REGULARITY OF SOLUTIONS TO THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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Abstract Most of the work on the Boltzmann equation is based on Grad's cutoff assumption. Even though the smoothing effect from the singular cross-section without cutoff coming from the grazing collision is expected, there is no general mathematical theory especially for the spatially inhomogeneous case. As a further study on the problem in the spatially homogeneous situation, in this paper, we will prove the Gevrey smoothing property of the solutions to the Cauchy problem for Maxwellian molecules without angular cutoff by using pseudo-differential calculus. Furthermore, we apply the similar analytic technique for the Sobolev space regularity to the nonlinear equation, and prove the smoothing property of solutions for the spatially homogeneous nonlinear Boltzmann equation with Debye-Yukawa potential.

Key words Boltzmann equation, Debye-Yukawa potential, Gevrey hypoellipticity, non-cutoff cross-sections.

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1. Introduction

Among the extensive studies on the Boltzmann equation, most of them are based on the Grad's cutoff assumption to avoid the mathematical difficulty from the grazing effects in the elastic collisions between particles. Recently, a lot of progress has made on the study on the non-cutoff problems, cf. [1, 2, 6, 7, 12] and references therein, which shows that the singularity of collision cross-section yields some gain of regularity on weak solutions. In some sense, this gives the hypo-ellipiticity property of the Boltzmann operator without angular cutoff. However, so far the study in this direction is still not satisfactory because there is no general theory especially for the spatially inhomogeneous problems.

This paper is concerned with the smoothing effects of the singular integral kenerl in the collision operator coming from the non-cutoff cross-sections in the Boltzmann equation. There are two main results in this paper. One is about the smoothing effect of the non-cutoff Debye-Yukawa potential which gives the gain of a fraction of the logarithm of Laplacian regularity. This is different from the non-cutoff inverse power laws which give the gain of a fraction of Laplacian regularity. Another problem is concerned with the Gevrey regularity of the non-cutoff inverse power laws. Even though both results are about the spatially homogeneous problem, it provides some new aspects of the regularity for the singular cross-sections.

Consider the Cauchy problem of spatially homogeneous nonlinear Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f,f),\ \ x\in\mathbb{R}^3,\ \ v\in\mathbb{R}^3,\ \ t>0\,; \qquad f|_{t=0}=f_0, \tag{1.1}$$

where $f = f(t,v) \geq 0$ on $[0,\infty)\times\mathbb{R}^3_v$ represents the particle distribution function. In the following, we assume that the initial datum $f_0 \not\equiv 0$ satisfies the natural boundedness on mass, energy and entropy, that is,

$$f_0 \geq 0, \quad \int_{\mathbb{R}^3} f_0(v)(1+|v|^2+\log(1+f_0(v)))dv < +\infty. \tag{1.2}$$

The Boltzmann quadratic operator $Q(g,f)$ is a bi-linear functional representing the change rate of the particle distribution through the elastic binary collisions which takes the form

$$Q(g,f) = \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-v_*,\sigma)\left\{g(v'_*)f(v') - g(v_*)f(v)\right\} d\sigma dv_*\,, \tag{1.3}$$

where for $\sigma \in \mathbb{S}^2$, and

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \tag{1.4}$$

are the relations between the post and pre collisional velocities. The non-negative function $B(z,\sigma)$ called the Boltzmann collision kernel depends only on $|z|$ and on the scalar product $< \frac{z}{|z|}, \sigma >$. In most cases, the collision kernel B can not be expressed explicitly, but to capture its main property, it can be assumed to be in the form

$$B(|v-v_*|,\cos\theta) = \Phi(|v-v_*|)b(\cos\theta), \quad \cos\theta = \langle \frac{v-v_*}{|v-v_*|}\,,\,\sigma\rangle,\ \ 0 \leq \theta \leq \frac{\pi}{2}.$$

In this paper, we consider only mathematical Maxwellian case, that is, we take $\Phi \equiv 1$. Except for hard sphere model, the function $b(\cos(\cdot))$ has a singularity at $\theta = 0$. For example, if the inter-molecule potential satisfies the inverse-power law potential $U(\rho) = \rho^{-(\gamma-1)}, \gamma > 2$, then

$$\sin\theta\, b(\cos\theta) \ \approx\ K\theta^{-1-2\alpha} \ \text{ when } \theta \to 0, \tag{1.5}$$

where $K > 0, 0 < \alpha = \frac{1}{\gamma-1} < 1$. The Maxwellian molecule case corresponds to $\gamma = 5$ and $\Phi = 1$. Notice that the Boltzmann collision operator is not well defined for the case when $\gamma = 2$ which is called the Coulomb potential. In the great majority of works on the Boltzmann equation, the angular singularity at $\theta = 0$ is removed by using the Grad's angular cutoff assumption so that B is locally integrable in σ variable.

We will first consdier a family of Debye-Yukawa type potentials where the potential function is given by

$$U(\rho) = \rho^{-1}e^{-\rho^s}, \ \text{ with } s > 0. \tag{1.6}$$

In some sense, it is a model between the Coulomb potential corresponding to $s = 0$ and the potential satisfying the inverse power law. In fact, the classical Debye-Yukawa potential is when $s = 1$.

In §2, we will show that the collision cross-section of this kind of potentials has the singularity in the following form, see (2.2),

$$b(\cos\theta) \ \approx\ K\theta^{-2}\left(\log\theta^{-1}\right)^{\frac{2}{s}-1}, \ \text{ when } \theta \to 0.$$

where $K > 0$ is constant, and further that this singularity endows the collision operator Q with the logalithmic regularity property, see Proposition 2.1. We mention that the logalithmic regularity theory was first introduced in [8] on the hypoellipticity of the infinitely degenerate elliptic operator and developed in [9, 10] on the logalithmic Sobolev estimates.

Suppose that there exists a weak solution to the Cauchy problem (1.1) with the following natural bound for some time $T > 0$, see the Defintion 3.1 of Section 3 and also [12],

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^3} f(t,v)(1+|v|^2+\log(1+f(t,v)))dv < +\infty. \tag{1.7}$$

In §3, we are going to prove the following theorem on the regularity of this weak solution.

Theorem 1.1. *Assume that the initial datum f_0 satisfies (1.2) and the collision cross-section satisfies*

$$B(|v-v_*|,\cos\theta)=b(\cos\theta)\ \approx\ K\theta^{-2}\left(\log\theta^{-1}\right)^m \quad \textit{when } \theta\to 0, \tag{1.8}$$

with $K>0, m>0$. Let f be a weak solution of Cauchy problem (1.1) satisfying (1.7). Then for any $0<t\le T$, we have $f(t,\cdot)\in H^{+\infty}(\mathbb{R}^3)$.

Remark 1.1: Note that $m>0$ corresponds to $0<s<2$ in (1.6). In [2, 7], the $H^{+\infty}(\mathbb{R}^3)$ regularity of weak solutions was proved under the condition (1.5). Notice that the condition (1.8) is much weaker than (1.5) and it still leads to $H^{+\infty}(\mathbb{R}^3)$ regularity on the weak solutions. Moreover, the following proof on the regularity of weak solutions is more straightforward and illustrative than the previous methods. Even though the assumption (1.8) on the cross-section is mathematical because the exact cross-section depends also on the relative velocity as given in (2.2), the following analysis reveals the smoothing effect of the singularity of the collision operator on the weak solution to the Boltzmann equation.

To have a more precise description on the regularity, in the second part of the paper, we consider the Gevrey regularity of solutions with cross-section satisfying (1.3). Notice that while local solutions having the Gevrey regularity have been constructed in [11] for initial data having the same Gevrey regularity, the result given here is concerned with the production of the Gevrey regularity for weak solutions whose initial data have no regularity.

Before stating the result, we now recall the definition of Gevrey regularity. For $s\ge 1$, $u\in G^s(\mathbb{R}^3)$ which is the Gevrey class function space with index s, if there exists $C>0$ such that for any $k\in\mathbb{N}$,

$$\|D^k u\|_{L^2}\le C^{k+1}(k!)^s,$$

or equivalently, there existe $\varepsilon_0>0$ such that $e^{\varepsilon_0<|D|>^{1/s}}u\in L^2$, where $L^2=L^2(\mathbb{R}^3)$ and

$$<|D|>=(1+|D_v|^2)^{1/2}, \qquad \|D^k u\|^2_{L^2}=\sum_{|\beta|=k}\|D^\beta u\|^2_{L^2}.$$

Note that $G^1(\mathbb{R}^3)$ is usual analytic function space. In this following discussion, we also adopt the following notations,

$$\|f\|_{L^k_\ell}=\left(\int_{\mathbb{R}^3}|f(v)|^k(1+|v|)^{k\ell}dv\right)^{\frac1k};\ \ \|f\|_{L\log L}=\int_{\mathbb{R}^3}|f(v)|\log(1+|f(v)|)dv.$$

What we are going to show in Section 4 is that the weak solutions to the linearized Boltzmann equation with the cross-section satisfying (1.3) are in the Gevrey class with index $\frac{1}{\alpha}$ for $t>0$. For this, we first linearize the Boltzmann equation near the absolute Maxwellian distribution

$$\mu(v)=(2\pi)^{-\frac32}e^{-\frac{|v|^2}{2}}. \tag{1.9}$$

Since $Q(\mu,\mu)=0$, we have

$$Q(\mu+g,\mu+g)=Q(\mu,g)+Q(g,\mu)+Q(g,g).$$

Set

$$Lg=Q(\mu,g)+Q(g,\mu),$$

where L is the usual linearized collision operator. Then, consider the linear Cauchy problem

$$\frac{\partial g}{\partial t}=Lg,\ \ v\in\mathbb{R}^3,\ \ t>0\,; \qquad g|_{t=0}=g_0. \tag{1.10}$$

The definition of weak solution for this linear equation is standard which is given precisely in Section 4. The result on Gevrey regularity can be stated as follows.

Theorem 1.2. *Assume that the initial perturbation in (1.10) satisfies* $g_0 \in L^1_2(\mathbb{R}^3)$*, and* Q *is defined by Maxwellian collision cross-section* B *satisfying (1.5) with* $0 < \alpha < 1$*. For* $T_0 > 0$*, if* $g \in L^1([0,T_0]; L^1_2(\mathbb{R}^3)) \cap L^\infty([0,T_0]; L^1(\mathbb{R}^3))$ *is a weak solution of the Cauchy problem (1.10), then* $g(t,\cdot) \in G^{1/\alpha}(\mathbb{R}^3) \cap L^1_2(\mathbb{R}^3)$ *for any* $0 < t \le T_0$.

The assumption in the above theorem does not seem strong enough to construct the weak solutions which meet the requirement of the theorem, but it is possible under additional assumptions. In §4, we will prove the

Proposition 1.1. *Suppose that* $0 < \alpha < 1/2$ *and* $g_0 \in L^2_\ell$ *for some* $\ell > 5/2$*. Then,* (1.10) *possesses a weak solution* $g \in L^\infty(0, T_0;\ L^2_\ell(\mathbb{R}^3))$ *for any* $T_0 > 0$.

