subscheme of $\mathbb{A}^2$ of length n. Remarkably it yields a crepant resolution of $(\mathbb{A}^2)^{(n)} = \mathbb{A}^2/\mathbf{S}_n$. It is also natural to ask if the equivariant Hilbert scheme, eventually its dynamical component $(\mathbb{A}^2)^n//\mathbf{S}_n$ leads to the same result i.e whether $\mathbf{H}_n \cong (\mathbb{A}^2)^n//\mathbf{S}_n$. An indication in that direction is the fact (3.16) that $\mathbf{H}_n$ is nothing but the Grassmann blow-up of $(\mathbb{A}^2)^{(n)}$ associated to a very particular isotypic factor of $\pi_*(\mathcal{O}_{(\mathbb{A}^2)^n})$ namely the factor corresponding to the signature $\chi = \epsilon$. It is a nice and difficult result proved by Haiman that for $(\mathbb{A}^2)^{(n)}$ this result holds true [30]. This result essentially solves the n! conjecture (see also Gordon's notes at this school [26]). Namely:

Theorem 4.13. *There is a natural isomorphism*

$$(\mathbb{A}^2)^n//\mathbf{S}_n \xrightarrow{\sim} \mathbf{H}_n \tag{4.18}$$

given by $\Sigma \in (\mathbb{A}^2)^n//\mathbf{S}_n \mapsto Z = \Sigma/\mathbf{S}_{n-1}$.

For the complete proof we refer to Haiman [31], [32] Theorem 2, and the references therein. The map that realizes the isomorphism is easy to find. Let Z be the universal $\mathbf{S}_n$-cluster with base $(\mathbb{A}^2)^n//\mathbf{S}_n$. It has a natural $\mathbf{S}_n$-action. The quotient[⋆] $Z/\mathbf{S}_{n-1}$ is

[⋆] $\mathbf{S}_{n-1}$ is the subgroup leaving fixed the last coordinate flat and finite of degree n over $(\mathbb{A}^2)^n//\mathbf{S}_n$, therefore yields a map

$$(\mathbb{A}^2)^n//\mathbf{S}_n \longrightarrow \mathbf{H}_n \tag{4.19}$$

On the other hand let X_n be the reduced fiber product, viz.

$$\begin{tikzcd} X_n \arrow[r] \arrow[d] & (\mathbb{A}^2)^n \arrow[d] \\ \mathbf{H}_n] \arrow[r, "\varphi"] & (\mathbb{A}^2)^n/\mathbf{S}_n \end{tikzcd} \tag{4.20}$$

Suppose the reduced scheme X_n is flat over $\mathbf{H}_n$, then finite flat of degree $n!$. The universal property of $(\mathbb{A}^2)^n//\mathbf{S}_n$ then yields a map

$$\mathbf{H}_n \to (\mathbb{A}^2)^n//\mathbf{S}_n \tag{4.21}$$

One of the main results in Haiman's approach of the $n!$ conjecture is the fact that the claim holds true, and as a consequence (4.19) and (4.21) both maps are isomorphism inverse each other.

5. ADE singularities

In this section we are going to comment, with or sometimes, without proof, on the structure of the minimal resolution of singular points of ADE type i.e. the rational double points (Example 1.1), with particular attention to the exceptional fiber.

5.1. **Reminder of Intersection theory on surfaces.** Our setting is the following. Let X be a normal surface, in most cases affine. There is no loss of generality to assume X has only one singular point, say p. Therefore $X - p$ is smooth. Recall that a desingularization of X means the following data:

$$\tilde{X} \xrightarrow{\pi} X \tag{5.1}$$

where $\tilde{X}$ is a smooth surface, and π is a proper morphism which is an isomorphism over $X - p$. It is known that the scheme theoretic fiber $\pi^{-1}(p)$ is purely of dimension one, and connected (see [34]). Thus we can identify $\pi^{-1}(p)$ to a dimension one cycle

$$\pi^{-1}(p) = \sum_{i=1}^{r} n_i E_i \tag{5.2}$$

the $E_i's$ being the irreducible reduced components, and $n_i \in \mathbb{N}_{>0}$. This fiber is called the *execptional fiber*. A key technical tool to study the exceptional fiber is the intersection pairing. Notice that any component $E = E_i$ is a complete curve, so for any coherent sheaf $\mathcal{F}$ on E, the Euler characteristic

$$\chi_E(\mathcal{F}) = \dim H^0(E, \mathcal{F}) - \dim H^1(E, \mathcal{F}) \tag{5.3}$$

is defined.

Recall that by an integral (or prime) divisor on $\tilde{X}$ we mean an integral curve. The group $\mathrm{Div}(\tilde{X})$ is the group freely generated by the prime divisors. There is a distinguished subgroup $\mathrm{Div}_\pi = \bigoplus_{i=1}^r \mathbb{Z}E_i$ called the subgroup of vertical divisors. A prime divisor $E \notin \{E_1, \cdots, E_r\}$ is horizontal. Let $x \in X$ be a (closed) point.

Since $\mathcal{O}_{\tilde{X},x}$ is a regular two-dimensional local ring, therefore a UFD, if $f, g \in \mathcal{M}_x$ are without a common prime factor, then $\dim \mathcal{O}_{\tilde{X},x}/(f,g) < \infty$. This follows from the fact that the only prime ideal containing f and g is $\mathcal{M}_x$. If f and g are the local equation of two positive divisors D and E through x, then this dimension is the intersection multiplicity of D and E at x, commonly denoted $(D,E)_x$. If this is the case for all intersection points $x \in |D| \cap |E|$, then the sum over all common points

$$D.E = \sum_{x \in |D| \cap |E|} (D.E)_x \in \mathbb{N} \tag{5.4}$$

is the the usual definition of the *intersection number* of D and E. Clearly $D.E = E.D$ whenever one of the members makes sense. The formula (5.4) does make sense if at somme point the intersection $D \cap E$ is not tranversal, i.e. if D and E have a common component. For this reason, and also due to the fact that $\tilde{X}$ is not necessarily complete, we must give another definition of the intersection multiplicity, and also restrict its domain of validity. If E is a complete curve, a purely of dimension one scheme, not necessarily irreducible, or even reduced, one knows how define the degree of a line bundle $\mathcal{L}$ ([34]):

Definition 5.1. *The degree of $\mathcal{L}$ is*

$$\deg(\mathcal{L}) = \chi(O_E) - \chi(\mathcal{L}^{-1}) \tag{5.5}$$

Let $E = \cup_{i=1}^r E_i$ the irreducible components of E, and let n_i denote the multiplicity of E along E_i, that is the length of the local ring of E at the generic point of E_i. Then one can shows that

$$\deg(\mathcal{L}) = \sum_i n_i \deg_{E_i}(\mathcal{L} \otimes \mathcal{O}_{E_i}) \tag{5.6}$$

Theorem 5.2. *There is a unique bilinear pairing* $\mathrm{Div}_\pi(\tilde{X}) \times \mathrm{Div}(\tilde{X}) \to \mathbb{Z}$, $\quad (E,D) \mapsto E.D$*, such that*

- *i) $E.D$ depends only on the linear equivalence class of D, i.e. of $\mathcal{O}(D)$. namely one has $(E.D) = \deg_E(\mathcal{O}(D))$.*
- *ii) If E, D are both vertical then*

$$(E.D) = \chi(\mathcal{O}_E) + \chi(\mathcal{O}_D) - \chi(\mathcal{O}_{E+D}) \tag{5.7}$$

in particular $(E.D) = (D.E)$. If furthermore E, D are non negative, then

$$(E.D) = \chi(\mathcal{O}_E \otimes_{\mathcal{O}_X} \mathcal{O}_D) - \chi(\mathcal{T}or^1_{\mathcal{O}_S}(\mathcal{O}_E, \mathcal{O}_D)) \tag{5.8}$$

- *iii) If E and D are non negative, and intersect at only finitely many points, then $(E.D) = \sum_{x \in E \cap D} \dim_k \mathcal{O}_{X,x}/(f_x, g_x)$, where f_x (resp. g_x) denotes a local equation of E (resp. D) at x.*

Let us return to a desingularization $\pi : \tilde{X} \to X$ of a normal surface X. We want to study the exceptional fiber over a singular point $p \in X$. There is no loss of generality to assume $X = \mathrm{Spec}\, R$, where R is a normal two dimensional algebra, and that p is the only singular point. Let $\mathcal{M} \subset R$ be the corresponding maximal ideal, and let

K denote the fraction field of R. The first information about the exceptional fiber $\pi^{-1}(p)$ is:

Proposition 5.3. *The exceptional fiber $\pi^{-1}(p)$ is connected.*

Proof:
The morphism π is an isomorphism over $X-p$, thus $\pi_*(\mathcal{O}_{\tilde{X}})$ is finitely a sub R-algebra of K, finitely generated as R-module, therefore must be equal to R. Then the connectedness theorem of Zariski yields the result (see [34]). $\square$

Let $E_1, \cdots, E_r$ denote the irreducible components of $\pi^{-1}(p)$. As a 1-cycle we can write $\pi^{-1}(p) = \sum_i n_i E_i$. It is very convenient to picture the exceptional fiber as a graph (the dual graph), with vertices the $E_i's$, and if $i \neq j$, there are $(E_i.E_j)$ edges between E_i and E_j. This graph is then connected. The following is very useful result due to Du Val and Mumford.

