On the Minimum Size of a Contraction-Universal Tree

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Abstract. A tree T_{uni} is *m-universal* for the class of trees if for every tree T of size m, T can be obtained from T_{uni} by successive contractions of edges. We prove that a m-universal tree for the class of trees has at least $m\ln(m) + (\gamma - 1)m + O(1)$ edges where γ is the Euler's constant and we build such a tree with less than m^c edges for a fixed constant $c = 1.984...$

1 Introduction

What is the minimum size of an object in which every object of size m embeds? Issued from the category theory, questions of this kind appeared in graph theory. For instance, R. Rado [1] proved the existence of an "initial countable graph". Recently, Z. Füredi and P. Komjàth [2] studied a connected question.

We use here the following definition : given a sub-class C of graphs (trees, planar graphs, etc.), a graph G_{uni} is *m-universal* for C if for every graph G of size m in C, G is a minor of G_{uni}, i.e. it can be obtained from G_{uni} by successive contractions or deletions of edges.

Inspired by the Robertson and Seymour work [3] on graph minors, P. Duchet asked whether a polynomial bound in m could be found for the size of a m-universal tree for the class of trees. We give here a positive sub-quadratic answer.

From an applied point of view, such an object would possibly allows us to define a tree from the representation of its contraction.

The main results of this paper are the following theorems which give bounds for the minimum size of a m-universal tree for the class of trees :

Theorem 1. *A m-universal tree for the class of trees has at least* $m\ln(m) + (\gamma - 1)m + O(1)$ *edges where* γ *is the Euler's constant.*

Theorem 2. *There exists a m-universal tree* T_{uni} *for the class of trees with less than* m^c *edges for a fixed constant* $c = 1.984...$

Our proof follows a recursive construction where large trees are obtained by some amalgamation process involving simpler trees. With this method, the constant c could be reduced to 1.88... but it seems difficult to improve this value.

We conclude the paper with related open questions.

2 Terminology

Our graphs are undirected and simple (with neither loops nor multiple edges). We denote by $G(V,E)$ a graph (its vertex set is $V(G)$ and its edge set is $E(G)$ (a subset of the family of all the $V(G)$-subsets of cardinality 2)). Referring to C. Thomassen [4], we recall some basic definitions that are useful for our purpose:

We denote by P_n the path of size n.

If x is a vertex then $d(x)$, the *degree* of x, is the number of edges incident to x.

Let e be an edge of $E(G)$, the graph denoted by $G-e$ is the graph on the vertex set of G, whose edge set is the edge set of G without e. We call classically this operation *deletion*.

Let $e=\{a,b\}$ be an edge of $G(V,E)$, we name *contraction of G along e*, the graph denoted by $G/e=H(V',E')$, with $V'=(V/\{a,b\})\cup\{c\}$ where c is a new vertex and E' the edge set which contains all the edges of the sub-graph G_1 on V/e and all the edges of the form $\{c,x\}$ for $\{a,x\}$ or $\{b,x\}$ belonging to E.

We say that H is a *minor* of G if and only if we can obtain it from G by successively deleting and /or contracting edges, in an other way, we can define the set $M(G)$ of minors of G by the recursive formula :

$$M(G)=G\cup\left(\bigcup_{e\in E(G)}M(G/e)\right)\cup\left(\bigcup_{e\in E(G)}M(G-e)\right)$$

The notion of minor induces a partial order on graphs. We write $A\preceq B$ to mean "A is a minor of B".

For technical reasons, we prefer to use the size of a tree (edge number) rather than its order (vertex number).

Finally, let us recall that, a graph G_{uni} is *m-universal for a sub-class C* of graphs if for every element G of C with m edges, G is a minor of G_{uni}.

3 A Lower Bound

In this section, we prove that a m-universal tree T_{uni} for the trees has asymptotically at least $m\ln(m)$ edges. We use the fact that T_{uni} has to contain all spiders of size m as minors. A *spider S on a vertex w* is a tree such that $\forall v\in V(S)\setminus\{w\}, d(v)\le 2$. We denote the spider constituted by paths of lengths $1\le m_1\le ...\le m_k$ by $Sp(m_1,...,m_k)$ (Fig.1).