Remark 1.2: (1) Evidently, $L^2_\ell(\mathbb{R}^3) \subset L^1_2(\mathbb{R}^3)$ when $\ell > 7/2$.
(2) Though the above results are given when the space dimension equals to three, they hold for any space dimensions with due modification.

2. Logarithmic regularity estimate

Firstly, following the computation given in [4, 13], we will give an asymptotic description of the Boltzmann collision kernel $B(z,\sigma)$ for the potential $U(\rho)$ defined in (1.6). Here, ρ is the distance between two interacting particles, $z = v - v_*$ is relative velocity, $\sigma \in \mathbb{S}^2$ and $< \frac{z}{|z|}, \sigma >= \cos(\pi - 2\vartheta)$, $\theta = \pi - 2\vartheta$ is the deviation angle. Let $p \ge 0$ be the impact parameter which is a function of ϑ and z. Then Boltzmann collision cross-section is defined by

$$B(|z|,\vartheta) = |z| s(|z|,\vartheta) = \frac{|z| s(|z|,\vartheta)}{4\cos\vartheta} = |z| \frac{p}{2\sin 2\vartheta}\frac{\partial p}{\partial \vartheta}, \tag{2.1}$$

where $s(|z|,\vartheta)$ is called the differential scattering cross-section.

If ρ and φ are the radial and angular coordinates in the plane of motion, then the impact parameter $p(V,\vartheta)$ is determined by the conservation of energy and angular momentum respectively:

$$\begin{cases} \frac{1}{2}\big(\dot\rho^2 + \rho^2\dot\varphi^2\big) + U(\rho) = \frac{1}{2}V^2 + U(\sigma), \qquad (\rho \le \sigma), \\ \rho^2\dot\varphi = pV^2. \end{cases}$$

where the relative speed is now denoted by $V = |v - v_*|$. As usual, it is impossible to give an explicit expression of solutions to this nonlinear system. Hence, in the following, we will study the singular behavior of the solutions around the grazing collisions, that it, when $\theta \sim 0$.

By using φ as independent variable to eliminate the time derivative, after integration, we have

$$\vartheta = \frac{1}{\sqrt{2}} V p \int_{\rho_0}^{\sigma} \rho^{-2}\Big[\frac{V^2}{2}\big(1 - \frac{p^2}{\rho^2}\big) - U(\rho) + U(\sigma)\Big]^{-1/2} d\rho + \sin^{-1}\Big(\frac{p}{\sigma}\Big),$$

where ρ_0 is the smallest distance between two particles which satisfies

$$\frac{1}{2}V^2\big(1 - \frac{p^2}{\rho_0^2}\big) = U(\rho_0) - U(\sigma) > 0.$$

Note that $p < \rho_0 < \rho \le \sigma$. By the transformation $u = \frac{p}{\rho}$, we have

$$\vartheta = \int_{p/\sigma}^{u_0} \Big[1 - u^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u}\big) - U(\sigma)\big)\Big]^{-1/2} du + \sin^{-1}\big(\frac{p}{\sigma}\big),$$

where $u_0 = p/\rho_0$ satisfies

$$1 - u_0^2 - \frac{2}{V^2}\big(U\big(\frac{p}{u_0}\big) - U(\sigma)\big) = 0.$$

Therefore

$$\begin{aligned}\frac{\theta}{2} &= \frac{\pi}{2} - \vartheta = \frac{\pi}{2} - \int_{p/\sigma}^{u_0} \left[1 - u^2 - \frac{2}{V^2}\big(U(\frac{p}{u}) - U(\sigma)\big)\right]^{-1/2} du - \sin^{-1}\big(\frac{p}{\sigma}\big) \\ &= \int_0^1 \frac{dt}{\sqrt{1-t^2}} - \int_0^{p/\sigma} \frac{dt}{\sqrt{1-t^2}} - \int_{p/\sigma}^{u_0} \left[1 - u^2 - \frac{2}{V^2}\big(U(\frac{p}{u}) - U(\sigma)\big)\right]^{-1/2} du.\end{aligned}$$

By setting $u = u_0 t$, we get

$$\begin{aligned}\frac{\theta}{2} &= \int_{p/\sigma}^1 \frac{dt}{\sqrt{1-t^2}} - \int_{\frac{p}{u_0\sigma}}^1 \left[1 - u_0^2 t^2 - \frac{2}{V^2}\big(U(\frac{p}{u_0 t}) - U(\sigma)\big)\right]^{-1/2} u_0 dt \\ &= \int_{p/\sigma}^1 \frac{dt}{\sqrt{1-t^2}} - \int_{\frac{p}{u_0\sigma}}^1 \left[1 - t^2 + \frac{2}{V^2 u_0^2}\big(U(\frac{p}{u_0}) - U(\frac{p}{u_0 t})\big)\right]^{-1/2} dt \\ &= -\int_{\frac{p}{u_0\sigma}}^{p/\sigma} \frac{dt}{\sqrt{1-t^2}} + \int_{\frac{p}{u_0\sigma}}^1 \frac{1}{\sqrt{1-t^2}}\left[1 - \big(1 + \frac{2U(\frac{p}{u_0}) - 2U(\frac{p}{u_0 t})}{(1-t^2)V^2 u_0}\big)^{-1/2}\right] dt,\end{aligned}$$

where we have used the fact that

$$\frac{1-u_0^2}{u_0^2} = \frac{2}{V^2 u_0^2}\big(U(\frac{p}{u_0}) - U(\sigma)\big).$$

It is clear that there is no explicit formula for $\theta = \theta(p, V)$. To study its asymptotic behavior when $\theta \sim 0$, we let $\sigma \to \infty$ which is equivalent to $p \to \infty$. Under this assumption, we have $u_0 \approx 1$ and

$$\big(1 + \frac{2U(\frac{p}{u_0}) - 2U(\frac{p}{u_0 t})}{(1-t^2)V^2 u_0}\big)^{-1/2} \approx 1 - \frac{U(\frac{p}{u_0}) - U(\frac{p}{u_0 t})}{(1-t^2)V^2 u_0}.$$

Thus,

$$\frac{\theta}{2} \approx \int_0^1 \frac{1}{\sqrt{1-t^2}} \frac{U(p) - U(\frac{p}{t})}{(1-t^2)V^2} dt.$$

By plugging $U(\rho) = \rho^{-1} e^{-\rho^s}$ into the above integral, we have

$$\frac{\theta}{2} \approx \frac{1}{V^2 p} e^{-p^s} \int_0^1 (1-t^2)^{-3/2} \left(1 - t e^{-p^s(t^{-s}-1)}\right) dt.$$

Since

$$\begin{aligned}0 &\le \frac{\partial}{\partial p}\left(\int_0^1 (1-t^2)^{-3/2}\left(1 - t e^{-p^s(t^{-s}-1)}\right) dt\right) \\ &= \int_0^1 (1-t^2)^{-3/2} t (t^{-s} - 1) s p^{s-1} e^{-p^s(t^{-s}-1)} dt \le C_s s p^{s-1},\end{aligned}$$

it holds that

$$0 < c_0 \le \int_0^1 (1-t^2)^{-3/2}\left(1 - t e^{-p^s(t^{-s}-1)}\right) dt \le C_s p^s + c_0.$$

where $c_0 = \int_0^1 (1-t^2)^{-3/2}(1-t)dt$. Finally, for $p \to \infty$ (equivalently $\theta \to 0$), we have

$$\log\theta \approx -K' p^s.$$

In summary, we have the Boltzmann collision cross-section for the Debye-Yukawa type potentials as

$$B(V,\theta) = -\frac{V}{\sin\theta}\frac{\partial p^2}{\partial\theta} \approx K V \theta^{-2}\left(\log\theta^{-1}\right)^{\frac{2}{s}-1}, \tag{2.2}$$

for some constant $K > 0$ when $\theta \to 0$. Note that the cross-section $B(V,\theta)$ satisfies for any $s > 0$,

$$\int_0^{\pi/2} B(V,\theta)\sin\theta d\theta = +\infty, \quad \text{and} \quad \int_0^{\pi/2} B(V,\theta)\sin^2\theta d\theta < +\infty.$$

We prove firstly the following logarithmic regularities estimate for the collision operator.

Proposition 2.1. *Assume that the collision kernel B satisfies the assumption (1.8) and $g \geq 0$, $g \in L^1_2 \bigcap L\log L$. Then there exists a constant C_g depending only on $B, \|g\|_{L^1_2}$ and $\|g\|_{L\log L}$ such that for any smooth function $f \in H^2(\mathbb{R}^3)$,*

$$\|\left(\log\Lambda\right)^{\frac{m+1}{2}} f\|^2_{L^2(\mathbb{R}^3)} \leq C_g\left\{(-Q(g,f),f)_{L^2(\mathbb{R}^3)} + \|f\|^2_{L^2(\mathbb{R}^3)}\right\}, \tag{2.3}$$

where $\Lambda = (e + |D_v|^2)^{1/2}$.

Remark 2.1: With hypothesis (1.5), we have the following sub-elliptic estimate (see [2, 5, 7])

$$\|\Lambda^\alpha f\|^2_{L^2(\mathbb{R}^3)} \leq C_g\left\{(-Q(g,f),f)_{L^2(\mathbb{R}^3)} + \|f\|^2_{L^2(\mathbb{R}^3)}\right\}. \tag{2.4}$$

And we will use this estimate to prove the Gevrey regularity stated in the Theorem 1.2.

Proof of Proposition 2.1.

For $f \in H^2(\mathbb{R}^3)$, we have

$$\begin{aligned}(-Q(g,f),f)_{L^2(\mathbb{R}^3)} &= -\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)f(v)\Big(f(v') - f(v)\Big)d\sigma dv_* dv \\ &= \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)\Big(f(v') - f(v)\Big)^2 d\sigma dv_* dv \\ &- \frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)\Big(f(v')^2 - f(v)^2\Big)d\sigma dv_* dv.\end{aligned}$$

According to cancellation lemma (Corollary 2 of [1]), we have

$$\left|\frac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)\Big(f(v')^2 - f(v)^2\Big)d\sigma dv_* dv\right| \leq C\|g\|_{L^1}\|f\|^2_{L^2}.$$

Notice that there is no weight in the norm of f in L^2 on the right hand side of the above equation because we consider the Maxwellian molecule type of cross-sections and it is a direct consequnce of

$$\int_{-\pi/2}^{\pi/2} \sin\theta b(\cos\theta)\left(\frac{1}{\cos^3(\theta/2)} - 1\right)d\theta < \infty.$$

Now the proof of Proposition 2.1 is reduced to the following lemma.

