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WEIGHTED MAXIMAL REGULARITY ESTIMATES AND SOLVABILITY OF NON-SMOOTH ELLIPTIC SYSTEMS

PASCAL AUSCHER AND ANDREAS AXELSSON

ABSTRACT. We develop new solvability methods for divergence form second order, real and complex, elliptic systems above Lipschitz graphs, with L_2 boundary data. Our methods yield full characterization of weak solutions, whose gradients have L_2 estimates of a non-tangential maximal function or of the square function, via an integral representation acting on the conormal gradient, with a singular operator-valued kernel.

The coefficients A may depend on all variables, but are assumed to be close to coefficients A_0 that are independent of the coordinate transversal to the boundary, in the Carleson sense $\|A - A_0\|_C$ defined by Dahlberg. We obtain a number of *a priori* estimates and boundary behaviour under finiteness of $\|A - A_0\|_C$. For example, the non-tangential maximal function of a weak solution is controlled in L_2 by the square function of its gradient. This estimate is new for systems in such generality, even for real non-symmetric equations in dimension 3 or higher. The existence of a proof *a priori* to well-posedness, is also a new fact. As corollaries, we obtain well-posedness of the Dirichlet, Neumann and Dirichlet regularity problems under smallness of $\|A - A_0\|_C$ and well-posedness for A_0 , improving earlier results for real symmetric equations. Our methods build on an algebraic reduction to a first order system first made for coefficients A_0 by the two authors and A. McIntosh in order to use functional calculus related to the Kato conjecture solution, and the main analytic tool for coefficients A is an operational calculus to prove weighted maximal regularity estimates.

MSC classes: 35J55, 35J25, 42B25, 47N20

Keywords: elliptic systems, maximal regularity, Dirichlet and Neumann problems, square function, non-tangential maximal function, Carleson measure, functional and operational calculus

1. INTRODUCTION

In this article, we present and develop new solvability methods for boundary value problems (BVPs) for divergence form second order, real and complex, elliptic systems. We look here at BVPs in domains Lipschitz diffeomorphic to the upper half space $\mathbf{R}_+^{1+n} := \{(t, x) \in \mathbf{R} \times \mathbf{R}^n ; t > 0\}$, $n \geq 1$. The same problems on bounded domains Lipschitz diffeomorphic to the unit ball, contain noticeable differences which we address in a forthcoming paper. Here, we focus on the fundamental scale-invariant estimates.

Consider first the equation

$$(1) \quad Lu^\alpha(t, x) = \sum_{i,j=0}^n \sum_{\beta=1}^m \partial_i \left(A_{i,j}^{\alpha,\beta}(t, x) \partial_j u^\beta(t, x) \right) = 0, \quad \alpha = 1, \dots, m$$

in \mathbf{R}_+^{1+n} , where $\partial_0 = \frac{\partial}{\partial t}$ and $\partial_i = \frac{\partial}{\partial x_i}$, $1 \leq i \leq n$. We assume

$$(2) \quad A = (A_{i,j}^{\alpha,\beta}(t, x))_{i,j=0,\dots,n}^{\alpha,\beta=1,\dots,m} \in L_\infty(\mathbf{R}^{1+n}; \mathcal{L}(\mathbf{C}^{(1+n)m})),$$

and that A is accretive on \mathcal{H} , meaning that there exists $\kappa > 0$ such that

$$(3) \quad \sum_{i,j=0}^n \sum_{\alpha,\beta=1}^m \int_{\mathbf{R}^n} \operatorname{Re}(A_{i,j}^{\alpha,\beta}(t,x) f_j^\beta(x) \overline{f_i^\alpha(x)}) dx \geq \kappa \sum_{i=0}^n \sum_{\alpha=1}^m \int_{\mathbf{R}^n} |f_i^\alpha(x)|^2 dx,$$

for all $f \in \mathcal{H}$ and a.e. $t > 0$. The definition of \mathcal{H} , a subspace of $L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$, will be given in Section 2.

We seek to prove well-posedness for (1), i.e. unique solvability in appropriate spaces given Dirichlet data $u|_{t=0}$, Neumann data $\partial_{\nu_A} u|_{t=0}$ or Dirichlet regularity data $\nabla_x u|_{t=0}$, assumed to satisfy an L_2 condition. Note that the continuity estimate required for well-posedness in the sense of Hadamard is not included in our notion of well-posedness, but will be shown to hold. For the Neumann and Dirichlet regularity problems, we will work in the class of weak solutions whose gradient $\nabla_{t,x} u$ has L_2 modified non-tangential maximal function in L_2 . Under our assumptions (see below), we shall describe the limiting behaviour of $\nabla_{t,x} u$ at $t = 0$ and ∞ and obtain well-posedness in this class. For the Dirichlet problem, it is more natural to work in the class of weak solutions with square function estimate $\iint_{\mathbf{R}_+^{1+n}} |\nabla_{t,x} u|^2 dt dx < \infty$ (and a natural condition to eliminate constants). Under our assumptions, we shall describe the limiting behaviour of u at $t = 0$ and ∞ and show non-tangential maximal estimates and L_2 estimates, and obtain well-posedness in this class.

Let us begin by pointing out that the coefficients depend on t , which makes these problems not always solvable in such generality. In Caffarelli, Fabes and Kenig [12], the necessity of a square Dini condition is pointed out. There has been a wealth of results for real symmetric equations (i.e. $m = 1$ and $A_{ij} = A_{ji} \in \mathbf{R}$, $\mathcal{H} = L_2(\mathbf{R}^n; \mathbf{C}^{1+n})$). In Fabes, Jerison and Kenig [21], the L_2 Dirichlet problem is solved under the square Dini condition and continuity. Dahlberg removed continuity and proved in [15] that if the discrepancy $A_1 - A_2$ of two matrices A_1, A_2 satisfies a small Carleson condition, then L_{p_1} -solvability of the Dirichlet problem with coefficients A_1 implies L_{p_2} -solvability of the Dirichlet problem with coefficients A_2 with $p_2 = p_1$. The smallness condition was removed in Fefferman, Kenig and Pipher [23], but then the value of p_2 becomes unspecified. R. Fefferman obtained in [22] the same conclusions as Dahlberg with $p_2 = p_1$, under large perturbation conditions of different nature. See also Lim [33]. Kenig and Pipher [27] proved that the L_p -Neumann and regularity problems are uniquely solvable if the discrepancy $A(t, x) - A(0, x)$ satisfies Dahlberg's small Carleson condition, depending on $p \in (1, 2 + \epsilon)$. Moreover, in [28] they proved small perturbation results for the Neumann and regularity problems analogous the result [15] for the Dirichlet problem, as well as large perturbation results for the regularity problem analogous to [23] for the Dirichlet problem.

Some related results of Dindos, Petermichl and Pipher [19] and Dindos and Rule [20] are obtained under smallness of a Carleson condition on $t \nabla_{t,x} A(t, x)$. Such an hypothesis does not compare to the one on $A(t, x) - A(0, x)$. See also Rios' work [34].

We note that these results are obtained for L_p data, for appropriate p 's, including $p = 2$. This is using all the available technology for *real scalar equations*, starting from the maximum principle, hence L -harmonic measure, and Green's functions. Moreover, as far as solvability is concerned, the main thrust of these works is to get $p = 2$ with non-tangential maximal estimates, using for this the classical variational solutions, or those obtained via the maximum principle.

Of course, t -dependent coefficients incorporate the t -independent ones. We refer to the book by Kenig [25] and references therein, and to Alfonseca, Auscher, Axelsson, Hofmann and Kim [4] for more recent results. See also below.

As the reader has observed, we consider complex systems and we wish to obtain L_2 solvability under conditions as general as possible. Hence we need other tools than those mentioned above. In fact, the tools we develop and that we describe next would not have been conceivable prior to the solution of the Kato problem and its extensions. In Auscher, Axelsson and McIntosh [8], a new method was presented for solving BVPs with t -independent coefficients, following an earlier setup designed in Auscher, Axelsson and Hofmann [6]. The main discovery in [8] is that the equation (1) becomes particularly simple when solving for the conormal gradient

$$f = \nabla_A u := \begin{bmatrix} \partial_{\nu_A} u \\ \nabla_x u \end{bmatrix},$$

where $\partial_{\nu_A} u$ denotes the conormal derivative (see Section 3), instead of the potential u itself. It is a set of generalized Cauchy–Riemann equations expressed as an autonomous first order system

$$(4) \quad \partial_t f + DBf = 0,$$

where D is a self-adjoint (but not positive) first order differential operator with constant coefficients that is elliptic in some sense and B is multiplication with a bounded matrix $B(x)$, which is accretive on the space \mathcal{H} in (3) and related to $A(x) = A(t, x)$, $t > 0$, by an explicit algebraic formula. The operator DB is a bisectorial operator and can be shown to have an L_2 -bounded holomorphic functional calculus for any (t -independent) matrix A satisfying (2) and (3). This fact was proved earlier by Axelsson, Keith and McIntosh [11, Theorem 3.1] elaborating on the technology for the solution of the Kato problem by Auscher, Hofmann, Lacey, McIntosh and Tchamitchian [9]; a more direct proof is proposed in Auscher, Axelsson and McIntosh [7]. As explained there, the main difficulty is the non-injectivity of D . The upshot is the possibility of solving (4) by a semi-group formula $f = e^{-t|DB|} f_0$ with f_0 in a suitable trace space, and such f has non-tangential and square function estimates. The BVP can then be solved in an appropriate class if and only if the map from the trace functions to boundary data is invertible. This is the scheme for the Neumann and regularity problems, for which the boundary data is simply the normal or tangential part of $\nabla_A u$. For the Dirichlet problem, it turns out that a “dual” scheme involving the operator BD can be used similarly. The one-to-one correspondence between trace functions f_0 and boundary data may fail, see Axelsson [10], and it is here that restrictions on A appear. It is known to hold if A is (complex) self-adjoint or block form (i.e. no cross derivatives $\partial_0 A \partial_i$ or $\partial_i A \partial_0$, $i \geq 1$, in (1)), or constant. Another consequence of this method, and this is why considering complex coefficients is useful, is that the set of t -independent A ’s for which solvability holds is open in L_∞ .

When A is t -dependent, our work takes the algebraic reduction to (4) as a starting point. This reduction can still be made in a distributional sense and the ODE becomes non-autonomous as B is also t -dependent. The simplest idea is to treat it in a perturbative way as

$$(5) \quad \partial_t f + DB_0 f = D(B_0 - B)f,$$

with B_0 t -independent, which leads to an implicit representation formula

$$(6) \quad f_t = e^{-t|DB_0|}h + S_A f_t,$$

for some function h on the boundary. The operator S_A is a highly singular integral operator, with an operator-valued kernel. We prove its boundedness on appropriate spaces invoking maximal regularity techniques and we see the Carleson condition from [14, 23, 27, 28] appearing in a very natural way.

Usual maximal regularity is the $L_2(dt; L_2)$ boundedness of the operator-valued singular integral operators S^\pm given formally by

$$(S^+ f)_t = \int_0^t \Lambda e^{-(t-s)\Lambda} f_s ds,$$

$$(S^- f)_t = \int_t^\infty \Lambda e^{-(s-t)\Lambda} f_s ds,$$

with $-\Lambda$ being the infinitesimal generator of a bounded holomorphic semigroup. This is originally due to de Simon [18]. See Kunstmann and Weis [30, Chapter 1] for an overview.

As we shall see, S_A can be expressed by means of S^+ and S^- , with $\Lambda = |DB_0|$, and multiplication by $B_t - B_0$ (which has the same behaviour as $A_t - A_0$). For the BVPs, we rather need to consider weighted spaces $L_2(t^\alpha dt; L_2)$ with $\alpha = \pm 1$, but boundedness fails for either S^+ or S^- (it holds if $-1 < \alpha < 1$, so $\alpha = \pm 1$ is critical). Thus an L_∞ control on $A_t - A_0$ is not enough. Our main estimates for S_A are

$$(7) \quad \|\tilde{N}_*(S_A f)\|_2 \lesssim \|A - A_0\|_C \|\tilde{N}_*(f)\|_2,$$

$$(8) \quad \|S_A f\|_{L_2(tdt; L_2)} \lesssim \|A - A_0\|_C \|f\|_{L_2(tdt; L_2)},$$

where $\|\cdot\|_C$ is the required Carleson control. Here \tilde{N}_* is a non-tangential maximal function (see Section 2), and the space defined by $\tilde{N}_*(f) \in L_2$ is slightly bigger than $L_2(t^{-1}dt; L_2)$ on which the analogue to (7) fails.

On a technical level, proper definition and handling of S_A is most efficiently done using operational calculus, and this avoids having to assume qualitatively that A is smooth in the calculations. We use this terminology, following the thesis [1] of Albrecht, for the extension of functional calculus when not only scalar holomorphic functions are applied to the underlying operator (in our case DB_0), but more general operator-valued holomorphic functions. The Hilbert space theory we use here, surveyed in Section 5.1, is a special case of the general theory developed in Albrecht, Franks and McIntosh [3, Section 4], Lancien, Lancien and LeMerdy [31], and Lancien and LeMerdy [32]. For further details and references, we refer to Kunstmann and Weis [30, Chapter 12].

It is quite clear from the estimates above that smallness of $\|A_t - A_0\|_C$ yields invertibility of $I - S_A$; (7) also enables to invert the boundary trace to data map for the Neumann as well as the regularity problem, provided the one for the t -independent matrix A_0 is invertible. For the Dirichlet problem, one uses instead (8). This is somehow a dual result (although we do not formalize this abstractly) to the one for the regularity problem, which is in agreement with the results of [27, 28] for real symmetric equations. See also Kilty and Shen [29], and Shen [35].

We do not know how to prove well-posedness under the finiteness of $\|A - A_0\|_C$ only. However, we do obtain a number of *a priori* estimates and boundary behaviour without knowing well-posedness for A or A_0 , thanks to our representation of solutions

to the equation (1). For example, we show that if $\|A - A_0\|_C < \infty$, all weak solutions to (1) such that $u_{t_0} \in L_2$, for some $t_0 > 0$, satisfy

$$\max(\|\tilde{N}_*(u)\|_2, \sup_{t>0} \|u_t\|_2) \lesssim \|\nabla_{t,x}u\|_{L_2(tdt;L_2)}.$$

Note in particular that this applies when $A = A_0$. (In that case, this is implicit in [8, Corollary 4.2] when restricted to the class of functions considered there.) Domination of the non-tangential maximal function $\|\tilde{N}_*(u)\|_2$ by the square function $\|\nabla_{t,x}u\|_{L_2(tdt;L_2)} \approx \|S(u)\|_2$, $S(u)(x) = (\int_{|y-x|<t} |\nabla_{t,y}u|^2 dt dy / t^{n-1})^{1/2}$, is reminiscent of the result of Dahlberg, Jerison and Kenig [16], and also of Dahlberg, Kenig, Pipher and Verchota [17]. But there is a difference. In [16] comparability of $\tilde{N}_*(u)$ and $S(u)$ is obtained for solutions of the equation (1) under (2) and (3), A real and $m = 1$, in all $L_q(\mathbf{R}^n; d\mu)$ spaces, $0 < q < \infty$, with μ a doubling A_∞ weight with respect to L -harmonic measure. If the Dirichlet problem in the class $\|\tilde{N}_*(u)\|_p < \infty$ is proved to be solvable for one $1 < p < \infty$, then Lebesgue measure is A_∞ of L -harmonic measure, hence $\|\tilde{N}_*(u)\|_q \approx \|S(u)\|_q$. This fact follows in particular from combining [24] and [21] under $\|A - A_0\|_C < \infty$ and A, A_0 real symmetric. In [17], comparability $\|\tilde{N}_*(u)\|_q \approx \|S(u)\|_q$, $0 < q < \infty$, is obtained for real symmetric constant elliptic (in the sense of Legendre–Hadamard) second order systems (and also higher order but the formulation becomes different) on bounded Lipschitz domains owing to the fact that L_2 solvability of the Dirichlet problem was known (see the introduction of [17]). This comparability also follows for real non-symmetric scalar equations in 2 dimensions combining the results of Kenig, Koch, Pipher and Toro in [26] and again [16]. Here, although we obtain only one part of the comparison, it is essential to note that this is an *a priori* estimate valid independently of well-posedness. The existence of an *a priori* proof is new, even for real symmetric scalar equations, and is permitted by the solution of the Kato square root problem and its extensions.

This is basically the type of results we obtain; precise statements are given in the text. The plan of the paper is as follows. In Section 3 we integrate the differential equation and generalize the setup for t -independent equations from [6, 8], to t -dependent equations. Section 5 provides the theory of operational calculus needed to estimate the singular integral operator S_A in Section 6, in the natural function spaces \mathcal{X} and \mathcal{Y} introduced in Section 4. The Neumann and regularity problems are solved in the non-tangential maximal function space \mathcal{X} in Section 7, and the Dirichlet problem is solved in the square function space \mathcal{Y} in Section 8. Complementary estimates of non-tangential maximal and square functions of the solutions are proved in Section 9. Through standard pull back arguments, these results extend to domains which are Lipschitz diffeomorphic to \mathbf{R}_+^{1+n} , and we state our results in this setting in Section 2.

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2. STATEMENT OF RESULTS

In this section we state our results concerning solvability of boundary value problems on domains $\Omega \subset \mathbf{R}^{1+n}$ which are Lipschitz diffeomorphic to the half space \mathbf{R}_+^{1+n} . Let $\rho : \mathbf{R}_+^{1+n} \rightarrow \Omega$ be the Lipschitz diffeomorphism. Denote the boundary by $\Sigma := \partial\Omega$ and the restricted boundary Lipschitz diffeomorphism by $\rho_0 : \mathbf{R}^n \rightarrow \Sigma$.

Let us first fix notation for \mathbf{R}^{1+n} . We write $\{e_0, e_1, \dots, e_n\}$ for the standard basis for \mathbf{R}^{1+n} with e_0 ‘‘upward’’ pointing into \mathbf{R}_+^{1+n} , and write $t = x_0$ for the vertical coordinate. For the vertical derivative, we write $\partial_0 = \partial_t$. For an m -tuple of vectors $v = (v_i^\alpha)_{0 \leq i \leq n}^{1 \leq \alpha \leq m}$, we write v_\perp and v_\parallel for the normal and tangential parts of v , i.e. $(v_\perp)_0^\alpha = v_0^\alpha$ and $(v_\perp)_i^\alpha = 0$ when $1 \leq i \leq n$, whereas $(v_\parallel)_i^\alpha = v_i^\alpha$ when $1 \leq i \leq n$ and $(v_\parallel)_0^\alpha = 0$. We write $f_t(x) := f(t, x)$ for functions in \mathbf{R}_+^{1+n} . As compared to [8], we here use subscript 0 to denote restriction to the boundary \mathbf{R}^n at $t = 0$, rather than the normal component of f . We also prefer to use small letters f, g, \dots to denote functions in \mathbf{R}_+^{1+n} , since this is where we work most of the time, not on the boundary as in [8].

For tuples of functions and vector fields, gradient and divergence act as $(\nabla_{t,x}u)_i^\alpha = \partial_i u^\alpha$ and $(\operatorname{div}_{t,x}f)^\alpha = \sum_{i=0}^n \partial_i f_i^\alpha$, with corresponding tangential versions $\nabla_x u = (\nabla_{t,x}u)_\parallel$ and $(\operatorname{div}_x f)^\alpha = \sum_{i=1}^n \partial_i f_i^\alpha$. With $\operatorname{curl}_{t,x}f = 0$ we understand that $\partial_j f_i^\alpha = \partial_i f_j^\alpha$, for all $i, j = 0, \dots, n$. Similarly, write $\operatorname{curl}_x f_\parallel = 0$ if $\partial_j f_i^\alpha = \partial_i f_j^\alpha$, for all $i, j = 1, \dots, n$.

Given a function $\tilde{u} : \Omega \rightarrow \mathbf{C}^m$, we pull it back to $u := \tilde{u} \circ \rho : \mathbf{R}_+^{1+n} \rightarrow \mathbf{C}^m$. By the chain rule, we have $\nabla u = \rho^*(\nabla \tilde{u})$, where the pullback of an m -tuple of vector fields f , is defined as $\rho^*(f)(x)^\alpha := \underline{\rho}^t(x) f^\alpha(\rho(x))$, with $\underline{\rho}^t$ denoting the transpose of Jacobian matrix $\underline{\rho}$. If \tilde{u} satisfies $\operatorname{div} \tilde{A} \nabla \tilde{u} = 0$ in Ω , with coefficients $\tilde{A} \in L_\infty(\Omega; \mathcal{L}(\mathbf{C}^{(1+n)m}))$, then u will satisfy $\operatorname{div} A \nabla u = 0$ in \mathbf{R}_+^{1+n} , where $A \in L_\infty(\mathbf{R}_+^{1+n}; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ is defined as

$$(9) \quad A(\mathbf{x}) := |J(\rho)(\mathbf{x})| (\underline{\rho}(\mathbf{x}))^{-1} \tilde{A}(\rho(\mathbf{x})) (\underline{\rho}^t(\mathbf{x}))^{-1}, \quad \mathbf{x} \in \mathbf{R}_+^{1+n}.$$

Here $J(\rho)$ is the Jacobian determinant of ρ . The accretivity assumption we require is that A satisfies (3), i.e.

$$\int_{\mathbf{R}^n} \operatorname{Re}(A(t, x) f(x), f(x)) dx \geq \kappa \int_{\mathbf{R}^n} |f(x)|^2 dx,$$

holds for some constant $\kappa > 0$, uniformly for $t > 0$ and all f belonging to the closed subspace

$$(10) \quad \mathcal{H} := \mathbf{N}(\operatorname{curl}_x) = \{g \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) ; \operatorname{curl}_x(g_\parallel) = 0\}.$$

For scalar equations, i.e. $m = 1$, (3) amounts to the pointwise condition

$$\operatorname{Re}(A(t, x)\zeta, \zeta) \geq \kappa |\zeta|^2, \quad \text{for all } \zeta \in \mathbf{C}^{1+n}, \text{ a.e. } (t, x) \in \mathbf{R}_+^{1+n}.$$

For systems, (3) is stronger than a strict Gårding inequality on \mathbf{R}_+^{1+n} (i.e. integration would be on \mathbf{R}_+^{1+n} and f such that $\operatorname{curl}_{t,x}f = 0$); still (3) is natural given the type of perturbation we consider here.

The boundary value problems we consider are to find $\tilde{u} : \Omega \rightarrow \mathbf{C}^m$ solving the divergence form second order elliptic system

$$\operatorname{div} \tilde{A} \nabla \tilde{u} = 0 \text{ in } \Omega, \quad \text{that is} \quad \operatorname{div} A \nabla u = 0 \text{ in } \mathbf{R}_+^{1+n},$$

with appropriate interior estimates and satisfying one of the following three natural boundary conditions.