Theorem 5.4. *The symmetric matrix $\|(E_i.E_j)\|$ is negative definite.*

Proof:
Let us choice $0 \neq f \in \mathcal{M}$. We can see f as a regular function on $\tilde{X}$. Its divisor is of the form

$$\operatorname{Div}(f) = \sum_i a_i E_i + D \tag{5.9}$$

where $a_i \in \mathbb{N}$, and D is a positive vertical divisor. Since $f \in \mathcal{M}$, then f vanishes identically on each E_i, therefore $a_i > 0$. Since π is an isomorphism over $X-p$, then we can identify D with the divisor of f computed in X. A well known theorem due to Krull says that any minimal prime ideal of R among those containing f is of height 1. Furthermore such an ideal contained in $\mathcal{M}$ must exist. This means that D must intersect one of the $E_i's$, i.e. that $(\sum_i E_i).D > 0$. Intersecting the equality (5.9) with E_i yields

$$\sum_j a_j(E_i.E_j) + (E_i.D) = 0 \Longrightarrow \sum_j a_j(E_i.E_j) \leq 0 \tag{5.10}$$

the inequality necessary strict for some index i. Set $c_{ij} = (E_i.E_j)$, and $s_i = \sum_j a_j c_{ij}$. Useful is the Coxeter trick: for any $x_1, \cdots, x_r \in \mathbb{R}$, we have the identity

$$\sum_{i,j} c_{ij}x_i x_j = \sum_k \frac{s_k x_k^2}{a_k} - \frac{1}{2}\sum_{i,k} a_i a_k c_{ik}\left(\frac{x_i}{a_i} - \frac{x_k}{a_k}\right)^2 \tag{5.11}$$

Notice $s_i \leq 0$, and even strictly negative for at least one index. Thus the left hand side of (5.11) is ≤ 0. We must check it is indeed stricly negative unless $x_1 = \cdots = x_r = 0$. Suppose it is zero. If for example $s_k < 0$, then this forces to have $x_k = 0$. Furthermore for all i with $c_{ik} \neq 0$, that is if the vertices E_k and E_i are connected by at leat an edge, we must have $x_i = 0$. The connectedness of the dual graph implies readily that all for all j, $x_j = 0$. This proves the Mumford result. $\square$

Notice a byproduct of the theorem is that for all $i, (E_i)^2 < 0$. Let $Z = \sum_{i=1}^r a_i E_i$ be a positive cycle ($a_i > 0$) such that $(Z, E_i) \leq 0\,(\forall i)$. Such a cycle exists by the construction above, but better [1]:

Lemma 5.5. *Among the positive cycles $Z = \sum_{i=1}^{r} a_i E_i$, $a_i > 0\,\forall i$, there exists a smallest one. This uniquely defined cycle is called the fundamental cycle.*

Proof:
Suppose $Z_1 = \sum_i a_i E_i$, $Z_2 = \sum_i b_i E_i$ are two such cycles. We set $c_i = \mathrm{Inf}(a_i, b_i) > 0$, then $Z_3 = \sum_i c_i E_i = \mathrm{Inf}(Z_1, Z_2)$. For a given j, if for example $c_j = a_j \leq b_j$, we have

$$(Z_3, E_j) = a_j(E_j)^2 + \sum_{i \neq j} c_i(E_i, E_j) \leq a_j(E_j)^2 + \sum_{i \neq j} a_i(E_i, E_j) = (Z_1, E_j) \leq 0 \tag{5.12}$$

The lemma follows easily. □

Definition 5.6. *The desingularization $\pi : \tilde{X} \to X$ is minimal if π don't factors through a smooth surface, i.e. in any factorization $\pi : \tilde{X} \xrightarrow{h} Y \to X$, with Y smooth, then h is an isomorphism.*

Recall that a component E_i of the exceptional fiber is of the first kind if $E_i \cong \mathbb{P}^1$ and $E_i^2 = -1$. It is a fundamental result (see [34]) that such curve can be contracted smoothly, i.e. there is factorization (a contraction) $\pi : \tilde{X} \xrightarrow{\nu} Y \to X$ with Y smooth, $\nu(E_i) = p_i \in Y$ a point, and $\nu : \tilde{X} - E_i \xrightarrow{\sim} Y - p_i$. Thus the desingularization $\pi : \tilde{X} \to X$ is minimal if and only if the irreducible components of the exceptional fiber are not of the first kind.

Proposition 5.7. *Assume the desingularization $\pi : \tilde{X} \to X$ crepant, then it is minimal.*

Proof:
Let E be an irreducible component of the exceptional fiber. The assumption amounts to $K_{\tilde{X}} \otimes \mathcal{O}_E \cong \mathcal{O}_E$. The adjunction formula yields $K_E = \mathcal{O}_E(E)$, which in turn gives

$$\deg K_E = (E^2) = \chi(K_E) - \chi(\mathcal{O}_E) = -2\chi(\mathcal{O}_E)$$

It is known that $\chi(\mathcal{O}_E) = 1 - p_a(E)$ where $p_a(E) = p_g(E) + \delta(E)$ is the arithmetic genus. Thus we get

$$2p_a(E) - 2 = (E^2) < 0$$

which implies that $p_a(E) = 0$. Thus $p_g(E) = \delta(E) = 0$ giving the fact that E is a smooth rational curve with self intersection -2. This show the resolution is minimal. □

We can now define the special but very workable class of *rational singularities* of normal surfaces ([1]).

Definition 5.8. *Let $X = \mathrm{Spec}(R)$ be a normal affine surface, say with only one singular point. Let $\pi : \tilde{X} \to X$ be a minimal resolution of the singularities of X. We say X has rational singularities if $\mathrm{R}^1\pi_*(\mathcal{O}_{\tilde{X}}) = 0$ (the first higher direct image is zero).*

It is a theorem of Artin ([1], theorem 3) that the singularity of X is rational if and only if $\chi(Z) = 1$, if furthermore $Z^2 = -2$ then the singular point is called a rational double point (loc.cit cor 3). A simple and well-known combinatorial lemma about negative definite integral matrix leads quickly to the classification of the dual graph attached to a rational double point.

Lemma 5.9. *Let $A = (a_{ij})$ be $n \times n$ integral symmetric matrix with $a_{ii} = -2$ for all $i = 1, \cdots, n$, and $a_{ij} \geq 0$ if $i \neq j$. We assume also that A is irreducible, which means there is no non trivial partition $[1, n] = I \sqcup J$ with $a_{ij} = 0$ for all $i \in I, j \in J$. Then there is a Dynkin graph of A, D, E type (i.e a simply laced root system) with incidence matrix C such that*[⋆]

$$A = C - 2I_n \tag{5.13}$$

Proof:
See [7]

□

It is a nice fact that, at least in characteristic zero, the rational double point are classified by means of the associated dual graph [44].

Theorem 5.10. *Let R be the local ring of a normal surface at a rational double point defined over an algebraically closed field of characteristic zero. Then $\hat{R}$ is isomorphic to one of the five type of singularities of the list 1.1.*

Such a singular point is in turn classified by the dual graph of its minimal resolution, and formally isomorphic to a quotient singularity $\mathbb{A}^2/G$ for a binary polyhedral group with same ADE label.