Lemma 2.1. *There exists a constant C_g, depending only on b, $\|g\|_{L^1_1}$ and $\|g\|_{L\log L}$ such that*

$$\|\left(\log\Lambda\right)^{\frac{m+1}{2}} f\|^2_{L^2} \leq C_g\Big\{\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)\Big(f(v') - f(v)\Big)^2 d\sigma dv_* dv + \|f\|^2_{L^2}\Big\}.$$

The proof of this lemma is similar to that of Theorem 1 in [1]. By taking the Fourier transform on collision operator and applying the Bobylev identity, we have

$$\begin{aligned}&\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} b(k\cdot\sigma)g(v_*)\Big(f(v) - f(v')\Big)^2 d\sigma dv_* dv \\ &= (2\pi)^{-3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{\hat g(0)|\hat f(\xi)|^2 + \hat g(0)|\hat f(\xi^+)|^2 - 2Re\,\hat g(\xi^-)\hat f(\xi^+)\bar{\hat f}(\xi)\Big\}d\sigma d\xi \\ &\geq \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}|\hat f(\xi)|^2\left\{\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)(\hat g(0) - |\hat g(\xi^-)|)d\sigma\right\}d\xi,\end{aligned}$$

where

$$\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}.$$

By using the condition that $g \geq 0, g \in L^1_1 \cap L \log L$ and the assumption (1.8), similar to the argument in [1], we can show that there exists a positive conatant C_g depending only on $\|g\|_{L^1_1}$ and $\|g\|_{L \log L}$ such that

$$\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big)(\hat{g}(0) - |\hat{g}(\xi^-)|)d\sigma \geq C_g^{-1}\big(\log < \xi > \big)^{m+1} - C_g.$$

And this completes the proof of lemma 2.1.

3. Smoothing effect for the nonlinear Cauchy problem

We will give the proof of Theorem 1.1 on the smoothing effect of the collision operator for the Debye-Yukawa type potentials in this section. Before that, let us recall the definition of weak solution for the Cauchy problem (1.1), cf. [12].

Definition 3.1. *Let $f_0(v) \geq 0$ be a function defined on $\mathbb{R}^3$ with finite mass, energy and entropy. $f(t,v)$ is called a weak solution of the Cauchy problem (1.1), if it satisfies the following conditions:*

$$f(t,v) \geq 0, \quad f(t,v) \in C(\mathbb{R}^+; \mathcal{D}'(\mathbb{R}^3)) \cap L^1([0,T]; L^1_2(\mathbb{R}^3)), \quad f(0,v) = f_0(v);$$

$$\int_{\mathbb{R}^3} f(t,v)\psi(v)dv = \int_{\mathbb{R}^3} f_0(v)\psi(v)dv \ \text{for}\ \psi = 1, v_j, |v|^2;$$

$$f(t,v) \in L^1(\mathbb{R}^3)\log L^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} f(t,v)\log f(t,v)dv \leq \int_{\mathbb{R}^3} f_0 \log f_0 dv, \quad \forall t \geq 0;$$

$$\int_{\mathbb{R}^3} f(t,v)\varphi(t,v)dv - \int_{\mathbb{R}^3} f_0\varphi(0,v)dv - \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau\varphi(\tau,v)dv$$
$$= \int_0^t d\tau \int_{\mathbb{R}^3} Q(f,f)(\tau,v)\varphi(\tau,v)dv,$$

where $\varphi(t,v) \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$. Here the last integral above on the right hand side is defined by

$$\int_{\mathbb{R}^3} Q(f,f)(v)\varphi(v)dv = \tfrac{1}{2}\int_{\mathbb{R}^6}\int_{\mathbb{S}^2} Bf(v_*)f(v)(\varphi(v') + \varphi(v'_*) - \varphi(v) - \varphi(v_*))dvdv_*d\sigma.$$

Hence, this integral is well defined for any test function $\varphi \in L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$(see p. 291 of [12]*).*

Since the existence of weak solution was proved in [12], we will then show the regularity of the weak solution. Let f be a weak solution of the Cauchy problem (1.1). For any fixed $T_0 > 0$, we know that $f(t) \in L^1(\mathbb{R}^3) \subset H^{-2}(\mathbb{R}^3)$ for all $t \in [0, T_0]$. For $t \in [0, T_0]$, $N > 0$ and $0 < \delta < 1$, set

$$M_\delta(t,\xi) = \Big(1 + |\xi|^2\Big)^{\frac{Nt-4}{2}} \times \Big(1 + \delta|\xi|^2\Big)^{-N_0},$$

with $N_0 = \frac{NT_0}{2} + 2$. Then, for any $\delta \in]0,1[$

$$M_\delta(t, D_v)f \in L^\infty([0,T_0]; W^{2,\infty}(\mathbb{R}^3)),$$

whose norm is estimated above from $C_\delta\|f_0\|_{L^1}$, in view of the mass conservation law.

By using Proposition 2.1, we have

$$\|\left(\log\Lambda\right)^{\frac{m+1}{2}} M_\delta(t,D_v)f\|^2_{L^2} \leq C_f\left\{(-Q(f, M_\delta f), M_\delta f)_{L^2} + \|M_\delta f\|^2_{L^2}\right\}, \tag{3.1}$$

where the constant C_f is independent of $\delta \in]0,1[$.

To apply this logarithmic regularity estimate to the nonlinear Boltzmann equation, we need to estimate the commutators of the pseudo-differtial operator $M_\delta(t, D_v)$ and the nonlinear operator $Q(f, \cdot)$ which is given in the following lemma.

Lemma 3.1. *Under the hypothesis of Theorem 1.1, we have that*

$$|(Q(f, M_\delta f), M_\delta f)_{L^2} - (Q(f, f), M_\delta^2 f)_{L^2}| \leq C_f \|M_\delta f\|_{L^2}^2 \tag{3.2}$$

with C_f independent of $0 < \delta < 1$.

Proof. By applying Proposition 2.1 to the function $M_\delta f \in H^2$, we have

$$\begin{aligned} & (-Q(f, M_\delta f), M_\delta f)_{L^2(\mathbb{R}^3)} + O(\|M_\delta f\|_{L^2}^2) \\ = \ & \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big\{ \hat{f}(0) M_\delta^2(t, \xi) |\hat{f}(\xi)|^2 + \hat{f}(0) M_\delta^2(t, \xi^+) |\hat{f}(\xi^+)|^2 \\ - \ & 2 Re\, \hat{f}(\xi^-) M_\delta(t, \xi^+) \hat{f}(\xi^+) M_\delta(t, \xi) \bar{\hat{f}}(\xi) \Big\} d\sigma d\xi, \end{aligned}$$

By the Bobylev identity, we also have

$$\begin{aligned} & \Big(Q(f, f), M_\delta^2 f\Big)_{L^2} = \int_{\mathbb{R}^6} \int_{\mathbb{S}^2} b(k \cdot \sigma) f(v_*) f(v) \Big(M_\delta^2 f(v') - M_\delta^2 f(v) \Big) dv_* d\sigma dv \\ = \ & \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big\{ \hat{f}(\xi^-) M_\delta^2(t, \xi) \hat{f}(\xi^+) \bar{\hat{f}}(\xi) - \hat{f}(0) M_\delta^2(t, \xi) |\hat{f}(\xi)|^2 \Big\} d\sigma d\xi. \end{aligned}$$

Thus,

$$\begin{aligned} & \Big(Q(f, f), M_\delta^2 f\Big)_{L^2} = - \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big\{ \hat{f}(0) M_\delta^2(t, \xi) |\hat{f}(\xi)|^2 \\ + \ & \hat{f}(0) M_\delta^2(t, \xi^+) |\hat{f}(\xi^+)|^2 - 2 Re\, \hat{f}(\xi^-) M_\delta(t, \xi^+) \hat{f}(\xi^+) M_\delta(t, \xi) \bar{\hat{f}}(\xi) \Big\} d\sigma d\xi \\ + \ & \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big\{ \hat{f}(0) M_\delta^2(t, \xi^+) |\hat{f}(\xi^+)|^2 - \hat{f}(0) M_\delta^2(t, \xi) |\hat{f}(\xi)|^2 \\ + \ & 2 Re\, \hat{f}(\xi^-) M_\delta(t, \xi) \hat{f}(\xi^+) \bar{\hat{f}}(\xi) \big[M_\delta(t, \xi) - M_\delta(t, \xi^+) \big] \Big\} d\sigma d\xi. \\ = \ & \Big(Q(f, M_\delta f), M_\delta f\Big)_{L^2} + O(\|M_\delta f\|_{L^2}^2) \\ + \ & \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|} \cdot \sigma\Big) \Big\{ \hat{f}(0) M_\delta^2(t, \xi^+) |\hat{f}(\xi^+)|^2 - \hat{f}(0) M_\delta^2(t, \xi) |\hat{f}(\xi)|^2 \\ + \ & 2 Re\, \hat{f}(\xi^-) M_\delta(t, \xi) \hat{f}(\xi^+) \bar{\hat{f}}(\xi) \big[M_\delta(t, \xi) - M_\delta(t, \xi^+) \big] \Big\} d\sigma d\xi. \end{aligned}$$

Hence, it remains to show that

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b \Big\{ \hat{f}(0) M_\delta^2(t, \xi^+) |\hat{f}(\xi^+)|^2 - \hat{f}(0) M_\delta^2(t, \xi) |\hat{f}(\xi)|^2 d\sigma d\xi \Big| \leq C_f \|M_\delta f\|_{L^2}^2 \,,$$

and

$$\Big| \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} b \Big\{ Re\, \hat{f}(\xi^-) M_\delta(t, \xi) \hat{f}(\xi^+) \bar{\hat{f}}(\xi) \big[M_\delta(t, \xi) - M_\delta(t, \xi^+) \big] \Big\} d\sigma d\xi \Big| \leq C_f \|M_\delta f\|_{L^2}^2 \,. \tag{3.3}$$

The first estimate can be obtained as for the cancellation lemma in [1] because

$$
\begin{aligned}
&|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big\{\hat f(0)M_\delta^2(t,\xi^+)|\hat f(\xi^+)|^2-\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2|\\
&=(2\pi)|\int_{\mathbb{R}^3}\int_{-\pi/2}^{\pi/2}\sin\theta\, b(\cos\theta)\hat f(0)M_\delta^2(t,\xi)|\hat f(\xi)|^2\big[\frac{1}{\cos^3(\theta/2)}-1\big]d\theta d\xi|\\
&\le C_0\|f\|_{L^1}\|M_\delta f\|_{L^2}^2.
\end{aligned}
$$

To prove the second estimate, we need to show that

$$
|M_\delta(t,\xi^+)-M_\delta(t,\xi)|\le N_0 2^{(NT_0+4)/2}\sin^2\frac{\theta}{2}M_\delta(t,\xi^+). \tag{3.4}
$$

For this, recall

$$
\xi^+=\frac{\xi+|\xi|\sigma}{2},\ \ |\xi^+|^2=|\xi|^2\cos^2\frac{\theta}{2},\ \ \frac{\xi}{|\xi|}\cdot\sigma=\cos\theta,
$$

and the collsion kernel is supported in $|\theta|\le\pi/2$. Then

$$
\frac{|\xi|^2}{2}\le|\xi^+|^2\le|\xi|^2,\quad |\xi|^2-|\xi^+|^2=|\xi^-|^2=\sin^2\frac{\theta}{2}|\xi|^2.
$$