- The Dirichlet condition $\tilde{u} = \tilde{\varphi}$ on Σ , or equivalently $u = \varphi := \tilde{\varphi} \circ \rho_0$ on \mathbf{R}^n , given $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^m)$.
- The Dirichlet regularity condition $\nabla_\Sigma \tilde{u} = \tilde{\varphi}$ on Σ (∇_Σ denoting the tangential gradient on Σ), or equivalently $\nabla_x u = \varphi := \rho_0^*(\tilde{\varphi})$ on \mathbf{R}^n , given $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^{nm})$ satisfying $\text{curl}_x \varphi = 0$.
- The Neumann condition $(\nu, \tilde{A} \nabla_\Omega \tilde{u}) = \tilde{\varphi}$ on Σ (contrary to tradition, ν being the inward unit normal vector field on Σ), or equivalently $(e_0, A \nabla_{t,x} u) = \varphi := |J(\rho_0)| \tilde{\varphi} \circ \rho_0$ on \mathbf{R}^n , given $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^m)$.

Definition 2.1. The *modified non-tangential maximal function* of a function f in \mathbf{R}_+^{1+n} is

$$\tilde{N}_*(f)(x) := \sup_{t>0} t^{-(1+n)/2} \|f\|_{L_2(W(t,x))}, \quad x \in \mathbf{R}^n,$$

where $W(t, x) := (c_0^{-1}t, c_0t) \times B(x; c_1t)$, for some fixed constants $c_0 > 1, c_1 > 0$. The *modified Carleson norm* of a function g in \mathbf{R}_+^{1+n} is

$$\|g\|_C := \left(\sup_Q \frac{1}{|Q|} \iint_{(0,l(Q)) \times Q} \sup_{W(t,x)} |g|^2 \frac{dt dx}{t} \right)^{1/2},$$

where the supremum is taken over all cubes Q in \mathbf{R}^n , with $l(Q)$ denoting their side lengths.

Note that different choices for c_0, c_1 will give different, but equivalent norms $\|\tilde{N}_*(f)\|_2$, as well as equivalent norms $\|g\|_C$. Furthermore, this maximal function is really non-tangential since $\tilde{N}_*(f)$ and the closely related maximal function $\sup_{|y-x|<t} t^{-(1+n)/2} \|f\|_{L_2(W(t,y))}$ have equivalent L_2 norms. The latter was introduced in [27]. The modified Carleson norm originates from Dahlberg [15].

For the Neumann and Dirichlet regularity problems, our result is the following.

Theorem 2.2. Consider $\tilde{A} \in L_\infty(\Omega; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ which pulls back to A as in (9), where $A \in L_\infty(\mathbf{R}_+^{1+n}; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ is accretive on \mathcal{H} .

- (i) *A priori estimates:* Consider $\tilde{u} : \Omega \rightarrow \mathbf{C}^m$ such that the pullback $u = \tilde{u} \circ \rho$ has gradient $\nabla_{t,x} u$ with estimate $\|\tilde{N}_*(\nabla_{t,x} u)\|_2 < \infty$, and where u satisfies (1) with the pulled back coefficients A in \mathbf{R}_+^{1+n} distributional sense. If there exists t -independent $A_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$, accretive on \mathcal{H} , such that $\|A - A_0\|_C < \infty$, then $\nabla_{t,x} u$ has limits

$$\lim_{t \rightarrow 0} t^{-1} \int_t^{2t} \|\nabla_{s,x} u_s - g_0\|_2^2 ds = 0 = \lim_{t \rightarrow \infty} t^{-1} \int_t^{2t} \|\nabla_{s,x} u_s\|_2^2 ds,$$

for some function $g_0 \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$, with estimate $\|g_0\|_2 \lesssim \|\tilde{N}_*(\nabla_{t,x} u)\|_2$.

- (ii) *Well-posedness:* By the Neumann problem with coefficients A (or A_0) being well-posed, we mean that given $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, there is a function $u : \mathbf{R}_+^{1+n} \rightarrow \mathbf{C}^m$, unique modulo constants, solving (1), with coefficients A (or A_0), and having estimates as in (i) and trace $g_0 = \lim_{t \rightarrow 0} \nabla_{t,x} u$ such that $(A_0 g_0)_\perp = \varphi$.

The following perturbation result holds. If the Neumann problem for A_0 is well-posed, then there exists $\epsilon > 0$ such that if $\|A - A_0\|_C < \epsilon$, then the Neumann problem is well-posed for A .

The corresponding result holds when the Neumann problem is replaced by the regularity problem and the boundary condition $(A_0 g_0)_\perp = \varphi$ is replaced by $(g_0)_\parallel = \varphi \in L_2(\mathbf{R}^n; \mathbf{C}^{nm})$, where φ satisfies $\text{curl}_x \varphi = 0$. Moreover, for both BVPs the solutions u have estimates

$$\|\tilde{N}_*(\nabla_{t,x}u)\|_2 \approx \|g_0\|_2 \approx \|\varphi\|_2.$$

- (iii) Further regularity: Assume that A_0 is as in (i), with $\|A - A_0\|_C$ sufficiently small, and consider solutions u as in (i).

If A satisfy the t -regularity condition $\|t\partial_t A\|_C < \infty$, then

$$\int_0^\infty \|\partial_t \nabla_{t,x}u\|_2^2 t dt \lesssim \sup_{t>0} \|\nabla_{t,x}u_t\|_2^2 \approx \|\tilde{N}_*(\nabla_{t,x}u)\|_2^2,$$

$t \mapsto \nabla_{t,x}u_t \in L_2$ is continuous and $\lim_{t \rightarrow 0} \|\nabla_{t,x}u_t - g_0\|_2 = 0 = \lim_{t \rightarrow \infty} \|\nabla_{t,x}u_t\|_2$. The converse estimate $\|\tilde{N}_*(\nabla_{t,x}u)\|_2^2 \lesssim \int_0^\infty \|\partial_t \nabla_{t,x}u\|_2^2 t dt$ holds provided $\|t\partial_t A\|_C$ is sufficiently small.

If $\max(\|t\partial_i A\|_C, \|t\partial_t A\|_C) < \infty$ holds for some $i = 1, \dots, n$, then

$$\int_0^\infty \|\partial_i \nabla_{t,x}u\|_2^2 t dt \lesssim \|\tilde{N}_*(\nabla_{t,x}u)\|_2^2.$$

The estimate $\|\tilde{N}_*(\nabla_{t,x}u)\|_2^2 \lesssim \sum_{i=1}^n \int_0^\infty \|\partial_i \nabla_{t,x}u\|_2^2 t dt$ holds provided $\|t\nabla_{t,x}A\|_C$ is sufficiently small.

Implicit constants in (i) and (ii) depend on n , m , $\|A\|_\infty$, κ and $\|A - A_0\|_C$. In (ii) they also depend on the ‘‘well-posedness’’ constants for A_0 , and in (iii) they also depend on the regularity assumptions on A . Note that in (ii), the uniqueness holds in the class defined by $\|\tilde{N}_*(\nabla_{t,x}u)\|_2 < \infty$.

For the Dirichlet problem, our main result is the following.

Theorem 2.3. Consider $\tilde{A} \in L_\infty(\Omega; \mathcal{L}(\mathbf{C}^{(1+n)m}))$, which pulls back to A as in (9), where $A \in L_\infty(\mathbf{R}_+^{1+n}; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ is accretive on \mathcal{H} .

- (i) *A priori estimates:* Consider $\tilde{u} : \Omega \rightarrow \mathbf{C}^m$ such that the pullback $u = \tilde{u} \circ \rho \in C(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$ has estimate $\int_0^\infty \|\nabla_{t,x}u_t\|_2^2 t dt < \infty$ of its gradient and satisfies (1) with the pulled back coefficients A , in \mathbf{R}_+^{1+n} distributional sense. If there exists t -independent $A_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$, accretive on \mathcal{H} , such that $\|A - A_0\|_C < \infty$, then u has L_2 limits

$$\lim_{t \rightarrow 0} \|u_t - u_0\|_2 = 0 = \lim_{t \rightarrow \infty} \|u_t\|_2,$$

for some $u_0 \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, and we have estimates

$$\max(\|\tilde{N}_*(u)\|_2^2, \sup_{t>0} \|u_t\|_2^2) \lesssim \int_0^\infty \|\nabla_{t,x}u\|_2^2 t dt.$$

- (ii) *Well-posedness:* By the Dirichlet problem with coefficients A (or A_0) being well-posed, we mean that given $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, there is a unique function $u : \mathbf{R}_+^{1+n} \rightarrow \mathbf{C}^m$ solving (1), with coefficients A (or A_0), and having estimates as in (i) and trace $u_0 = \varphi$.

The following perturbation result holds. If the Dirichlet problem for A_0 is well-posed, then there exists $\epsilon > 0$ such that if $\|A - A_0\|_C < \epsilon$, then the Dirichlet problem is well-posed for A . Moreover, these solutions u have estimates

$$\|\tilde{N}_*(u)\|_2^2 \approx \sup_{t>0} \|u_t\|_2^2 \approx \int_0^\infty \|\nabla_{t,x}u\|_2^2 t dt \approx \|\varphi\|_2^2.$$

Note that by the square function estimate, the condition $u \in C(\mathbf{R}_+; L_2)$ in (i) may be replaced by $u_t \in L_2$ at some Lebesgue point $t > 0$, possibly redefining $t \mapsto u_t$ on a null set. Implicit constants in (i) and (ii) depend on $n, m, \|A\|_\infty, \kappa$ and $\|A - A_0\|_C$. In (ii), they also depend on the “well-posedness” constants for A_0 . Note that in (ii), uniqueness holds in the class defined by $\int_0^\infty \|\nabla_{t,x}u\|_2^2 t dt < \infty$ and $u_{t_0} \in L_2$ at some Lebesgue point $t_0 > 0$.

We remark that the hypothesis on well-posedness of the boundary value problems with t -independent coefficients A_0 is satisfied, for all three BVPs, for Hermitean coefficients, i.e. $A_0(x)^* = A_0(x)$, for block form coefficients, i.e. $(A_0)_{\perp\parallel} = 0 = (A_0)_{\parallel\perp}$, and for constant coefficients, i.e. $A_0(x) = A_0$, as well as for sufficiently small t -independent $L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ perturbations thereof. This was proved in [8, Theorem 2.2]. That the notions of well-posedness of these BVPs used in [8] coincide with the ones here, for t -independent coefficients, follows from Corollaries 7.3 and 8.3.

Note that we do not assume pointwise bounds on the solutions, hence we use \tilde{N}_* instead of the usual non-tangential maximal function.

When $m = 1$ and A, A_0 are real symmetric (and \mathbf{R}_+^{1+n} replaced by the unit ball), Theorem 2.2(ii) is in [27], and Theorem 2.3(ii) is in [14] (and [16] for the square function estimate). The rest of Theorems 2.2 and 2.3 are mostly new.

Proof of Theorems 2.2 and 2.3. The divergence form elliptic system for \tilde{u} with coefficients \tilde{A} in Ω , and boundary data $\tilde{\varphi}$ on Σ , is pulled back to the system for u with coefficients A in \mathbf{R}_+^{1+n} , and boundary data φ on \mathbf{R}^n , as described above.

For the Neumann and regularity problems in \mathbf{R}_+^{1+n} , part (i) follows from Theorem 7.2, part (ii) follows from Corollary 7.4, and part (iii) is proved in Theorem 9.4.

For the Dirichlet problem in \mathbf{R}_+^{1+n} , part (i) follows from Theorem 8.2 and part (ii) follows from Corollary 8.4, except for the estimate of the non-tangential maximal function, which is proved in Theorem 9.1. \square

3. INTEGRATION OF THE DIFFERENTIAL EQUATION

Following [8], we construct solutions u to the divergence form system (1), by replacing u by its gradient g as the unknown function. Consequently (1) for u is replaced by (11) below for g . Proposition 3.1 reformulates this first order system (11) further, by solving for the t -derivatives, as the vector-valued ODE (12) for the conormal gradient

$$f = \nabla_A u = [\partial_{\nu_A} u, \nabla_x u]^t, \quad \text{where } [\alpha, v]^t := \begin{bmatrix} \alpha \\ v \end{bmatrix}$$

for $\alpha \in \mathbf{C}^m$ and $v \in \mathbf{C}^{nm}$, and $\partial_{\nu_A} u := (A \nabla_{t,x} u)_\perp$ denotes the (inward!) conormal derivative of u .

According to the decomposition of m -tuples into normal and tangential parts as introduced in Section 2, we split the matrix as

$$A(t, x) = \begin{bmatrix} A_{\perp\perp}(t, x) & A_{\perp\parallel}(t, x) \\ A_{\parallel\perp}(t, x) & A_{\parallel\parallel}(t, x) \end{bmatrix}.$$

Note that with our assumption that A be accretive on \mathcal{H} for a.e. $t > 0$, the matrix $A_{\perp\perp}$ is invertible.

Proposition 3.1. *The pointwise transformation*

$$A \mapsto \hat{A} := \begin{bmatrix} A_{\perp\perp}^{-1} & -A_{\perp\perp}^{-1}A_{\perp\parallel} \\ A_{\parallel\perp}A_{\perp\perp}^{-1} & A_{\parallel\parallel} - A_{\parallel\perp}A_{\perp\perp}^{-1}A_{\perp\parallel} \end{bmatrix}$$

is a self-inverse bijective transformation of the set of bounded matrices which are accretive on \mathcal{H} .

For a pair of coefficient matrices $A = \hat{B}$ and $B = \hat{A}$, the pointwise map $g \mapsto f = [(Ag)_{\perp}, g_{\parallel}]^t$ gives a one-one correspondence, with inverse $g = [(Bf)_{\perp}, f_{\parallel}]^t$, between solutions $g \in L_2^{\text{loc}}(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}))$ to the equations

$$(11) \quad \operatorname{div}_{t,x}(Ag) = 0 = \operatorname{curl}_{t,x}g$$

and solutions $f \in L_2^{\text{loc}}(\mathbf{R}_+; \mathcal{H})$ to the generalized Cauchy–Riemann equations

$$(12) \quad \partial_t f + DBf = 0,$$

where the derivatives are taken in \mathbf{R}_+^{1+n} distributional sense, and $D := \begin{bmatrix} 0 & \operatorname{div}_x \\ -\nabla_x & 0 \end{bmatrix}$.

This was proved in [8, Section 3], but for completeness we sketch a proof of this important result. Note that $\overline{\mathbf{R}(D)} = \mathcal{H}$.

Proof. The stated properties of the matrix transformation are straightforward to verify, using the observation that $\operatorname{Re}(Ag, g) = \operatorname{Re}(Bf, f)$. Equations (11) are equivalent to

$$(13) \quad \begin{cases} \partial_t f_{\perp} + \operatorname{div}_x(A_{\parallel\perp}g_{\perp} + A_{\parallel\parallel}f_{\parallel}) = 0, \\ \partial_t f_{\parallel} - \nabla_x g_{\perp} = 0, \\ \operatorname{curl}_x f_{\parallel} = 0. \end{cases}$$

Inserting $g_{\perp} = (Bf)_{\perp} = A_{\perp\perp}^{-1}(f_{\perp} - A_{\perp\parallel}f_{\parallel})$, this becomes Equation (12), together with the constraint $f_t \in \mathcal{H}$, when written on matrix form. \square

Let us recall the situation when $B(t, x) = B_0(x)$ does not depend on the t -variable. In this case, we view B_0 as a multiplication operator in the boundary function space $L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$. Define closed and open sectors and double sectors in the complex plane by

$$\begin{aligned} S_{\omega+} &:= \{\lambda \in \mathbf{C} ; |\arg \lambda| \leq \omega\} \cup \{0\}, & S_{\omega} &:= S_{\omega+} \cup (-S_{\omega+}), \\ S_{\nu+}^o &:= \{\lambda \in \mathbf{C} ; \lambda \neq 0, |\arg \lambda| < \nu\}, & S_{\nu}^o &:= S_{\nu+}^o \cup (-S_{\nu+}^o), \end{aligned}$$

and define the *angle of accretivity* of B_0 to be

$$\omega := \sup_{f \neq 0, f \in \mathcal{H}} |\arg(B_0 f, f)| < \pi/2.$$

The method for constructing solutions to the elliptic divergence form system, developed in [6, 8], uses holomorphic functional calculus of the *infinitesimal generator* DB_0 appearing in the ODE (12), and the following was proved.

- (i) The operator DB_0 is a closed and densely defined ω -bisectorial operator, i.e. $\sigma(DB_0) \subset S_\omega$, where ω is the angle of accretivity of B_0 . Moreover, there are resolvent bounds $\|(\lambda - DB_0)^{-1}\| \lesssim 1/\text{dist}(\lambda, S_\omega)$ when $\lambda \notin S_\omega$.
- (ii) The function space splits topologically as

$$L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}) = \mathcal{H} \oplus \mathbf{N}(DB_0),$$

and the restriction of DB_0 to $\mathcal{H} = \overline{\mathbf{R}(D)}$ is a closed, densely defined and injective operator with dense range in \mathcal{H} , with same estimates on spectrum and resolvents as in (i).

- (iii) The operator DB_0 has a bounded holomorphic functional calculus in \mathcal{H} , i.e. for each bounded holomorphic function $b(\lambda)$ on a double sector S_ν° , $\omega < \nu < \pi/2$, the operator $b(DB_0)$ in \mathcal{H} is bounded with estimates

$$\|b(DB_0)\|_{\mathcal{H} \rightarrow \mathcal{H}} \lesssim \|b\|_{L_\infty(S_\nu^\circ)}.$$

The construction of the operators $b(DB_0)$ is explained in detail in Section 5.1, in the more general case of operational calculus. The two most important functions $b(\lambda)$ here are the following.

- The characteristic functions $\chi^+(\lambda)$ and $\chi^-(\lambda)$ for the right and left half planes, which give the generalised *Hardy projections* $E_0^\pm := \chi^\pm(DB_0)$.
- The exponential functions $e^{-t|\lambda|}$, $t > 0$, which give the operators $e^{-t|DB_0|}$. Here $|\lambda| := \lambda \text{sgn}(\lambda)$ and $\text{sgn}(\lambda) := \chi^+(\lambda) - \chi^-(\lambda)$.

A key result that we make use of frequently, is that the boundedness of the projections E_0^\pm shows that there is a topological splitting

$$(14) \quad \mathcal{H} = E_0^+ \mathcal{H} \oplus E_0^- \mathcal{H}$$

of $\mathcal{H} = \overline{\mathbf{R}(D)} = \overline{\mathbf{R}(DB_0)}$ into complementary closed subspaces $E_0^\pm \mathcal{H} := \mathbf{R}(E_0^\pm)$.

Solutions to the elliptic equation $\partial_t f + DB_0 f = 0$ are constructed as follows. Given $f_0 \in \mathcal{H}$, this is the boundary trace of a solution to the ODE which decays at infinity, if and only if f_0 belongs to the positive spectral subspace of DB_0 , i.e. $f_0 \in E_0^+ \mathcal{H}$. In this case the Cauchy extension of f_0 , i.e. the solution to the ODE with this boundary trace, is

$$(15) \quad f_t := e^{-t|DB_0|} f_0, \quad t > 0.$$

Now consider more general t -dependent coefficients $B(t, x)$. Fix some t -independent coefficients B_0 , accretive on \mathcal{H} . (This B_0 should be thought of as the boundary trace of B , acting in \mathbf{R}_+^{1+n} independently of t .) To construct solutions to the ODE, we rewrite it as

$$(16) \quad \partial_t f + DB_0 f = D\mathcal{E}f, \quad \text{where } \mathcal{E}_t := B_0 - B_t.$$

However, while $\partial_t f + DB_0 f = 0$ can be interpreted in the strong sense with $f \in C^1(\mathbf{R}_+; L_2) \cap C^0(\mathbf{R}_+; \mathbf{D}(DB_0))$ (the class of solutions used in [8]), (16) will be understood in the sense of distributions. The following proposition rewrites this equation in integral form. It uses modified Hardy projections \widehat{E}_0^\pm , defined as

$$(17) \quad \widehat{E}_0^\pm := E_0^\pm B_0^{-1} P_{B_0 \mathcal{H}},$$

where $P_{B_0 \mathcal{H}}$ denotes the projection onto $B_0 \mathcal{H}$ in the topological splitting $L_2 = B_0 \mathcal{H} \oplus \mathcal{H}^\perp$ and B_0^{-1} is the inverse of $B_0 : \mathcal{H} \rightarrow B_0 \mathcal{H}$. Beware that B_0^{-1} is not necessarily a multiplication operator and is only defined on the subspace $B_0 \mathcal{H}$.

Proposition 3.2. *If $f \in L_2^{\text{loc}}(\mathbf{R}_+; \mathcal{H})$ satisfies $\partial_t f + DBf = 0$ in \mathbf{R}_+^{1+n} distributional sense, then*

$$\begin{aligned} - \int_0^t \eta'_+(s) e^{-(t-s)|DB_0|} E_0^+ f_s ds &= \int_0^t \eta_+(s) DB_0 e^{-(t-s)|DB_0|} \widehat{E}_0^+ \mathcal{E}_s f_s ds, \\ - \int_t^\infty \eta'_-(s) e^{-(s-t)|DB_0|} E_0^- f_s ds &= \int_t^\infty \eta_-(s) DB_0 e^{-(s-t)|DB_0|} \widehat{E}_0^- \mathcal{E}_s f_s ds, \end{aligned}$$

for all $t > 0$ and smooth bump functions $\eta_\pm(s) \geq 0$, where η_+ is compactly supported in $(0, t)$, and η_- is compactly supported in (t, ∞) .

Proof. By assumption

$$\int_0^\infty \left((-\partial_s \phi_s, f_s) + (D\phi_s, B_0 f_s) \right) ds = \int_0^\infty (D\phi_s, \mathcal{E}_s f_s) ds,$$

for all $\phi \in C_0^\infty(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m})$. Let $\phi_0 \in \mathcal{H}$ be any boundary function, and choose $\phi_s := \eta_\pm(s) (e^{-|(t-s)DB_0|} E_0^\pm)^* \phi_0 \in C_0^\infty(\mathbf{R}_+; \mathbf{D}(D))$. With a limiting argument, approximating ϕ by $C_0^\infty(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m})$ functions through \mathbf{R}^n -mollification, we may use this ϕ as test function. This yields

$$- \left(\phi_0, \int_0^\infty \eta'_\pm(s) e^{-|(t-s)DB_0|} E_0^\pm f_s ds \right) = \left(\phi_0, \int_0^\infty \eta_\pm(s) e^{-|(t-s)DB_0|} E_0^\pm D \mathcal{E}_s f_s ds \right).$$

Since this holds for all ϕ_0 and since $E_0^\pm D = E_0^\pm DP_{B_0 \mathcal{H}} = E_0^\pm (DB_0) B_0^{-1} P_{B_0 \mathcal{H}} = DB_0 \widehat{E}_0^\pm$, the proposition follows. In particular, $e^{-|(t-s)DB_0|} E_0^\pm D$ extends by continuity to a bounded operator on L_2 for $s \neq t$. \square

Formally, if we let η_\pm approximate the characteristic functions for $(0, t)$ and (t, ∞) respectively, we obtain in the limit from Proposition 3.2 that

$$\begin{aligned} E_0^+ f_t - e^{-t|DB_0|} E_0^+ f_0 &= \int_0^t DB_0 e^{-(t-s)|DB_0|} \widehat{E}_0^+ \mathcal{E}_s f_s ds, \\ 0 - E_0^- f_t &= \int_t^\infty DB_0 e^{-(s-t)|DB_0|} \widehat{E}_0^- \mathcal{E}_s f_s ds, \end{aligned}$$

if $\lim_{t \rightarrow 0} f_t = f_0$ and $\lim_{t \rightarrow \infty} f_t = 0$ in appropriate sense. Subtraction yields $f_t = e^{-t|DB_0|} E_0^+ f_0 + S_A f_t$, which we wish to solve as

$$(18) \quad f = (I - S_A)^{-1} C_0^+ f_0,$$

where the integral operator S_A is

$$(19) \quad S_A f_t := \int_0^t DB_0 e^{-(t-s)|DB_0|} \widehat{E}_0^+ \mathcal{E}_s f_s ds - \int_t^\infty DB_0 e^{-(s-t)|DB_0|} \widehat{E}_0^- \mathcal{E}_s f_s ds$$

and the generalized Cauchy integral C_0^+ is

$$(C_0^+ f_0)(t, x) := (e^{-t|DB_0|} E_0^+ f_0)(x).$$

We remark that we view C_0^+ as an operator mapping functions on \mathbf{R}^n to functions in \mathbf{R}_+^{1+n} . The equation (18) can also be viewed as a generalized Cauchy integral formula, for t -dependent coefficients A , and we shall see that, given any $f_0 \in L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m})$, it constructs a solution f_t to the elliptic equation. However, for this one needs to have that $I - S_A$ is bounded and invertible in a suitable space of functions in \mathbf{R}_+^{1+n} .