Definition 1. *Let T be a tree, we denote by ∂T the subtree of T with $V(\partial T)=V(T)\backslash A$, where A is the set of the leaves of T. Also, we denote by ∂^k the k-th iteration of ∂.*

Lemma 1. *$Sp(m_1,...,m_k)\preceq T$ involves that $\partial Sp(m_1,...,m_k)\preceq\partial T$. Moreover, if for all i, $m_i=1$ then $\partial Sp(m_1,...,m_k)$ is a vertex. Otherwise, put a the first value such that $m_a>1$, we have $\partial Sp(m_1,...,m_k)=Sp(m_a-1,...,m_k-1)$ excepted for $k=1$, in this last case we have $\partial Sp(m_1)=Sp(m_1-2)$.*

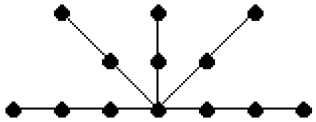


Fig.1. $Sp(2,2,2,3,3)$

Proof. This just follows from an observation. □

Lemma 2. *For every tree T, $Sp(m_1,...,m_k) \preceq T \Rightarrow T$ has at least k leaves.*

Proof. Trivial. □

Theorem 3. *A m-universal tree T_{uni} for the class of trees has at least $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor$ edges.*

Proof. A m-universal tree T_{uni} for the class of trees has to contain as minors all spiders of size m. So, for all p it contains as minors the spiders $Sp(p,...,p)$ where we have $\left\lfloor \frac{m}{p} \right\rfloor$ times the letter p. By the lemma 1, for all $p \leq \frac{m}{2}$, $Sp(1,...,1) \preceq \partial^{p-1}T_{uni}$ and if m is odd, $Sp(1) \preceq \partial^{\lfloor \frac{m}{2} \rfloor -1}T_{uni}$. Moreover, it is clear that the terminal edges of the $\partial^p T_{uni}$ constitute a partition of T_{uni}. By the lemma 2, this involves that T_{uni} has at least $\sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \left\lfloor \frac{m}{i} \right\rfloor$ edges if m is even and $1 + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \left\lfloor \frac{m}{i} \right\rfloor$ edges if m is odd. An easy calculation proves that these values are always equal to $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor$. □

Proof. (of the theorem 1) it follows from the usual estimate $\sum_{i=1}^{n} \frac{1}{i} \sim \ln(n) + \gamma + O\left(\frac{1}{n}\right)$ and the inequality $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor \geq 1 + \sum_{i=1,i\neq 2}^{m-1} \left(\frac{m}{i} - 1\right)$. □

What the above proof shows, in fact, is the following :

Corollary 1. *A minimum m-universal spider for the class of spiders has $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor$ edges.*

Proof. The spider $Sp\left(\left\lfloor \frac{m}{m} \right\rfloor, \left\lfloor \frac{m}{m-1} \right\rfloor, ..., \left\lfloor \frac{m}{2} \right\rfloor, \left\lceil \frac{m}{2} \right\rceil\right)$ is clearly a m-universal spider of size $\sum_{i=1,i\neq 2}^{m} \left\lfloor \frac{m}{i} \right\rfloor$ for the class of spiders, and by theorem 3 it is a minimum value. □

4 The Main Stem

In the sequel, we deal with *rooted graph*, i.e. graph G where we can distinguish a special vertex denoted by $r(G)$, called the *root*. Conventionally, any contracted graph G' of same rooted graph G will be rooted at the unique vertex which is the image of the root under the contraction mapping, we say in this case that the rooted graph G' is a *rooted contraction* of G. Note that, the contraction operator suffices to obtain all minor trees of a tree. So, we can now define the following new notion for sub-classes of rooted trees : a rooted tree T_{uni} is *strongly m-universal for a sub-classes C of rooted trees* if for every rooted tree T *in* C of size m, T is a rooted contraction of T_{uni}. The concept of root is introduced to avoid problems with graph isomorphisms that, otherwise would greatly impede our inductive proof.

For every edge e of a tree T, the forest $T \backslash e$ has two connected components. We call *e-branch*, denoted by B_e, the connected component of T' which does not contain $r(T)$, we define the root of B_e as $e \cap V(B_e)$.

A *main stem* of a rooted tree of size m is defined as a path P which is issued from the root and such that for all e-branches B_e with $e \notin E(C)$, we have $|E(B_e)| < \lfloor \frac{m}{2} \rfloor$ (Fig.2).

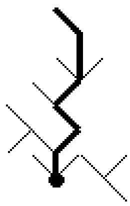


Fig.2. A main stem in bold

The following lemma suggests the procedure which will be used to find a subquadratic upper bound for universal trees. Roughly speaking, it endows every tree with some recursive structure constructed with the help of main stems.