Example 5.1. *Resolution of the A_{n-1} point*

There are many ways to resolve the rational double points [48], [53]. Here we work by hand the A_n case. We first resolve the A_1 singularity. Let us blow up the x-y-z space at $x = z = 0$. Namely, we replace the x-y-z space by a union of two spaces with coordinates $(x, y, \widetilde{z})$ and $(\widetilde{x}, y, z)$ which are mapped to the x-y-z space by $(x, y, z) = (x, y, x\widetilde{z}) = (z\widetilde{x}, y, z)$. The x-y-$\widetilde{z}$ and the $\widetilde{x}$-y-z spaces are glued by $\widetilde{z}\widetilde{x} = 1$ and $z = x\widetilde{z}$. The equation $xy = z^2$ of the A_1 singularity looks as $x(y - x\widetilde{z}^2) = 0$ in the x-y-$\widetilde{z}$ space and $z(\widetilde{x}y - z) = 0$ in the $\widetilde{x}$-y-z space. If we ignore the piece described by $x = 0$ and $z = 0$ which is mapped to the y-axis $x = z = 0$, we obtain a union of two smooth surfaces

$$U_1 = \{y = x\widetilde{z}^2\} \cup U_2 = \{\widetilde{x}y = z\}$$

The surfaces U_1 and U_2 are coordinatized by $(x, \widetilde{z})$ and $(\widetilde{x}, y)$ respectively and are glued together by $\widetilde{z}\widetilde{x} = 1$ and $x\widetilde{z} = \widetilde{x}y$. Thus, we obtain a smooth surface. This surface is mapped onto the original singular by $(x, y, z) = (x, x\widetilde{z}^2, x\widetilde{z})$ on U_1 and $(x, y, z) = (\widetilde{x}^2 y, y, \widetilde{x}y)$ on U_2. The inverse image of the singular point $x = y = z = 0$ is described by $x = 0$ in U_1 and by $y = 0$ in U_2. It is coordinatized by $\widetilde{z}$ and $\widetilde{x}$ which are related by $\widetilde{z}\widetilde{x} = 1$, and thus is a projective line $\mathbf{P}^1$.

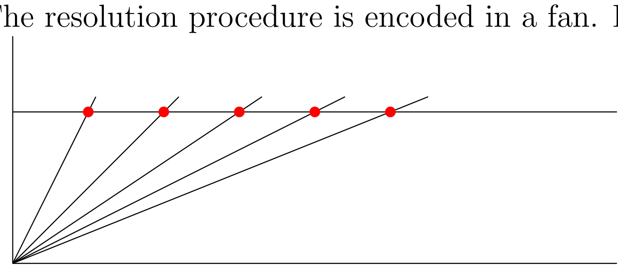
If we started with higher A_{n-1} singularity, the equation $xy = z^n$ looks as $y = x^{n-1}\widetilde{z}^n$ in the x-y-$\widetilde{z}$ space and $\widetilde{x}y = z^{n-1}$ in the $\widetilde{x}$-y-z space (ignoring the trivial piece $x = 0$ and $z = 0$). It is smooth in the x-y-$\widetilde{z}$ plane but the part is the $\widetilde{x}$-y-z has the A_{n-2}

[⋆] $-A$ is the Cartan matrix of the root system singularity at $\widetilde{x} = y = z = 0$. Thus, the surface is not yet resolved but n has decreased by one. We can further decrease $n-1$ by one by blowing up the $\widetilde{x}$-z plane at $\widetilde{x} = z = 0$. Iterating this process, we can finally resolve the singular A_{n-1} surface. It is straightforward to see that the resolved space is covered by n planes U_1, U_2, U_3, ..., U_n with coordinates $(x_1, z_1) = (x, \widetilde{z})$, $(x_2 = \widetilde{x}, z_2)$, (x_3, z_3), ..., $(x_n, z_n = y)$ which are mapped to the singular A_{n-1} surface by

$$U_i \ni (x_i, z_i) \longmapsto \begin{cases} x = x_i^i z_i^{i-1} \\ y = x_i^{n-i} z_i^{n+1-i} \\ z = x_i z_i \end{cases} \tag{5.14}$$

The planes U_i are glued together by $z_i x_{i+1} = 1$ and $x_i z_i = x_{i+1} z_{i+1}$. The map onto the singular A_{n-1} surface is isomorphic except at the inverse image of the singular point $x = y = z = 0$. The inverse image consists of $n-1$ $\mathbf{P}^1$s C_1, C_2, ..., C_{n-1} where C_i is the locus of $x_i = 0$ in U_i and $z_{i+1} = 0$ in U_{i+1}, and is coordinatized by z_i and x_{i+1} that are related by $z_i x_{i+1} = 1$. C_i and C_j do not intersect unless $j = i \pm 1$, and C_{i-1} and C_i intersect transversely at $x_i = z_i = 0$. It is also possible to show that the self-intersection of C_i is -2. Thus, we see that the intersection matrix of the components $C_1, \ldots, C_{n-1}$ is the opposite of the A_{n-1} Cartan matrix.

A more aesthetic way to produce this resolution is by mean of a toric methods. The resolution procedure is encoded in a fan. For $n = 6$ it looks like:



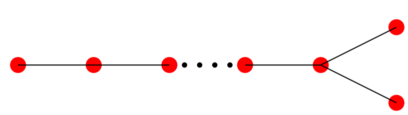
fan for $\mathbb{A}^2/\mathbb{Z}_6$

One can work out likewise the other singular points, and produce the dual graph of the exceptional curve in the resolved space.

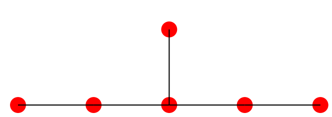
Dynkin$(A - D - E)$ *diagrams*

A_n

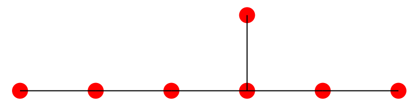
D_n



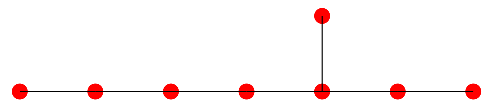
E_6



E_7



E_8



5.1.1. *Reflexive modules on a rational singularity.* Throughout this subsection R means the completion of the local ring at a rational singular point of some surface. By $\mathcal{M}$ we denote the maximal ideal, and by k the residue field. The ring R is normal and $U = \operatorname{Spec} R - \{\mathcal{M}\}$ is a regular one dimensional scheme. The closed points of U are the height one prime ideals of R. The category we are interested in is a suitable full subcategory of R-Mod. Here is the definition of the objects*.

Definition 5.11. *Let M be a finitely generated module. If the canonical map $M \to M^{**} = \operatorname{Hom}(M, \operatorname{Hom}(M, R))$ is an isomorphism, then we say M is reflexive.*

A free module of finite rank is reflexive, a dual N^* is reflexive. There is a wellknown and easy criterion to check a module M is reflexive [16].

Proposition 5.12. *The conditions below are equivalent:*

1. *The module M is reflexive,*
2. $\operatorname{depth}(M) = 2$ *i.e M is a maximal Cohen-Macaulay module,*
3. *M is torsion free and $M = \cap_{\wp} M_{\wp}$ where the intersection runs over the height one prime ideals,*
4. *there exists a linear map $\varphi : R^p \to R^q$ with $M = \ker(\varphi)$.*

Proof:
We limit ourselves to the last assertion. Assume M is reflexive. Let $R^q \to R^p \to M^* \to 0$ be a presentation of the dual module. Then applying the right exact functor $\operatorname{Hom}_R(-, R)$ we get an exact sequence

$$0 \to M = M^{**} \to R^p \to R^q$$

as expected. Conversely let M be a kernel $M = \ker(L = R^p \to R^q)$. Then M is torsion free so $M \subset M^{**} \subset L^{**} = L$. The module M^{**}/M is torsion and a submodule of $L/M \subset R^q$. Then $M = M^{**}$. $\square$

The definition extends immediatly on an arbitrary scheme. On a two dimensional regular ring (or regular scheme) a reflexive module (sheaf) is locally free. This is a direct consequence of the Auslander-Buchbaum formula [16].

On the other hand if R is the local ring at a two dimensional isolated singularity it is not difficult to identify the category of reflexive R-modules with the category of vector bundles on the open curve $U = \operatorname{Spec} R - \{\mathcal{M}\}$:

*In this definition R is any commutative ring

Lemma 5.13. *Let F be a vector bundle on U (locally free module of finite rank), then the R-module $M = \Gamma(U, F)$ is reflexive. The functor $F \mapsto \Gamma(U, F)$ yields an equivalence between the category of vector bundles on U, and the category of reflexive R-modules. The inverse functor is $M \mapsto \tilde{M}_{|U}$.*

Proof:
Let $\imath : U \hookrightarrow \operatorname{Spec} R$ be the injection. It is known that we can find for a given vector bundle (or coherent sheaf) F on U a finitely generated R-module N with $F = \imath^*(\tilde{N})$. Then $\Gamma(U, F) = \Gamma(U, \tilde{N})$. It suffices to show that $\Gamma(U, \tilde{N}) = N^{**}$ the double dual. The module N is locally free in codimension one, because F is, so we may take N^{**} in place of N, and assume N is reflexive. Since $\operatorname{depth}(N) = 2$ (5.12) the restriction map

$$N \longrightarrow \Gamma(U, \tilde{N})$$

is bijective. This proves the lemma. $\square$

Assume now that R is the complete local ring at the singular point of $\mathbb{A}^2/G$, where $G \subset \mathbf{SU}(2)$ as in 1.1. Here we assume $k = \mathbb{C}$, but all work more generally over an algebraically closed field of characteristic zero, and $G \subset \mathbf{SL}(2, k)$. In this setting there is another description of the previous category. We set $S = \hat{\mathcal{O}}_{\mathbb{A}^2,0}$, so $R = S^G$.