4. NATURAL FUNCTION SPACES

It is well known that solutions g to (11) with L_2 boundary data typically satisfy certain square function estimates, as well as non-tangential maximal function estimates. In this section, we study the basic properties of some natural function spaces related to BVPs with L_2 boundary data.

Definition 4.1. In \mathbf{R}_+^{1+n} , define the Banach/Hilbert spaces

$$\begin{aligned}\mathcal{X} &:= \{f : \mathbf{R}_+^{1+n} \rightarrow \mathbf{C}^{(1+n)m} ; \tilde{N}_*(f) \in L_2(\mathbf{R}^n)\}, \\ \mathcal{Y} &:= \{f : \mathbf{R}_+^{1+n} \rightarrow \mathbf{C}^{(1+n)m} ; \int_0^\infty \|f_t\|_{L_2(\mathbf{R}^n)}^2 t dt < \infty\},\end{aligned}$$

with the obvious norms. Here \tilde{N}_* denotes the modified non-tangential maximal function from Definition 2.1. By $\mathcal{Y}^* = L_2(\mathbf{R}_+^{1+n}, dt dx/t; \mathbf{C}^{(1+n)m})$ we denote the dual space of \mathcal{Y} , relative to $L_2(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m})$.

In Sections 7 and 8 we demonstrate that the maximal function space \mathcal{X} is the natural space to solve the Neumann and regularity problems in, whereas \mathcal{Y} is natural for the Dirichlet problem. That the spaces \mathcal{Y} and \mathcal{X} are relevant for $L_2(\mathbf{R}^n)$ boundary value problems with t -independent coefficients is clear from the following theorem. For proofs, we refer to [8, Proposition 2.3] and [6, Proposition 2.56].

Theorem 4.2. *Let f_0 belong to the spectral subspace $E_0^+ \mathcal{H}$. Then $f_t := e^{-t|DB_0|} f_0$ gives a solution to $\partial_t f_t + DB_0 f_t = 0$, in the strong sense $f \in C^1(\mathbf{R}_+; L_2) \cap C^0(\mathbf{R}_+; D(DB_0))$, with L_2 limits $\lim_{t \rightarrow 0} f_t = f_0$ and $\lim_{t \rightarrow \infty} f_t = 0$. This solution has estimates*

$$\|\partial_t f\|_{\mathcal{Y}} \approx \|f\|_{\mathcal{X}} \approx \sup_{t>0} \|f_t\|_2 \approx \|f_0\|_2.$$

We will show in Corollary 7.3 that any distributional solution $f \in \mathcal{X}$ to $\partial_t f_t + DB_0 f_t = 0$ is of the form $f_t := e^{-t|DB_0|} f_0$ for some $f_0 \in E_0^+ \mathcal{H}$.

Clearly $\mathcal{Y} \subset L_2^{\text{loc}}(\mathbf{R}_+; L_2)$. The following lemma shows that \mathcal{X} is locally L_2 inside \mathbf{R}_+^{1+n} as well, and is quite close to \mathcal{Y}^* .

Lemma 4.3. *There are estimates*

$$\sup_{t>0} \frac{1}{t} \int_t^{2t} \|f_s\|_2^2 ds \lesssim \|\tilde{N}_*(f)\|_2^2 \lesssim \int_0^\infty \|f_s\|_2^2 \frac{ds}{s}.$$

In particular $\mathcal{Y}^ \subset \mathcal{X}$.*

Proof. The second inequality follows by integrating the pointwise estimate

$$\tilde{N}_*(f)(x)^2 \approx \sup_{t>0} \iint_{W(t,x)} |f(s,y)|^2 \frac{ds dy}{s^{1+n}} \leq \iint_{|y-x| < c_0 c_1 s} |f(s,y)|^2 \frac{ds dy}{s^{1+n}}.$$

For the lower bound on $\|\tilde{N}_*(f)\|_2$, it suffices to estimate $t^{-1} \int_t^{c_0 t} \|f_s\|_2^2 ds$, uniformly for $t > 0$. To this end, split $\mathbf{R}^n = \bigcup_k Q_k$, where Q_k all are disjoint cubes with diagonal lengths $c_1 t$. Then

$$t^{-1} \int_t^{c_0 t} \int_{Q_k} |f(s,y)|^2 ds dy \lesssim |Q_k| \inf_{x \in Q_k} |\tilde{N}_*(f)(x)|^2 \lesssim \int_{Q_k} |\tilde{N}_*(f)(x)|^2 dx.$$

Summation over k gives the stated estimate. \square

The space \mathcal{Y}^* is a subspace of \mathcal{X} of functions with zero trace at the boundary \mathbf{R}^n , in the square L_2 -Dini sense $\lim_{t \rightarrow 0} t^{-1} \int_t^{2t} \|f_s\|_2^2 ds = 0$. The following lemma gives a sufficient Carleson condition for a multiplication operator to map into this subspace.

Lemma 4.4. *For functions $\mathcal{E} : \mathbf{R}_+^{1+n} \rightarrow \mathcal{L}(\mathbf{C}^{(1+n)m})$, we have estimates*

$$\|\mathcal{E}\|_\infty \lesssim \|\mathcal{E}\|_* \lesssim \|\mathcal{E}\|_C,$$

where $\|\mathcal{E}\|_* := \|\mathcal{E}\|_{\mathcal{X} \rightarrow \mathcal{Y}^*} = \sup_{\|f\|_{\mathcal{X}}=1} \|\mathcal{E}f\|_{\mathcal{Y}^*}$ denotes the multiplier norm, and $\|\mathcal{E}\|_C$ denotes the modified Carleson norm from Definition 2.1.

Proof. For the first estimate, fix t and consider only f supported on $(t, 2t)$ in the definition of $\|\mathcal{E}\|_{\mathcal{X} \rightarrow \mathcal{Y}^*}$. Lemma 4.3 shows that

$$\sup \|\mathcal{E}f\|_{\mathcal{Y}^*} / \|f\|_{\mathcal{X}} \approx \sup (t^{-1/2} \|\mathcal{E}f\|_{L_2}) / (t^{-1/2} \|f\|_{L_2}) = \sup_{t < s < 2t} \|\mathcal{E}_s\|_\infty.$$

Taking supremum over t shows the estimate $\|\mathcal{E}\|_\infty \lesssim \|\mathcal{E}\|_*$.

For the second estimate, we calculate

$$\begin{aligned} \|\mathcal{E}f\|_{\mathcal{Y}^*}^2 &\approx \iint_{\mathbf{R}_+^{1+n}} \left(\frac{1}{t^{1+n}} \iint_{W(t,x)} ds dy \right) |\mathcal{E}(t,x)f(t,x)|^2 \frac{dt dx}{t} \\ &\approx \iint_{\mathbf{R}_+^{1+n}} \left(\frac{1}{s^{1+n}} \iint_{W(s,y)} |\mathcal{E}(t,x)f(t,x)|^2 \frac{dt dx}{t} \right) ds dy \\ &\lesssim \iint_{\mathbf{R}_+^{1+n}} \left(\frac{1}{s} \sup_{W(s,y)} |\mathcal{E}|^2 \right) \left(\frac{1}{s^{1+n}} \iint_{W(s,y)} |f(t,x)|^2 dt dx \right) ds dy \lesssim \|\mathcal{E}\|_C^2 \|f\|_{\mathcal{X}}^2, \end{aligned}$$

where the final estimate is by Carleson's theorem. \square

We have not been able to identify the $\|\cdot\|_*$ norm, which lies between the standard and the modified Carleson norm. Indeed, choosing f as the characteristic function for the Carleson box $(0, l(Q)) \times Q$ (times a unit vector field) in the estimate $\|\mathcal{E}f\|_{\mathcal{Y}^*} \leq \|\mathcal{E}\|_* \|f\|_{\mathcal{X}}$, shows that

$$\sup_Q \frac{1}{|Q|} \iint_{(0, l(Q)) \times Q} |\mathcal{E}(t,x)|^2 \frac{dt dx}{t} \lesssim \|\mathcal{E}\|_*^2.$$

Furthermore, it is straightforward to see that the modified Carleson norm is dominated by the corresponding modified square Dini norm

$$\|\mathcal{E}\|_C^2 \lesssim \int_0^\infty \sup_{c_0^{-1}t < s < c_0 t} \|\mathcal{E}_s\|_\infty^2 \frac{dt}{t}.$$

5. HOLOMORPHIC OPERATIONAL CALCULUS

Throughout this section Λ denotes a closed, densely defined ω -sectorial operator in an arbitrary Hilbert space \mathcal{H} , i.e. $\sigma(\Lambda) \subset S_{\omega+}$, and we assume resolvent bounds $\|(\lambda - \Lambda)^{-1}\|_{\mathcal{H} \rightarrow \mathcal{H}} \lesssim 1/\text{dist}(\lambda, S_{\omega+})$. For simplicity, we assume throughout that Λ is injective, and therefore has dense range. In our applications Λ will be $|DB_0|$, and \mathcal{H} will be the Hilbert space from (10). See Section 6.

The goal in this section is to develop the theory needed to make rigorous the limiting argument following Proposition 3.2. To this end, we study uniform boundedness and convergence of model operators

$$(20) \quad S_\epsilon^+ f_t := \int_0^t \eta_\epsilon^+(t, s) \Lambda e^{-(t-s)\Lambda} f_s ds,$$

$$(21) \quad S_\epsilon^- f_t := \int_t^\infty \eta_\epsilon^-(t, s) \Lambda e^{-(s-t)\Lambda} f_s ds,$$

acting on functions $f_t(x) = f(t, x)$ in a Hilbert space $L_2(\mathbf{R}_+, d\mu(t); \mathcal{H})$. For uniform boundedness issues, it suffices that the bump functions $\eta_\epsilon^+(t, s)$ and $\eta_\epsilon^-(t, s)$ are uniformly bounded and compactly supported within $\{(s, t) ; 0 < s < t\}$ and $\{(s, t) ; 0 < t < s\}$ respectively. For convergence issues and to link to the ODE, they should approximate the characteristic functions of the above sets. A convenient choice which we shall use systematically is the following. Define $\eta^0(t)$ to be the piecewise linear continuous function with support $[1, \infty)$, which equals 1 on $(2, \infty)$ and is linear on $(1, 2)$. Then let $\eta_\epsilon(t) := \eta^0(t/\epsilon)(1 - \eta^0(2\epsilon t))$ and

$$\eta_\epsilon^\pm(t, s) := \eta^0(\pm(t-s)/\epsilon) \eta_\epsilon(t) \eta_\epsilon(s).$$

We study the operators S_ϵ^\pm from the point of view of operational calculus. This means for example that we view $S_\epsilon^+ = F(\Lambda)$ as obtained from the underlying operator Λ (acting horizontally, i.e. in the variable x) by applying the operator-valued function $\lambda \mapsto F(\lambda)$, where

$$(F(\lambda)f)_t := \int_0^t \eta_\epsilon^+(t, s) \lambda e^{-(t-s)\lambda} f_s ds,$$

which depends holomorphically on λ in a sector $S_{\nu+}^o$ containing the spectrum of Λ . Note that each of these vertically acting, i.e. acting in the t -variable, operators $F(\lambda)$ commute with Λ .

5.1. Operational calculus in Hilbert space. Consider Λ as above. Let $\mathcal{K} := L_2(\mathbf{R}_+, d\mu(t); \mathcal{H})$ for some Borel measure μ . We extend the resolvents $(\lambda - \Lambda)^{-1} \in \mathcal{L}(\mathcal{H})$, $\lambda \notin S_{\omega+}$, to bounded operators on \mathcal{K} (and we use the same notation, letting $((\lambda - \Lambda)^{-1}f)_t := (\lambda - \Lambda)^{-1}(f_t)$ for all $f \in \mathcal{K}$ and a.e. $t > 0$). These extensions of the resolvents to \mathcal{K} clearly inherit the bounds from \mathcal{H} . We may think of them as being the resolvents of an ω -sectorial operator $\Lambda = \Lambda_{\mathcal{K}}$, although this extended unbounded operator $\Lambda_{\mathcal{K}}$ is not needed below.

Define the commutant of Λ to be

$$\Lambda' := \{T \in \mathcal{L}(\mathcal{K}) ; (\lambda - \Lambda)^{-1}T = T(\lambda - \Lambda)^{-1} \text{ for } \lambda \notin S_{\omega+}\}.$$

Fix $\omega < \nu < \pi/2$, and consider classes of operator-valued holomorphic functions

$$H(S_{\nu+}^o; \Lambda') := \{\text{holomorphic } F : S_{\nu+}^o \rightarrow \Lambda'\},$$

$$\Psi(S_{\nu+}^o; \Lambda') := \{F \in H(S_{\nu+}^o; \Lambda') ; \|F(\lambda)\| \lesssim \min(|\lambda|^a, |\lambda|^{-a}), \text{ some } a > 0\},$$

$$H_\infty(S_{\nu+}^o; \Lambda') := \{F \in H(S_{\nu+}^o; \Lambda') ; \sup_{\lambda \in S_{\nu+}^o} \|F(\lambda)\| < \infty\}.$$

Through Dunford calculus, we define for $F \in \Psi(S_{\nu+}^o; \Lambda')$ the operator

$$(22) \quad F(\Lambda) := \frac{1}{2\pi i} \int_\gamma F(\lambda) (\lambda - \Lambda)^{-1} d\lambda,$$

where γ is the unbounded contour $\{re^{\pm i\theta} ; r > 0\}$, $\omega < \theta < \nu$, parametrized counter clockwise around $S_{\omega+}$. This yields a bounded operator $F(\Lambda)$, since the bounds on F and the resolvents guarantee that the integral converges absolutely.

Remark 5.1. Functional calculus of the operator Λ is a special case of this operational calculus (22). Applying a scalar holomorphic function $f(\lambda)$ to Λ with functional calculus is the same as applying the operator-valued holomorphic function $F(\lambda) = f(\lambda)I$ to Λ with operational calculus. For the functional calculus, we write $\Psi(S_{\nu+}^o)$ and $H_\infty(S_{\nu+}^o)$ for the corresponding classes of scalar symbol functions.

We also remark that a more general functional and operational calculus for bisectorial operators like DB_0 are developed entirely similar to that of sectorial operators Λ , replacing the sector $S_{\omega+}$ by the bisector S_ω .

The following three propositions contain all the theory of operational calculus that we need. To be self-contained and illustrate their simplicity, we give full proofs, although the propositions are proved in exactly the same way as for functional calculus, and can be found in [2].

Proposition 5.2. *If $F, G \in \Psi(S_{\nu+}^o; \Lambda')$, then*

$$F(\Lambda)G(\Lambda) = (FG)(\Lambda).$$

Note that we need not assume that $F(\lambda)$ and $G(\mu)$ commute for any $\lambda, \mu \in S_{\nu+}^o$.

Proof. We use contours γ_1 and γ_2 , with angles $\omega < \theta_1 < \theta_2 < \pi/2$, so that γ_2 encircles γ_1 . Cauchy's theorem now yields

$$\begin{aligned} (2\pi i)^2 F(\Lambda)G(\Lambda) &= \left(\int_{\gamma_1} \frac{F(\lambda)}{\lambda - \Lambda} d\lambda \right) \left(\int_{\gamma_2} \frac{G(\mu)}{\mu - \Lambda} d\mu \right) \\ &= \int_{\gamma_1} \int_{\gamma_2} F(\lambda)G(\mu) \frac{1}{\mu - \lambda} \left(\frac{1}{\lambda - \Lambda} - \frac{1}{\mu - \Lambda} \right) d\lambda d\mu \\ &= \int_{\gamma_1} \frac{F(\lambda)}{\lambda - \Lambda} \left(\int_{\gamma_2} \frac{G(\mu)}{\mu - \lambda} d\mu \right) d\lambda - \int_{\gamma_2} \left(\int_{\gamma_1} \frac{F(\lambda)}{\mu - \lambda} d\lambda \right) \frac{G(\mu)}{\mu - \Lambda} d\mu \\ &= \int_{\gamma_1} \frac{F(\lambda)}{\lambda - \Lambda} 2\pi i G(\lambda) d\lambda - 0 = (2\pi i)^2 (FG)(\Lambda), \end{aligned}$$

using the resolvent equation. □

Proposition 5.3. *Assume that Λ satisfies square function estimates, i.e. assume that*

$$\int_0^\infty \|\psi(t\Lambda)u\|_{\mathcal{H}}^2 \frac{dt}{t} \approx \|u\|_{\mathcal{H}}^2, \quad \text{for all } u \in \mathcal{H}$$

and some fixed $\psi \in \Psi(S_{\nu+}^o)$. Then there exists $C < \infty$ such that

$$\|F(\Lambda)\| \leq C \sup_{\lambda \in S_{\nu+}^o} \|F(\lambda)\|, \quad \text{for all } F \in \Psi(S_{\nu+}^o; \Lambda').$$

We remark that if square function estimates for Λ hold with one such ψ , then they hold for any non-zero $\psi \in \Psi(S_{\nu+}^o)$.

Proof. Note that the square function estimates extend to $u \in \mathcal{K}$, with $\|\cdot\|_{\mathcal{K}}$ instead of $\|\cdot\|_{\mathcal{H}}$. We drop \mathcal{K} in $\|\cdot\|_{\mathcal{K}}$. Using the resolution of identity $\int_0^\infty \psi^2(s\Lambda)uds/s = cu$,

where $0 < c < \infty$ is a constant, and the square function estimates, we calculate

$$\begin{aligned} \|F(\Lambda)u\|^2 &\approx \int_0^\infty \|\psi(t\Lambda)F(\Lambda)u\|^2 \frac{dt}{t} \\ &\approx \int_0^\infty \left\| \int_0^\infty (\psi(t\Lambda)F(\Lambda)\psi(s\Lambda))(\psi(s\Lambda)u) \frac{ds}{s} \right\|^2 \frac{dt}{t} \\ &\lesssim \sup_{S_{\nu+}^o} \|F(\lambda)\|^2 \int_0^\infty \left(\int_0^\infty \eta(t/s) \frac{ds}{s} \right) \left(\int_0^\infty \eta(t/s) \|\psi(s\Lambda)u\|^2 \frac{ds}{s} \right) \frac{dt}{t} \\ &\lesssim \sup_{S_{\nu+}^o} \|F(\lambda)\|^2 \int_0^\infty \|\psi(s\Lambda)u\|^2 \frac{ds}{s} \lesssim \sup_{S_{\nu+}^o} \|F(\lambda)\|^2 \|u\|^2. \end{aligned}$$

We have used the estimate

$$\|\psi(t\Lambda)F(\Lambda)\psi(s\Lambda)\| \lesssim \int_\gamma \|F(\lambda)\| |\psi(t\lambda)\psi(s\lambda)\lambda^{-1}d\lambda| \lesssim \sup_{\lambda \in S_{\nu+}^o} \|F(\lambda)\| \eta(t/s),$$

where $\eta(x) := \min\{x^a, x^{-a}\}(1 + |\log x|)$ for some $a > 0$. □

Proposition 5.4. *Assume that Λ satisfies square function estimates as in Proposition 5.3. Let $F_n \in \Psi(S_{\nu+}^o; \Lambda')$, $n = 1, 2, \dots$, satisfy $\sup_{n,\lambda} \|F_n(\lambda)\| < \infty$, and let $F \in H_\infty(S_{\nu+}^o; \Lambda')$. Assume that for each fixed $v \in \mathcal{K}$ and $\lambda \in S_{\nu+}^o$, we have strong convergence $\lim_{n \rightarrow \infty} \|F_n(\lambda)v - F(\lambda)v\| = 0$. Then the operators $F_n(\Lambda)$ converge strongly to a bounded operator $F(\Lambda)$, i.e.*

$$F_n(\Lambda)u \rightarrow F(\Lambda)u, \quad \text{for all } u \in \mathcal{K}, \text{ as } n \rightarrow \infty.$$

Proof. Since $\sup_n \|F_n(\Lambda)\| < \infty$ by Proposition 5.3, it suffices to consider $u = \psi(\Lambda)v$ for some fixed $\psi \in \Psi(S_{\nu+}^o) \setminus \{0\}$, since $\mathcal{R}(\psi(\Lambda))$ is dense in \mathcal{K} . From (22), we get

$$\|F_n(\Lambda)u - F_m(\Lambda)u\| \lesssim \int_\gamma \|(F_n(\lambda) - F_m(\lambda))v\| |\psi(\lambda)\lambda^{-1}d\lambda|,$$

where $\|(F_n(\lambda) - F_m(\lambda))v\| \lesssim \|v\|$ and $|\psi(\lambda)|/|\lambda|$ is integrable. The dominated convergence theorem applies and proves the proposition. □

Propositions 5.2, 5.3 and 5.4 show that we have a continuous Banach algebra homomorphism

$$H_\infty(S_{\nu+}^o; \Lambda') \rightarrow \mathcal{L}(\mathcal{K}) : F \mapsto F(\Lambda),$$

provided that Λ satisfies square function estimates as in Proposition 5.3. This is the operational calculus that we need. Note that with $F(\Lambda)$ defined in this way for all $F \in H_\infty(S_{\nu+}^o; \Lambda')$, Proposition 5.4 continues to hold for any $F_n \in H_\infty(S_{\nu+}^o; \Lambda')$.

5.2. Maximal regularity estimates. Here, we apply the operational calculus from Section 5.1 to prove weighted bounds on the operators S_ϵ^\pm from (20) and (21).

Theorem 5.5. *The operators S_ϵ^+ are uniformly bounded and converge strongly as $\epsilon \rightarrow 0$ on the weighted space $L_2(t^\alpha dt; \mathcal{H})$ if $\alpha < 1$. The operators S_ϵ^- are uniformly bounded and converge strongly as $\epsilon \rightarrow 0$ on the weighted space $L_2(t^\alpha dt; \mathcal{H})$ if $\alpha > -1$.*

Note that the case $\alpha = 0$ is the usual maximal regularity result in $L_2(\mathbf{R}_+; \mathcal{H})$. The methods here provide a proof of it.