Denote

$$
\tilde M_\delta(t,s)=(1+s)^{\frac{Nt-4}{2}}\times(1+\delta s)^{-N_0},\ \ s=|\xi|^2,\quad s_+=|\xi^+|^2,
$$

so that

$$
M_\delta(t,\xi)=\tilde M_\delta(t,|\xi|^2).
$$

Then, there exists $s^+<\tilde s<s$ such that

$$
\tilde M_\delta(t,s)-\tilde M_\delta(t,s_+)=\frac{\partial\tilde M_\delta}{\partial s}(t,\tilde s)(s-s_+).
$$

Note that $s-s_+=s\sin^2\frac{\theta}{2}$ and

$$
\frac{\partial\tilde M_\delta}{\partial s}(t,s)=\Big\{(Nt-4)\frac{1}{2(1+s)}-N_0\frac{\delta}{1+\delta s}\Big\}\tilde M_\delta(t,s).
$$

By using

$$
\frac{s}{1+s},\quad\frac{\delta s}{1+\delta s}\le 1,
$$

and

$$
\left|\frac{\tilde M_\delta(t,\tilde s)}{\tilde M_\delta(t,s^+)}\right|\le 2^{(NT_0+4)/2},
$$

we have

$$
|\tilde M_\delta(t,s)-\tilde M_\delta(t,s_+)|\le N_0 2^{(NT_0+4)/2}\sin^2\frac{\theta}{2}\tilde M_\delta(t,s^+),
$$

which gives (3.4). Now the second estimate in (3.3) can be proved as follows,

$$
\begin{aligned}
&\left|\int b\Big\{Re\,\hat f(\xi^-)M_\delta(t,\xi)\hat f(\xi^+)\bar{\hat f}(\xi)\big[M_\delta(t,\xi)-M_\delta(t,\xi^+)\big]\Big\}d\sigma d\xi\right|\\
\le\ & C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\sin^2\frac{\theta}{2}|\hat f(\xi^-)|M_\delta(t,\xi^+)|\hat f(\xi^+)|M_\delta(t,\xi)|\hat f(\xi)|d\sigma d\xi\\
\le\ & C\int_{\mathbb{R}^3}\int_{-\pi/2}^{\pi/2} b(\cos\theta)\sin^2\frac{\theta}{2}\sin\theta|\hat f(\xi^-)|M_\delta(t,\xi^+)|\hat f(\xi^+)|M_\delta(t,\xi)|\hat f(\xi)|d\theta d\xi\\
\le\ & C\|f\|_{L^1}\|M_\delta f\|_{L^2}^2.
\end{aligned}
$$

□

If $M_\delta(t,D_v)$ is replaced by the differential operator D_v^k, then the commutator is given by Leibniz formula. Therefore, in some sense, this lemma is a microlocal version of the computation given in [7]. We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Firstly, we note from Definition 3.1 that any weak solution f enjoys the following properties; $M_\delta^2 f \in L^\infty([0,T_0]; W^{2,\infty}(\mathbb{R}^3))$,

$$M_\delta f \in C([0,T_0]; L^2(\mathbb{R}^3)) \tag{3.5}$$

and that

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}^3} f(t) M_\delta^2(t) f(t) dv - \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big) f(\tau) dv d\tau \\ = \frac{1}{2}\int_{\mathbb{R}^3} f_0 M_\delta^2(0) f_0 dv + \int_0^t \Big(Q(f,f)(\tau), M_\delta^2(\tau) f(\tau)\Big)_{L^2} d\tau.\end{aligned} \tag{3.6}$$

holds for any $t \in]0,T_0]$. The proof will be given at the end of this section.

On the other hand, it follows from (3.1) and (3.2) that

$$\| (\log\Lambda)^{\frac{m+1}{2}} M_\delta f\|_{L^2}^2 \leq C_f \left\{ (-Q(f,f), M_\delta^2 f)_{L^2} + \|M_\delta f\|_{L^2}^2 \right\}. \tag{3.7}$$

Since

$$(\partial_t M_\delta)(t,\xi) = N \log <\xi> M_\delta(t,\xi),$$

we obtain

$$\left|\int_0^t \int_{\mathbb{R}^3} f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big) f(\tau) dv d\tau\right| \leq 2N \int_0^t \| (\log\Lambda)^{\frac{1}{2}} (M_\delta f)(\tau)\|_{L^2}^2 d\tau.$$

This, together with (3.6) and (3.7), implies

$$\begin{aligned}\|(M_\delta f)(t)\|_{L^2}^2 + \tfrac{1}{2C_f}\textstyle\int_0^t \| (\log\Lambda)^{\frac{m+1}{2}} (M_\delta f)(\tau)\|_{L^2}^2 d\tau \leq \\ \|M_\delta(0) f_0\|_{L^2}^2 + 2N \textstyle\int_0^t \| (\log\Lambda)^{\frac{1}{2}} (M_\delta f)(\tau)\|_{L^2}^2 d\tau + \int_0^t \|(M_\delta f)(\tau)\|_{L^2}^2 d\tau.\end{aligned}$$

For $m>0$, by interpolation inequality implies that for any $\varepsilon>0$,

$$\begin{aligned}\|(M_\delta f)(t)\|_{L^2}^2 + \left(\frac{1}{2C_f} - \varepsilon\right)\int_0^t \| (\log\Lambda)^{\frac{m+1}{2}} (M_\delta f)(\tau)\|_{L^2}^2 d\tau \\ \leq \|M_\delta(0) f_0\|_{L^2}^2 + C_{\varepsilon,N}\int_0^t \|(M_\delta f)(\tau)\|_{L^2}^2 d\tau.\end{aligned}$$

By choosing $\varepsilon = \frac{1}{4C_f}$, there exists $C_{f,N}>0$ depending only on C_f, N, T_0 and being independent of $\delta \in]0,1[$, such that for any $t \in]0,T_0]$,

$$\|M_\delta(t) f(t)\|_{L^2}^2 \leq \|M_\delta(0) f_0\|_{L^2}^2 + C_{f,N}\int_0^t \|M_\delta(\tau) f(\tau)\|_{L^2}^2 d\tau.$$

Then Gronwall inequality yields

$$\|(M_\delta f)(t)\|_{L^2}^2 \leq e^{C_{f,N} t}\|M_\delta(0) f_0\|_{L^2}^2.$$

Since $\|M_\delta(t) f(t)\|_{L^2}^2 = \|(1-\delta\triangle)^{-N_0} f(t)\|_{H^{Nt-4}(\mathbb{R}^3)}^2$, and

$$\|M_\delta(0) f_0\|_{L^2}^2 = \|(1-\delta\triangle)^{-N_0} f_0\|_{H^{-4}(\mathbb{R}^3)}^2 \leq \|f_0\|_{H^{-4}(\mathbb{R}^3)}^2 \leq C_0\|f_0\|_{L^1}^2,$$

we obtain

$$\|(1-\delta\triangle)^{-N_0} f(t)\|_{H^{Nt-4}(\mathbb{R}^3)}^2 \leq \tilde{C} e^{C_{f,N} t}\|f_0\|_{L^1}^2,$$

where the constant $\tilde{C} > 0$ is independent of δ. Finally, for any given $t > 0$, since N can be arbitrarily large, by letting $\delta \to 0$, we have

$$f(t) \in H^{+\infty}(\mathbb{R}^3).$$

And this completes the proof of Theorem 1.1.

The rest of this section is devoted to prove (3.5) and (3.6). In Definition 3.1, taking $\varphi(t,v) = \psi(v) \in C_0^\infty(\mathbb{R}^3)$ we get

$$\int_{\mathbb{R}^3} f(t)\psi dv - \int_{\mathbb{R}^3} f(s)\psi dv = \int_s^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau))\psi dv\,,\ \ 0 \le s \le t \le T_0\,.$$

We can put $\psi = M_\delta^2 f(t), M_\delta^2 f(s)$ because they belong to $L^\infty([0,T]; W^{2,\infty}(\mathbb{R}^3))$. Taking the sum, we obtain

$$\int_{\mathbb{R}^3} f(t) M_\delta^2 f(t) dv - \int_{\mathbb{R}^3} f(s) M_\delta^2 f(s) dv = \int_{\mathbb{R}^3} f(t)\left(M_\delta^2(t) - M_\delta^2(s)\right) f(s) dv$$

$$+ \int_s^t d\tau \int_{\mathbb{R}^3} Q(f(\tau), f(\tau))\left(M_\delta^2 f(t) + M_\delta^2 f(s)\right) dv\,.$$

Since the integrand of the first term on the right is estimated by $|t-s|C_\delta'||f_0||_{L^1} f(t)$ and since the collision integral term is bounded by $C_\delta\prime\prime ||f_0||_{L^1}||f||^2_{L^1_2}$, we obtain (3.5), namely $M_\delta f \in C([0,T_0]; L^2(\mathbb{R}^3))$. In Definition 3.1 , we may write the term

$$\int_0^t d\tau \int_{\mathbb{R}^3} f(\tau,v)\partial_\tau \varphi(\tau,v) dv = \lim_{h\to 0} \int_0^t d\tau \int_{\mathbb{R}^3} (f(\tau,v) + f(\tau+h,v)) \frac{\varphi(\tau+h,v) - \varphi(\tau,v)}{2h} dv$$

for $\varphi(t,v) \in C^1(\mathbb{R}^+; C_0^\infty(\mathbb{R}^3))$, by noting $f \in C(\mathbb{R}^+; \mathcal{D}')$. Putting into the right hand side

$$\varphi(t) \equiv M_\delta^2(t) f(t)$$

we see that the right hand side equals

$$\lim_{h\to 0}\left\{ \int_0^t d\tau \int_{\mathbb{R}^3} \frac{(M_\delta f)^2(\tau+h) - (M_\delta f)^2(\tau)}{2h} dv \right.$$

$$\left. + \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau) f(\tau+h) \frac{(M_\delta)^2(\tau+h) - (M_\delta)^2(\tau)}{2h} dv \right\}.$$

It follows from (3.5) that

$$\lim_{h\to 0} \int_0^t d\tau \int_{\mathbb{R}^3} \frac{(M_\delta f)^2(\tau+h) - (M_\delta f)^2(\tau)}{2h} dv$$

$$= \lim_{h\to 0} \frac{1}{2h}\left\{ \int_t^{t+h} d\tau - \int_0^h d\tau \right\} \int_{\mathbb{R}^3} (M_\delta f)^2(\tau) dv = \frac{1}{2}\int_{\mathbb{R}^3} (M_\delta f)^2(t) dv - \frac{1}{2}\int_{\mathbb{R}^3} (M_\delta f)^2(0) dv.$$

Hence we obtain (3.6) because the Lebesgue convergence theorem shows

$$\lim_{h\to 0} \int_0^t d\tau \int_{\mathbb{R}^3} f(\tau) \frac{(M_\delta)^2(\tau+h) - (M_\delta)^2(\tau)}{2h} f(\tau+h) dv = \frac{1}{2}\int_0^t \int_{\mathbb{R}^3} f(\tau)\Big(\partial_t M_\delta^2(\tau)\Big) f(\tau) dv d\tau.$$

4. Gevrey regularity for linear Cauchy problem

In this section, we will consider the Gevrey class property of the solutions to the Boltzmann equation for potentials satisfying the inverse power laws. The following analysis only applies to the linearized problem and the nonlinear problem will be pursued in the future. Consider the Cauchy problem for the linearized Boltzmann equation

$$\frac{\partial g}{\partial t}=Lg=Q(\mu,g)+Q(g,\mu),\ \ v\in\mathbb{R}^3,\ \ t>0\,;\qquad g|_{t=0}=g_0, \tag{4.1}$$

where μ is the normalized Maxwellian distribution given in the introduction. The definition of the weak solutions is similar to that in Definition 3.1.