Proposition 5.14. *The map $V \mapsto (S \otimes V)^G$ induces a one to one correspondance between reflexive R-modules and representations of G. In particular indecomposable reflexives modules are in bijection with the irreps of G.*

Proof:
Let M be a reflexive R-module (5.12). We can see M as the kernel of $\varphi : R^p \to R^q$. We extend this map to a S-linear map

$$\varphi : S^p \longrightarrow S^q$$

The entries of φ are in R, so $N = \ker(S^p \to S^q)$ is a G-submodule of S^p. It is also a reflexive S-module. But S is regular so the homological dimension of N is $\dim R - \operatorname{depth}(N) = 0$, so N is free. The characteristic of the base field is zero so it is easy to see that as (S, G)-module

$$N = S \otimes_k V \quad (V = N \otimes_S k) \tag{5.15}$$

where $V = N \otimes_S k$, and finally $M = N^G$. It is not difficult to check that conversely for any G-module V, the R-module $M = (S \otimes_k V)^G$ is reflexive. Indeed if π : $\operatorname{Spec} S - \{0, 0\} \to U$ denotes the etale cover with base the open curve U, then $M = \Gamma(U, \pi_*(\mathcal{O} \otimes V)^G)$, which in turn yields the result. The fact that the correspondance is one to one follows from the remark that we can recover V from $M = (S \otimes V)^G$. Indeed it is esay to check that $S \otimes V \cong (M \otimes_R S)^{**}$.

$\square$

Corollary 5.15. *Under the same assumptions as before there is only a finite number of indecomposable reflexive R-modules.*

In [3], [4], Auslander and Reiten proved a much more difficult result, valid for all rational double point over any algebraically closed field k, even when there is no reference to a finite group. This should be the case in char $= 2, 3, 5$. The McKay quiver must be replaced by the AR quiver (Auslander-reiten quiver)

Theorem 5.16. *Let R be a normal two dimensional noetherian complete local ring, defining a rational double point. Then the category of reflexive modules is of finite representation type (the number of indecomposable reflexive modules is finite). The corresponding AR quiver is an A-D-E quiver (once the vertex corresponding to R removed).*

$\square$

Example 5.2. *A $\mathbb{Z}/2\mathbb{Z}$ quotient singularity in characteristic two*

To motivate the fact that in characteristic two the McKay graph (quiver) collapses and must be replaced by the AR-quiver, not defined in these notes, we can use the Mumford example. In this example k is an algebraically closed field of characteristic two, the group $G = \mathbb{Z}/2\mathbb{Z}$ acts non linearly on $S = k[[u, v]]$ according to

$$\sigma(u) = \frac{u}{1+u}, \sigma(v) = \frac{v}{1+v} \tag{5.16}$$

It is not difficult to check that the invariant subring is $R = k[[x, y, z]]$ where

$$x = \frac{u^2}{1/u}, \ y = \frac{v^2}{1+v}, \ \ z = uy + vx \tag{5.17}$$

The ring if invariants is an hypersurface algebra

$$R = k[[X, Y, Z]]/(Z^2 + XYZ + XY(X + Y)) \tag{5.18}$$

Therefore Spec R is is normal with an isolated singularity at $\mathcal{M} = (x, y, z)$. The corresponding singularity is a D_4 rational double point [1]. From the remark above one can find exactly three indecomposables reflexive non free modules of rank one showing the difference with the characteristic zero case. Viewing S as a R-module defines a rank two indecomposable reflexive R-module.

On the other hand, and in the case of an arbitrary two dimensional rational singularity, it is important to figure out some relationship between on one hand the reflexives modules on R, and on other hand suitable sheaves lying on a minimal resolution $\tilde{X} \to X = \operatorname{Spec} R$. The result is a key step in the geometric proof of the McKay correspondance by Gonzalez-Sprinberg and Verdier [25]. Obviously when R is an invariant ring of an ADE group, and in characteristic zero, the result is simply reminiscent to the fact to be proved below that the G-Hilbert scheme yields the minimal resolution. Even when the group interpretation collapses, such a relationship remains, the minimal resolution is the universal scheme that makes the strict transform of the reflexive R-modules flat.

The result below, see for example [20], makes a bridge between the reflexive modules over R and a special class of vector bundles on the minimal resolution. This is a key step to understand the geometry behind the McKay remark. Obviously with the knowledge that the minimal resolution is given by the G-Hilbert scheme, some part of the theorem are tautological.

Theorem 5.17. *Let R be a two-dimensional (complete) normal noetherian local ring with a rational singularity*[⋆]. *Let $\pi : \tilde{X} \to X = \mathrm{Spec}(R)$ be the minimal resolution. Let M be a reflexive R-module, then its strict transform (1.4) $M^\sharp = \pi^*(M)/(tors)$ enjoys the following properties:*

1. *$M^\sharp$ is locally free and generated by its global sections,*
2. *One has $\Gamma(\tilde{X}, M^\sharp) = M$ and $H^1(\tilde{X}, M^{\sharp *}) = 0$.*

The map $M \mapsto M^\sharp$ yields an equivalence between the category of reflexive R-modules and the category of special vector bundles on $\tilde{X}$, i.e the vector bundles for which 1) and 2) holds true.

Proof:
Throughout M will denote either the module or the associated sheaf on $\mathrm{Spec}\, R$. If we write M as a quotient of the free module R^n, then $\pi^*(M)$ is a quotient of $\mathcal{O}^n_{\tilde{X}}$. Then we get a surjective map $\mathcal{O}^n_{\tilde{X}} \to M^\sharp$. Since $\mathbf{R}^2\pi_* = 0$, an obvious exact sequence, taking into account of the fact that $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$, yields $\mathbf{R}^1(M^\sharp) = 0$. It is easy to check that $\mathbf{R}\pi_*(M^\sharp) = M$. Indeed let N denote the R-module $\Gamma(\tilde{X}, M^\sharp)$. It is finitely generated since π is proper. Pulling back the map $\pi^*(M) \to M^\sharp$, yields a map $\pi_*\pi^*(M) \to \pi_*(M^\sharp)$, which together with the canonical map $M \to \pi^*\pi^*(M)$, gives us a map

$$M \to N \tag{5.19}$$

This map is obviously the identity over the open set U. Now we have the commutative diagram

$$\begin{array}{ccc} M & \longrightarrow & n \\ \downarrow & & \downarrow \\ \Gamma(U, M) & \xrightarrow{=} & \Gamma(U, N) \end{array} \tag{5.20}$$

The left vertical map is bijective since M is Cohen-Macaulay, and the right vertical map is into since N is torsion free. Therefore $M = N$.

Now we are going to prove 1). That $M^\sharp$ is generated by its global sections is clear from the contruction. We now check $M^\sharp$ is locally free, equivalently reflexive, together the last part of 2). This part of the proof uses a bit of homological algebra, the local and global duality theorem as exposed by Hartshorne [35]. To check the stalk of $M^\sharp$ at some point $x \in \tilde{X}$ is Cohen-Macaulay amounts to see the vanishing property of local cohomology groups $\mathbf{H}^j_x(M^\sharp_x) = 0$ for $j < 2$. The local duality theorem applied to a finitely generated $\mathcal{O}_c$-module F gives a canonical isomorphism[⋆]

$$\mathbf{H}^j_x(F) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_x}(\mathrm{Ext}^{2-j}(F, \mathcal{O}_c), I) \tag{5.21}$$

where I denotes the injective hull of $k = k(x)$. So we must check that for all $x \in \tilde{X}$ we have $\mathrm{Ext}^k(M^\sharp_x, \mathcal{O}_c) = 0$ for $k > 0$. This is equivalent to the vanishing of the sheaves $\mathcal{E}xt^k(M^\sharp, \mathcal{O}_{\tilde{X}})$ for $k > 0$ equivalently the complex $\mathbf{R}\mathrm{Hom}(M^\sharp, \mathcal{O}_{\tilde{X}})$ on $\tilde{X}$