To establish boundedness of the integral operators $F(\lambda)$, we rely on the following version of Schur's lemma. The proof is straightforward using Cauchy–Schwarz' inequality.

Lemma 5.6. *Consider the integral operator $f_t \mapsto \int_0^\infty k(t, s)f_s ds$, with \mathbf{C} -valued kernel $k(t, s)$. If the kernel has the bounds*

$$\sup_t \frac{1}{t^{\beta_2 - \alpha}} \int_0^\infty |k(t, s)|s^{\beta_1} ds = C_1 < \infty, \quad \sup_s \frac{1}{s^{\beta_1 + \alpha}} \int_0^\infty |k(t, s)|t^{\beta_2} dt = C_2 < \infty,$$

for some $\beta_1, \beta_2 \in \mathbf{R}$, then the integral operator is bounded on $L_2(t^\alpha dt; \mathbf{C})$ with norm at most $\sqrt{C_1 C_2}$.

The second result that we need shows that when the integral operators $F(\lambda)$ define a holomorphic function in $\Psi(S_{\nu+}^\circ; \mathcal{L}(\mathcal{K}))$, then the resulting operator $F(\Lambda)$ can be represented as an integral operator which operator-valued kernel.

Lemma 5.7. *Consider a family of integral operators $F(\lambda)f_t = \int_0^\infty k_\lambda(t, s)f_s ds$ such that the \mathbf{C} -valued kernels have the bounds*

$$\sup_t \frac{1}{t^{\beta_2 - \alpha}} \int_0^\infty |k_\lambda(t, s)|s^{\beta_1} ds \leq \eta(\lambda), \quad \sup_s \frac{1}{s^{\beta_1 + \alpha}} \int_0^\infty |k_\lambda(t, s)|t^{\beta_2} dt \leq \eta(\lambda).$$

If $\sup_{\lambda \in S_{\nu+}^\circ} \eta(\lambda) < \infty$, if $\lambda \mapsto k_\lambda(t, s)$ is holomorphic in $S_{\nu+}^\circ$ for a.e. (t, s) , and if $\iint_K |\partial_\lambda k_\lambda(t, s)| dt ds$ is locally bounded in λ for each compact set K , then $F \in H_\infty(S_{\nu+}^\circ; \mathcal{L}(L_2(t^\alpha dt; \mathcal{H})))$.

If furthermore $\eta(\lambda) \lesssim \min(|\lambda|^a, |\lambda|^{-a})$ for $\lambda \in S_{\nu+}^\circ$ and some $a > 0$, then $F \in \Psi(S_{\nu+}^\circ; \mathcal{L}(L_2(t^\alpha dt; \mathcal{H})))$, and

$$F(\Lambda)f_t = \int_0^\infty k_\Lambda(t, s)f_s ds, \quad \text{for all } f \in L_2(t^\alpha dt; \mathcal{H}) \text{ and a.e. } t,$$

where the operator-valued kernels $k_\Lambda(t, s)$ are defined through (22) for a.e. (t, s) .

Proof. Schur's lemma 5.6 provides the bounds on $F(\lambda)$. To show that the operator-valued function F is holomorphic, by local boundedness it suffices to show that the scalar function

$$\lambda \mapsto \iint (h_t, k_\lambda(t, s)f_s) ds dt$$

is holomorphic, for all bounded and compactly supported f, h . The hypothesis on $\partial_\lambda k_\lambda(t, s)$ guarantees this.

To prove the representation formula for $F(\Lambda)$, it suffices to show that for each $f \in L_2(t^\alpha dt; \mathcal{H})$, $v \in \mathcal{H}$, and a.e. t , changing order of integration is possible in

$$\iint (v, k_\lambda(t, s)(\lambda - \Lambda)^{-1}f_s) ds d\lambda.$$

Thus, by Fubini, one needs to show

$$\iint |k_\lambda(t, s)| \|f_s\| ds \frac{|d\lambda|}{|\lambda|} < \infty, \quad \text{for a.e. } t.$$

The bounds on $k_\lambda(t, s)$ in the hypothesis guarantee this. \square

Proof of Theorem 5.5. Since S_ϵ^+ in $L_2(t^\alpha dt; \mathcal{H})$ and S_ϵ^- in $L_2(t^{-\alpha} dt; \mathcal{H})$, with Λ replaced by Λ^* , are adjoint operators, it suffices to consider S_ϵ^+ . Let

$$F_\epsilon(\lambda)f_t := \int_0^t \eta_\epsilon^+(t, s) \lambda e^{-(t-s)\lambda} f_s ds.$$

Uniform boundedness of the integral operators $F_\epsilon(\lambda)$ follows from Lemma 5.6 with $\beta_1 = -\alpha$, $\beta_2 = 0$, using the estimate $\int_0^y e^x x^{-\alpha} dx \lesssim e^y y^{-\alpha}$, which holds if and only if $\alpha \in (-\infty, 1)$. Indeed, since $\lambda \in S_{\nu^+}^o$ with $\nu < \pi/2$, we have $\lambda_1 := \operatorname{Re} \lambda \approx |\lambda|$ and

$$\int_0^t |\lambda e^{-\lambda(t-s)}| s^{-\alpha} ds \approx \int_0^t \lambda_1 e^{-\lambda_1(t-s)} s^{-\alpha} ds = \lambda_1^\alpha e^{-\lambda_1 t} \int_0^{\lambda_1 t} e^x x^{-\alpha} dx \lesssim t^{-\alpha},$$

Similarly, $\int_s^\infty |\lambda e^{-\lambda(t-s)}| dt \lesssim e^{\lambda_1 s} \int_{\lambda_1 s}^\infty e^{-x} dx = 1$.

Again using Lemma 5.6, we note for fixed $\epsilon > 0$ the crude estimate $\|F_\epsilon(\lambda)\| \lesssim |\lambda| e^{-\epsilon \operatorname{Re} \lambda}$, and with Lemma 5.7 we verify that $F_\epsilon \in \Psi(S_{\nu^+}^o; \mathcal{L}(L_2(t^\alpha dt; \mathcal{H})))$, and

$$F_\epsilon(\Lambda)f_t = \int_0^t \eta_\epsilon^+(t, s) \Lambda e^{-(t-s)\Lambda} f_s ds = S_\epsilon^+ f_t, \quad \text{for a.e. } t.$$

To prove strong convergence, by Proposition 5.4 it suffices to show strong convergence of $F_\epsilon(\lambda)$ as $\epsilon \rightarrow 0$, for fixed $\lambda \in S_{\nu^+}^o$. By uniform boundedness of $F_\epsilon(\lambda)$, it suffices to show that $F_\epsilon(\lambda)f$ converges in $L_2(t^\alpha dt; \mathcal{H})$ as $\epsilon \rightarrow 0$ for each f in the dense set $\bigcup_{\delta > 0} L_2((\delta, \delta^{-1}), t^\alpha dt; \mathcal{H})$. This will follow from norm convergence of $F_\epsilon(\lambda)$ in $\mathcal{L}(L_2((\delta, \delta^{-1}), t^\alpha dt; \mathcal{H}), L_2(t^\alpha dt; \mathcal{H}))$ for each fixed $\delta > 0$. To see this, we use Lemma 5.6 with $\beta_1 = -\alpha$ and $\beta_2 = 0$. As above C_1 is uniformly bounded. One verifies decay to 0 as $\epsilon \rightarrow 0$ of

$$\sup_{s \in (\delta, \delta^{-1})} \int_{(2\epsilon)^{-1}}^\infty \lambda_1 e^{-(t-s)\lambda_1} dt \quad \text{and} \quad \sup_{s \in (\delta, \delta^{-1})} \int_s^{s+2\epsilon} \lambda_1 e^{-(t-s)\lambda_1} dt.$$

This shows that $C_2 \rightarrow 0$ as $\epsilon \rightarrow 0$, which proves the strong convergence and the theorem. \square

5.3. Endpoint cases. The operators S_ϵ^- are not uniformly bounded on $L_2(t^\alpha dt; \mathcal{H})$ when $\alpha \leq -1$, and therefore no limit operator S^- exists in these spaces. Indeed, if $\eta(t)$ is a smooth approximation to the Dirac delta at $t = 1$ and $f \in \mathcal{H}$, then $S_\epsilon^-(\eta f)_t$ is independent of ϵ for $\epsilon < t/2$, with non-zero value $\approx \Lambda e^{-\Lambda} f \in \mathcal{H}$ for $t \approx 0$. Thus $\sup_{\epsilon > 0} \int_0^\infty \|S_\epsilon^-(\eta f)_t\|_{\mathcal{H}}^2 t^\alpha dt = \infty$ if $\alpha \leq -1$. By duality S_ϵ^+ cannot be uniformly bounded when $\alpha \geq 1$.

In this section we study the action on the endpoint space $Y^* := L_2(t^{-1} dt; \mathcal{H})$. To obtain a uniform boundedness result for S_ϵ^- , we introduce an auxiliary Banach space X with continuous embeddings

$$(23) \quad Y^* \subset X \subset L_2^{\text{loc}}(dt; \mathcal{H}),$$

i.e. $\int_a^b \|f_t\|_{\mathcal{H}}^2 dt \lesssim \|f\|_X^2 \lesssim \int_0^\infty \|f_t\|_{\mathcal{H}}^2 dt/t$ hold for each fixed $0 < a < b < \infty$, and such that the map $u \mapsto (e^{-t\Lambda} u)_{t>0}$ is bounded $\mathcal{H} \rightarrow X$, i.e.

$$(24) \quad \|e^{-t\Lambda} u\|_X \lesssim \|u\|_{\mathcal{H}}, \quad \text{for all } u \in \mathcal{H}.$$

The spaces Y^* , X and $Y := L_2(t dt; \mathcal{H})$ we view as abstract versions of \mathcal{Y}^* , \mathcal{X} and \mathcal{Y} from Definition 4.1.

Theorem 5.8. *Consider the model operators S_ϵ^+ and S_ϵ^- from (20-21).*

The operators S_ϵ^+ are uniformly bounded on Y^ and converge strongly to a limit operator $S^+ \in \mathcal{L}(Y^*, Y^*)$ as $\epsilon \rightarrow 0$.*

The operators S_ϵ^- are uniformly bounded $Y^ \rightarrow X$, and there is a limit operator $S^- \in \mathcal{L}(Y^*, X)$ such that $\lim_{\epsilon \rightarrow 0} \|S_\epsilon^- f - S^- f\|_{L_2(a,b;\mathcal{H})} = 0$ for any fixed $0 < a < b < \infty$ and $f \in Y^*$.*

For the proof, we shall need the first part of the following lemma. The second part will be required in Propositions 6.1 and 6.2 below.

Lemma 5.9. *The operators*

$$\int_0^\infty \eta_\epsilon(s) \Lambda e^{-s\Lambda} f_s ds : Y^* \rightarrow \mathcal{H}$$

are bounded, uniformly in ϵ , and converge strongly as $\epsilon \rightarrow 0$. Let $U_s : \mathcal{H} \rightarrow \mathcal{H}$ be bounded operators such that $U_s^ e^{-s\Lambda^*} : \mathcal{H} \rightarrow Y^*$ is bounded. Then the operators*

$$\int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} U_s f_s ds : Y \rightarrow \mathcal{H}$$

are bounded, uniformly in ϵ , and converge strongly as $\epsilon \rightarrow 0$.

Proof. For the first operator, square function estimates for Λ^* give

$$\left\| \int_0^\infty \eta_\epsilon(s) \Lambda e^{-s\Lambda} f_s ds \right\|_{\mathcal{H}} = \sup_{\|h\|_2=1} \left| \int_0^\infty (s\Lambda^* e^{-s\Lambda^*} h, f_s) \eta_\epsilon(s) \frac{ds}{s} \right| \lesssim \|\eta_\epsilon f\|_{Y^*} \lesssim \|f\|_{Y^*}.$$

For the second operator

$$\begin{aligned} \left\| \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} U_s f_s ds \right\|_{\mathcal{H}} &\lesssim \sup_{\|h\|_2=1} \left| \int_0^\infty (U_s^* e^{-s\Lambda^*} h, f_s) \eta_\epsilon(s) ds \right| \\ &\lesssim \sup_{\|h\|_2=1} \|U_s^* e^{-s\Lambda^*} h\|_{Y^*} \|\eta_\epsilon f\|_Y \lesssim \|\eta_\epsilon f\|_Y \lesssim \|f\|_Y, \end{aligned}$$

where in the second last estimate the hypothesis is used. (Note that the \mathcal{H} -bound on U_s is not used quantitatively.)

To see the strong convergence, replace η_ϵ by $\eta_\epsilon - \eta_{\epsilon'}$ and use the dominated convergence theorem. \square

Proof of Theorem 5.8. The result for S_ϵ^+ is contained in Theorem 5.5, so it suffices to consider S_ϵ^- . Write

$$\begin{aligned} (25) \quad S_\epsilon^- f_t &= \int_t^\infty \eta_\epsilon^-(t, s) \Lambda e^{-(s-t)\Lambda} f_s ds = \int_t^\infty \eta_\epsilon^-(t, s) \Lambda (e^{-(s-t)\Lambda} - e^{-(s+t)\Lambda}) f_s ds \\ &\quad - \int_0^{t+2\epsilon} (\eta_\epsilon(t) \eta_\epsilon(s) - \eta_\epsilon^-(t, s)) \Lambda e^{-(s+t)\Lambda} f_s ds \\ &\quad + \eta_\epsilon(t) e^{-t\Lambda} \int_0^\infty \eta_\epsilon(s) \Lambda e^{-s\Lambda} f_s ds = I - II + III. \end{aligned}$$

We show that it is only the last term which is singular in the sense that it is not uniformly bounded on Y^* . Consider the term I and the symbol $F_\epsilon^I(\lambda) u_t = \int_t^\infty \eta_\epsilon^-(t, s) k_\lambda(t, s) u_s ds$, where $k_\lambda(t, s) := \lambda e^{-(s-t)\lambda} (1 - e^{-2t\lambda})$. Boundedness of $F_\epsilon^I(\lambda)$

on Y^* , uniformly in λ and ϵ follows from Lemma 5.6 and the estimates $\int_t^\infty |k_\lambda(t, s)|s ds \lesssim t$ and $\int_0^s |k_\lambda(t, s)|dt \lesssim 1$. For example

$$\int_t^\infty |k_\lambda(t, s)|s ds \lesssim \min(1, t\lambda_1)e^{t\lambda_1} \int_t^\infty \lambda_1 e^{-s\lambda_1} s ds = t \min(1, t\lambda_1)(1 + 1/(t\lambda_1)) \lesssim t,$$

with $\lambda_1 := \operatorname{Re} \lambda$. On the other hand, for fixed $\epsilon > 0$, it is straightforward to verify with Lemma 5.6 that $\|F_\epsilon^I(\lambda)\|_{Y^* \rightarrow Y^*} \lesssim |\lambda|e^{-\epsilon \operatorname{Re} \lambda}$, and with Lemma 5.7 that $F_\epsilon^I \in \Psi(S_{\nu+}^o; \mathcal{L}(Y^*))$ and

$$F_\epsilon^I(\Lambda)f_t = \int_t^\infty \eta_\epsilon^-(t, s)\Lambda(e^{-(s-t)\Lambda} - e^{-(s+t)\Lambda})f_s ds, \quad \text{for a.e. } t.$$

To prove strong convergence, as in the proof of Theorem 5.5, by uniform boundedness it suffices to show norm convergence of $F_\epsilon^I(\lambda)$ in $\mathcal{L}(L_2((\delta, \delta^{-1}), t^{-1}dt; \mathcal{H}), Y^*)$ for each fixed $\delta > 0$. This follows from Lemma 5.6, where one verifies decay to 0 as $\epsilon \rightarrow 0$ of $\sup_{s \in (\delta, \delta^{-1})} \int_0^{2\epsilon} |k_\lambda(t, s)|dt$ and $\sup_{s \in (\delta, \delta^{-1})} \int_{s-2\epsilon}^s |k_\lambda(t, s)|dt$, and hence $C_2 \rightarrow 0$, for fixed $\lambda \in S_{\nu+}^o$. Together with the uniform bound $\sup_t t^{-1} \int_t^\infty |k_\lambda(t, s)|s ds < \infty$, this proves the strong convergence for the term I .

Consider next the term II and the symbol

$$F_\epsilon^{II}(\lambda)u_t = \int_0^{t+2\epsilon} (\eta_\epsilon(t)\eta_\epsilon(s) - \eta_\epsilon^-(t, s))\lambda e^{-(s+t)\lambda} u_s ds.$$

Boundedness of $F_\epsilon^{II}(\lambda)$ on Y^* , uniformly in λ and ϵ follows from Lemma 5.6 and the estimates $\int_0^{3t} |\lambda e^{-(s+t)\lambda}|s ds \lesssim t$ and $\int_{s/3}^\infty |\lambda e^{-(s+t)\lambda}|dt \lesssim 1$. On the other hand, for fixed $\epsilon > 0$, we verify with Lemma 5.6 that $\|F_\epsilon^{II}(\lambda)\|_{Y^* \rightarrow Y^*} \lesssim |\lambda|e^{-\epsilon \operatorname{Re} \lambda}$, and with Lemma 5.7 that $F_\epsilon^{II} \in \Psi(S_{\nu+}^o; \mathcal{L}(Y^*))$ and

$$F_\epsilon^{II}(\Lambda)f_t = \int_0^{t+2\epsilon} (\eta_\epsilon(t)\eta_\epsilon(s) - \eta_\epsilon^-(t, s))\Lambda e^{-(s+t)\Lambda} u_s ds, \quad \text{for a.e. } t.$$

With the same technique as for the term I , the strong convergence of the term II follows from the decay to 0 as $\epsilon \rightarrow 0$ of $\sup_{s \in (\delta, \delta^{-1})} \int_{s-2\epsilon}^s |\lambda e^{-(s+t)\lambda}|dt$.

It remains to estimate the principal term III . Since the variables t and s separate, we can factor this term through the boundary space \mathcal{H} as a composition $Y^* \rightarrow \mathcal{H} \rightarrow X$, where Lemma 5.9 and the assumed bounds $e^{-t\Lambda} : \mathcal{H} \rightarrow X$ prove boundedness, uniform in ϵ , as well as strong convergence as maps $Y^* \rightarrow \mathcal{H} \rightarrow L_2(a, b; \mathcal{H})$. This completes the proof. \square

6. ESTIMATES OF THE INTEGRAL OPERATORS S_A AND \tilde{S}_A

Consider the operator DB_0 from Section 3. We set $\Lambda = |DB_0| := DB_0 \operatorname{sgn}(DB_0)$ on $\mathcal{H} = \overline{\mathbf{R}(D)}$, and see that Λ satisfies the assumptions of Section 5. It is a closed, densely defined, injective operator with $\sigma(\Lambda) \subset S_{\omega+}$ and $\|(\lambda - \Lambda)^{-1}\|_{\mathcal{H} \rightarrow \mathcal{H}} \lesssim 1/\operatorname{dist}(\lambda, S_{\omega+})$ (this follows from the resolvent bounds on DB_0). In Section 5.3, we set $Y^* := \mathcal{Y}^* \cap L_2^{\operatorname{loc}}(\mathbf{R}_+; \mathcal{H})$, $X := \mathcal{X} \cap L_2^{\operatorname{loc}}(\mathbf{R}_+; \mathcal{H})$ and $Y := \mathcal{Y} \cap L_2^{\operatorname{loc}}(\mathbf{R}_+; \mathcal{H})$. Note that the continuous embeddings (23) follow from Lemma 4.3 and the boundedness hypothesis (24) on $e^{-t|DB_0|} : \mathcal{H} \rightarrow X$ follows from Theorem 4.2 (and the analogous result for the lower half space \mathbf{R}_-^{1+n} , i.e. $f_0 \in E_0^- \mathcal{H}$ giving a solution of $\partial_t f + DB_0 f = 0$ for $t < 0$).

We shall use the operational calculus of Λ to estimate S_A in (19). Before doing so, let us describe the strategy for the Dirichlet problem, which leads to introduce

a second operator. We seek to estimate the potential u and for this we will also need to apply functional calculus of B_0D . The key idea is to write the equation $\nabla_A u = e^{-t\tilde{\Lambda}}h^+ + S_A\nabla_A u$ from (18), with $h^+ := E_0^+(\nabla_A u|_{\mathbf{R}^n})$, as $\nabla_A u = Dv$, where v is the vector-valued potential $v := e^{-t\tilde{\Lambda}}\tilde{h}^+ + \tilde{S}_A\nabla_A u$, $h^+ = D\tilde{h}^+$ and

$$\tilde{S}_A f_t := \int_0^t e^{-(t-s)\tilde{\Lambda}}\tilde{E}_0^+(\mathcal{E}_s f_s) ds - \int_t^\infty e^{-(s-t)\tilde{\Lambda}}\tilde{E}_0^-(\mathcal{E}_s f_s) ds,$$

where $\tilde{\Lambda} := |B_0D|$ and $\tilde{E}_0^\pm := \chi^\pm(B_0D)$. Note that $B_0\hat{E}_0^\pm = B_0E_0^\pm B_0^{-1}P_{B_0\mathcal{H}} = \tilde{E}_0^\pm P_{B_0\mathcal{H}} = \tilde{E}_0^\pm$ by (26) below. Since $\nabla_x u = (\nabla_A u)_\parallel = -\nabla_x v_\perp$, we have $u_t = -(v_t)_\perp$ since u_t is an L_2 function, as will v_t be.

If B_0 were invertible on all L_2 , then DB_0 and B_0D would be similar operators, but this is not the case in general. Still, whenever B_0 is accretive on \mathcal{H} , it is true that B_0D is an ω -bisectorial operator with resolvent bounds. Furthermore, the function space splits

$$L_2 = B_0\mathcal{H} \oplus \mathcal{H}^\perp$$

(cf. (17)) and B_0D restricts to an injective operator with dense range in $B_0\mathcal{H}$. This operator has square function estimates, and therefore bounded functional and operational calculus in $B_0\mathcal{H}$, as in Section 5.1. For proofs and further details, see [7]. Unless otherwise stated, we extend an operator $b(B_0D)$ in the functional calculus to an operator on all L_2 , by letting $b(B_0D) = 0$ on $\mathcal{H}^\perp = \mathbf{N}(B_0D)$. With this notation $\tilde{E}_0^\pm(B_0\mathcal{H}) = \tilde{E}_0^\pm L_2$, and we shall prefer the latter to ease the notation.

A important relation between the functional calculus of DB_0 and B_0D is

$$(26) \quad B_0b(DB_0) = b(B_0D)B_0,$$

where we also extend operators $b(DB_0)$ to all L_2 , letting $b(DB_0)|_{\mathbf{N}(DB_0)} := 0$. The equation (26) clearly holds for resolvents $b(z) = (\lambda - z)^{-1}$. The general case follows from Dunford integration (22) and taking strong limits as in Proposition 5.4 (adapted to bisectorial operators). Note that (26) in particular shows that for appropriate b and u

$$b(DB_0)Du = Db(B_0D)u.$$

Recall that $\Lambda = |DB_0|$ and $\tilde{\Lambda} = |B_0D|$, and note that $\tilde{\Lambda}^* = |DB_0^*|$ and $\Lambda^* = |B_0^*D|$.