Lemma 3. *Every rooted tree has a main stem.*

Proof. By induction on the size of the rooted tree. Let T be a rooted tree, if T has one or two edges, it is trivial. Otherwise let us consider the sub-graph $T \backslash r(T)$, which is a forest. We choose a connected component T_1 with maximum size and we denote by b_1 the unique vertex of T_1 which is adjacent to $r(T)$. Tree T_1, rooted in b_1, has, by the induction hypothesis, a main stem B. Then the path $(V(B) \cup \{r(T)\}, E(B) \cup \{\{r(T), b_1\}\})$ is a main stem of T. □

Remark 1. A tree may possess in general several main stems. Let us notice also that a main stem is not necessarily one of the longest paths which contain the root.

5 The Upper Bound

We need some new definitions. A *rooted brush* (Fig.3) is a rooted tree such that the vertices of degree greater than 2 are on a same path P issued from the root.

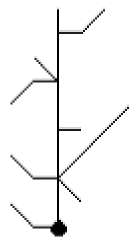


Fig.3. A rooted brush

A *rooted comb* X (Fig.4) is a rooted brush with $d\left(r\left(X\right)\right) \leq 2$ and $\forall v \in V\left(X\right)$, $d\left(v\right) \leq 3$.

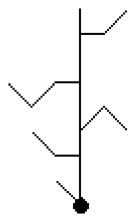


Fig.4. A rooted comb

The *length of a rooted comb* corresponds to the length of the longest path P issued from the root which contains all vertices of degree greater than 2.

To obtain an upper bound, we consider two building processes : the first one, a brushing M_B, maps rooted trees with a main stem into rooted brushes, the second one, a ramifying M_T, consists in obtaining a sequence of rooted trees, assuming that we have an increasing sequence of rooted combs. We note M_T^k the k-th element of the sequence. These building processes will possess the following fundamental property:

Property 1. Let (T, σ) a rooted tree with a main stem σ and $\left(X_n\right)_{n\in\mathbb{N}}$ a sequence of rooted combs :

$$\left(\forall T' \preceq T, M_B\left(T', \sigma\right) \preceq X_{|E(T')|}\right) \Rightarrow T \preceq M_T^{|E(T)|}\left(\left(X_n\right)_{n\in\mathbb{N}}\right).$$

Lemma 4. *If building processes verify the property 1 and if for all i, the rooted comb X_i is strongly i-universal for the class of rooted brushes then the rooted tree $M_T^m\left((X_n)_{n\in\mathbb{N}}\right)$ is strongly m-universal for the class of rooted trees.*

Proof. It is just an interpretation of the property. □

We now establish the existence of building processes which satisfy property 1.

Brushing M_B (Fig.5). Let T be a rooted tree with a main stem σ. We are going to associate a rooted brush B with it, denoted $M_B(T,\sigma)$ of the same size built from the same main stem σ with the following process: every e-branch B_e connected to the main stem by edge e is replaced by a path of length $|E(B_e)|$ connected by the same edge.

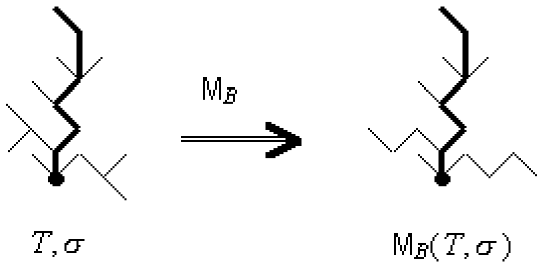


Fig.5.

Ramifying M_T^k**.** For the second building process we work in two steps :

First step. Given rooted trees $T_1, ..., T_k$ with disjoint vertex sets, we build another rooted tree T, denoted $[T_1, ..., T_k]$, in the following way :

$$V(T) = \bigcup_{i=1}^{k} V(T_i) \cup \{v_1, ..., v_{k+1}\},$$

$$E(T) = \bigcup_{i=1}^{k} E(T_i) \cup \{\{v_1, r(T_1)\}, ..., \{v_k, r(T_k)\}\} \cup \{\{v_1, v_2\}, ..., \{v_k, v_{k+1}\}\},$$

and $r(T) = v_1$.

If $T_i = \emptyset$, conventionally $\{v_i, r(T_i)\} = \emptyset$.