Definition 4.1. *For an initial datum $g_0(v)\in L^1_2(\mathbb{R}^3)$, $g(t,v)$ is called a weak solution of the Cauchy problem (4.1) if it satisfies:*

$$g(t,v)\in C(\mathbb{R}^+;\mathcal{D}'(\mathbb{R}^3))\cap L^1([0,T_0];L^1_2(\mathbb{R}^3))\cap L^\infty([0,T_0];L^1(\mathbb{R}^3));\quad g(0,v)=g_0;$$
$$\int_{\mathbb{R}^3}g(t,v)\varphi(t,v)dv-\int_{\mathbb{R}^3}g_0(v)\varphi(0,v)dv-\int_0^t d\tau\int_{\mathbb{R}^3}g(\tau,v)\partial_\tau\varphi(\tau,v)dv$$
$$=\int_0^t d\tau\int_{\mathbb{R}^3}L(g)(\tau,v)\varphi(\tau,v)dv,$$

for any test function $\varphi(t,v)\in C^1(\mathbb{R}^+;C_0^\infty(\mathbb{R}^3))$. The right hand side of the last integral above is defined as the one in Definition 3.1 and it makes a sense for any $\varphi\in L^\infty([0,T_0];W^{2,\infty}(\mathbb{R}^3))$.

Notice that in the linear case, the nonnegativity $g\geq 0$ cannot be assumed, so that the mass-energy conservation law, though it holds, does not implies $g(t,\cdot)\in L^1_2$.

From now on, we are going to show that the weak solution $g(t,\cdot)$ is $\in G^{\frac{1}{\alpha}}(\mathbb{R}^3)$ for $0<t\leq T_0$. The existence of weak solutions in the above class will be discussed at the end of this section.

Under the assumption (1.5) on the collision cross-section, the following sub-elliptic estimate is known, cf. [1]:

$$\|\Lambda^\alpha f\|^2_{L^2}\leq C_h\left\{(-Q(h,f),f)_{L^2}+\|f\|^2_{L^2}\right\}, \tag{4.2}$$

for any $f\in H^2$ and $h\geq 0, h\in L^1_1\cap L\log L$. Here, the constant C_h depends only on $\|h\|_{L^1_1}$ and $\|h\|_{L\log L}$. For $0<\delta<1$, set

$$G_\delta(t,\xi)=\frac{e^{t\langle|\xi|\rangle^\alpha}}{1+\delta e^{t\langle|\xi|\rangle^\alpha}}.$$

Then if $g\in L^1([0,T_0];L^1_2(\mathbb{R}^3))$, we have

$$G_\delta(t,D_v)\langle|D_v|\rangle^{-4}g\in L^1([0,T_0];H^2(\mathbb{R}^3))\,;\quad G^2_\delta(t,D_v)\langle|D_v|\rangle^{-8}g\in L^1([0,T_0];W^{2,\infty}(\mathbb{R}^3)),$$

and

$$\begin{aligned}\|\Lambda^\alpha G_\delta(t,D_v)\langle|D|\rangle^{-4}g\|^2_{L^2}\ \ &\leq\ \ C_\mu\Big\{(-Q(\mu,G_\delta\langle|D|\rangle^{-4}g),G_\delta\langle|D|\rangle^{-4}g)_{L^2}\\&\qquad+\|G_\delta(t,D_v)\langle|D|\rangle^{-4}g\|^2_{L^2}\Big\},\end{aligned} \tag{4.3}$$

where the constant C_μ is independent on δ.

As in the previous section, the following lemma gives the estimate on the commutator of the pseudo-differtial operator $G_\delta(t,D_v)\langle|D_v|\rangle^{-4}$ and the collision operator $Q(\mu,\cdot)$.

Lemma 4.1. *For the g and notations given above, we have*

$$\begin{aligned}&|(Q(\mu,G_\delta\langle|D|\rangle^{-4}g),G_\delta\langle|D|\rangle^{-4}g)_{L^2}-(Q(\mu,g),G^2_\delta\langle|D|\rangle^{-8}g)_{L^2}|\\&\leq C_\mu\|G_\delta\langle|D|\rangle^{-4}g\|_{L^2}\|\Lambda^\alpha G_\delta\langle|D|\rangle^{-4}g\|_{L^2},\end{aligned} \tag{4.4}$$

and

$$|(Q(g,\mu),G^2_\delta\langle|D|\rangle^{-8}g)_{L^2}|\leq C_\mu\|g\|_{L^1_2}\|G_\delta\langle|D|\rangle^{-4}g\|_{L^2}, \tag{4.5}$$

where C_μ is independent of $0<\delta<1$.

Proof. For (4.4), similar to the proof of Lemma 3.1, we choose $G_\delta(t,D_v)\left\langle |D|\right\rangle^{-4} g\in H^2(\mathbb{R}^3)$ as the test function. Without loss of generality and simplicity of notations, we drop the regularizied operator $\left\langle |D|\right\rangle^{-4}$ in the following calculation because it does not create extra difficulty. In fact, the main issue to estimate following term as (3.3),

$$\Big|\int b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\Big\{Re\,\hat\mu(\xi^-)\hat g(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)\big[G_\delta(t,\xi)-G_\delta(t,\xi^+)\big]\Big\}d\sigma d\xi\Big|.$$

Notice that the weight $G_\delta(t,\xi)$ is an exponential function so that an estimate like (3.4) fails. Instead, we will show the following estimate

$$|G_\delta(t,\xi^+)-G_\delta(t,\xi)|\le C\sin^2\frac{\theta}{2}\left\langle\xi\right\rangle^\alpha G_\delta(t,\xi^-)G_\delta(t,\xi^+),\tag{4.6}$$

where the constant $C>0$ depends only on α and T_0. For this, set

$$\tilde G_\delta(s)=\frac{s}{1+\delta s}.$$

Note that $\frac{d}{ds}\tilde G_\delta(s)>0$ and

$$G_\delta(t,\xi)=\tilde G_\delta\left(e^{t(1+|\xi|^2)^{\alpha/2}}\right).$$

By recalling $|\xi|^2=|\xi^+|^2+|\xi^-|^2$ and $|\xi^-|^2=|\xi|^2\sin^2\frac{\theta}{2}$, we have

$$\begin{aligned}&|G_\delta(t,\xi^+)-G_\delta(t,\xi)|=\Big|\int_0^1\frac{\exp t(1+|\xi|^2+\tau(|\xi^+|^2-|\xi|^2))^{\alpha/2}}{\big(1+\delta\exp t(1+|\xi|^2+\tau(|\xi^+|^2-|\xi|^2))^{\alpha/2}\big)^2}\\&\times\frac{t\alpha}{2}(1+|\xi|^2+\tau(|\xi^+|^2-|\xi|^2))^{\alpha/2-1}d\tau\Big|\,|\xi^-|^2\\&\le CG_\delta(t,\xi)(1+|\xi|^2)^{\alpha/2}\sin^2\frac{\theta}{2},\end{aligned}$$

where we have used $\frac12|\xi|^2\le|\xi|^2+\tau(|\xi^+|^2-|\xi|^2)\le|\xi|^2$. Notice that for $0<\alpha<1, 0<\delta<1$, and for any $a,b\ge 0$, we have

$$(1+a+b)^\alpha\le(1+a)^\alpha+(1+b)^\alpha\,,\quad(1+\delta e^a)(1+\delta e^b)\le 3(1+\delta e^{a+b}).$$

Then

$$\tilde G_\delta\left(e^{t(1+|\xi^+|^2+|\xi^-|^2)^{\alpha/2}}\right)\le\tilde G_\delta\left(e^{t(1+|\xi^+|^2)^{\alpha/2}+t(1+|\xi^-|^2)^{\alpha/2}}\right)\le 3G_\delta(t,\xi^+)G_\delta(t,\xi^-).$$

Hence

$$\begin{aligned}&\Big|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b\big\{Re\,\hat\mu(\xi^-)\hat g(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)\big[G_\delta(t,\xi)-G_\delta(t,\xi^+)\big]\big\}d\sigma d\xi\Big|\\ \le\;& C\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}b\,\sin^2\frac{\theta}{2}|G_\delta(t,\xi^-)\hat\mu(\xi^-)|G_\delta(t,\xi^+)|\hat g(\xi^+)|\left\langle\xi\right\rangle^\alpha G_\delta(t,\xi)|\hat g(\xi)|d\sigma d\xi\\ \le\;& C\|G_\delta\mu\|_{L^1}\|G_\delta g\|_{L^2}\|\Lambda^\alpha G_\delta g\|_{L^2},\end{aligned}$$

which gives (4.6) and then (4.4).