Proposition 6.1. *Assume that $\mathcal{E} : \mathbf{R}_+^{1+n} \rightarrow \mathcal{L}(\mathbf{C}^{(1+n)m})$ satisfies $\|\mathcal{E}\|_* < \infty$, and define operators*

$$S_A^\epsilon f_t := \int_0^t \eta_\epsilon^+(t, s)\Lambda e^{-(t-s)\Lambda}\hat{E}_0^+\mathcal{E}_s f_s ds + \int_t^\infty \eta_\epsilon^-(t, s)\Lambda e^{-(s-t)\Lambda}\hat{E}_0^-\mathcal{E}_s f_s ds.$$

Then $\|S_A^\epsilon\|_{\mathcal{X} \rightarrow \mathcal{X}} \lesssim \|\mathcal{E}\|_$ and $\|S_A^\epsilon\|_{\mathcal{Y} \rightarrow \mathcal{Y}} \lesssim \|\mathcal{E}\|_*$, uniformly for $\epsilon > 0$. In the space \mathcal{X} there is a limit operator $S_A = S_A^\mathcal{X} \in \mathcal{L}(\mathcal{X}; \mathcal{X})$ such that*

$$\lim_{\epsilon \rightarrow 0} \|S_A^\epsilon f - S_A^\mathcal{X} f\|_{L_2(a, b; L_2)} = 0, \quad \text{for any } f \in \mathcal{X}, 0 < a < b < \infty.$$

In the space \mathcal{Y} , there is a limit operator $S_A = S_A^\mathcal{Y} \in \mathcal{L}(\mathcal{Y}; \mathcal{Y})$ such that

$$\lim_{\epsilon \rightarrow 0} \|S_A^\epsilon f - S_A^\mathcal{Y} f\|_{\mathcal{Y}} = 0, \quad \text{for any } f \in \mathcal{Y}.$$

Proof. The result on \mathcal{X} is a direct consequence of Theorem 5.8, since $\|\mathcal{E}f\|_{\mathcal{Y}^*} \leq \|\mathcal{E}\|_* \|f\|_{\mathcal{X}}$ and $S_A^\epsilon = S_\epsilon^+ \hat{E}_0^+ \mathcal{E} + S_\epsilon^- \hat{E}_0^- \mathcal{E}$. Note that $\mathbf{R}(\hat{E}_0^\pm) \subset \mathcal{H} \subset L_2$.

Consider now the space \mathcal{Y} . The second term $S_\epsilon^- \widehat{E}_0^- \mathcal{E}$ is bounded on \mathcal{Y} , uniformly in ϵ , and converges strongly on \mathcal{Y} . This follows from Theorem 5.5 and the boundedness $\|\widehat{E}_0^- \mathcal{E}\|_{\mathcal{Y} \rightarrow \mathcal{Y}} \lesssim \|\mathcal{E}\|_\infty \lesssim \|\mathcal{E}\|_* < \infty$. The term $S_\epsilon^+ \widehat{E}_0^+ \mathcal{E}$ we split as

$$\begin{aligned} \int_0^t \eta_\epsilon^+(t, s) \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds &= \int_0^t \eta_\epsilon^+(t, s) \Lambda (e^{-(t-s)\Lambda} - e^{-(t+s)\Lambda}) \widehat{E}_0^+ \mathcal{E}_s f_s ds \\ &\quad - \int_{t-2\epsilon}^\infty (\eta_\epsilon(t) \eta_\epsilon(s) - \eta_\epsilon^+(t, s)) \Lambda e^{-(t+s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds \\ &\quad + \eta_\epsilon(t) \Lambda e^{-t\Lambda} \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds. \end{aligned}$$

The result for the first two terms follows from the proof of Theorem 5.8 by duality, only using the boundedness of \mathcal{E} on \mathcal{Y} . For the last term, as the variables t and s split, it suffices to show uniform boundedness and convergence of

$$L_2 \rightarrow \mathcal{Y} : h \mapsto \eta_\epsilon(t) \Lambda e^{-t\Lambda} h$$

and

$$\mathcal{Y} \rightarrow L_2 : f_t \mapsto \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds$$

separately. For the first operator, this follows directly from the square function estimates for Λ . To handle the second, it suffices to estimate $B_0 \int_0^\infty \eta_\epsilon(s) e^{-s\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds = \int_0^\infty \eta_\epsilon(s) e^{-s\tilde{\Lambda}} \widetilde{E}_0^+ \mathcal{E}_s f_s ds$, since B_0 is accretive on $\mathcal{H} \supset \mathcal{R}(e^{-s\Lambda} \widehat{E}_0^+)$. To this end, we apply Lemma 5.9 with $U_s := \widetilde{E}_0^+ \mathcal{E}_s P_{\mathcal{H}}$, where $P_{\mathcal{H}}$ is orthogonal projection onto \mathcal{H} , and Λ replaced by $\tilde{\Lambda}$. The hypothesis there on boundedness of

$$\mathcal{H} \rightarrow Y^* : h \mapsto U_s^* e^{-s\tilde{\Lambda}^*} h = P_{\mathcal{H}} \mathcal{E}_s^* e^{-t|DB_0^*|} \chi^+(DB_0^*) h,$$

follows from the maximal estimate in Theorem 4.2 (with B_0 replaced by B_0^*), the assumed boundedness of $\mathcal{E}^* : \mathcal{X} \rightarrow \mathcal{Y}^*$ and L_2 boundedness of $\chi^+(DB_0^*)$ and $P_{\mathcal{H}}$. This completes the proof. \square

By inspection of the proofs above, the limit operator S_A , both for $f \in \mathcal{X}$ and $f \in \mathcal{Y}$, is seen to be

$$S_A f_t = \lim_{\epsilon \rightarrow 0} \left(\int_\epsilon^{t-\epsilon} \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds + \int_{t-\epsilon}^{\epsilon^{-1}} \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds \right),$$

with convergence in $L_2(a, b; L_2)$ for any $0 < a < b < \infty$. This holds since we may equally well choose to work with the characteristic function $\eta^0(t) = \chi_{(1, \infty)}(t)$ instead of the piecewise linear function η^0 defined below (20-21). The only places we need the continuity of η^0 are in Theorem 7.2 and 8.2 below.

For the non-singular integral operator \widetilde{S}_A , our result is the following. Write $C_b(X, V)$ for the space of bounded and continuous functions on X with values in V .

Proposition 6.2. *The operators*

$$\widetilde{S}_A^\epsilon f_t := \int_0^t \eta_\epsilon^+(t, s) e^{-(t-s)\tilde{\Lambda}} \widetilde{E}_0^+ \mathcal{E}_s f_s ds - \int_t^\infty \eta_\epsilon^-(t, s) e^{-(s-t)\tilde{\Lambda}} \widetilde{E}_0^- \mathcal{E}_s f_s ds$$

are bounded $\mathcal{Y} \rightarrow C_b(\overline{\mathbf{R}}_+; L_2)$, with $\sup_{t>0} \|\widetilde{S}_A^\epsilon f_t\|_2 \lesssim \|\mathcal{E}\|_* \|f\|_{\mathcal{Y}}$, uniformly for $\epsilon > 0$, and there is a limit operator $\widetilde{S}_A \in \mathcal{L}(\mathcal{Y}, C_b(\overline{\mathbf{R}}_+; L_2))$ such that $\lim_{\epsilon \rightarrow 0} \|\widetilde{S}_A^\epsilon f_t - \widetilde{S}_A f_t\|_2 = 0$ locally uniformly for $t \in (0, \infty)$, for any $f \in \mathcal{Y}$. The limit operator

satisfies $S_A f = D\tilde{S}_A f$ in \mathbf{R}_+^{1+n} distributional sense, where $S_A = S_A^\mathcal{Y}$ is the operator from Proposition 6.1, and has limits

$$\lim_{t \rightarrow 0} \|\tilde{S}_A f_t - \tilde{h}^-\|_2 = 0 = \lim_{t \rightarrow \infty} \|\tilde{S}_A f_t\|_2,$$

where $\tilde{h}^- := -\int_0^\infty e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s f_s ds \in \tilde{E}_0^- L_2$, for any $f \in \mathcal{Y}$.

Note that $\tilde{S}_A^\epsilon f_t = 0$ when $t \notin (\epsilon, \epsilon^{-1})$, so convergence $\tilde{S}_A^\epsilon f_t \rightarrow \tilde{S}_A f_t$ is not uniform up to $t = 0$. By inspection of the proof below, the limit operator is seen to be

$$\tilde{S}_A f_t = \int_0^t e^{-(t-s)\tilde{\Lambda}} \tilde{E}_0^+ \mathcal{E}_s f_s ds - \int_t^\infty e^{-(s-t)\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s f_s ds,$$

where the integrals are weakly convergent in L_2 for all $f \in \mathcal{Y}$ and $t > 0$.

Proof. The estimates for \tilde{S}_A^ϵ are more straightforward than those for S_A^ϵ since there is no singularity at $s = t$. For the $(0, t)$ -integral, split it as

$$\int_0^t \eta_\epsilon^+(t, s) e^{-(t-s)\tilde{\Lambda}} (I - e^{-2s\tilde{\Lambda}}) \tilde{E}_0^+ \mathcal{E}_s f_s ds + e^{-t\tilde{\Lambda}} \int_0^t \eta_\epsilon^+(t, s) e^{-s\tilde{\Lambda}} \tilde{E}_0^+ \mathcal{E}_s f_s ds.$$

For the first term, we write $e^{-(t-s)\tilde{\Lambda}} (I - e^{-2s\tilde{\Lambda}}) = \frac{s}{t-s} ((t-s)\tilde{\Lambda} e^{-(t-s)\tilde{\Lambda}}) ((I - e^{-2s\tilde{\Lambda}}) / (s\tilde{\Lambda}))$ to obtain the estimate $\|e^{-(t-s)\tilde{\Lambda}} (I - e^{-2s\tilde{\Lambda}})\| \lesssim s/t$. From this uniform boundedness and convergence, locally uniformly in t , as $\epsilon \rightarrow 0$ follows by Cauchy–Schwarz inequality. For the second term we use uniform boundedness of $e^{-t\tilde{\Lambda}}$ and duality to estimate it by

$$\sup_{\|h\|_2=1} \left| \int_0^t (\mathcal{E}_s^* e^{-s\tilde{\Lambda}^*} (\tilde{E}_0^+)^* h, f_s) \eta_\epsilon^+(t, s) ds \right| \lesssim \|\mathcal{E}^*\|_* \|f\|_{\mathcal{Y}},$$

using Lemma 5.9 as in the proof of Proposition 6.1. Moreover, the L_2 difference between the integral at ϵ and ϵ' is bounded by $\int_0^t \|f_s\|_2^2 |\eta_\epsilon^+(t, s) - \eta_{\epsilon'}^+(t, s)|^2 ds \rightarrow 0$ as $\epsilon, \epsilon' \rightarrow 0$ for fixed t , which proves the convergence.

The proof for the (t, ∞) -integral in \tilde{S}_A^ϵ is similar, splitting it as

$$\int_t^\infty \eta_\epsilon^-(t, s) e^{-(s-t)\tilde{\Lambda}} (I - e^{-2t\tilde{\Lambda}}) \tilde{E}_0^- \mathcal{E}_s f_s ds + e^{-t\tilde{\Lambda}} \int_t^\infty \eta_\epsilon^-(t, s) e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s f_s ds,$$

and using the estimate $\|e^{-(s-t)\tilde{\Lambda}} (I - e^{-2t\tilde{\Lambda}})\| \lesssim t/s$ for the first term and Lemma 5.9 for the second.

Since clearly $\tilde{S}_A^\epsilon f \in C_b(\mathbf{R}_+; L_2)$, its locally uniform limit $\tilde{S}_A f$ also belongs to $C_b(\mathbf{R}_+; L_2)$. To find the limits of $\tilde{S}_A f_t$ at 0 and ∞ , since $\tilde{S}_A : \mathcal{Y} \rightarrow C_b(\mathbf{R}_+; L_2)$ is bounded it suffices to consider $f \in \mathcal{Y}$ such that $f_t = 0$ for $t \notin (a, b)$, with $0 < a < b < \infty$ fixed but arbitrary. In this case,

$$\tilde{S}_A f_t = \int_{a < s < \min(t, b)} e^{-(t-s)\tilde{\Lambda}} \tilde{E}_0^+ \mathcal{E}_s f_s ds - \int_{\max(t, a) < s < b} e^{-(s-t)\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s f_s ds$$

satisfies $\tilde{E}_0^+ \tilde{S}_A f_t = 0$ for $t < a$ and $\tilde{E}_0^- \tilde{S}_A f_t = 0$ when $t > b$, from which the two limits $\lim_{t \rightarrow 0} \tilde{E}_0^+ \tilde{S}_A f_t = 0 = \lim_{t \rightarrow \infty} \tilde{E}_0^- \tilde{S}_A f_t$ follow. For the remaining two limits $\lim_{t \rightarrow \infty} \tilde{E}_0^+ \tilde{S}_A f_t$ and $\lim_{t \rightarrow 0} \tilde{E}_0^- \tilde{S}_A f_t$, we use that

$$\lim_{t \rightarrow \infty} \int_a^b \|e^{-(t-s)\tilde{\Lambda}} \tilde{E}_0^+ \mathcal{E}_s f_s\|_2 ds = 0 = \lim_{t \rightarrow 0} \int_a^b \|(e^{-(s-t)\tilde{\Lambda}} - e^{-s\tilde{\Lambda}}) \tilde{E}_0^- \mathcal{E}_s f_s\|_2 ds$$

by dominated convergence.

To verify the identity $S_A = D\tilde{S}_A$, note that $\int_0^\infty (\phi_t, S_A^\epsilon f_t) dt = \int_0^\infty (D\phi_t, \tilde{S}_A^\epsilon f_t) dt$ for all $f \in \mathcal{Y}$ and $\phi \in C_0^\infty(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m})$. Let $\epsilon \rightarrow 0$ and use S_A^ϵ and \tilde{S}_A^ϵ convergence. This completes the proof. \square

7. THE NEUMANN AND REGULARITY PROBLEMS

Throughout this section, A denotes t -dependent coefficients satisfying (2) and (3), and $A_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ denotes t -independent coefficients which are accretive on \mathcal{H} . We let $B := \hat{A}$ and $B_0 := \hat{A}_0$ be the transformed accretive coefficients from Proposition 3.1, and define $\mathcal{E} := B_0 - B$.

For the Neumann and regularity problems, one seeks estimates of the gradient $g = \nabla_{t,x} u$ rather than the potential u . With a slight abuse of notation, we say below that g solves the divergence form equation when u does so.

Definition 7.1. By an \mathcal{X} -solution to the divergence form equation, with coefficients A , we mean a function $g \in L_2^{\text{loc}}(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^{(1+n)m}))$, with estimate $\|\tilde{N}_*(g)\|_2 < \infty$, which satisfies

$$\operatorname{div}_{t,x} Ag = 0 = \operatorname{curl}_{t,x} g$$

in \mathbf{R}_+^{1+n} distributional sense.

Note that the boundary behaviour of g is not specified in this definition; we show existence of a limit in appropriate sense (see also Section 10). This will allow us to formulate in what sense the boundary data is prescribed. We first prove the following representation and regularity result for \mathcal{X} -solutions.

Theorem 7.2. Assume that $\|\mathcal{E}\|_* < \infty$. Then g is an \mathcal{X} -solution to the divergence form equation with coefficients A if and only if the corresponding conormal gradient $f = [(Ag)_\perp, g_\parallel]^t \in \mathcal{X}$ satisfies the equation

$$f_t = e^{-t\Lambda} h^+ + S_A f_t, \quad \text{for some } h^+ \in E_0^+ \mathcal{H}.$$

In this case, f has limits

$$(27) \quad \lim_{t \rightarrow 0} t^{-1} \int_t^{2t} \|f_s - f_0\|_2^2 ds = 0 = \lim_{t \rightarrow \infty} t^{-1} \int_t^{2t} \|f_s\|_2^2 ds,$$

where $f_0 := h^+ + h^-$ and $h^- := \int_0^\infty \Lambda e^{-s\Lambda} \hat{E}_0^- \mathcal{E}_s f_s ds \in E_0^- \mathcal{H}$, with estimates

$$\max(\|h^+\|_2, \|h^-\|_2) \approx \|f_0\|_2 \approx \|g_0\|_2 \lesssim \|f\|_{\mathcal{X}} \approx \|g\|_{\mathcal{X}}.$$

The limits analogous to (27), replacing f by g and f_0 by $g_0 := [(B_0 f_0)_\perp, (f_0)_\parallel]^t$, hold. If furthermore $\|\mathcal{E}\|_*$ is sufficiently small, then there are estimates

$$\|h^-\|_2 \lesssim \|h^+\|_2 \approx \|f_0\|_2 \approx \|g_0\|_2 \approx \|f\|_{\mathcal{X}} \approx \|g\|_{\mathcal{X}}.$$

Note that these limits for \mathcal{X} -solutions are stronger than L_2 convergence of Cesaro means $t^{-1} \int_t^{2t} g_s ds$, and that we do get limits for g and f , i.e. for the full gradient and conormal gradient. That $\|g_0\|_2 \approx \|f_0\|_2$ and $\|g\|_{\mathcal{X}} \approx \|f\|_{\mathcal{X}}$ are easy consequences of Proposition 3.1.

Proof. (i) Assume that g is an \mathcal{X} -solution, and consider f . To show that $f_t = e^{-t\Lambda}h^+ + S_A f_t$, we choose η_ϵ^\pm for η^\pm in Proposition 3.2 and subtract the equations to obtain

$$(28) \quad - \int_0^t (\partial_s \eta_\epsilon^+)(t, s) e^{-(t-s)\Lambda} E_0^+ f_s ds + \int_t^\infty (\partial_s \eta_\epsilon^-)(t, s) e^{-(s-t)\Lambda} E_0^- f_s ds \\ = \int_0^t \eta_\epsilon^+(t, s) \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s ds + \int_t^\infty \eta_\epsilon^-(t, s) \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds.$$

Note that $DB_0 = \pm|DB_0| = \pm\Lambda$ on $E_0^\pm \mathcal{H}$. We fix $0 < a < b < \infty$ and consider the equation in $L_2(a, b; \mathcal{H})$. By Proposition 6.1, the right hand side converges to $S_A f$ in $L_2(a, b; \mathcal{H})$. When $t \in (a, b)$ and ϵ is small, the left hand side equals

$$(29) \quad \epsilon^{-1} \int_\epsilon^{2\epsilon} e^{-s\Lambda} (E_0^+ f_{t-s} + E_0^- f_{t+s}) ds \\ - \epsilon^{-1} \int_\epsilon^{2\epsilon} e^{-(t-s)\Lambda} E_0^+ f_s ds - 2\epsilon \int_{(2\epsilon)^{-1}}^{\epsilon^{-1}} e^{-(s-t)\Lambda} E_0^- f_s ds.$$

To prove that the first term converges to f in $L_2(a, b; \mathcal{H})$, adding and subtracting the term $\epsilon^{-1} \int_\epsilon^{2\epsilon} e^{-s\Lambda} f_t ds = e^{-\epsilon\Lambda} (\epsilon\Lambda)^{-1} (I - e^{-\epsilon\Lambda}) f_t$ shows that the square of the $L_2(a, b; \mathcal{H})$ norm of the difference is bounded by

$$\int_a^b \left\| \left(I - e^{-\epsilon\Lambda} \frac{I - e^{-\epsilon\Lambda}}{\epsilon\Lambda} \right) f_t \right\|_2^2 dt + \int_a^b \epsilon^{-1} \int_\epsilon^{2\epsilon} \|f_t - E_0^+ f_{t-s} - E_0^- f_{t+s}\|_2^2 ds dt \rightarrow 0$$

as $\epsilon \rightarrow 0$, using Proposition 5.4 for the functional calculus, dominated convergence, and the identity $f_t = E_0^+ f_t + E_0^- f_t$.

Next consider the last term in (29). For any $\phi \in L_2(a, b; \mathcal{H})$, we have

$$\int_a^b \left(\epsilon \int_{(2\epsilon)^{-1}}^{\epsilon^{-1}} e^{-(s-t)\Lambda} E_0^- f_s ds, \phi_t \right) dt \\ = \epsilon \int_{(2\epsilon)^{-1}}^{\epsilon^{-1}} \left(f_s, \int_a^b (e^{-(s-t)\Lambda^*} - e^{-s\Lambda^*}) (E_0^-)^* \phi_t dt + e^{-s\Lambda^*} (E_0^-)^* \int_a^b \phi_t dt \right) ds.$$

From the sup $-L_2$ estimate in Lemma 4.3 for f , the estimate $\|e^{-(s-t)\Lambda^*} - e^{-s\Lambda^*}\| \lesssim t/s$ and the strong limit $\lim_{s \rightarrow \infty} e^{-s\Lambda^*} (E_0^-)^* = 0$, it follows that the last term in (29) converges weakly to 0. Hence the middle term must converge weakly in $L_2(a, b; L_2)$ as well, and we may replace $e^{-(t-s)\Lambda}$ by $e^{-t\Lambda}$ since $\|e^{-(t-s)\Lambda} - e^{-t\Lambda}\| \lesssim s/t$. We get that

$$\int_a^b \left(e^{-t\Lambda} (\epsilon^{-1} \int_\epsilon^{2\epsilon} E_0^+ f_s ds), \phi_t \right) dt = \left(\epsilon^{-1} \int_\epsilon^{2\epsilon} E_0^+ f_s ds, \int_a^b e^{-t\Lambda^*} \phi_t dt \right)$$

converges for all $\phi \in L_2(a, b; L_2)$. Since $\epsilon^{-1} \int_\epsilon^{2\epsilon} E_0^+ f_s ds$ are uniformly bounded in \mathcal{H} by Lemma 4.3, and since functions $\int_a^b e^{-t\Lambda^*} \phi_t dt$ are dense in $B_0 \mathcal{H} \approx \mathcal{H}^*$ (for example $\int_\epsilon^{2\epsilon} e^{-t\Lambda^*} \epsilon^{-1} \phi dt \rightarrow P_{B_0 \mathcal{H}} \phi$), it follows that $\epsilon^{-1} \int_\epsilon^{2\epsilon} E_0^+ f_s ds$ converges weakly to a function $h^+ \in E_0^+ \mathcal{H}$, and that the weak limit of the middle term in (29) is $e^{-t\Lambda} h^+$. In total, this proves that $f_t - e^{-t\Lambda} h^+ = S_A f_t$.