Prosaically, from a path $P_k = [v_1, ..., v_{k+1}]$ of size k and from k rooted trees $T_1, ..., T_k$, we build a rooted tree joining a branch T_i to the vertex v_i of P (Fig.6).

Second step. By convention, $P_{-1} = \emptyset$.

We are going to construct rooted trees T_k in the following way :

$T_{-1} = \emptyset$, $T_0 = X_0$, and $\forall i,\ 1 \le i \le k, T_i = \left[T_{\min(u_1, i-1)}, ..., T_{\min\left(u_{n_i}, i-1\right)}\right]$ if $X_i = \left[P_{u_1}, ..., P_{u_{n_i}}\right]$.

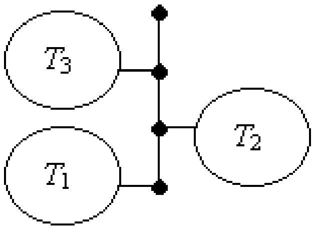


Fig.6. A rooted comb $[T_1, T_2, T_3]$

We can now define M_T^k :

$$M_T^k\left((X_n)_{n\in\mathbb{N}}\right) = T_k.$$

Lemma 5. *The building processes described above verify the property 1.*

Proof. First, note that $M_T\left((X_n)_{n\in\mathbb{N}}\right)$ is an increasing sequence. We prove the lemma by recurrence on the size m of T. When $m = 0$ or $m = 1$, this is trivial. We suppose the property is verified for T with size $m < m_0$. Let T be a rooted tree of size m_0 with a stem σ, we note $e_1, ..., e_k$ the edges of T issued from σ which do not belong to σ. To each e-branch of T with $e \in \{e_1, ..., e_k\}$ corresponds by M_B a e-branch (it is a path of same size) in $M_B(T, \sigma)$. So there exists k distinct e-branches $R_1, ..., R_k$ in X_{m_0} that we can respectively contract to obtain each e-branch with $e = e_1, ..., e_k$ in $M_B(T, \sigma)$. By recurrence hypothesis, we have for $1 \le i \le k, B_{e_i} \preceq M_T^{\left|E\left(B_{e_i}\right)\right|}\left((X_n)_{n\in\mathbb{N}}\right)$ and we have also $M_T^{\left|E\left(B_{e_i}\right)\right|}\left((X_n)_{n\in\mathbb{N}}\right) \preceq M_T^{|E(R_i)|}\left((X_n)_{n\in\mathbb{N}}\right)$. So each e-branch of T is a minor contraction of $M_T^{|E(R_i)|}\left((X_n)_{n\in\mathbb{N}}\right)$. By associativity of contraction map, we have $T \preceq M_T^{|E(T)|}\left((X_n)_{n\in\mathbb{N}}\right)$. □

In this phase, we determine a sequence of rooted combs $(X_i)_{i\in\mathbb{N}}$ such that the rooted combs X_i are strongly i-universal for the rooted brushes.

In order to achieve this result, we define F_p as the set of functions $f : \{1, ..., p\} \to \left\{1, ..., \left\lfloor\frac{p}{2}\right\rfloor\right\}$ satisfying the following property :

$$(\forall n \in \{1, ..., p\})\left(\forall i \le \left\lfloor\frac{n}{2}\right\rfloor\right)(\exists k \in \mathbb{N})\,(n - i + 1 \le k \le n \text{ and } f(k) \ge i)$$

Lemma 6. *F_p is not empty, it contains the following function φ_p, defined for $1 \le i \le p$ by :*

$$\varphi_p(i) = \min\left(2^{\upsilon_2(i)+1} - 1, \left\lfloor\frac{p}{2}\right\rfloor, i - 1\right)$$

where $\upsilon_2(k)$ is the 2-valuation of k (i.e. the greatest power of 2 dividing k).

Proof. The verification is obvious. □

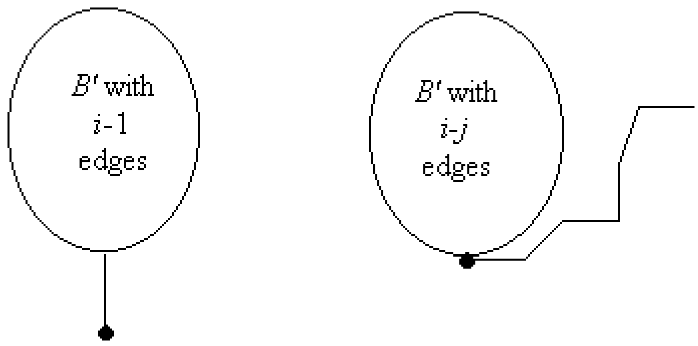
Lemma 7. *For every sequence $F = (f_1, f_2, ...)$ of functions such that $f_i \in F_i$ for $i \geq 1$ and $f_i(k) \leq f_{i+1}(k)$ for all $i \geq 1$ and $1 \leq k \leq i$, the rooted comb defined by $Comb_m^F = [Pf_1^m, ..., Pf_m^m]$ where Pf_i^m designs the path of size $f_m(m+1-i)-1$, for $1 \leq i \leq m$ is strongly m-universal for the rooted brushes.*