[⋆]the complete local ring at a singular point of a normal surface

[⋆]$\tilde{X}$ is regular so the local dualizing complex at x is $\mathcal{O}_{\tilde{X},x}$

has only cohomology in degree zero. But $M^\sharp$ is torsion free, so is locally free in codimension one, which in turn said that the higher cohomology of our complex is concentrated in finitely many points of the exceptional fiber. To check the vanishing proiperty is therefore equivalent to the vanishing in degree $k > 0$ of the cohomology of $\mathrm{R}\pi_*(\mathbf{R}\mathrm{Hom}(M^\sharp, \mathcal{O}_{\tilde{X}}))$. But now the global duality theorem [35] yields an isomorphism

$$\pi_*(\mathbf{R}\mathrm{Hom}(M^\sharp, \mathcal{O}_{\tilde{X}})) \xrightarrow{\sim} \mathbf{R}\mathrm{Hom}_{\mathcal{O}_X}(\mathrm{R}\pi_*(M^\sharp), \omega_X)) \tag{5.22}$$

where ω_X denote the dualizing sheaf of X. We known that $\mathbf{R}\pi_*(M\sharp) = M$. This is a Cohen-Macaulay module so the previous local-duality argument shows that the complexe on the right has only cohomology in degree zero. The spectral sequence in degree $p + q = 1$

$$\mathrm{E}_2^{p,q} = \mathbf{R}^p\pi_*(\mathrm{Ext}^q(M^\sharp, \mathcal{O}_{\tilde{X}})) \Longrightarrow \mathrm{Ext}^1(M, \omega_R) = 0 \tag{5.23}$$

degenerates and yields what we want

$$\pi_*(\mathrm{Ext}^1(M^\sharp, \mathcal{O}_{\tilde{X}})) = \mathbf{R}^1\pi_*(M^\sharp) = 0 \tag{5.24}$$

The last thing to check is the fact that a *special* vector bundle on $\tilde{X}$ comes from a reflexive module. This is clear because if F is such vector bundle, the candidate is $M = \Gamma(\tilde{X}, F) = \mathbf{R}\pi_*(F)$. From 1) there is a surjective map

$$\pi^*(M) \longrightarrow F$$

with kernel the torsion subsheaf T of $\pi^*(M)$, therefore $F = M^\sharp$. □

5.1.2. *Matrix factorizations.* There are indeed two results in the theorem 5.16. The first says that for isolated singularities in any dimension $d \geq 2$, the full subcategory of maximal Cohen-Macaulay modules, i.e the Cohen-Macaulay modules with dimension d, [16] has almost-split (A-R) sequences [3]. For rational double points this category has finite representation type, a companion result to Gabriel's theorem [10]. As a byproduct of the proof, the minimal number of generators of a reflexive module is twice its rank.

The category of reflexive modules (when $d = 2$), or maximal Cohen-Macaulay when $d \geq 2$, $\mathbf{MCM}(R/(w))$, is enhanced if we think it in the stable sense. In the stable category the set of morphisms $\underline{\mathrm{Hom}}(M, N)$ is the quotient of $\mathrm{Hom}_R(M, N)$ be the submodule consisting of linear maps which factors through a free module. This rather abstract category has a rich structure. It is a Krull-Schmidt category and is in a natural way a triangulated category. It is nice fact that we can interpret this triangulated category as the homotopy category of the $\mathbb{Z}/2\mathbb{Z}$-graded differential category of *matrix factorizations* of the defining equation of the singularity, the so-called *super potential* [56].

The definition goes as follows. Let $w(x_1, \cdots, x_r) \in R = k[x_1, \cdots, x_r]$ be a polynomial, the so-called (super)potential in the physics litterature. Even if in most applications $w(x)$ is choosen homogeneous, or quasi-homogeneous, it is not necessary to impose such a restriction in this introduction . Throughout 1_n denotes the $n \times n$ identity matrix.

Definition 5.18. *By a rank n matrix factorization (MF in short) of w, we mean the data of a pair of $n \times n$ square matrices P, Q with polynomial entries, such that*

$$PQ = QP = w(x).1_n \tag{5.25}$$

Alternatively we can see 5.25 as a diagram with two arrows between free R-modules of rank n, with the periodicity condition, source of φ equal target of ψ, and $\psi\varphi = \varphi\psi = w.1$.

$$(\mathbf{M}) \quad M_0 \cong R^n \xrightarrow{\varphi} M_1 \cong R^n \xrightarrow{\psi} M_0 \tag{5.26}$$

Indeed put $M_0 = M_1 = R^n$, and take for φ and ψ the linear mapping with respective matrix P and Q. In the sequel we will use 5.25 and 5.26 interchangeably, and will denote by a bold letter $\mathbf{M}$ such MF. Notice at this stage there no need to work with a polynomial ring. So the properties listed below are equally valid with an arbitrary commutative ring R as ground ring, and $w \in R$ a non zero divisor. Another classical choice for deformation theoretic reasons is $R = k[[x_1, \cdots, x_r]]$ a power series ring.

A MF $\mathbf{M}$ is trivial if either φ or ψ is an isomorphism, and said reduced if $P(0) = Q(0) = 0$. Troughout a MF is non trivial and reduced. Notice that 5.25 yields $\det P \det Q = w(x)^n$, so the linear mapping φ and ψ are injective.

It is important to see a MF $\mathbf{M}$ as a $\mathbb{Z}/2\mathbb{Z}$-graded module $\mathbf{M} = M_0 \oplus M_1$, endowed with as an odd operator $\mathbf{Q}$ which is a square root of w, precisely

$$\mathbf{Q} = \begin{pmatrix} 0 & \psi \\ \varphi & 0 \end{pmatrix} : \mathbf{M} \to \mathbf{M} \quad (\mathbf{Q}^2 = w.1) \tag{5.27}$$

We hope there is no trouble to denote by the same bold letter $\mathbf{M}$ a MF, and the same object viewed as a R-graded module equipped with the odd operator Q. Hereafter a bold character will always means a graded object. Let us define the morphisms between two MF $\mathbf{M}$ and $\mathbf{M}'$.

Definition 5.19. *Let $\mathbf{M}, \mathbf{M}'$ be two MF of the same potential $w \in R$. A MF-morphism $f = (A, D) : \mathbf{M} \to \mathbf{M}'$ is the data of a commutative diagram*

$$\begin{tikzcd} M_0 \arrow[r, "\varphi"] \arrow[d, "A"'] & M_1 \arrow[r, "\psi"] \arrow[d, "D"'] & M_0 \arrow[d, "A"'] \\ M_0' \arrow[r, "\varphi'"] & M_1' \arrow[r, "\psi'"] & M_0' \end{tikzcd}$$

where A, D are R-linear maps, i.e given by matrices with polynomials entries. The MF morphism f is an isomorphism if and only if A and D are.

Said differently an isomorphism (A, D) between the MF $\mathbf{M} = (P, Q)\, \mathbf{M}' = (P', Q')$ is a simultaneous similarity $P' = DPA^{-1}$, $Q' = AQD^{-1}$.

In other hand, and more generally, the module $\hom_R(\mathbf{M}, \mathbf{M}')$ of all R-linear mapping $\mathbf{M} \to \mathbf{M}'$ can be seen in a natural manner as a $\mathbb{Z}/2\mathbb{Z}$-graded module, i.e a module with an even (resp. odd) part. The notation $\hom_R^{\bullet}(\mathbf{M}, \mathbf{M}')$ i.e the dot, will refer to this graded structure. It is important to notice $\hom_R^{\bullet}(\mathbf{M}, \mathbf{M}')$ has a richer structure, it may be endowed with a $\mathbb{Z}/2\mathbb{Z}$-differential graded module struture with differential the graded bracket

$$D(\xi) = \mathbf{Q}'\xi - (-1)^k \xi \mathbf{Q} = [\mathbf{Q}, \xi] \quad (k = \deg \xi \in \mathbb{Z}/2\mathbb{Z}) \tag{5.28}$$

An easy check yields $D^2 = 0$. Furthermore $\hom_R^\bullet(\mathbf{M}, \mathbf{M})$ is easily seen to be a dg-algebra. With this structure in mind, the MF-morphisms as previously defined, are in this setting the even morphisms annihilated by D, i.e are the even cocycles. Two MF-morphisms f, g such that $f - g = D(h)$ for some odd morphism h are said homotopic. For convenience of the reader we include a diggest of homological algebra. By a dg-module (a module is an R-module) it is meant a $\mathbb{Z}/2\mathbb{Z}$-graded module $C = C^0 \oplus C^1$, together with an odd operator $D : C \to C$ with $D^2 = 0$, called the differential. The (co)homology module $\mathbf{H}(C) = \ker D/\mathrm{Im} D$ is also clearly graded, thus $\mathbf{H}(C) = \mathbf{H}^0(C) \oplus \mathbf{H}^1(C)$. A dg-algebra is a dg-module in the previous sense, endowed with an even multiplication $C \otimes C \to C$, $a \otimes b \to ab$, such that the following Leibniz rule holds

$$D(ab) = D(a)b + (-1)^{\deg a} a D(b)$$

A dg-morphism $f : C \to C'$ between dg-modules is an even morphism commuting with the differentials. A dg-isomorphism is defined accordingly*

With the MF with fixed potential w as objects we can build various categories. The first one is obtained if we take as morphisms the MF-morphisms as previously defined. The previous observation showing $\hom_R^\bullet(\mathbf{M}, \mathbf{M}')$ is a dg-module, leads to the fact that $\mathcal{MF}(w)$, the category whose objects are the MF, and with morphisms between $\mathbf{M}$ and $\mathbf{M}'$ the dg-module $\hom_R^\bullet(\mathbf{M}, \mathbf{M}')$, is a $\mathbb{Z}/2\mathbb{Z}$-dg-category.