(ii) Conversely, assume that $f \in \mathcal{X}$ solves $f_t = e^{-t\Lambda}h^+ + S_A f_t$. To verify that f satisfies the differential equation, note that $(\partial_t + DB_0)e^{-t\Lambda}h^+ = 0$. It suffices to show that for $\phi \in C_0^\infty(\mathbf{R}_+^{1+n}; \mathbf{C}^{(1+n)m})$ there is convergence

$$\int (-\partial_t \phi_t + B_0^* D\phi_t, f_t^\epsilon) dt \rightarrow \int (D\phi_s, \mathcal{E}_s f_s) ds, \quad \epsilon \rightarrow 0,$$

where $f_t^\epsilon := S_A^\epsilon f_t$. For the term $S_\epsilon^+ \widehat{E}_0^+ \mathcal{E} f$, Fubini's theorem and integration by parts give

$$\begin{aligned} & \int_0^\infty \int_0^t \eta_\epsilon^+(t, s) ((-\partial_t + \Lambda^*)\phi_t, \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s f_s) ds dt \\ &= - \int_0^\infty \left(\int_s^\infty \eta_\epsilon^+(t, s) \partial_t (e^{-(t-s)\Lambda^*} \Lambda^* \phi_t) dt, \widehat{E}_0^+ \mathcal{E}_s f_s \right) ds \\ &= \int_0^\infty \left(\int_s^\infty (\partial_t \eta_\epsilon^+)(t, s) e^{-(t-s)\Lambda^*} \Lambda^* \phi_t dt, \widehat{E}_0^+ \mathcal{E}_s f_s \right) ds \\ &\rightarrow \int_0^\infty (\Lambda^* \phi_s, \widehat{E}_0^+ \mathcal{E}_s f_s) ds = \int_0^\infty (D\phi_s, \widetilde{E}_0^+ \mathcal{E}_s f_s) ds. \end{aligned}$$

Adding the corresponding limit for the term $S_\epsilon^- \widehat{E}_0^- \mathcal{E} f$ gives the stated result. Note that $\widetilde{E}_0^+ + \widetilde{E}_0^- = P_{B_0\mathcal{H}}$ and $DP_{B_0\mathcal{H}} = D$.

(iii) To show the limits, note that $E_0^+ f - e^{-t\Lambda}h^+ = S^+ \widehat{E}_0^+ \mathcal{E} f \in \mathcal{Y}^*$, and by inspection of the proof of Theorem 5.8 we see that $E_0^- f - e^{-t\Lambda}h^+ = \int_0^\infty \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds \in \mathcal{Y}^*$. From this, the limits for f follow. To see the limit for g at $t = 0$, write $B_t f_t - B_0 f_0 = B_0(f_t - f_0) + \mathcal{E}_t f_t$. Since $\mathcal{E}_t f_t \in \mathcal{Y}^*$, we have $\lim_{t \rightarrow 0} t^{-1} \int_t^{2t} \|\mathcal{E}_s f_s\|_2^2 ds = 0$. The limit of $B_0(f_t - f_0)$ at $t = 0$, as well as that of g at $t = \infty$, is immediate from the limits of f .

(iv) It remains to prove the estimates. Note that (14) and Lemma 4.3 show that

$$\max(\|h^+\|_2^2, \|h^-\|_2^2) \approx \|f_0\|_2^2 = \lim_{t \rightarrow 0} t^{-1} \int_t^{2t} \|f_s\|_2^2 ds \lesssim \|f\|_{\mathcal{X}}^2.$$

Proposition 6.1 shows that $\|S_A\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq 1/2$ if $\|\mathcal{E}\|_*$ is sufficiently small. In this case $I - S_A$ is an isomorphism on \mathcal{X} with $\|(I - S_A)^{-1}\|_{\mathcal{X} \rightarrow \mathcal{X}} \leq 2$. Using this together with Theorem 4.2, we get estimates $\|f\|_{\mathcal{X}} = \|(I - S_A)^{-1} e^{-t\Lambda}h^+\|_{\mathcal{X}} \approx \|h^+\|_2$. This proves the stated estimates and completes the proof. \square

We note the following immediate corollary to Theorem 7.2. Write $h^+ = h$ below.

Corollary 7.3. *Assume that coefficients $A = A_0$ are t -independent. Then g is an \mathcal{X} -solution to the divergence form equation if and only if the associated conormal gradient f can be represented*

$$f_t = e^{-t\Lambda}h, \quad \text{for some } h \in E_0^+\mathcal{H}.$$

In particular, the class of \mathcal{X} -solutions in Definition 7.1 coincides with the class of solutions in [8, Definition 2.1(i-ii)] for t -independent coefficients.

That the solutions in [8] are of this form was shown in the proof of [8, Theorem 2.3]. Note that the operator $T_A|_{\mathcal{H}}$ used in [8] is similar to our operator $DB_0|_{\mathcal{H}}$, as in [8, Definition 3.1].

For t -dependent coefficients A , Theorem 7.2 shows that if $\|\mathcal{E}\|_*$ is small enough, then g is an \mathcal{X} -solution to the divergence form equation with coefficients A if and only if the corresponding conormal gradient f can be represented as

$$(30) \quad f = (I - S_A)^{-1} e^{-t\Lambda} h,$$

for some $h \in E_0^+ \mathcal{H}$. (Here $h \mapsto e^{-t\Lambda} h$ is viewed as a map $E_0^+ \mathcal{H} \rightarrow \mathcal{X}$.) As noted above, in the case of t -independent coefficients $A = A_0$, this simplifies to $f = e^{-t\Lambda} h$, where $h = \lim_{t \rightarrow 0} f_t$. We recall that for the class of solutions used in [8], with t -independent coefficients A_0 , well-posedness of the Neumann and regularity problems was shown to be equivalent to the maps

$$\begin{aligned} E_0^+ \mathcal{H} &\rightarrow L_2(\mathbf{R}^n; \mathbf{C}^m) : h \mapsto h_\perp, \\ E_0^+ \mathcal{H} &\rightarrow \{f \in L_2(\mathbf{R}^n; \mathbf{C}^{nm}) ; \operatorname{curl}_x(f_\parallel) = 0\} : h \mapsto h_\parallel, \end{aligned}$$

being isomorphisms respectively. From Corollary 7.3, it is equivalent to well-posedness in the class of \mathcal{X} -solutions.

Corollary 7.4. *Assume that the Neumann problem for A_0 is well-posed. Then there exists $\epsilon > 0$ such that for any t -dependent coefficient matrix A with $\|\mathcal{E}\|_* < \epsilon$, the Neumann problem is well-posed for A in the following sense.*

Given any function $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, there is a unique \mathcal{X} -solution g to the divergence form equation with coefficients A , whose trace g_0 satisfies $(A_0 g_0)_\perp = \varphi$. Moreover, this solution has estimates

$$\|\tilde{N}_*(g)\|_2 \approx \|g_0\|_2 \approx \|\varphi\|_2.$$

The same holds true when the Neumann problem is replaced by the regularity problem and the boundary condition $(A_0 g_0)_\perp = \varphi$ is replaced by $(g_0)_\parallel = \varphi \in L_2(\mathbf{R}^n; \mathbf{C}^{nm})$ such that $\operatorname{curl}_x \varphi = 0$.

Proof. Throughout the proof, we assume that $\|\mathcal{E}\|_*$ is small enough, so that $I - S_A$ is invertible on \mathcal{X} by Proposition 6.1. To solve the Neumann problem, we make the ansatz (30) for the solution f and calculate its full trace

$$f_0 = h + \int_0^\infty \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds,$$

using Theorem 7.2. We see that f satisfies the Neumann boundary condition $(f_0)_\perp = \varphi$ if and only if h solves the equation $\Gamma_A h = \varphi$, where $\Gamma_A : E_0^+ \mathcal{H} \rightarrow L_2(\mathbf{R}^n; \mathbf{C}^m)$ is the operator

$$\Gamma_A : h \mapsto \left(h + \int_0^\infty \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds \right)_\perp.$$

Note that $\|\Gamma_A - \Gamma_{A_0}\|_{L_2 \rightarrow L_2} \lesssim \|A - A_0\|_*$ and that $\Gamma_{A_0} h = h_\perp$. By assumption Γ_{A_0} is an invertible operator, and thus Γ_A remains an isomorphism whenever $\|A - A_0\|_*$ is sufficiently small. Thus, in this case we can, given φ , calculate $h = \Gamma_A^{-1} \varphi$ with $\|h\|_2 \approx \|\varphi\|_2$ and find a unique solution f to the Neumann problem, with estimates $\|g\|_{\mathcal{X}} \approx \|g_0\|_2 \approx \|h\|_2 \approx \|\varphi\|_2$.

For the regularity problem, we proceed as for the Neumann problem, but instead solve for h in the equation $\left(h + \int_0^\infty \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s f_s ds \right)_\parallel = \varphi$. \square

Remark 7.5. Inspection of the proofs of Theorem 5.8 and Theorem 7.2 reveals that $S_A f_t = e^{-t\Lambda} h^- + \hat{f}_t$ with $\hat{f} \in \mathcal{Y}^*$. Hence, if g is an \mathcal{X} -solution, the corresponding conormal gradient f can be represented (assuming $\|\mathcal{E}\|_* < \infty$) as $f_t = e^{-t\Lambda} f_0 + \hat{f}_t$, since $f_0 = h^+ + h^-$. Note in particular that $f - e^{-t\Lambda} f_0 \in \mathcal{Y}^* \not\subseteq \mathcal{X}$, i.e. the free evolution $e^{-t\Lambda} f_0$ is the term responsible for f belonging to \mathcal{X} and not better.

8. THE DIRICHLET PROBLEM

Throughout this section, A denotes t -dependent coefficients satisfying (2) and (3), and $A_0 \in L_\infty(\mathbf{R}^n; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ denotes t -independent coefficients which are accretive on \mathcal{H} . We let $B := \hat{A}$ and $B_0 := \hat{A}_0$ be the transformed accretive coefficients from Proposition 3.1, and define $\mathcal{E} := B_0 - B$.

Definition 8.1. By a \mathcal{Y} -solution to the divergence form equation, with coefficients A , we mean a function $u \in C(\mathbf{R}_+; L_2(\mathbf{R}^n; \mathbf{C}^m))$, with estimate $\int_0^\infty \|g_t\|_2^2 t dt < \infty$ of its gradient $g := \nabla_{t,x} u$ which satisfies $\operatorname{div}_{t,x} A g = 0$ in \mathbf{R}_+^{1+n} distributional sense.

Note that we do not assume any limits of u at $t = 0$ or $t = \infty$, but will prove such below. This will allow us to formulate in what sense the boundary values are prescribed. Since $g \in \mathcal{Y}$ implies that $\partial_t u \in L_1^{\text{loc}}(\mathbf{R}_+; L_2)$, the condition $u \in C(\mathbf{R}_+; L_2)$ may be replaced by $u_t \in L_2$ at some Lebesgue point $t > 0$, possibly redefining $t \mapsto u_t$ on a null set.

Our representation and regularity result for \mathcal{Y} -solutions is the following.

Theorem 8.2. Assume that $\|\mathcal{E}\|_* < \infty$. Consider $u \in C(\mathbf{R}_+; L_2)$ with conormal gradient $f = [(A \nabla_{t,x} u)_\perp, \nabla_x u]^t \in \mathcal{Y}$. Then f solves $\partial_t f + D B f = 0$ in \mathbf{R}_+^{1+n} distributional sense if and only if it satisfies the equation

$$f_t = D e^{-t\tilde{\Lambda}} \tilde{h}^+ + S_A f_t, \quad \text{for some } \tilde{h}^+ \in \tilde{E}_0^+ L_2.$$

In this case, let $v_t := e^{-t\tilde{\Lambda}} \tilde{h}^+ + \tilde{S}_A f_t$. Then $f = D v$ and v_t has L_2 limits

$$(31) \quad \lim_{t \rightarrow 0} \|v_t - v_0\|_2 = 0 = \lim_{t \rightarrow \infty} \|v_t\|_2,$$

where $v_0 := \tilde{h}^+ + \tilde{h}^-$ and $\tilde{h}^- := -\int_0^\infty e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s f_s ds \in \tilde{E}_0^- L_2$, with estimates

$$\max(\|\tilde{h}^+\|_2, \|\tilde{h}^-\|_2) \approx \|v_0\|_2 \lesssim \sup_{t>0} \|v_t\|_2 \lesssim \|f\|_{\mathcal{Y}}.$$

We have $u = -v_\perp$, and in particular the limits (31) hold with v and v_0 replaced by the potential u and $u_0 := -(v_0)_\perp$, and

$$\|u_0\|_2 \leq \sup_{t>0} \|u_t\|_2 \lesssim \|\nabla_{t,x} u\|_{\mathcal{Y}}.$$

If furthermore $\|\mathcal{E}\|_*$ is sufficiently small, then there are estimates

$$\max(\|\tilde{h}^-\|_2, \sup_{t>0} \|u_t\|_2) \lesssim \|\tilde{h}^+\|_2 \approx \sup_{t>0} \|v_t\|_2 \approx \|f\|_{\mathcal{Y}} \approx \|\nabla_{t,x} u\|_{\mathcal{Y}}.$$

Proof. (i) Assume that $f \in \mathcal{Y}$ satisfies the differential equation. As in the proof of Theorem 7.2, we aim to take limits $\epsilon \rightarrow 0$ in equation (28). By Proposition 6.1, the right hand side converges in \mathcal{Y} to $S_A f$. Fix $0 < a < b < \infty$. For $t \in (a, b)$ and small

ϵ , the left hand side equals

$$(32) \quad \epsilon^{-1} \int_{\epsilon}^{2\epsilon} e^{-s\Lambda} (E_0^+ f_{t-s} + E_0^- f_{t+s}) ds \\ - \epsilon^{-1} \int_{\epsilon}^{2\epsilon} e^{-(t-s)\Lambda} E_0^+ \tilde{f}_s ds - 2\epsilon \int_{(2\epsilon)^{-1}}^{\epsilon^{-1}} e^{-(s-t)\Lambda} E_0^- f_s ds.$$

As in the proof of Theorem 7.2, the first term converges to f in $L_2(a, b; L_2)$. The L_2 -norm of the last term is bounded by $\epsilon \int_{(2\epsilon)^{-1}}^{\epsilon^{-1}} \|f_s\|_2 ds \lesssim \epsilon (\int_{(2\epsilon)^{-1}}^{\epsilon^{-1}} \|f_s\|_2^2 ds)^{1/2}$, and hence converges to 0, uniformly for $t \in (a, b)$.

We conclude that $\tilde{f}_t^\epsilon := \epsilon^{-1} \int_{\epsilon}^{2\epsilon} e^{-(t-s)\Lambda} E_0^+ f_s ds$ converges in $L_2(a, b; L_2)$ as $\epsilon \rightarrow 0$. In fact, since $\sup_{t>0} \|e^{-t\Lambda}\|_{L_2 \rightarrow L_2} < \infty$ we have

$$\|\tilde{f}_{t_0}^\epsilon - \tilde{f}_{t_0}^{\epsilon'}\|_2 \leq \frac{1}{b-a} \int_a^b \|e^{-(t_0-t)\Lambda} (\tilde{f}_t^\epsilon - \tilde{f}_t^{\epsilon'})\|_2 dt \lesssim \left(\int_a^b \|\tilde{f}_t^\epsilon - \tilde{f}_t^{\epsilon'}\|_2^2 dt \right)^{1/2},$$

when $t_0 > b$. Hence, since (a, b) is arbitrary, \tilde{f}_t^ϵ converges in L_2 , locally uniformly in t . Call the limit \tilde{f} and note that it coincides with $f - S_A f \in \mathcal{Y}$ for a.e. $t > 0$. Fix $t_0 > 0$ and note that $\tilde{f}_{t+t_0} = \lim_{\epsilon \rightarrow 0} e^{-t\Lambda} \tilde{f}_{t_0}^\epsilon = e^{-t\Lambda} \tilde{f}_{t_0}$ and that in fact $\tilde{f}_{t_0} \in E_0^+ \mathcal{H}$ by the definition of \tilde{f}_{t_0} . The estimate

$$\sup_{t_0 > 0} \int_0^\infty \|e^{-t\Lambda} \tilde{f}_{t_0}\|_2^2 t dt \leq \|\tilde{f}\|_{\mathcal{Y}}^2 \lesssim \|f\|_{\mathcal{Y}}^2$$

follows. Consider the restriction Λ_+ of Λ to $E_0^+ \mathcal{H}$, which is a closed and injective operator with dense domain and range. We claim that $\tilde{f}_{t_0} \in \mathcal{D}(\Lambda_+^{-1})$. To see this, by duality it suffices to show that

$$|((\Lambda_+^{-1})^* \phi, \tilde{f}_{t_0})| \lesssim \|\phi\|_2, \quad \text{for all } \phi \in \mathcal{D}((\Lambda_+^{-1})^*).$$

As in the proof of Proposition 5.3, we use an identity $\int_0^\infty (t\Lambda_+ e^{-t\Lambda_+})^2 \tilde{f}_{t_0} dt/t = 4^{-1} \tilde{f}_{t_0}$ to estimate

$$|((\Lambda_+^{-1})^* \phi, \tilde{f}_{t_0})| \approx \left| \int_0^\infty (t\Lambda_+^* e^{-t\Lambda_+^*} \phi, t e^{-t\Lambda_+} \tilde{f}_{t_0}) \frac{dt}{t} \right| \lesssim \|\phi\|_2 \|f\|_{\mathcal{Y}}.$$

Hence the claim. As $\mathcal{D}(\Lambda_+^{-1}) = \mathcal{R}(\Lambda_+) \subset \mathcal{R}(D)$, this shows that $\tilde{f}_{t_0} = D\tilde{h}_{t_0}^+$, where $\tilde{h}_{t_0}^+ \in B_0 E_0^+ \mathcal{H} = \tilde{E}_0^+ L_2$ has bounds $\|\tilde{h}_{t_0}^+\|_2 \lesssim \|f\|_{\mathcal{Y}}$, uniformly in t_0 . From the identity $\tilde{f}_{t+t_0} = e^{-t\Lambda} \tilde{f}_{t_0} = e^{-t\Lambda} D\tilde{h}_{t_0}^+$, we get

$$\int_a^b (\phi_t, \tilde{f}_{t+t_0}) dt = \left(\int_a^b D e^{-t\Lambda} \phi_t dt, \tilde{h}_{t_0}^+ \right),$$

for any $\phi \in L_2(a, b; L_2)$. Here the left hand side converges as $t_0 \rightarrow 0$, and the functions $\int_a^b D e^{-t\Lambda} \phi_t dt$ are dense in \mathcal{H} . (For example $\int_\epsilon^{2\epsilon} D e^{-t\Lambda} \epsilon^{-1} \phi dt \rightarrow D\phi$.) Since $\|\tilde{h}_{t_0}^+\|_2$ is uniformly bounded, it follows that $\tilde{h}_{t_0}^+ \rightarrow \tilde{h}^+$ weakly in $\tilde{E}_0^+ L_2$ as $t_0 \rightarrow 0$. Letting $t_0 \rightarrow 0$ in $\tilde{f}_{t+t_0} = e^{-t\Lambda} D\tilde{h}_{t_0}^+ = D e^{-t\Lambda} \tilde{h}_{t_0}^+$, we obtain $f_t - S_A f_t = \tilde{f}_t = D e^{-t\Lambda} \tilde{h}^+$ for a.e. $t > 0$.

(ii) Conversely, assume that $f = Dv$, where $v_t = e^{-t\Lambda} \tilde{h}^+ + \tilde{S}_A f_t$. As in Theorem 7.2, we verify that f satisfies the differential equation, and we omit the details. The stated limits follow from Propositions 5.4 and 6.2.

To prove the estimates, note that the square function estimates for B_0D and the accretivity of B_0 on \mathcal{H} show that

$$\|\tilde{h}^+\|_2 \approx \|B_0De^{-t\tilde{\Lambda}}\tilde{h}^+\|_{\mathcal{Y}} \approx \|De^{-t\tilde{\Lambda}}\tilde{h}^+\|_{\mathcal{Y}} \lesssim \|f\|_{\mathcal{Y}} + \|S_A f\|_{\mathcal{Y}} \lesssim \|f\|_{\mathcal{Y}}.$$

From Proposition 6.2, we also obtain the estimates $\max(\|\tilde{h}^+\|_2, \|\tilde{h}^-\|_2) \approx \|v_0\|_2 \leq \sup_{t>0} \|v_t\|_2 \lesssim \|\tilde{h}^+\|_2 + \|f\|_{\mathcal{Y}} \lesssim \|f\|_{\mathcal{Y}}$, where we have used the topological splitting $B_0\mathcal{H} = \tilde{E}_0^+L_2 \oplus \tilde{E}_0^-L_2$. In particular $\|u_0\|_2 \leq \sup_{t>0} \|u_t\|_2 \lesssim \|\nabla_{t,x}u\|_{\mathcal{Y}}$, since $|u| \leq |v|$.

Finally, Proposition 6.1 shows that $\|S_A\|_{\mathcal{Y} \rightarrow \mathcal{Y}} \leq 1/2$ if $\|\mathcal{E}\|_*$ is sufficiently small. In this case $I - S_A$ is an isomorphism on \mathcal{Y} , giving the estimate

$$\|f\|_{\mathcal{Y}} \lesssim \|De^{-t\tilde{\Lambda}}\tilde{h}^+\|_{\mathcal{Y}}.$$

As $\|De^{-t\tilde{\Lambda}}\tilde{h}^+\|_{\mathcal{Y}} \approx \|\tilde{h}^+\|_2$, this proves the stated estimates and completes the proof. \square

We note the following immediate corollary to Theorem 8.2. Write $\tilde{h}^+ = h$ below.

Corollary 8.3. *Assume that coefficients $A = A_0$ are t -independent. Then u is a \mathcal{Y} -solution to the divergence form equation if and only if it can be represented*

$$u_t = (e^{-t\tilde{\Lambda}}h)_{\perp}, \quad \text{for some } h \in \tilde{E}_0^+L_2.$$

In particular, the class of \mathcal{Y} -solutions in Definition 8.1 coincides with the class of solutions in [8, Definition 2.1(iii)] for t -independent coefficients.

That the solutions considered in [8] are of this form follows from [8, Lemma 4.2] and the proof of [8, Theorem 2.3]. Note that the operator $T_A|_{\mathcal{H}}$ used in [8] is similar to our operator $B_0D|_{B_0\mathcal{H}}$, as in [8, Definition 3.1]. This corollary also shows that the results in [5] concerning the domain of the Dirichlet semi-group, apply to \mathcal{Y} -solutions.

For t -dependent coefficients A , Theorem 8.2 shows that any \mathcal{Y} -solution to the divergence form equation with coefficients A can be represented

$$(33) \quad u = \left((I + \tilde{S}_A(I - S_A)^{-1}D)e^{-t\tilde{\Lambda}}h \right)_{\perp}.$$

for some $h \in \tilde{E}_0^+L_2$, provided $\|\mathcal{E}\|_*$ is small enough. (Here $h \mapsto De^{-t\tilde{\Lambda}}h$ is viewed as a map $\tilde{E}_0^+L_2 \rightarrow \mathcal{Y}$.) We remark that the tangential part v_{\parallel} of the vector-valued potential $v = (I + \tilde{S}_A(I - S_A)^{-1}D)e^{-t\tilde{\Lambda}}h$ can be viewed as a set of generalized conjugate functions to u . Our proof of Theorem 8.2 above eliminates the need of the technical condition on these conjugate functions which was required in [8, Definition 3.1].

We recall that for the class of solutions used in [8], with t -independent coefficients A_0 , well-posedness of the Dirichlet problem was shown to be equivalent to the maps

$$\tilde{E}_0^+L_2 \rightarrow L_2(\mathbf{R}^n; \mathbf{C}^m) : h \mapsto h_{\perp}$$

being an isomorphism. From Corollary 8.3, it is equivalent to well-posedness in the class of \mathcal{Y} -solutions.