We now turn to prove (4.5). By using Bobylev identity, and $\hat\mu(\xi) = \hat\mu(\xi^+)\hat\mu(\xi^-)$, $\hat\mu(0) = 1$, we have

$$
\begin{aligned}
&|(Q(g,\mu), G_\delta^2 g)_{L^2}| = \left|\int b\left(\hat g(\xi^-)\hat\mu(\xi^+) - \hat g(0)\hat\mu(\xi)\right) G_\delta^2(t,\xi)\bar{\hat g}(\xi) d\sigma d\xi\right| \\
=& \left|\int b\left(\hat g(\xi^-) - \hat g(0)\hat\mu(\xi^-)\right) G_\delta(t,\xi)\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\sigma d\xi\right| \\
\le& \left|\int b\,\hat g(0)\left(\hat\mu(\xi^-) - \hat\mu(0)\right) G_\delta(t,\xi)\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\sigma d\xi\right| \\
+& \left|\int b\left(\hat g(\xi^-) - \hat g(0)\right) G_\delta(t,\xi)\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\sigma d\xi\right|.
\end{aligned}
$$

For the first term in the last inequality, since

$$
|\hat\mu(\xi^-) - \hat\mu(0)| \le |\xi^-|^2 \le |\xi|^2 \sin^2\frac{\theta}{2},
$$

we have

$$
\begin{aligned}
&\left|\int b\,\hat g(0)\left(\hat\mu(\xi^-) - \hat\mu(0)\right) G_\delta(t,\xi)\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\sigma d\xi\right| \\
\le& \|g\|_{L^1}\left|\int b(\cos\theta)\sin^2\frac{\theta}{2} G_\delta(t,\xi)|\xi|^2\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\theta d\xi\right| \\
\le& C_{T_0}\|g\|_{L^1}\|G_\delta g\|_{L^2},
\end{aligned}
$$

where $C_{T_0} = 4\|G_\delta(2T_0, D)|D|^2\mu\|_{L^2}$. While for the second term, when $0 < \alpha < 1/2$, the estimate

$$
|\hat g(\xi^-) - \hat g(0)| \le \|\bigtriangledown \hat g\|_{L^\infty}|\xi^-| \le \|g\|_{L^1_1}|\xi|\sin\frac{\theta}{2},
$$

gives

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\,\left(\hat g(\xi^-) - \hat g(0)\right) G_\delta(t,\xi)\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\sigma d\xi\right| \\
\le& \|g\|_{L^1_1}\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\left|\sin\frac{\theta}{2}\right| G_\delta(t,\xi)|\xi|\hat\mu(\xi^+) G_\delta(t,\xi)\bar{\hat g}(\xi) d\theta d\xi\right| \\
\le& C_{T_0}\|g\|_{L^1_1}\|G_\delta g\|_{L^2}.
\end{aligned}
$$

On the other hand, when $1/2 \le \alpha < 1$, the above simple calculation does not work. Instead, we need to use the symmetry in the integral according to the geometric structure of

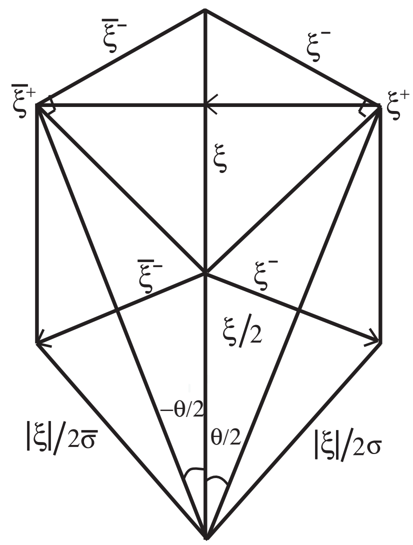
$$
\xi^+ = \frac{\xi + |\xi|\sigma}{2}, \quad \xi^- = \frac{\xi - |\xi|\sigma}{2}, \quad \cos\theta = |\xi|^{-1}\langle\xi,\ \sigma\rangle.
$$

For a fixed $\xi \ne 0$, denote the unit vector $\sigma = R_\theta\left(\frac{\xi}{|\xi|}\right)$ as a rotation of the unit vector $\frac{\xi}{|\xi|}$ by an angle θ. Moreover, denote $\bar\sigma = R_{-\theta}\left(\frac{\xi}{|\xi|}\right)$ and

$$
\bar{\xi}^+ = \frac{\xi + |\xi|\bar\sigma}{2}, \quad \bar{\xi}^- = \frac{\xi - |\xi|\bar\sigma}{2}.
$$

Then we have, cf. Figure 1,

$$
|\xi^+| = |\bar{\xi}^+|, \quad |\xi^-| = |\bar{\xi}^-|, \quad |\xi|^{-1}\langle\xi,\ \bar\sigma\rangle = \cos\theta.
$$



With these notations, the integral can be estimated as follows,

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\big(\hat g(\xi^-)-\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi\\
=&\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\Big(\frac{\xi}{|\xi|}\cdot\bar\sigma\Big)\big(\hat g(\bar\xi^-)-\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\bar\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\bar\sigma d\xi\\
=&\frac12\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\big(\hat g(\xi^-)+\hat g(\bar\xi^-)-2\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi.
\end{aligned}
$$

Here we have used the fact that $d\bar\sigma = d\sigma$ and $\mu(\xi^+) = \mu(\bar\xi^+)$. Notice that ξ^- and $\bar\xi^-$ are symmetric with respect to ξ so that we can denote them by

$$\xi^- = \vec a+\vec b,\ \ \bar\xi^- = \vec a-\vec b,$$

with

$$|\vec a| = \sin\frac{\theta}{2}|\xi^-| = \sin^2\frac{\theta}{2}|\xi|,\quad |\vec b| = \sin\frac{\theta}{2}|\xi^+| = \sin\frac{\theta}{2}\cos\frac{\theta}{2}|\xi|.$$

Thus,

$$
\begin{aligned}
&|\hat g(\xi^-)+\hat g(\bar\xi^-)-2\hat g(0)| = |\hat g(\vec a+\vec b)-2\hat g(\vec a)+\hat g(\vec a-\vec b)+2(\hat g(\vec a)-\hat g(0))|\\
&\le \|g\|_{L^1_2}|\vec b|^2+2\|g\|_{L^1_1}|\vec a|.
\end{aligned}
$$

Finally, for $1/2\le\alpha<1$, we have

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b\ \big(\hat g(\xi^-)-\hat g(0)\big)G_\delta(t,\xi)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\sigma d\xi\right|\\
\le\ & (\|g\|_{L^1_1}+\|g\|_{L^1_2})\left|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\theta)\right|\sin^2\frac{\theta}{2}\left|G_\delta(t,\xi)(|\xi|+|\xi|^2)\hat\mu(\xi^+)G_\delta(t,\xi)\bar{\hat g}(\xi)d\theta d\xi\right|\\
\le\ & C_{T_0}(\|g\|_{L^1_1}+\|g\|_{L^1_2})\|G_\delta g\|_{L^2}.
\end{aligned}
$$

Therefore, we have obtained (4.5) and then completes the proof of the lemma. □

We are now ready to prove the second main result in this paper.

Proof of Theorem 1.2. By the same argument as in Section 3, we see that $G_\delta^2(t,D_v)\langle|D_v|\rangle^{-8}g\in L^\infty([0,T_0];W^{2,+\infty}(\mathbb{R}^3))$ whoes norm is estimated by $C_\delta\sup_{[0,T_0]}\|g(t)\|_{L^1}$, and moreover

$$G_\delta(t,D_v)\langle|D_v|\rangle^{-4}g\in C([0,T_0];L^2(\mathbb{R}^3)),$$

by noting $g\in L^1([0,T_0],L^1_2)$. Hence, setting

$$\varphi(t)=G_\delta^2(t,D_v)\langle|D_v|\rangle^{-8}g(t,v)$$

in the last equation of Definition 4.1 we obtain, by means of the same argument as for (3.6),

$$\begin{aligned}(4.7)\qquad &\tfrac12\int_{\mathbb{R}^3}|G_\delta(t)\langle|D|\rangle^{-4}g(t)|^2dv-\tfrac12\int_{\mathbb{R}^3}|G_\delta(0)\langle|D|\rangle^{-4}g_0|^2dv\\ &-\tfrac12\int_0^t\int_{\mathbb{R}^3}g(\tau)\big(\partial_tG_\delta^2(\tau)\big)\langle|D|\rangle^{-8}g(\tau)dvd\tau=\int_0^t\big(Lg(\tau),G_\delta^2(\tau)\langle|D|\rangle^{-8}g(\tau)\big)_{L^2}d\tau,\end{aligned}$$

for any $t\in[0,T_0]$. On the other hand, it follows from (4.3), (4.4) and (4.5) that

$$(4.8)\qquad \|\Lambda^\alpha G_\delta\langle|D|\rangle^{-4}g\|^2_{L^2}\le C_\mu\left\{(-Lg,G_\delta^2\langle|D|\rangle^{-8}g)_{L^2}+\|G_\delta\langle|D|\rangle^{-4}g\|^2_{L^2}+\|g\|^2_{L^1_2}\right\}.$$

Combining (4.7) and (4.8) implies

$$\begin{aligned}&\|G_\delta(t)\langle|D|\rangle^{-4}g(t)\|^2_{L^2}+\frac{1}{2C_\mu}\int_0^t\|\Lambda^\alpha G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|^2_{L^2}d\tau\le\\ &\|G_\delta(0)\langle|D|\rangle^{-4}g_0\|^2_{L^2}+\left|\int_0^t\int_{\mathbb{R}^3}g(\tau)\big(\partial_tG_\delta^2(\tau)\big)\langle|D|\rangle^{-8}g(\tau)dvd\tau\right|\\ &+\int_0^t\|G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|^2_{L^2}d\tau+\int_0^t\|g(\tau)\|^2_{L^1_2}d\tau.\end{aligned}$$

Since

$$|\partial_tG_\delta(t,\xi)|\le G_\delta(t,\xi)<\xi>^\alpha,$$

we have

$$\left|\int_0^t\int_{\mathbb{R}^3}g(\tau)\big(\partial_tG_\delta^2(\tau)\big)\langle|D|\rangle^{-8}g(\tau)dvd\tau\right|\le2\int_0^t\|\Lambda^\alpha G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|_{L^2}\|G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|_{L^2}d\tau,$$

and

$$\|G_\delta(0)\langle|D|\rangle^{-4}g_0\|^2_{L^2}\le\|\langle|D|\rangle^{-4}g_0\|^2_{L^2}\le C\|g_0\|^2_{L^1}.$$

Thus, for any $\varepsilon>0$, we have

$$\begin{aligned}&\|G_\delta\langle|D|\rangle^{-4}g(t)\|^2_{L^2}+\Big(\frac{1}{2C_\mu}-\varepsilon\Big)\int_0^t\|\Lambda^\alpha G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|^2_{L^2}d\tau\\ &\le C_0\|g_0\|^2_{L^1}+C_\varepsilon\int_0^t\|G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|^2_{L^2}d\tau+C_1\int_0^t\|g(\tau)\|^2_{L^1_2}d\tau.\end{aligned}$$

By choosing $\varepsilon=\frac{1}{4C_\mu}$, the above inequality shows that there exists a constant $C_2>0$ independent of $\delta\in]0,1[$, such that for any $t\in]0,T_0]$

$$\|G_\delta(t)\langle|D|\rangle^{-4}g(t)\|^2_{L^2}\le C_0\|g_0\|^2_{L^1}+C_2\left\{\int_0^t\|G_\delta(\tau)\langle|D|\rangle^{-4}g(\tau)\|^2_{L^2}d\tau+\int_0^t\|g(\tau)\|^2_{L^1_2}d\tau\right\}.$$

Then the Gronwall inequality yields

$$\|G_\delta(t)\langle|D|\rangle^{-4}g(t)\|^2_{L^2}\le C_0e^{C_2t}\|g_0\|^2_{L^1}+C_2e^{C_2t}\int_0^te^{-C_1\tau}\|g(\tau)\|^2_{L^1_2}d\tau,$$

where the positive constants C_0,C_2 are independent of δ. Hence, by noticing

$$e^{-t\langle|\xi|\rangle^\alpha}\langle|\xi|\rangle^8\le C_\alpha t^{-\frac{8}{\alpha}},$$

for any fixed $0 < t \le T_0$ fixed, we have

$$e^{\frac{1}{2}t\langle |D_v|\rangle^\alpha} g(t,v) \in L^2.$$

And this completes the proof of Theorem 1.2.