We can perform inside this category another basic contruction. Let $\mathbf{M}$ be a MF of w. We set

$$\mathbf{M}[1] : \quad M_1 \xrightarrow{-\psi} M_0 \xrightarrow{-\varphi} M_1 \tag{5.29}$$

Clearly $\mathbf{M}[1]$ is a MF of w. The functor $\mathbf{M} \to \mathbf{M}[1]$ is the shift (involutive) functor $\mathcal{MF}(w) \to \mathcal{MF}(w)$. Finally we can also consider morphisms, even or odd, up to homotopy. Indeed taking cohomology in the graded differential module $\hom^\bullet(\mathbf{M}, \mathbf{M}')$ yields a graded cohomology module

$$\mathrm{H}^\bullet(\hom^\bullet(\mathbf{M}, \mathbf{M}')) = \mathrm{Ext}^\bullet(\mathbf{M}, \mathbf{M}') = \frac{\ker D}{\mathrm{Im} D} = \mathrm{Ext}^0(\mathbf{M}, \mathbf{M}') \oplus \mathrm{Ext}^1(\mathbf{M}, \mathbf{M}') \tag{5.30}$$

The module $\mathrm{Ext}^0(\mathbf{M}, \mathbf{M}')$ is nothing but the module of MF-morphisms taken up homotopy, and $\mathrm{Ext}^1(\mathbf{M}, \mathbf{M}') = \mathrm{Ext}^0(\mathbf{M}, \mathbf{M}'[1])$ is the module of odd morphisms up homotopy. The category we are interested in is described as follows. Let us denote $\mathbf{MF}_R(w)$ (or $\mathbf{MF}(w)$) the homotopic category of MF for a fixed potential w, that is the homotopic category derived from the dg-category $\mathcal{MF}(w)$. In this category the objects are the MF, and the morphisms between $\mathbf{M}$, $\mathbf{M}'$, are the homotopic classes of even MF-morphisms, i.e the elements of $\mathrm{Ext}^0(\mathbf{M}, \mathbf{M}')$. It is not difficult to check

*In a more abstract setting, there is a concept of dg-category which extends the notion of dg-algebra. A dg-category means an R-linear category such that the morphisms set $\hom(a, b)$ are endowed with a dg-module structure, the composition map $\hom(a, b) \otimes \hom(b, c) \to \hom(a, c)$ being compatible in an obvious sense with these dg-structures.

that $\mathbf{MF}(w)$ can be endowed with a structure of triangulated category (see exercice below).

Exercise 5.1. *Let $f : \mathbf{M} \to \mathbf{M}'$ be a morphism. Define the cone of f as the graded object $C(f) = \mathbf{M}[1] \oplus \mathbf{M}'$ together with the following operators*

$$\Phi = \begin{pmatrix} \psi & 0 \\ \varphi & \varphi' \end{pmatrix}, \quad \Psi = \begin{pmatrix} \varphi & 0 \\ A & \psi' \end{pmatrix} \tag{5.31}$$

where $f = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$. A distinguished triangle in $\mathbf{MF}_R(w)$ is one isomorphic to $\mathbf{M} \xrightarrow{f} \mathbf{M}' \to C(f) \to \mathbf{M}[1]$. Check we get in this way a triangulated structure on $\mathbf{MF}_R(w)$.

Exercise 5.2. *In this exercise we define the tensor product in the category $\mathbf{MF}_R(w+w')$. Suppose the potential splits as a sum $w + w'$, $(w, w' \in R)$. If we are given two MF $\mathbf{M}$, $\mathbf{M}'$ with respective potential w, w', then show the graded tensor product $\mathbf{M} \otimes_R \mathbf{M}'$ yields a MF with potential $w + w'$. As graded module $(\mathbf{M} \otimes \mathbf{M}')_0 = M_0 \otimes M_0' \oplus M_1 \otimes M_1'$ and $(\mathbf{M} \otimes \mathbf{M}')_1 = M_0 \otimes M_1' \oplus M_1 \otimes M_0'$. Then we set*

$$\Phi = \begin{pmatrix} 1 \otimes \varphi' & \psi \otimes 1 \\ \varphi \otimes 1 & -1 \otimes \psi' \end{pmatrix}, \qquad \Psi = \begin{pmatrix} 1 \otimes \psi' & \psi \otimes 1 \\ \varphi \otimes 1 & -1 \otimes \varphi' \end{pmatrix} \tag{5.32}$$

Check $\Phi\Psi = \Psi\Phi = (w + w')1$.

The matrix factorizations of w are relevant to our interest in maximal Cohen-Macaulay modules. Indeed we have the following result, where as indicated before $\mathbf{MCM}_{R/(w)}$ denotes the stable category of maximal Cohen-Macaulay on the hypersurface (isolated) singularity $R/(w)$. For any $\mathbf{M}$, we set $\mathrm{Cok}(\mathbf{M}) = \mathrm{coker}(\varphi)$. Clearly $w\mathrm{Cok}(\mathbf{M}) = 0$, so we can see $\mathrm{Cok}(\mathbf{M})$ as a $R/(w)$-module.

Theorem 5.20. *We have an equivalence of triangulated categories*

$$\mathrm{Cok} : \mathbf{MF}_R(w) \xrightarrow{\sim} \mathbf{MCM}_{R/(w)} \tag{5.33}$$

Proof:
The exact sequence $0 \to M_0 \xrightarrow{\varphi} M_1 \to \mathrm{Cok}(\mathbf{M}) \to 0$ shows that the depth of $\mathrm{Cok}(\mathbf{M})$ is the same as $\dim R/(w)$, so it is maximal Cohen-Macaulay. It is an easy exercise to check that $\mathrm{Cok}(-)$ defines a functor $\mathbf{MF}_R(w) \longrightarrow \mathbf{MCM}_{R/(w)}$. This functor is essentially surjective. Indeed, let E be a maximal Cohen-Macaulay module over $R/(w)$. Then as R-module it must have homological dimension one (Auslander-Buchsbaum formula). If $0 \to M_0 \xrightarrow{\varphi} M_1 \to E \to 0$ is a free resolution of E over R, then due to the fact that w annihilates E, it is easy to check that we can find $\psi : M_1 \to M_0$ making (φ, ψ) a matrix factorization.

□

Here we give one example of a matrix factorization, which in turn yields a presentation of a reflexive module. For simplicity we limit ourselves to the A_{n-1} case, i.e. $G = \mathbb{Z}/n\mathbb{Z}$, see [25] for more examples. Then $w(x, y, z) = z^n - xy$. The rank two indecomposable matrix factorizations of w are the following

$$w.1_2 = \begin{pmatrix} z^k & x \\ y & z^{n-k} \end{pmatrix} \begin{pmatrix} z^{n-k} & -x \\ -y & z^k \end{pmatrix} \quad (k = 1, \cdots, n-1) \tag{5.34}$$

Exercise 5.3. *Complete the proof of theorem 5.20, that is prove that* Cok *induces a group isomorphism* $\mathrm{Hom}_{\mathbf{MF}_{R}(w)} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{MCM}(R/(w)}$.

Exercise 5.4. *Express the matrix factorization of example 5.34 as a tensor product of two rank one factorizations of z^n and xy respectively.*

5.2. **ADE world and the McKay correspondence.** In this section we come back to our preferred setting, that is $X = \mathbb{A}^2$ the affine plane over an algebraically closed field k of characteristic zero, and G is a finite subgroup of $\mathbf{SL}_2(k)$, classically known as a binary polyedral group. The quotient surface $\mathbb{A}^2/G$ is normal with an isolated singularity at the origin, which from the point of view of singularity theory is a rational double point. We previously saw there is a minimal resolution of singularities, which is unique up isomorphism. The type of this singularity i.e the dual graph of the exceptional fiber in the minimal resolution is the same as the ADE type of G (see (example 1.1).