Corollary 8.4. *Assume that the Dirichlet problem for A_0 is well-posed. Then there exists $\epsilon > 0$ such that for any t -dependent coefficient matrix A with $\|\mathcal{E}\|_* < \epsilon$, the Dirichlet problem is well-posed for A in the following sense.*

Given any function $\varphi \in L_2(\mathbf{R}^n; \mathbf{C}^m)$, there is a unique \mathcal{Y} -solution u to the divergence form equation with coefficients A , with boundary trace $u_0 = \varphi$. Moreover, this solution has estimates

$$\|\nabla_{t,x}u\|_{\mathcal{Y}} \approx \sup_{t>0} \|u_t\|_2 \approx \|\varphi\|_2.$$

Proof. Throughout the proof, we assume that $\|\mathcal{E}\|_*$ is small enough, so that $I - S_A$ is invertible on \mathcal{Y} by Proposition 6.1. To solve the Dirichlet problem, we make the ansatz (33) for u . We see from Theorem 8.2 that the Dirichlet boundary condition $u_0 = \varphi$ is satisfied if and only if h solves the equation $\tilde{\Gamma}_A h = \varphi$, where $\tilde{\Gamma}_A : \tilde{E}_0^+ L_2 \rightarrow L_2(\mathbf{R}^n; \mathbf{C}^m)$ is the operator

$$\tilde{\Gamma}_A : h \mapsto \left(h - \int_0^\infty e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s f_s ds \right)_\perp,$$

where $f := (I - S_A)^{-1} D e^{-t\tilde{\Lambda}} h$. Note that $\|\tilde{\Gamma}_A - \tilde{\Gamma}_{A_0}\|_{L_2 \rightarrow L_2} \lesssim \|A - A_0\|_*$ and that $\tilde{\Gamma}_{A_0} h = h_\perp$. By assumption $\tilde{\Gamma}_{A_0}$ is an invertible operator, and thus $\tilde{\Gamma}_A$ remains an isomorphism whenever $\|\mathcal{E}\|_*$ is sufficiently small. Thus, in this case we can, given φ , calculate $h = \tilde{\Gamma}_A^{-1} \varphi$ with $\|h\|_2 \approx \|\varphi\|_2$ and find a unique solution u to the Dirichlet problem. From Theorem 8.2, we get estimates

$$\|\varphi\|_2 \leq \sup_{t>0} \|u_t\|_2 \lesssim \|\nabla_{t,x}u\|_{\mathcal{Y}} \approx \|h\|_2 \approx \|\varphi\|_2.$$

This proves the theorem. \square

9. FURTHER ESTIMATES

In Section 7, we constructed solutions, with estimates on the modified non-tangential maximal function, to the Neumann and regularity problems with L_2 boundary data, and in Section 8 we constructed solutions, with estimates on the square function, to the Dirichlet problem with L_2 boundary data. In this section, we prove two theorems which give modified non-tangential maximal function estimates for the Dirichlet problem, and square function estimates for the Neumann/regularity problems.

9.1. Maximal function estimates for \mathcal{Y} -solutions.

Theorem 9.1. *Let A_0 be t -independent coefficients which are accretive on \mathcal{H} , and assume that $\|A - A_0\|_C < \infty$. Then any \mathcal{Y} -solution u to the divergence form equation, with boundary trace u_0 , has modified non-tangential maximal estimates*

$$\|u_0\|_2 \lesssim \|\tilde{N}_*(u)\|_2 \lesssim \|\nabla_{t,x}u\|_{\mathcal{Y}}.$$

The core of the proof reduces to the following estimate of the operator \tilde{S}_A .

Lemma 9.2. *For any fixed $p \in [1, 2)$, the operator \tilde{S}_A has estimates*

$$\|\tilde{N}_*^p((\tilde{S}_A h)_\perp)\|_2 \lesssim \|\mathcal{E}\|_C \|h\|_{\mathcal{Y}}.$$

Here $\tilde{N}_*^p(h)(x) := \sup_{t>0} t^{-(1+n)/p} \|h\|_{L_p(W(t,x))}$ is an L_p modified non-tangential maximal function.

Proof of Theorem 9.1 modulo Lemma 9.2. As in Theorem 8.2, any \mathcal{Y} -solution u can be written

$$u_t = (e^{-t\tilde{\Lambda}}\tilde{h}^+ + \tilde{S}_A f_t)_\perp, \quad \tilde{h}^+ \in \tilde{E}_0^+ L_2, f \in \mathcal{Y}.$$

From Poincaré's inequality $\|u - u_{W(t,x)}\|_{L_2(W(t,x))} \lesssim t \|\nabla_{s,y} u\|_{L_2(W(t,x))}$, where $u_{W(t,x)}$ denotes the average, we obtain the estimate $\|\tilde{N}_*(u)\|_2 \lesssim \|\tilde{N}_*^1(u)\|_2 + \|\nabla_{t,x} u\|_{\mathcal{Y}}$. Theorem 4.2, Lemma 9.2 and Theorem 8.2 now apply to give the estimate

$$\|\tilde{N}_*^1(u)\|_2 \lesssim \|\tilde{h}^+\|_2 + \|f\|_{\mathcal{Y}} \approx \|\nabla_{t,x} u\|_{\mathcal{Y}}.$$

To see the first estimate, write $\tilde{h}^+ = B_0 h^+$ with $h^+ \in E_0^+ \mathcal{H}$, and apply Theorem 4.2 to get $\|e^{-t\tilde{\Lambda}} B_0 h^+\|_{\mathcal{X}} = \|B_0 e^{-t\tilde{\Lambda}} h^+\|_{\mathcal{X}} \lesssim \|h^+\|_2 \approx \|\tilde{h}^+\|_2$. The lower estimate follows from Lemma 4.3 since

$$\|\tilde{N}_*(u)\|_2^2 \gtrsim \lim_{t \rightarrow 0} t^{-1} \int_t^{2t} \|u_s\|_2^2 ds = \|u_0\|_2^2.$$

□

Proof of Lemma 9.2. Before we start, we remark that $p \mapsto \|\tilde{N}_*^p((\tilde{S}_A h)_\perp)\|_2$ is increasing, so it suffices to consider p close to 2. We shall select such a p later. Next it suffices to prove the inequality for $t \mapsto h_t$ compactly supported in \mathbf{R}_+ . Indeed, combining Lemma 4.3 and Proposition 6.2, for all $\epsilon > 0$ and $h \in \mathcal{Y}$ we have (since $p \leq 2$)

$$\begin{aligned} \|\tilde{N}_*^p(\chi_{(\epsilon, \epsilon^{-1})}(t)(\tilde{S}_A h)_\perp)\|_2^2 &\leq \|\tilde{N}_*(\chi_{(\epsilon, \epsilon^{-1})}(t)(\tilde{S}_A h)_\perp)\|_2^2 \\ &\lesssim \int_\epsilon^{\epsilon^{-1}} \|(\tilde{S}_A h)_\perp\|_2^2 \frac{dt}{t} \lesssim \ln \epsilon \sup_{t>0} \|\tilde{S}_A h\|_2^2 \lesssim \ln \epsilon \|h\|_{\mathcal{Y}}^2. \end{aligned}$$

Thus, if $h_\delta := \chi_{(\delta, \delta^{-1})}(t)h$ for $h \in \mathcal{Y}$, we have for fixed $\epsilon > 0$

$$\|\tilde{N}_*^p(\chi_{(\epsilon, \epsilon^{-1})}(t)(\tilde{S}_A h)_\perp)\|_2 \leq \liminf_{\delta \rightarrow 0} \|\tilde{N}_*^p(\chi_{(\epsilon, \epsilon^{-1})}(t)(\tilde{S}_A h_\delta)_\perp)\|_2.$$

Now our assumption gives

$$\|\tilde{N}_*^p(\chi_{(\epsilon, \epsilon^{-1})}(t)(\tilde{S}_A h_\delta)_\perp)\|_2 \lesssim \|\mathcal{E}\|_C \|h_\delta\|_{\mathcal{Y}} \lesssim \|\mathcal{E}\|_C \|h\|_{\mathcal{Y}},$$

uniformly in ϵ , so for all $h \in \mathcal{Y}$ and $\epsilon > 0$ we obtain

$$\|\tilde{N}_*^p(\chi_{(\epsilon, \epsilon^{-1})}(t)(\tilde{S}_A h)_\perp)\|_2 \lesssim \|\mathcal{E}\|_C \|h\|_{\mathcal{Y}}.$$

It remains to let $\epsilon \rightarrow 0$ and apply the monotone convergence theorem.

(i) We now fix $t \mapsto h_t$ compactly supported in \mathbf{R}_+ and write

$$\tilde{S}_A h_t = \int_0^t e^{-(t-s)\tilde{\Lambda}} \tilde{E}_0^+ \mathcal{E}_s h_s ds - \int_t^\infty e^{-(s-t)\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s h_s ds =: I - II.$$

Most of the time we use the pointwise inequality $\tilde{N}_*^p \leq \tilde{N}_*$. It is only for one term, estimated in (iii) below, that we require $p < 2$.

Split the integral I as

$$I = \int_0^t e^{-(t-s)\tilde{\Lambda}} (I - e^{-2s\tilde{\Lambda}}) \tilde{E}_0^+ \mathcal{E}_s h_s ds + e^{-t\tilde{\Lambda}} \int_0^t e^{-s\tilde{\Lambda}} \tilde{E}_0^+ \mathcal{E}_s h_s ds = I_1 + I_2.$$

As in the proof of Proposition 6.2, the kernel of I_1 has bounds s/t , giving the estimate

$$(34) \quad \|\tilde{N}_*(I_1)\|_2^2 \lesssim \int_0^\infty \|I_1\|_2^2 \frac{dt}{t} \lesssim \int_0^\infty \left(\int_0^t \frac{s}{t} \frac{ds}{s} \right) \left(\int_0^t \frac{s}{t} \|\mathcal{E}_s h_s\|_2^2 ds \right) \frac{dt}{t} \\ \lesssim \int_0^\infty \|\mathcal{E}_s h_s\|_2^2 s ds \leq \|\mathcal{E}\|_\infty^2 \|h\|_{\mathcal{Y}}^2.$$

Similarly we split $II = II_1 + II_2$ by writing $e^{-(s-t)\tilde{\Lambda}} = e^{-(s-t)\tilde{\Lambda}}(I - e^{-2t\tilde{\Lambda}}) + e^{-t\tilde{\Lambda}}e^{-s\tilde{\Lambda}}$, and a Schur estimate similar to (34) give the bound for II_1 . Next we write

$$II_2 = e^{-t\tilde{\Lambda}} \int_0^\infty e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s h_s ds - e^{-t\tilde{\Lambda}} \int_0^t e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s h_s ds =: II_3 - II_4.$$

By Theorem 4.2, the term II_3 has bound

$$\left\| \tilde{N}_* \left(B_0 e^{-t\tilde{\Lambda}} B_0^{-1} P_{B_0 \mathcal{H}} \int_0^\infty e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s h_s ds \right) \right\|_2 \lesssim \left\| \int_0^\infty e^{-s\tilde{\Lambda}} \tilde{E}_0^- \mathcal{E}_s h_s ds \right\|_2 \\ = \sup_{\|f\|_2=1} \left| \int_0^\infty (\mathcal{E}_s^* e^{-s\tilde{\Lambda}^*} (\tilde{E}_0^-)^* f, h_s) ds \right| \lesssim \|\mathcal{E}\|_* \|h\|_{\mathcal{Y}}.$$

(ii) It remains to consider $I_2 + II_4 = (\tilde{E}_0^+ + \tilde{E}_0^-) e^{-t\tilde{\Lambda}} \int_0^t e^{-s\tilde{\Lambda}} \mathcal{E}_s h_s ds$. Note that $(\tilde{E}_0^+ + \tilde{E}_0^-) = P_{B_0 \mathcal{H}}$. Since we only consider the normal component of $I_2 + II_4$ and $(P_{B_0 \mathcal{H}} \tilde{h})_\perp = (\tilde{h})_\perp$ for any \tilde{h} , it remains to estimate $e^{-t\tilde{\Lambda}} \int_0^t e^{-s\tilde{\Lambda}} \mathcal{E}_s h_s ds$. To make use of off-diagonal estimates (see Lemma 9.3), we need to replace $e^{-t\tilde{\Lambda}}$ by the resolvents $(I + itB_0D)^{-1}$. To this end, define $\psi_t(z) := e^{-t|z|} - (1 + itz)^{-1}$ and split the integral

$$e^{-t\tilde{\Lambda}} \int_0^t e^{-s\tilde{\Lambda}} \mathcal{E}_s h_s ds = \psi_t(B_0D) \int_0^\infty e^{-s\tilde{\Lambda}} \mathcal{E}_s h_s ds - \int_t^\infty \psi_t(B_0D) e^{-s\tilde{\Lambda}} \mathcal{E}_s h_s ds \\ + \int_0^t (I + itB_0D)^{-1} (e^{-s\tilde{\Lambda}} - I) \mathcal{E}_s h_s ds + (I + itB_0D)^{-1} \int_0^t \mathcal{E}_s h_s ds.$$

For the first term, square function estimates show that $\psi_t(B_0D) : L_2 \rightarrow \mathcal{Y}^* \subset \mathcal{X}$ is continuous, and a duality argument like for II_3 gives the bound. For the second and third terms, we note the operator estimates

$$\|\psi_t(B_0D) e^{-s\tilde{\Lambda}}\| = \left\| \frac{t e^{-t|B_0D|} - (I + itB_0D)^{-1}}{s} (sB_0D) e^{-s|B_0D|} \right\| \lesssim t/s,$$

and

$$\|(I + itB_0D)^{-1} (e^{-s\tilde{\Lambda}} - I)\| \lesssim \left\| \frac{s}{t} \frac{tB_0D}{I + itB_0D} \frac{e^{-s|B_0D|} - I}{sB_0D} \right\| \lesssim s/t.$$

Schur estimates similar to (34) give the \tilde{N}_* bounds.

(iii) It remains to prove the estimate

$$\left\| \tilde{N}_*^p \left((I + itB_0D)^{-1} \int_0^t \mathcal{E}_s h_s ds \right) \right\|_2 \lesssim \|\mathcal{E}\|_C \|h\|_{\mathcal{Y}}.$$

To show this, fix a Whitney box $W(t_0, x_0)$, take $f \in L_q(W(t_0, x_0); \mathbf{C}^{(1+n)m})$, and let $f = 0$ outside $W(t_0, x_0)$. Here $1/p + 1/q = 1$, $p < 2$ and $q > 2$. To bound the

$L_p(W(t_0, x_0))$ norm, we do the duality argument

$$\begin{aligned} & \frac{1}{t_0} \int_{c_0^{-1}t_0}^{c_0t_0} \left((I + itB_0D)^{-1} \int_0^t \mathcal{E}_s h_s ds, f_t \right) dt \\ &= \int_0^{c_0t_0} \left(\mathcal{E}_s h_s, \frac{1}{t_0} \int_{\max(c_0^{-1}t_0, s)}^{c_0t_0} (I - itDB_0^*)^{-1} f_t dt \right) ds \\ &\leq \int_{\mathbf{R}^n} \int_0^{c_0t_0} |\mathcal{E}(s, y)| |h(s, y)| F(y) ds dy, \end{aligned}$$

where

$$F(y) := \frac{1}{t_0} \int_{c_0^{-1}t_0}^{c_0t_0} |(I - itDB_0^*)^{-1} f_t(y)| dt.$$

To handle the tails of $(I - itDB_0^*)^{-1} f_t$, we split the space into annular regions $\mathbf{R}^n = \bigcup_{k=0}^{\infty} A_k$, where $A_0 := B(x_0; t_0)$ and $A_k := (2^k A_0) \setminus (2^{k-1} A_0)$ for $k \geq 1$. Define $h_k(s, y) := \chi_{(0, c_0 t_0)}(s) \chi_{A_k}(y) h(s, y)$ and $F_k(y) := \chi_{A_k}(y) F(y)$. Then Whitney averaging as in the proof of Lemma 4.4 gives

$$\begin{aligned} & \int_{\mathbf{R}^n} \int_0^{c_0t_0} |\mathcal{E}(s, y)| |h(s, y)| F(y) ds dy \leq \sum_{k=0}^{\infty} \iint_{\mathbf{R}_+^{1+n}} |\mathcal{E}(s, y)| |s| h_k(s, y) |F_k(y)| \frac{ds dy}{s} \\ &\approx \sum_{k=0}^{\infty} \iint_{\mathbf{R}_+^{1+n}} \left(\frac{1}{t^{1+n}} \iint_{W(t, x)} |\mathcal{E}(s, y)| |s| h_k(s, y) |F_k(y)| ds dy \right) \frac{dt dx}{t} \\ &\lesssim \sum_{k=0}^{\infty} \iint_{\mathbf{R}_+^{1+n}} \sup_{W(t, x)} |\mathcal{E}| \left(\frac{1}{t^{1+n}} \iint_{W(t, x)} |sh_k|^2 \right)^{1/2} \left(\frac{1}{t^{1+n}} \iint_{W(t, x)} |F_k|^2 \right)^{1/2} \frac{dt dx}{t} \\ &\lesssim \sum_{k=0}^{\infty} \|\mathcal{E}\|_C \int_{\mathbf{R}^n} \mathcal{A} \left(\frac{1}{\sqrt{t^{1+n}}} \|sh_k\|_{L_2(W(t, x))} \frac{1}{\sqrt{t^{1+n}}} \|F_k\|_{L_2(W(t, x))} \right) (z) dz \\ &\lesssim \sum_{k=0}^{\infty} \|\mathcal{E}\|_C \int_{\mathbf{R}^n} \mathcal{A} \left(\frac{1}{\sqrt{t^{1+n}}} \|sh_k\|_{L_2(W(t, x))} \right) (z) N_* \left(\frac{1}{\sqrt{t^{1+n}}} \|F_k\|_{L_2(W(t, x))} \right) (z) dz \\ &\lesssim \sum_{k=0}^{\infty} \|\mathcal{E}\|_C \|\mathcal{A}(sh_k)\|_{L_p(\mathbf{R}^n)} \|M(|F_k|^2)^{1/2}\|_{L_q(\mathbf{R}^n)}. \end{aligned}$$

Here \mathcal{A} denotes the area function $\mathcal{A}g(z) := (\iint_{|y-z| < cs} |g(s, y)|^2 s^{-(1+n)} ds dy)^{1/2}$ and $N_*g(z) := \sup_{|y-z| < cs} |g(s, x)|$ is the non-tangential maximal function, where $c \in (0, \infty)$ is some fixed constant, and M is the Hardy–Littlewood maximal function. On the fourth line we used the tent space estimate by Coifman, Meyer and Stein in [13, Theorem 1(a)]. Since $M : L_{q/2} \rightarrow L_{q/2}$ is bounded, we have

$$\begin{aligned} \|M(|F_k|^2)^{1/2}\|_{L_q(\mathbf{R}^n)} &\lesssim \|F\|_{L_q(A_k)} \leq \frac{1}{t_0} \int_{c_0^{-1}t_0}^{c_0t_0} \|((I - itDB_0^*)^{-1} f_t)\|_{L_q(A_k)} dt \\ &\lesssim 2^{-km} \frac{1}{t_0} \int_{c_0^{-1}t_0}^{c_0t_0} \|f_t\|_{L_q(B(x_0, c_0 t_0))} dt \lesssim 2^{-km} \|f\|_{L_q(W(t_0, x_0))}. \end{aligned}$$

The third estimate uses Lemma 9.3 below, and thus is where we choose $p < 2$ sufficiently close to 2 so that $2 < q < 2 + \delta$. We obtain the maximal function

estimate

$$\begin{aligned} \tilde{N}_*^p \left((I + itB_0D)^{-1} \int_0^t \mathcal{E}_s h_s ds \right) (x_0) &\lesssim \|\mathcal{E}\|_C \sup_{t_0 > 0} \sum_{k=0}^{\infty} 2^{-km} t_0^{n/q-n} \|\mathcal{A}(sh_k)\|_{L_p(\mathbf{R}^n)} \\ &\lesssim \|\mathcal{E}\|_C \sum_{k=0}^{\infty} 2^{-k(m-n/p)} \sup_{t_0 > 0} \left(\frac{1}{(2^k t_0)^n} \int_{B(x_0; (2^k + cc_0)t_0)} |\mathcal{A}(sh)|^p dx \right)^{1/p} \\ &\lesssim \|\mathcal{E}\|_C M(\mathcal{A}(sh)^p)^{1/p}(x_0), \end{aligned}$$

where c is the constant from the definition of \mathcal{A} and $m > n/p$. Since $M : L_{2/p} \rightarrow L_{2/p}$ is bounded, this yields

$$\begin{aligned} \left\| \tilde{N}_*^p \left((I + itB_0D)^{-1} \int_0^t \mathcal{E}_s h_s ds \right) \right\|_2 \\ \lesssim \|\mathcal{E}\|_C \|M(\mathcal{A}(sh)^p)^{1/p}\|_2 \lesssim \|\mathcal{E}\|_C \|\mathcal{A}(sh)\|_2 \approx \|\mathcal{E}\|_C \|h\|_y. \end{aligned}$$

This completes the proof of the maximal function estimate. \square

The following lemma, which we used above, is contained in [6, Lemma 2.57]. However, we give a more direct proof here, since the algebraic setup in [6] was quite different.

Lemma 9.3. *Let B_0 be t -independent coefficients, accretive on $\mathcal{H} = \overline{R(D)}$. Then for each positive integer m , there is $C_m < \infty$ and $\delta > 0$ such that*

$$\|(1 + itDB_0)^{-1} f\|_{L_q(E)} \leq \frac{C_m}{(1 + \text{dist}(E, F)/t)^m} \|f\|_{L_q(F)}$$

for all $t > 0$ and sets $E, F \subset \mathbf{R}^n$ such that $\text{supp } f \subset F$, and all q such that $|q-2| < \delta$. Here $\text{dist}(E, F) := \inf\{|x-y| ; x \in E, y \in F\}$.