The rest of this section is devoted to

Proof of Proposition 1.1. The construction of the weak solutions which meet the requirement of Theorem 1.2 is based on the following estimate of the operator L in the weighted space $L^2_\ell = L^2_\ell(\mathbb{R}^3)$.

Proposition 4.1. *Assume* $0 < \alpha < 1/2$ *and let* $\ell \in \mathbb{N}$. *Then, there exists a positive constant* C *such that for any* $g \in L^2_\ell$, *it holds that*

$$(Lg, g)_{L^2_\ell} \le C\|g\|_{L^2_\ell}(\|g\|_{L^1_1} + \|g\|_{L^2_\ell}).$$

Actually, this is just part of the coercivity estimate of L stated in Remark 4.1 below, but is enough for the present purpose. Notice that the restriction $0 < \alpha < 1/2$ comes from (4.18). The proof of this proposition will be given at the end of this section, and we now proceed to the proof of Proposition 1.1.

Let L_δ denote the operator L with a cutoff kernel

$$b_\delta(\cos\theta) = \chi(|\theta| > \delta) b(\cos\theta),$$

where χ is the usual characteristic function. Although this is not a bounded operator on $L^2 = L^2_0$, so is the operator $L_{R,\delta} = \mathcal{I}_R L_\delta \mathcal{I}_R$ where $\mathcal{I}_R$ is a smooth cutoff function

$$\mathcal{I}_R \in C_0^\infty(\mathbb{R}^3), \quad 0 \le \mathcal{I}_R(x) \le 1, \quad \mathcal{I}_R(v) = \begin{cases} 1 & (|v| \le R), \\ 0 & (|v| \ge R+1). \end{cases}$$

The proof is easy and hence omitted. Thus, $L_{R,\delta}$ is a generator of C_0 semi-group $e^{tL_{R,\delta}}$ on L^2. For any $g_0 \in L^2$, define

$$h_{R,\delta}(t) = e^{tL_{R,\delta}} g_0.$$

Since $L_{R,\delta}$ is a bounded operator, $h_{R,\delta}(t)$ enjoys the strong t-regularity

$$h_{R,\delta} \in C^\infty([0,\infty)]; L^2),$$

and gives a unique strong solution to the Cauchy problem

$$\frac{dh_{R,\delta}}{dt} = L_{R,\delta} h_{R,\delta} \quad \text{in} \quad L^2 \quad (t \ge 0), \qquad h_{R,\delta}(0) = g_0. \tag{4.9}$$

Moreover, the following holds.

Lemma 4.2. *For any* $\ell \in \mathbb{N}$ *and* $g_0 \in L^2_\ell$, $h_{R,\delta}(t)$ *is in* $C^\infty([0,\infty); L^2_\ell)$ *and satisfies* (4.9) *strongly in* L^2_ℓ.

Proof. Put $W_\ell(v) = (1+|v|)^\ell$. Since $W_\ell\, \mathcal{I}_R \in L^\infty(\mathbb{R}^3)$ and since $L_{R,\delta}$ is a bounded operator, the series

$$W_\ell h_{R,\delta} = W_\ell e^{t\mathcal{I}_R L^\delta \mathcal{I}_R} g_0 = W_\ell g_0 + \sum_{k=1}^\infty \frac{t^k}{k!} W_\ell \mathcal{I}_R (\mathcal{I}_R L^\delta \mathcal{I}_R)^k g_0$$

converges in the norm of L^2 as well as the series obtained by the term-by-term differentiation in t. This completes the proof of the lemma.

On the other hand, it is easy to see, in the course of its proof, that Proposition 4.1 applies also to $L_{R,\delta}$ with the same constant C which is independent of R, δ. This and Lemma 4.2 then yield

$$\begin{aligned} \frac{d}{dt}\|h_{R,\delta}\|^2_{L^2_\ell} &= 2(\frac{dh_{R,\delta}}{dt}, h_{R,\delta})_{L^2_\ell} = 2(L_{R,\delta}h_{R,\delta}, h_{R,\delta})_{L^2_\ell} \\ &\le C\|h_{R,\delta}\|_{L^2_\ell}(\|h_{R,\delta}\|_{L^1_1} + \|h_{R,\delta}\|_{L^2_\ell}). \end{aligned} \tag{4.10}$$

From now on, assume $\ell > 5/2$ so that $L^2_\ell \subset L^1_1$. Then (4.10) yields

$$\|h_{R,\delta}(t)\|_{L^2_\ell} \le e^{Ct}\|g_0\|_{L^2_\ell}, \tag{4.11}$$

for all $t \ge 0$.

To simplify the notation, put

$$X_\ell = L^\infty([0,T];\ L^2_\ell), \quad Y_\ell = L^2([0,T];\ L^2_\ell).$$

It follows from (4.11) that

$$h_{R,\delta} \in X_\ell \cap Y_\ell, \quad \|h_{R,\delta}\|_{X_\ell} \le e^{CT}\|g_0\|_{L^2_\ell}, \quad \|h_{R,\delta}\|_{Y_\ell} \le Te^{CT}\|g_0\|_{L^2_\ell},$$

for any $T > 0$.

Now, fix $\delta > 0$ and let $R \to \infty$. It is clear from the above that there exist a function h_δ and a subsequence $\{h_{R,\delta}\}$ (with abuse of notation) such that for any $T > 0$,

$$\begin{aligned} &h_\delta \in X_\ell \cap Y_\ell, \quad \|h_\delta\|_{X_\ell} \le e^{CT}\|g_0\|_{L^2_\ell}, \quad \|h_\delta\|_{Y_\ell} \le Te^{CT}\|g_0\|_{L^2_\ell}, \\ &h_{R,\delta} \to h_\delta \qquad \text{weakly* in } X_\ell \text{ and weakly in } Y_\ell. \end{aligned}$$

Consider the weak formulation of (4.9):

$$-(g_0, \phi(0))_{L^2} - \int_0^T (h_{R,\delta}(\tau), \phi_\tau(\tau))_{L^2} d\tau = \int_0^T (h_{R,\delta}(\tau), \mathcal{I}_R L^*_\delta \mathcal{I}_R \phi(t))_{L^2} d\tau, \tag{4.12}$$

where ϕ is any test function in $C^\infty_0([0,T] \times \mathbb{R}^3)$ subject to the final condition $\phi(T) = 0$ and L^* is the adjoint operator of L defined in the same sense as in Definition 3.1. Take the limit of (4.12) as $R \to \infty$. Clearly,

$$W_\ell^{-1}\mathcal{I}_R L^*_\delta \mathcal{I}_R \phi \to W_\ell^{-1} L^*_\delta \phi \quad \text{strongly in } Y_0,$$

so that we have

$$-(g_0, \phi(0))_{L^2} - \int_0^T (h_\delta(\tau), \phi_\tau(\tau))_{L^2} d\tau = \int_0^T (W_\ell h_\delta(\tau), W_\ell^{-1} L^*_\delta \phi(t))_{L^2} d\tau. \tag{4.13}$$

Now, let $\delta \to 0$. Then, there exist a function g and a subsequence $\{h_\delta\}$ (again with abuse of notation) such that

$$\begin{aligned} &g \in X_\ell \cap Y_\ell, \quad \|g\|_{X_\ell} \le e^{CT}\|g_0\|_{L^2_\ell}, \quad \|g\|_{Y_\ell} \le Te^{CT}\|g_0\|_{L^2_\ell}, \\ &h_\delta \to g \qquad \text{weakly* in } X_\ell \text{ and weakly in } Y_\ell. \end{aligned}$$

This g is a desired weak solution. To see this, note that

$$W_\ell^{-1} L^*_\delta \phi \to W_\ell^{-1} L^* \phi \quad \text{strongly in } Y_0,$$

and take the limit of (4.13), to deduce

$$-(g_0, \phi(0))_{L^2} - \int_0^T (g(\tau), \phi_\tau(\tau))_{L^2} d\tau = \int_0^T (W_\ell g(\tau), W_\ell^{-1} L^* \phi(\tau))_{L^2} d\tau. \tag{4.14}$$

Finally, set

$$\phi(t,v) = \int_t^T \eta(s) ds \psi(t,v), \quad \eta \in C^\infty([0,T]),\ \psi \in C^\infty_0([0,T] \times \mathbb{R}^N).$$

Then (4.14) is deduced to

$$\int_0^T \eta(t)\Big\{(g(t),\psi(t))_{L^2}-(g_0,\psi(0))_{L^2}-\int_0^t (g(\tau),\psi_\tau(\tau))_{L^2}d\tau \\ -\int_0^t (W_\ell g(\tau),W_\ell^{-1}L^*\psi(\tau))_{L^2}d\tau\Big\}dt=0,$$

which implies

$$\{\cdots\}=0 \quad \text{a.a.} \quad t.$$

This is just the last equation in Definition 4.1, which, then, gives for any test function of the form $\psi(t,v)=\bar\psi(v)\in C_0^\infty(\mathbb{R}^N)$,

$$(g(t),\bar\psi)_{L^2}=(g_0,\bar\psi)_{L^2}+\int_0^t w(\tau)d\tau,$$

where $w(t)=(W_\ell g(t),W_\ell^{-1}L^*\bar\psi)_{L^2}\in L^1(0,T)$. Thus, $g\in C(\mathbb{R}^+,\mathcal{D}')$. To summarize, g meets all the requirement stated in Definition 4.1. The proof of Proposition 1.1 is now complete, except for

Proof of Proposition 4.1. First, consider $L_1g=Q(\mu,g)$. Recall $W_\ell(v)=(1+|v|)^\ell$ and use the notation $W_\ell'=W_\ell(v')$, etc, to deduce

$$\begin{aligned}(L_1g,g)_{L^2_\ell}&=(L_1g,W_\ell^2g)_{L^2}=\int_{\mathbb{R}^6\times S^2} b(\mu_*'g'-\mu_*g)W_\ell^2g\,dvdv_*d\sigma\\ &=\int_{\mathbb{R}^6\times S^2} b\mu_*g\{(W_\ell')^2g'-W_\ell^2g\}dvdv_*d\sigma\\ &=\int_{\mathbb{R}^6\times S^2} b\mu_*g(W_\ell W_\ell'g'-W_\ell^2g)dvdv_*d\sigma+\int_{\mathbb{R}^6\times S^2} b\mu_*g(W_\ell'-W_\ell)W_\ell'g'dvdv_*d\sigma\\ &=A_1+A_2.\end{aligned}$$

We note that

$$\begin{aligned}A_1&=-\int_{\mathbb{R}^6\times S^2} b\mu_*(W_\ell^2g^2-W_\ell gW_\ell'g')dvdv_*d\sigma\\ &=-\frac{1}{2}\int_{\mathbb{R}^6\times S^2} b\mu_*(W_\ell g-W_\ell'g')^2dvdv_*d\sigma-\frac{1}{2}\int_{\mathbb{R}^6\times S^2} b\mu_*\{(W_\ell g)^2-(W_\ell'g')^2\}dvdv_*d\sigma\\ &=A_{11}+A_{12}.\end{aligned}$$