5.2.1. *The G-Hilbert scheme of an A-D-E group.* From the point of view of the equivariant Hilbert scheme, we have a very nice result. It says the G-Hilbert scheme yields precisely the minimal desingularization. The McKay correspondance, at some level, amounts to identify the dual graph of exceptional fiber with the corresponding A-D-E Dynkin graph in group theoretical terms. The first result is an observation of J. McKay that this graph can be built with the group G alone, without the knowledge of the resolution. Here is a way to build a graph or a quiver starting with G, the so called McKay quiver. Let us denote V the representation of G which follows from the inclusion $G \subset \mathbf{SU}(2)$, and set α the character of V. Let $\{1 = \chi_0, \cdots, \chi_r\}$ be the ordered list of irreducible characters of G.

Definition 5.21. *The McKay graph (resp. quiver) associated to G is the non oriented (oriented) graph where*

- *the vertices are the non trivial irreducible characters of G,*
- *The number of edges (oriented edges) between χ_i and χ_j is $a_{ij} = (\chi_i, \alpha\chi_j)$, that is the number of times χ_i is contained in $\alpha\chi_j$.*

The extended McKay graph (resp. quiver) is the one obtained by adding the trivial representation.

The observation of McKay is

Proposition 5.22. *The McKay graph is an A-D-E graph. More precisely the incidence matrix of the McKay graph is $2I - C$ with C the Cartan matrix of an A-D-E root system. Likewise the extended McKay graphs (quivers) correspond to the extended (affine) $\tilde{A} - \tilde{D} - \tilde{E}$ root systems.*

Proof:
The result follows readily from a more general group theoretical result as suggested by T. Springer:

Lemma 5.23. *Let G be a finite group with center $Z(G)$. Let $G \to \mathbf{GL}(V)$ be a faithful representation with character $\alpha = \overline{\alpha}$ i.e with real values, and $n = \dim V$.*

Assume that $V^{Z(G)} = 0$. *Let* $\{\xi_0, \cdots, \xi_r\}$ *be the ordered set of irreductible characters of* G. *Define a* $(r+1) \times (r+1)$ *integral matrix by*

$$a_{ij} = (\alpha\chi_i, \chi_j) - n\delta_{i,j} \tag{5.35}$$

The integral matrix $|a_{ij}|$ *is symmetric semi-definite negative with kernel the line span by the vector* $(\chi_0(1), \cdots, \chi_r(1))$ *i.e the regular representation.*

Proof:
To begin with notice from the definition of the scalar product on $R(G) \otimes \mathbb{C}$ that

$$(\alpha\chi_i, \chi_j) = (\chi_i, \overline{\alpha}_j) = (\chi_i, \alpha\chi_j) = (\alpha\chi_j, \chi_i)$$

so the matrix (5.27) is symmetric. Furthermore $a_{ij} \geq 0$ if $i \neq j$. To evaluate the diagonal entries $a_{ii} = (\alpha\chi_i, \chi_i) - n$ the key point is the fact that

$$(\alpha\chi_i, \chi_i) = 0 \quad (i = 0, \cdots, r)$$

To check this, we restrict all the representations to $Z(G)$. So assume G acts on V_i (the representation space of χ_i) by the one dimensional character μ_i (Schur's lemma), and $Z(G)$ acts on V through the characters $\lambda_1, \cdots, \lambda_n$. Note that by our hypothesis $\lambda_k \neq 1$. Then as $Z(G)$ module $V \otimes V_i$ is the sum of $\chi_i(1)$ copies of V, the action on V being twisted by μ_i. If V_i is contained in $V \otimes V_i$ then we must have $\mu_i\lambda_k = \mu_i$ for some k, so $\lambda_k = 1$, contrary to our hypothesis. Thus $a_{ii} = -n$.

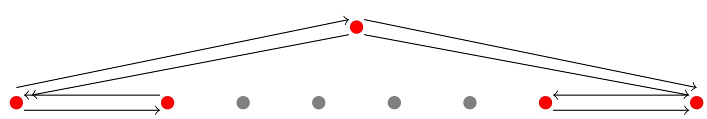
The kernel is the set of vectors $\xi = \sum_j n_j\chi_j \in R(G) \otimes \mathbb{C}$ such that $\sum_j a_{ij}n_j = 0$ for all i. Equivalently

$$((\alpha - n1)\chi_i, \xi) = (\chi_i, (\alpha - n1)\xi) = 0$$

so $(\alpha - n1)\xi = 0$. The representation of G in V is faithful so $\alpha(s) < n$ when $s \neq 1$. This implies $\xi(s) = 0$, so ξ is a multiple of the regular representation. The proof is complete.

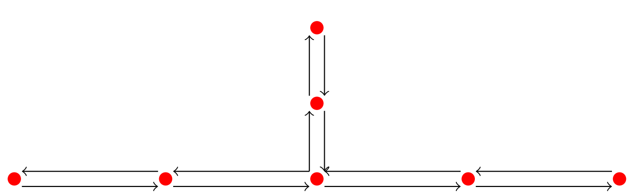
□

$G = \mathbb{Z}/(n+1)\mathbb{Z}$



It is easy to validate this picture. The irreps of $G = \mathbb{Z}/(n+1)\mathbb{Z}$ are labelled as $(\chi_0 = 1, \chi_1, \cdots, \chi_n)$ where $\chi(1) = e^{2i\pi/n+1}$. Then $V = \chi_1 \oplus \chi_n$ (notation of lemma 5.23). Clearly if $i \neq j$, $a_{ij} = 1 \iff |i - j| = 1$, 0 otherwise.

$\tilde{E}_6$ quiver



To get the Dynkin graph from the McKay quiver one has to replace $\bullet \rightleftarrows \bullet$ by an unoriented arrow $\bullet$ —— $\bullet$.

It is important to interpret the G-Hilbert scheme as a representation variety of the McKay quiver, construction analogous to the ALE space construction of Kronheimer, see [39]. This is a simple extension of the previous description of $\mathbf{H}_n$ as a representation variety. With the notations of subsection 3.2.4 a G-cluster can be obtained from a G-linear map

$$\Phi : R \longrightarrow T \otimes R \tag{5.36}$$

where $T\Omega^*$ is the representation dual to the standard representation of $G \subset \mathbf{SL}(\Omega)^{\star}$, and R stands for the regular representation. We have $R = \oplus_{i=0}^{r} R_i \otimes V_i$ the action of G being on V_i, and $\dim R_i = \deg \chi_i = \dim V_i$. Now the representation $T \otimes R$ splits as $T \otimes R = \bigoplus_i R_i \otimes T \otimes V_i$. Thus $\mathrm{Hom}_G(R, T \otimes R) =$

$$\begin{aligned} \bigoplus_{k,j} \mathrm{Hom}_G(R_k \otimes V_k, R_j \otimes T \otimes V_j) &= \bigoplus_{k,j} \mathrm{Hom}_k(R_k, R_j) \otimes \mathrm{Hom}_G(V_k, T \otimes V_j) \\ &= \bigoplus_{k,j} \mathrm{Hom}_k(R_k, R_j)^{a_{k,j}} \end{aligned} \tag{5.37}$$

where $a_{k,j}$ equal the number of times V_k is contained in $T \otimes V_j$. This show a G-cluster in the affine space $\Omega = \mathbb{A}^2$ is the same thing as a class of representations of the McKay quiver with the relation $\Phi \wedge \Phi = 0$. The gauge group is $\prod_{i=0}^{r} \mathbf{GL}(R_i)$.

The McKay quiver has arrows in both directions, we see it is the double $\overline{Q}$ of the corresponding quiver Dynkin quiver Q, i.e. the Dynkin graph together with the choice of an orientation. The equation $\Phi \wedge \Phi = 0$ is encoded in the defining relation of the path algebra $\Pi(Q)$. The dimension vector is $\delta = (\delta_i = \dim V_i)$. The last thing to do is to translate the fact that the distinguished vector given by the trivial character generates R as a stability condition. We refer to Ito-Nakajima [39] or to Ginzburg's notes ([24], 4.6 ; Thm 4.6.2) for a complete discussion.