Proof. For $q = 2$, these off-diagonal estimates can be proved as in [7, Proposition 5.1], using estimates on commutators with bump functions (and replacing the operator B_0D there by DB_0). By interpolation, it suffices to estimate $\|(1 + itDB_0)^{-1} f\|_{L_q(\mathbf{R}^n) \rightarrow L_q(\mathbf{R}^n)}$, uniformly for t and q in a neighbourhood of 2. To this end, assume that $(I + itDB_0)\tilde{f} = f$. As in Proposition 3.1, but replacing ∂_t by $(it)^{-1}$, this equation is equivalent to

$$\begin{cases} (A_0\tilde{g})_{\perp} + it\text{div}_x(A_0\tilde{g})_{\parallel} = (A_0g)_{\perp}, \\ \tilde{g}_{\parallel} - it\nabla_x\tilde{g}_{\perp} = g_{\parallel}, \end{cases}$$

where A_0, g, \tilde{g} are related to B_0, f, \tilde{f} , respectively, as in Proposition 3.1. Using the second equation to eliminate \tilde{g}_{\parallel} in the first, shows that \tilde{g}_{\perp} satisfies the divergence form equation

$$L\tilde{g}_{\perp} := \begin{bmatrix} 1 & it\text{div}_x \end{bmatrix} A_0(x) \begin{bmatrix} 1 \\ it\nabla_x \end{bmatrix} \tilde{g}_{\perp} = \begin{bmatrix} 1 & it\text{div}_x \end{bmatrix} \begin{bmatrix} A_{\perp\perp}g_{\perp} \\ -A_{\parallel\parallel}g_{\parallel} \end{bmatrix}.$$

By the stability result of Šneĭberg [36] it follows that the divergence form operator L is an isomorphism $L : W_q^1(\mathbf{R}^n) \rightarrow W_q^{-1}(\mathbf{R}^n)$ for $|q-2| < \delta$, giving us the desired estimate

$$\|\tilde{f}\|_q \approx \|\tilde{g}\|_q \lesssim \|\tilde{g}_{\perp}\|_q + t\|\nabla_x\tilde{g}_{\perp}\|_q + \|g_{\parallel}\|_q \lesssim \|g\|_q \approx \|f\|_q.$$

\square

9.2. Square function estimates for \mathcal{X} -solutions under t -regularity for the coefficients. Staring at the equation $\operatorname{div}_{t,x} Ag = 0 = \operatorname{curl}_{t,x} g$, there is no reason to expect that \mathcal{X} -solutions g would in general satisfy the square function estimate $\int_0^\infty \|\partial_t g_t\|_2^2 dt < \infty$, i.e. $\partial_t g_t \in \mathcal{Y}$, when A is t -dependent. We show in the next result that this can be obtained upon a further t -regularity assumption on A . This also improves the regularity of g_t itself. This regularity assumption is akin to the one in [19, 20] but is not directly comparable: the assumptions of the cited works are rather of perturbation type “small Lipschitz constant” while we are looking at perturbations of “good” t -independent coefficients. Besides, we do not need smallness in this regularity assumption. The result is as follows.

Theorem 9.4. *Let A_0 be t -independent coefficients which are accretive on \mathcal{H} and assume that $\|A - A_0\|_*$ is sufficiently small.*

If A satisfies the t -regularity condition

$$\|t\partial_t A\|_* < \infty,$$

then any \mathcal{X} -solution g to the divergence form equation with boundary trace g_0 has regularity $\partial_t g_t \in L_2^{\text{loc}}(\mathbf{R}_+; L_2)$ with estimates

$$\|\partial_t g_t\|_{\mathcal{Y}} \lesssim \|g\|_{\mathcal{X}}.$$

We also have estimates $\sup_{t>0} \|g_t\|_2 \approx \|g\|_{\mathcal{X}}$, and $t \mapsto g_t \in L_2$ is continuous with limits $\lim_{t \rightarrow 0} \|g_t - g_0\|_2 = 0 = \lim_{t \rightarrow \infty} \|g_t\|_2$. The converse estimate $\|g\|_{\mathcal{X}} \lesssim \|\partial_t g\|_{\mathcal{Y}}$ holds for all \mathcal{X} -solutions g , provided $\|t\partial_t A\|_$ is sufficiently small.*

If $\max(\|t\partial_i A\|_, \|t\partial_t A\|_*) < \infty$ for some $i = 1, \dots, n$, then $\partial_i g_t \in L_2^{\text{loc}}(\mathbf{R}_+; L_2)$ for any \mathcal{X} -solution g to the divergence form equation, with estimates $\|\partial_i g_t\|_{\mathcal{Y}} \lesssim \|g\|_{\mathcal{X}}$. The estimate $\|g\|_{\mathcal{X}} \lesssim \|\nabla_x g\|_{\mathcal{Y}}$ holds for all \mathcal{X} -solutions g , provided $\|t\nabla_{t,x} A\|_*$ is sufficiently small.*

It is not know whether the smallness assumptions are needed for the converse estimates to hold. We also remark that the same conclusion holds for the conormal gradient f , as is clear from the proof below.

Lemma 9.5. *If $h \in \mathcal{X}$ has distribution derivative $\partial_t h \in \mathcal{Y}$, then $\partial_t(S_A h) \in \mathcal{Y}$ with estimates*

$$\|\partial_t(S_A h)\|_{\mathcal{Y}} \lesssim (\|\mathcal{E}\|_* + \|t\partial_t \mathcal{E}\|_*) \|h\|_{\mathcal{X}} + \|\mathcal{E}\|_{\infty} \|\partial_t h\|_{\mathcal{Y}}.$$

Proof of Theorem 9.4 modulo Lemma 9.5. (i) As in the proof of Corollary 7.4, any \mathcal{X} -solution can be written $g = [(Bf)_{\perp}, f_{\parallel}]^t$, where

$$(I - S_A)f = e^{-t\Lambda} h^+, \quad \text{for some } h^+ \in E_0^+ \mathcal{H}.$$

Introduce the auxiliary Banach space $Z := \{h \in \mathcal{X} ; \partial_t h \in \mathcal{Y}\} \subset \mathcal{X}$, with norm $\|h\|_Z := \|h\|_{\mathcal{X}} + a\|\partial_t h\|_{\mathcal{Y}}$. By Proposition 6.1 and Lemma 9.5 we have estimates $\|S_A h\|_{\mathcal{X}} \leq C\|h\|_{\mathcal{X}}$ and $\|\partial_t(S_A h)\|_{\mathcal{Y}} \leq D\|h\|_{\mathcal{X}} + C\|\partial_t h\|_{\mathcal{Y}}$, where we assume $C < 1$, and we choose the parameter $a > 0$ small enough so that

$$\|S_A\|_{Z \rightarrow Z} \leq C + aD < 1.$$

Hence $I - S_A$ is invertible on both \mathcal{X} and Z . Since $e^{-t\Lambda} h^+ \in Z$ by Theorem 4.2, we conclude that $f \in Z$ with estimates $\|\partial_t f\|_{\mathcal{Y}} \lesssim \|f\|_Z \approx \|e^{-t\Lambda} h^+\|_Z \approx \|h^+\|_2$. For the gradient g , this gives the bound $\|\partial_t g\|_{\mathcal{Y}} \lesssim \|t\partial_t B\|_* \|f\|_{\mathcal{X}} + (\|B\|_{\infty} + 1)\|\partial_t f\|_{\mathcal{Y}} \lesssim \|h^+\|_2 \approx \|f\|_{\mathcal{X}} \approx \|g\|_{\mathcal{X}}$.

(ii) To prove the sup $-L_2$ estimate and trace result for g_t , write $\int s\eta(s)\partial_s g_s ds = \int (\eta(s) + s\eta'(s))g_s ds$, for some $\eta \in C_0^\infty(\mathbf{R}_+)$. Take the limit as η approaches the characteristic function for $(0, t)$ to get

$$g_t = \frac{1}{t} \int_0^t g_s ds + \frac{1}{t} \int_0^t \partial_s g_s s ds, \quad \text{a.e. } t > 0.$$

The last term has bound $(\int_0^t \|\partial_s g_s\|^2 s ds)^{1/2}$, whereas the first term satisfies

$$\left\| \frac{1}{t} \int_0^t g_s ds - g_0 \right\|_2^2 \leq \sum_{k=1}^{\infty} 2^{-k} \left(\frac{1}{2^{-k}t} \int_{2^{-k}t}^{2^{1-k}t} \|g_s - g_0\|_2^2 ds \right) \rightarrow 0$$

as $t \rightarrow 0$. Hence the trace claims follow from the square function estimates $\|\partial_t g_t\|_{\mathcal{Y}} < \infty$. Moreover, the estimate $\sup_{t>0} \|g_t\|_2 \lesssim \|g\|_{\mathcal{X}} + \|\partial_t g\|_{\mathcal{Y}} \lesssim \|g\|_{\mathcal{X}}$ follows. The converse estimate follows from Theorem 7.2.

An integration by part, similar to above, shows that

$$2g_{2t} = g_t + \frac{1}{t} \int_t^{2t} g_s ds + \frac{1}{t} \int_t^{2t} \partial_s g_s s ds, \quad \text{a.e. } t > 0.$$

Taking $\limsup_{t \rightarrow \infty}$ of both sides, shows $2 \limsup_{t \rightarrow \infty} \|g_t\|_2 = \limsup_{t \rightarrow \infty} \|g_t\|_2$. Since $\|g_t\|_2$ is bounded, we conclude that $\lim_{t \rightarrow \infty} \|g_t\|_2 = 0$.

(iii) To show $\|g\|_{\mathcal{X}} \lesssim \|\partial_t g\|_{\mathcal{Y}}$, consider f satisfying $e^{-t\Lambda} h^+ = f_t - S_A f_t$. Theorem 4.2 and Lemma 9.5 give

$$\|h^+\|_2 \approx \|\partial_t e^{-t\Lambda} h^+\|_{\mathcal{Y}} \lesssim \|\partial_t f\|_{\mathcal{Y}} + (\|\mathcal{E}\|_* + \|t\partial_t A\|_*) \|f\|_{\mathcal{X}} + \|\mathcal{E}\|_{\infty} \|\partial_t f\|_{\mathcal{Y}},$$

where by Theorem 7.2 we have $\|f\|_{\mathcal{X}} \approx \|h^+\|_2$ as $\|\mathcal{E}\|_*$ is assumed small enough. If in addition $\|t\partial_t A\|_*$ is sufficiently small, then we obtain $\|f\|_{\mathcal{X}} \lesssim \|\partial_t f\|_{\mathcal{Y}}$. As in (i), again using smallness of $\|t\partial_t A\|_*$, this implies $\|g\|_{\mathcal{X}} \lesssim \|\partial_t g\|_{\mathcal{Y}}$.

(iv) To prove the x -regularity result, consider the equation $\partial_t f + DBf = 0$, which implies

$$\|\partial_t f\|_{\mathcal{Y}} = \|DP_{\mathcal{H}} Bf\|_{\mathcal{Y}} \approx \sum_{i=1}^n \|(P_{\mathcal{H}} B)(\partial_i f) + P_{\mathcal{H}}(\partial_i B)f\|_{\mathcal{Y}}$$

since $D = DP_{\mathcal{H}}$ and the operator D has estimates $\|Dh\|_2 \approx \sum_{i=1}^n \|\partial_i h\|_2$ for all $h \in \mathbf{D}(D) \cap \mathcal{H}$. (The latter is straightforward to verify with the Fourier transform.) Here $P_{\mathcal{H}}$ denotes orthogonal projection onto \mathcal{H} ; it commutes with ∂_i . This yields the bound

$$\|\partial_i f\|_{\mathcal{Y}} \approx \|(P_{\mathcal{H}} B)\partial_i f\|_{\mathcal{Y}} \lesssim \|\partial_i f\|_{\mathcal{Y}} + \|t(\partial_i B)f\|_{\mathcal{Y}^*} \lesssim (1 + \|t\partial_i B\|_*) \|f\|_{\mathcal{X}} \lesssim \|f\|_{\mathcal{X}}$$

if $\max(\|t\partial_i A\|_*, \|t\partial_t A\|_*) < \infty$, where we used that $P_{\mathcal{H}} B_t : \mathcal{H} \rightarrow \mathcal{H}$ is an isomorphism in the first comparison. Conversely, if $\|t\partial_t A\|_*$ is sufficiently small, then

$$\|f\|_{\mathcal{X}} \lesssim \|\partial_t f\|_{\mathcal{Y}} \lesssim \sum_{i=1}^n (\|\partial_i f\|_{\mathcal{Y}} + \|t\partial_i B\|_* \|f\|_{\mathcal{X}}),$$

where the first estimate is by (iii). Using next that $\sum_{i=1}^n \|t\partial_i B\|_*$ is small enough, this implies $\|f\|_{\mathcal{X}} \lesssim \|\nabla_x f\|_{\mathcal{Y}}$.

As in (i) above, these estimates translate to $\|\partial_i g\|_{\mathcal{Y}} \lesssim \|g\|_{\mathcal{X}}$ and $\|g\|_{\mathcal{X}} \lesssim \|\nabla_x g\|_{\mathcal{Y}}$ respectively. \square

Proof of Lemma 9.5. Assume that the coefficients A satisfy $\|A - A_0\|_* < \infty$ and has distribution derivative $\partial_t A \in L_\infty^{\text{loc}}(\mathbf{R}_+^{1+n}; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ such that $\|t\partial_t A\|_* < \infty$. Fix $h \in \mathcal{X}$ with distribution derivative $\partial_t h \in \mathcal{Y}$. By Theorem 6.1, $\int_a^b \|S_A h_t - S_A^\epsilon h_t\|_2^2 dt \rightarrow 0$ as $\epsilon \rightarrow 0$, where

$$S_A^\epsilon h_t := \int_0^t \eta_\epsilon^+(t, s) \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s h_s ds - \int_t^\infty \eta_\epsilon^-(t, s) \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- \mathcal{E}_s h_s ds = I - II.$$

Hence it suffices to bound $\|\partial_t(S_A^\epsilon h)\|_{\mathcal{Y}}$, uniformly for $\epsilon > 0$.

(i) Differentiate I and write

$$\begin{aligned} t\partial_t(I) &= \int_0^t (t\partial_t \eta_\epsilon^+) \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s h_s ds - \int_0^t \eta_\epsilon^+(t-s) \Lambda^2 e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s h_s ds \\ &\quad - \int_0^t \eta_\epsilon^+(\partial_s \Lambda e^{-(t-s)\Lambda}) \widehat{E}_0^+ (s \mathcal{E}_s h_s) ds = \int_0^t (t\partial_t \eta_\epsilon^+ + s\partial_s \eta_\epsilon^+) \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s h_s ds \\ &\quad - \int_0^t \eta_\epsilon^+(t-s) \Lambda^2 e^{-(t-s)\Lambda} \widehat{E}_0^+ \mathcal{E}_s h_s ds + \int_0^t \eta_\epsilon^+ \Lambda e^{-(t-s)\Lambda} \widehat{E}_0^+ \partial_s (s \mathcal{E}_s h_s) ds = I_1 - I_2 + I_3. \end{aligned}$$

Note that in I_3 the distribution derivative $\partial_s (s \mathcal{E}_s h_s)$ extends its action to test functions $s \mapsto (\eta_\epsilon^+(t, s) \lambda e^{-(t-s)\Lambda} \widehat{E}_0^+)^* \phi$, for any $\phi \in \mathcal{H}$. Theorem 5.5 and Lemma 4.4 give the estimate

$$\|I_3\|_{\mathcal{Y}^*} \lesssim \|\partial_t(t \mathcal{E}_t h_t)\|_{\mathcal{Y}^*} \lesssim (\|\mathcal{E}\|_* + \|t\partial_t \mathcal{E}\|_*) \|h\|_{\mathcal{X}} + \|\mathcal{E}\|_\infty \|\partial_t h\|_{\mathcal{Y}}.$$

To bound I_2 , we apply Lemma 5.7, using the bounds

$$\int_0^t |(t-s) \lambda^2 e^{-(t-s)\lambda}| ds \lesssim t \quad \text{and} \quad \int_s^\infty |(t-s) \lambda^2 e^{-(t-s)\lambda}| dt \lesssim 1,$$

which shows $\|I_2\|_{\mathcal{Y}^*} \lesssim \|\mathcal{E}h\|_{\mathcal{Y}^*} \lesssim \|\mathcal{E}\|_* \|h\|_{\mathcal{X}}$. To estimate I_1 , we calculate

$$(t\partial_t + s\partial_s) \eta_\epsilon^+(t, s) = \frac{t-s}{\epsilon} (\eta^0)' \left(\frac{t-s}{\epsilon} \right) \eta_\epsilon(t) \eta_\epsilon(s) + \eta^0 \left(\frac{t-s}{\epsilon} \right) (t\eta_\epsilon'(t) \eta_\epsilon(s) + s\eta_\epsilon(t) \eta_\epsilon'(s)).$$

From this, we verify that $|(t\partial_t + s\partial_s) \eta_\epsilon^+| \lesssim \chi_{\text{supp } \nabla \eta_\epsilon^+} \leq 1$. Hence an estimate as in the proof of Theorem 5.5 shows that $\|I_1\|_{\mathcal{Y}^*} \lesssim \|\mathcal{E}\|_* \|h\|_{\mathcal{X}}$.

(ii) Next we differentiate II and write

$$\begin{aligned} t\partial_t(II) &= \int_t^\infty (t\partial_t \eta_\epsilon^-) \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- \mathcal{E}_s h_s ds - \int_t^\infty t\eta_\epsilon^- (\partial_s \Lambda e^{-(s-t)\Lambda}) \widehat{E}_0^- \mathcal{E}_s h_s ds \\ &= \int_t^\infty t(\partial_t \eta_\epsilon^- + \partial_s \eta_\epsilon^-) \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- \mathcal{E}_s h_s ds + \int_t^\infty \eta_\epsilon^- \frac{t}{s} \Lambda e^{-(s-t)\Lambda} \widehat{E}_0^- s \partial_s (\mathcal{E}_s h_s) ds \\ &= II_1 + II_2. \end{aligned}$$

To bound II_2 , we apply Lemma 5.7 using the bounds

$$\int_t^\infty |(t/s) \lambda e^{-(s-t)\lambda}| ds \lesssim t \quad \text{and} \quad \int_0^s |(t/s) \lambda e^{-(s-t)\lambda}| dt \lesssim 1,$$

which shows $\|II_2\|_{\mathcal{Y}^*} \lesssim \|t\partial_t \mathcal{E}\|_* \|h\|_{\mathcal{X}} + \|\mathcal{E}\|_\infty \|\partial_t h\|_{\mathcal{Y}}$. To estimate II_1 , we calculate

$$t(\partial_t + \partial_s) \eta_\epsilon^+(t, s) = t\eta^0 \left(\frac{t-s}{\epsilon} \right) (\eta_\epsilon'(t) \eta_\epsilon(s) + \eta_\epsilon(t) \eta_\epsilon'(s)).$$

The last term is supported on $s \in (1/(2\epsilon), 1/\epsilon)$, $t \in (\epsilon, s - \epsilon)$, where it is bounded by $et \lesssim t/s$. Thus estimates as for II_2 apply. The first term is supported on $t \in (\epsilon, 2\epsilon)$, $s \in (t + \epsilon, 1/\epsilon)$ (and another component which can be taken together with the last

term) and is bounded by 1. Splitting this remaining term as in (25), it suffices to estimate

$$\begin{aligned} & \left\| \chi_{(\epsilon, 2\epsilon)}(t) t \eta'_\epsilon(t) e^{-t\Lambda} \int_0^\infty \eta_\epsilon(s) \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s h_s ds \right\|_{\mathcal{Y}^*} \\ & \lesssim \left(\frac{1}{\epsilon} \int_\epsilon^{2\epsilon} \left\| e^{-t\Lambda} \int_0^\infty \eta_\epsilon(s) \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s h_s ds \right\|_2^2 dt \right)^{1/2} \\ & \lesssim \left\| \int_0^\infty \eta_\epsilon(s) \Lambda e^{-s\Lambda} \widehat{E}_0^- \mathcal{E}_s h_s ds \right\|_2 \lesssim \|\mathcal{E}h\|_{\mathcal{Y}^*} \lesssim \|\mathcal{E}\|_* \|h\|_X, \end{aligned}$$

using the uniform boundedness of $e^{-t\Lambda}$ and Lemma 5.9. This completes the proof. \square

10. MISCELLANEOUS REMARKS AND OPEN QUESTIONS

(i) The condition $\widetilde{N}_*(\nabla_{t,x}u) \in L_2$ implies that Whitney averages $\frac{1}{|W(t,y)|} \iint_{W(t,y)} u$ converge non-tangentially for almost every x , i.e. with $|y-x| < \alpha t$ for some $\alpha < \infty$, to some $u_0(x)$ with u_0 belonging to the closure of $C_0^\infty(\mathbf{R}^n)$ with respect to $\|\nabla_x f\|_2 < \infty$. Furthermore, $t^{-1} \int_t^{2t} \nabla_x u_s ds$ converges weakly to $\nabla_x u_0$ in L_2 as $t \rightarrow 0$ (compare Theorem 2.2(i)). In particular $\|\nabla_x u_0\|_2 \lesssim \|\widetilde{N}_*(\nabla_{t,x}u)\|_2$. This is essentially in [27, p. 461-462], where it is done on the unit ball instead of the upper half space, and with pointwise values instead of averages, working with u 's solving a real symmetric equation. However, the result has nothing to do with BVPs, but is a result on a function space.

(ii) Assume that $A \in L_\infty(\mathbf{R}_+^{1+n}; \mathcal{L}(\mathbf{C}^{(1+n)m}))$ and that $\widetilde{N}_*(\nabla_{t,x}u) \in L_2$ with u satisfying (1) in \mathbf{R}_+^{1+n} distributional sense. Then there exists $g \in \dot{H}^{-1/2}(\mathbf{R}^n; \mathbf{C}^m)$ such that

$$(35) \quad \iint_{\mathbf{R}_+^{1+n}} (A \nabla_{t,x} u, \nabla_{t,x} \phi) dt dx = (g, \phi|_{\mathbf{R}^n}), \quad \text{for all } \phi \in C_0^\infty(\mathbf{R}^{1+n}; \mathbf{C}^m).$$

If $\partial_{\nu_A} u(s, x) := (A \nabla_{t,x} u(s, x))_\perp$ for all $s > 0$, $x \in \mathbf{R}^n$, then $t^{-1} \int_t^{2t} \partial_{\nu_A} u_s ds$ converges weakly to $-g$ in L_2 as $t \rightarrow 0$. In particular $\|g\|_2 \lesssim \|\widetilde{N}_*(\nabla_{t,x}u)\|_2$. This is again essentially [27] for the unit ball. See [4, Lemma 4.3(iii)] for an argument in \mathbf{R}_+^{1+n} . The equality (35) justifies that g is called the Neumann data. This result has nothing to do with accretivity of A , boundedness suffices. Compare again Theorem 2.2(i).

(iii) Theorem 2.3(i) contains *a priori* estimates on \mathcal{Y} -solutions. A natural question is to reverse the *a priori* estimates for such systems. Does a weak solution to (1) with $\|A - A_0\|_C < \infty$ and $\widetilde{N}_*(u) \in L_2$ satisfy $\|\nabla_{t,x}u\|_{\mathcal{Y}} \lesssim \|\widetilde{N}_*(u)\|_2$? Same question replacing $\widetilde{N}_*(u) \in L_2$ with $\sup_{t>0} \|u_t\|_2 < \infty$. The smallness of $\|A - A_0\|_C$, which implies well-posedness of the Dirichlet problem for \mathcal{Y} -solutions, yields *a posteriori* such estimates. It would be interesting to have positive answers *a priori* (i.e. independently of well-posedness) when $\|A - A_0\|_C < \infty$.

(iv) Is there existence of \mathcal{X} -solutions to the Neumann and regularity problems with L_2 data under $\|A - A_0\|_C < \infty$ (or even under the stronger $\int_0^\infty \omega_A(t)^2 dt/t < \infty$, where $\omega_A(t) := \sup_{0 < s < t} \|A_s - A_0\|_\infty$)? Is there uniqueness under the same constraint on A , provided existence holds? Recall that tools such as Green's functions are not available here.

(v) Same questions for \mathcal{Y} -solutions and the Dirichlet problem with L_2 data.

(vi) It is likely that \mathcal{Y} -solutions have the a.e. non-tangential convergence property for averages: $\frac{1}{|W(t,y)|} \iint_{W(t,y)} u \rightarrow u_0(x)$ for a.e. $x \in \mathbf{R}^n$ and $(t,y) \rightarrow (0,x)$ in $|y-x| < \alpha t$. This requires an argument which we leave open.

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