Clearly,

$$A_{11}\le 0 \tag{4.15}$$

while A_{22} can be computed just by the cancellation lemma in [1] with $\gamma=0$, yielding

$$A_{12}=-\frac{1}{2}\int_{\mathbb{R}^3}\mu_*\{S*_v(W_\ell g)^2\}dv_*,$$

where "$*_v$" is the convolution in v and S is the function introduced in [1], which is a constant function in our case, that is,

$$S=2\pi\int_0^{\pi/2}\sin\theta\Big[(\cos\theta)^{-3}-1\Big]b(\cos\theta)d\theta,$$

whence

$$|A_{12}|=\frac{S}{2}\int_{\mathbb{R}^3}\mu_*\Big\{\int_{\mathbb{R}^3}(W_\ell g)^2dv\Big\}dv_*=C\|W_\ell g\|^2=C\|g\|_\ell^2. \tag{4.16}$$

In order to evaluate A_2, first, we compute

$$|W'_\ell - W_\ell| = ||v'| - |v|| \sum_{k=0}^{\ell-1} W'_{\ell-1-k} W_k \le C|v' - v|(W'_{\ell-1} + W_{\ell-1}) \le C\theta(|v| + |v_*|)(W'_{\ell-1} + W_{\ell-1}), \tag{4.17}$$

where we used

$$|v' - v|^2 = \frac{1}{2}|v - v_*|^2(1 - \cos\theta),$$

which comes from (1.4). Therefore, recalling that μ is a Maxwellian, we have

$$\mu_*|W'_\ell - W_\ell| \le C\theta\mu_*^{1/2}(|v| + 1)(W'_{\ell-1} + W_{\ell-1}) \le C\theta\mu_*^{1/2}(W'_\ell + W_\ell) \le C\theta\mu_*^{1/4}W_\ell(\mu_*^{1/4}W_\ell^{-1}W'_\ell + 1) \le C\theta\mu_*^{1/4}W_\ell,$$

because

$$\frac{\mu^r(v_*)W_\ell(v')}{W_\ell(v)} \le C\frac{W_\ell(v')}{W_\ell(v_*)W_\ell(v)} \le C, \quad (r > 0),$$

by virtue of the conservation law $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$.

Now, we need to assume that $0 < \alpha < 1/2$ so that

$$\int_{S^2} \theta b(k \cdot \sigma)d\sigma < +\infty \qquad (k \in S^2), \tag{4.18}$$

can hold for the collision cross section B satisfying (1.5). Then, we have

$$|A_2| \le C\int_{\mathbb{R}^3} \mu_*^{1/4}\Big\{\int_{\mathbb{R}^3}(W_\ell g)(W'_\ell g')dv\Big\}dv_*.$$

By the Schwarz inequality,

$$|\int_{\mathbb{R}^3}(W_\ell g)(W'_\ell g')dv|^2 \le \int_{\mathbb{R}^3}|W_\ell g|^2dv\int_{\mathbb{R}^3}|(W_\ell g)(v')|^2dv = \|g\|^2_{L^2_\ell}\int_{\mathbb{R}^3}|(W_\ell g)(v')|^2dv,$$

while the change of variables

$$v \mapsto v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma \tag{4.19}$$

for fixed v_* and σ whose Jacobian is shown in [1] to satisfy

$$\Big|\frac{\partial v}{\partial v'}\Big| = \frac{4}{\cos^2(\theta/2)} \le 8, \qquad \theta \in [0, \frac{\pi}{2}], \tag{4.20}$$

yields

$$\int_{\mathbb{R}^3}|(W_\ell g)(v')|^2dv \le C\int_{\mathbb{R}^3}|(W_\ell g)(v')|^2dv' = C\|g\|^2_{L^2_\ell},$$

whence follows

$$|A_2| \le C\|g\|^2_\ell. \tag{4.21}$$

.

We shall now estimate $L_2 g = Q(g,\mu)$. Write

$$
\begin{aligned}
(L_2 g,g)_{L^2_\ell} &= (L_2 g, W_\ell^2 g)_{L^2} = \int_{\mathbb{R}^6\times S^2} b(g'_*\mu' - g_*\mu)W_\ell^2 g dv dv_* d\sigma \\
&= \int_{\mathbb{R}^6\times S^2} b g_*\mu\{(W'_\ell)^2 g' - W_\ell^2 g\} dv dv_* d\sigma \\
&= \int_{\mathbb{R}^6\times S^2} b g_*\mu W_\ell (W'_\ell g' - W_\ell g) dv dv_* d\sigma + \int_{\mathbb{R}^6\times S^2} b g_*\mu (W'_\ell - W_\ell) W'_\ell g' dv dv_* d\sigma \\
&= A_3 + A_4.
\end{aligned}
$$

In order to evaluate A_3, we again invoke Bobylev's idenity [3] with $\hat g(\xi) = \mathcal{F}(g)$, $\Phi(\xi) = \mathcal{F}(W_\ell g)$, $\Psi(\xi) = \mathcal{F}(W_\ell \mu)$, to deduce

$$
\begin{aligned}
A_3 &= \int_{\mathbb{R}^6\times S^2} b\Big(\frac{v-v_*}{|v-v_*|}\cdot\sigma\Big)\{g'_*(W'_\ell\mu') - g_*(W_\ell\mu)\} W_\ell g dv dv_* d\sigma \\
&= \int_{\mathbb{R}^3\times S^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\{\hat g(\xi^-)\Psi(\xi^+) - \hat g(0)\Psi(\xi)\}\overline{\Phi(\xi)} d\xi d\sigma,
\end{aligned}
$$

where

$$
\xi^+ = \frac{1}{2}(\xi + |\xi|\sigma), \qquad \xi^- = \frac{1}{2}(\xi - |\xi|\sigma).
$$

Split A_3 as follows.

$$
\begin{aligned}
A_3 &= \int_{\mathbb{R}^3\times S^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\hat g(\xi^-)\{\Psi(\xi^+) - \Psi(\xi)\}\overline{\Phi(\xi)} d\xi d\sigma \\
&\quad + \int_{\mathbb{R}^3\times S^2} b\Big(\frac{\xi}{|\xi|}\cdot\sigma\Big)\{\hat g(\xi^-) - \hat g(0)\}\Psi(\xi)\overline{\Phi(\xi)} d\xi d\sigma = A_{31} + A_{32}.
\end{aligned}
$$

Without loss of generality, we may take $W_\ell(v) = (1+|v|^2)^\ell$. Then, since $\hat\mu(\xi) = (2\pi)^{3/2}\mu(\xi)$ for the absolute Maxwellian (1.9), we have

$$
\Psi(\xi) = \mathcal{F}(W_\ell\mu) = (I - \Delta_\xi)^\ell \hat\mu(\xi) = P(\xi)\mu(\xi),
$$

where $P(\xi)$ is a polynomial in ξ of order 2ℓ. Noticing that

$$
|\xi^+ - \xi| = |\xi||\sin\frac{\theta}{2}| \qquad \frac{|\xi|}{\sqrt{2}} \le |\xi^+| \le |\xi|,
$$

with $(\xi\cdot\sigma)/|\xi| = \cos\theta$, $\theta\in[0,\pi/2]$, we get

$$
\begin{aligned}
|\Psi(\xi^+) - \Psi(\xi)| &\le |P(\xi^+)||\mu(\xi^+) - \mu(\xi)| + |P(\xi^+) - P(\xi)|\mu(\xi) \\
&\le |P(\xi^+)||\mu^{1/2}(\xi^+) - \mu^{1/2}(\xi)||\mu^{1/2}(\xi^+) + \mu^{1/2}(\xi)| + |P(\xi^+) - P(\xi)|\mu(\xi) \\
&\le C(1+|\xi|^{2\ell})\theta|\xi|\mu^{1/4}(\xi) \le C\theta\mu^{1/8}(\xi),
\end{aligned}
$$

which yields, together with (4.18),

$$
\begin{aligned}
|A_{31}| &\le C\int_{\mathbb{R}^3} |\hat g(\xi^-)|\mu^{1/8}(\xi)|\Phi(\xi)| d\xi \\
&\le C\|\hat g\|_{L^\infty_\xi}\|\mu^{1/8}(\xi)\|_{L^2_\xi}\|\Phi\|_{L^2_\xi} \le C\|g\|_{L^1}\|W_\ell g\|_{L^2}.
\end{aligned} \tag{4.22}
$$

On the other hand, recalling the estimate

$$
|\hat g(\xi^-) - \hat g(0)| \le |(\nabla_\xi \hat g)(\tilde\xi)||\xi^-| \le C|\xi||\theta|\|\nabla_\xi \hat g\|_{L^\infty_\xi} \le C|\xi||\theta|\||v|g\|_{L^1},
$$

we get

$$|A_{32}| \le C\||v|g\|_{L^1}\int_{\mathbb{R}^3}|\Psi(\xi)||\Phi(\xi)|d\xi \qquad (4.23)$$
$$\le C\||v|g\|_{L^1}\|W_\ell\mu\|_{L^2}\|W_\ell g\|_{L^2} \le C\|g\|_{L^1_1}\|g\|_{L^2_\ell}.$$

It remains to evaluate A_4. It follows from (4.17) that

$$\mu|W' - W| \le C\theta\mu^{1/2}(1+|v_*|))\{W'_{\ell-1}+1\}$$
$$\le C\theta\mu^{1/4}(v)\{W_\ell(v_*)+W_\ell(v')\}$$
$$\le C\theta\mu^{1/8}(v)W_\ell(v_*)\Big\{1+\frac{\mu^{1/8}(v)W_\ell(v')}{W_\ell(v_*)}\Big\} \le C\theta\mu^{1/8}(v)W_\ell(v_*),$$

the last inequality coming again from the conservation law $|v|^2+|v_*|^2 = |v'|^2+|v'_*|^2$. Consequently, proceeding just as in (4.21) for A_2, we obtain

$$|A_4| \le C\int \mu^{1/8}(v)\Big\{\int (W_*g_*)(Wg)'dv_*\Big\}dv \le C\|Wg\|^2, \qquad (4.24)$$

where, instead of (4.19), the change of variables

$$v_* \mapsto v' = \frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma \qquad (4.25)$$

was used for fixed v and σ, for which the Jacobian has the same estimate (4.20) because v' is symmetric with respect to v and v_* both in (4.19) and (4.25).

Putting together (4.15), (4.16), (4.21), (4.22), (4.23), and (4.24) completes the proof of Proposition 4.1.

Remark 4.1: A_{11} has the coercivity estimate,

$$-A_{11} \ge C_1\||D_v|^\alpha(W_\ell g)\|_{L^2} - C_2\|g\|^2_{L^2_\ell},$$

which comes from [1, §6] and leads to a generalized version of the sub-elliptic esimate (4.2) in the weighted space L^2_ℓ.

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