Theorem 5.24. *For a group G of ADE type, the G-Hilbert scheme $G - \mathbf{H}_{\mathbb{A}^2}$ is smooth irreducible so coincides with $\mathbf{H}_G(\mathbb{A}^2)$, and the equivariant Hilbert-Chow morphism $\varphi : G - \mathbf{H}_{\mathbb{A}^2} \longrightarrow \mathbb{A}^2/G$ is the minimal desingularization. Moreover this is a crepant desingularization.*

Proof:
The G-Hilbert scheme is the fixed point subscheme for the natural action of G on $\mathbf{H}_n$. Since $\mathbf{H}_n$ $(n = |G|)$ is smooth, it follows that $G - H_{\mathbb{A}^2}$ is smooth, but perhaps not connected. It is not clear in this set-up to see if the fixed point set is connected or not. The fact that this is indeed the case is more transparent in the quiver varietiy language where more powerful results are available ([24], Thm 4.6.2). Then the equivariant Hilbert-Chow map

$$\varphi : G - \mathbf{H}_{\mathbb{A}^2} \longrightarrow \mathbb{A}^2/G \tag{5.38}$$

is birational, therefore yields a desingularisation of the normal surface $\mathbb{A}^2/G$. To check this is a minimal desingularization, it suffices to check it is crepant, i.e. that the canonical bundel of $G - \mathbf{H}_{\mathbb{A}^2}$ is trivial. To see why this is sufficient, one has to

*Indeed Ω is self-dual.

use the adjonction formula [34]. If $E \subset S := G - \mathbf{H}_{\mathbb{A}^2}$ is a smooth rational curve, this formula says that

$$K_E \cong (K_S)_{|E} \otimes N_{E/S} \tag{5.39}$$

where $N_{E/S}$ denote the normal bundle. There fore $E^2 = \deg(N_{E/S}) = \deg K_E = -2$. Thus E cannot be an exceptional curve of the first kind, i.e. contracted to a smooth point.

Thus the rest of the proof amounts to check that the canonical class of $S = G - \mathbf{H}_{\mathbb{A}^2}$ is trivial. Luchily we already know that $\mathbf{H}_{\mathbb{A}^2}$ has a symplectic structure. Therefore it suffices that this structure induces also a symplectic structure on the fixed point subset of G. The symplectic structure of $\mathbf{H}_{\mathbb{A}^2}$ comes from the obvious 2-form on $\mathbb{A}^2$, viz. $dx \wedge dy$. This form is G-invariant since G is a sugroup of $\mathbf{SL}_2(k)$. As a consequence the symplectic 2-form $\sum_i dx_i \wedge dy_i$ on $\mathbb{A}^2$ is G-invariant, meaning that G acts symplectically on $\mathbb{A}^n$. The conclusion will follows from the easy lemma:

Lemma 5.25. *Let Z be a smooth variety equipped with a non degenerate 2-form ω. Let G be a finite group that acts on Z, such that for all $g \in G$, we have $g^*(\omega) = \omega$. Let $Y = Z^G$ the loci of fixed points (a smooth subvariety). Then $\omega_{|Y}$ is non degenerate.*

Proof:
This is a purely algebraic lemma. Namely let V be a symplectic vector space, with symplectic form ω. Suppose G is a finite subgroup of the symplectic group of V. Then the restriction of ω to V^G is non degenerate. Let V_G be the subspace of coinvariants, i.e. span by the $gv - v\,(g \in G, v \in V)$. Then $V = V^G \oplus V_G$, is an orthogonal direct sum. This is obvious. Therefore if $v \in V^G$ is such that $\omega(v, V^G) = 0$, then $\omega(v, V) = 0$, and $v = 0$. The lemma is proved, and also the theorem. □

The very interesting problem is at this stage to identify the exceptional locus in terms of G-clusters. As said before, this is a McKay type problem. Such a description, case by case has been made by Ito-Nakamura [40]. Here is an example to made more explicit the formal point of view used in our presentation.

Example 5.3. *(The singularity A_n)*

In this example $G = \mathbb{Z}/n\mathbb{Z}$. Let ζ be a primitive n-root of 1. The group G acts on the affine plane according to the rule $(x, y) \mapsto (\zeta x, \zeta^{-1} y)$. Then the ring of invariant polynomials is $k[x, y]^G = k[x^n, y^n, xy] \cong k[u, v, w]/(uv - w^n)$. The following general lemma will help us.

Lemma 5.26. *Let G be a finite subgroup of $\mathbf{GL}_n(k)$. Then G acts in obvious way on $R = k[x_1, \cdots, x_n]$ i.e. $\mathbb{A}^n$. Let Z be any point of $G - \mathbf{H}_{\mathbb{A}^n}$ with support the origin $(0, \cdots, 0)$. Then I_Z contains all invariant polynomials $P \in R^G$, with $P(0) = 0$.*

Proof:
As G-module R/I_Z is the regular representation. Therefore it contains only one copy of the trivial representation, i.e. $(R/I_Z)^G = k.1$ the constants. If $P \in R^G$ with $P(0) = 0$, then its residue class $\overline{P} \in R/I_Z$ must be zero.

□

The lemme shows that I must contain the ideal $R_G \subset R$ generated by the invariant polynomials without constant term. The quotient algebra R/R_G is called the *coinvariant ring*. It is a finitely dimensional algebra. In turn this show that the points of the G-Hilbert scheme that lie over the origin, i.e. the exceptional fiber, form the "local" G-Hilbert scheme of the coinvariant algebra.

We now apply this to our example. The coinvariant algebra reduces to

$$k[x,y]/(x^n, y^n, xy) = (k[x,y]/(xy))/(x^n, y^n) \tag{5.40}$$

This reduces our search of G-clusters with support the origin, to find the G-clusters of the nodal algebra that contain x^n and y^n. It is not difficult to check that any point of the n-Hilbert scheme of the nodal algebra is good, and also is G-invariant. This follows from the precise description of these ideals Proposition 2.23. Therefore the exceptional fiber of the Hilbert-Chow map $G - \mathbf{H}_{\mathbb{A}^2} \to \mathbb{A}^2/G$ is identical to the n-Hilbert scheme of the nodal algebra. But we have shown that this scheme is exactly a string of $n-1$-smooth rational curves 2.1, therefore it yields the diagram A_{n-1}. $\square$

5.2.2. *The geometric McKay correspondance.* It is a very interesting problem to understand the geometry behind McKay's observation. Many speculations about this question are collected in the Bourbaki seminar by M. Reid [52]. To give some appetite to a reader, we state and discuss two answers. To simplify the ideas we assume the ground field is of characteristic zero, thus the reflexive modules are in one to one correspondance with the representations of G. The first answer is the following result due to Gonzalez-Sprinberg and Verdier [25]. If F is a rank n vector bundle on a scheme, its determinant, or first Chern class is the line bundle $c_1(F) = \wedge^n F$.

Theorem 5.27. *Let $G \subset \mathbf{SU}_2$ be an A-D-E group. Let $\pi : \tilde{X} \to X = \operatorname{Spec} R$ be the minimal resolution of the corresponding singularity. For any irreducible character $\chi \in \mathbf{Irrep(G)}$ there is an irreducible component $E_{i(\chi)} \in \{E_1, \cdots, E_r\}$ characterized by*

$$(c_1(M^\sharp_\chi).E_j) = \begin{cases} 1 & \text{if } j = i(\chi) \\ 0 & \text{otherwise} \end{cases}$$

The map $\chi \mapsto i(\chi)$ from $\mathbf{Irrep(G)}$ to the set of irreducible components of the exceptional fiber of π is a one to one, and induces an isomorphism between the McKay graph and the dual graph of the exceptional fiber.

This result as already said survives for any rational double point and in any characteristic [2], Theorem 1.11, [3]. Since in some cases i.e in characteristic 2,3,5 the quotient singularity set-up collapses, the result involves indecomposable reflexive modules.

Theorem 5.28. *Let R be a complete normal noetherian local ring such that $X = \operatorname{Spec} R$ has a rational double point as singularity at the closed point. Let $\pi : \tilde{X} \to X$ be the minimal resolution.*

(1) *The first Chern class $c_1(M^\sharp)$ establisches a 1-1 correspondance between isomorphism classes of non free indecomposable reflexive R-modules and irreducible components of the exceptional fiber.*
(2) *The previous map identifies the A-R quiver ([3]) with the A-D-E quiver associated to the exceptional divisor.*

One way to understand this result at least when X is a quotient singularity $\mathbb{A}^2/G$, i.e $char(k) = 0$, is to use the fact that the G-Hilbert scheme is the minimal resolution $\tilde{X}$. Then the universal G-cluster $Z \subset (\mathbb{A}^2//G) \times \mathbb{A}^2$interpolates between $\tilde{X}$ and $\mathbb{A}^2$

(5.41)
$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathbb{A}^2//G = \tilde{X} & & \mathbb{A}^2 \end{tikzcd}$$

We can use this correspondance to define a morphism at the derived category level

(5.42)
$$\mathbf{R}q_* \circ p^* : D(\mathbb{A}^2//G) \longrightarrow D^G(\mathbb{A}^2)$$

here $D(-)$ means the derived category of complexes of coherent sheaves with bounded cohomology, a substitute of the K-theory, $D^G(-)$ stands for the same objet but with G-sheaves instead of ordinary sheaves.

The main result of [9] states that this kind of Fourier-Mukai transform is an equivalence of derived categories. From this we can rederive (5.27).

Exercise 5.5. *Using the matrix factorization (5.34), perform the Grassmann blow-up of the corresponding reflexive module, et check in the A_n case the McKay correspondance.*

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