Proof. By induction on m : $Comb_1^F$ is strongly 1-universal for the rooted brushes.

Suppose that $Comb_i^F$ has all rooted brushes with $i-1$ edges as rooted contractions.

We consider two cases depending on the shape of a rooted brush B of size i :

case 1 case 2



Brushes of case 1 are clearly rooted contractions of the rooted comb $Comb_i^F$ ($B' \preceq Comb_{i-1}^F$, so $B \preceq \left[P_0, Pf_1^{i-1}, ..., Pf_{i-1}^{i-1}\right] \preceq Comb_i^F$). Let us study case 2 : B' is by induction hypothesis a rooted contraction of the rooted comb $Comb_{i-j}^F$, moreover $Comb_{i-j}^F \preceq \left[Pf_{j+1}^i, ..., Pf_i^i\right]$. Finally, by the property of f_i, there exists $1 \leq \alpha \leq j$, such that Pf_α^i has more than j edges. Linking these two points, we can conclude that the rooted brush B is always a rooted contraction of the rooted comb $Comb_i^F$. □

The rooted comb built as in lemma 7 will be said to be *associated to the sequence* F and denoted by $Comb_m^F$.

Theorem 4. *A minimum strongly m-universal rooted brush for the rooted brushes has $O(m \ln(m))$ edges.*

Proof. Proceeding as for theorem 1, we obtain, mutatis mutandis, that a m-universal brush for the brushes has at least $m\ln(m) + O(m)$ edges. This order of magnitude is precisely the size of the strongly m-universal rooted comb $Comb_m^F$ for the class of rooted brushes. □

We have this immediate corollary :

Corollary 2. *A minimum m-universal brush for the brushes has $O(m\ln(m))$ edges.*

By convention, we put $Comb_0^F = P_0$ (tree reduced in a vertex)

We define $Tree_m^F = M_T^m\left(\left(Comb_n^F\right)_{n\in\mathbb{N}}\right)$.

As before, we will say that the tree built in such a way is *recursively associated to the sequence F* and denoted by $Tree_m^F$.

Thus, we have :

Theorem 5. *The rooted tree $Tree_m^F$ is strongly m-universal for the class of rooted trees.*

We now analyze the size of $Tree_m^F$.

Proposition 1. *Let $F = (f_1, f_2, ...)$ be a sequence of functions such that $f_i \in F_i$ for $i \geq 1$. The size of a m-universal tree constructed from the sequence is given by the following recursive formula :*

$$u_{-1} = -1, u_0 = 0 \text{ and } u_k = 2k - 1 + \sum_{i=1}^{k} u_{f_k(i)-1}$$

Proof. It derives from the following observation :
m edges constitute the main stem, we have to add $m-1$ edges to link branches to the main stem and $\sum_{i=1}^{k} u_{f_k(i)-1}$ edges for the branches. □

Theorem 6. *There is a sequence of functions $G = (g_1, g_2, ...)$ such that $g_i \in F_i$ and $\left|E\left(Tree_m^G\right)\right| < (2m)^c$ where $c = 1.984...$ is the unique positive solution of the equation $\frac{1}{2^c} + \frac{1}{2^{2c}} + \frac{1}{2^{(c-1)}-1} - \frac{1}{2^c-1} = 1$.*

Proof. We take the following sequence of functions :
$g_m(i) = \min\left(2^{\upsilon_2(i)+1}, i\right)$ if $i < m$ and i even, $g_m(i) = 1$ if i odd and $g_m(m) = \left\lfloor\frac{m}{4}\right\rfloor$. It is clear that, if m is a power of 2, the comb $Comb_m^G$ is strongly m-universal for the brushes.

In fact, the function g_m takes the value $2^{\upsilon_2(i)+1}$ when i is not a power of 2, otherwise it is equal to i. Thanks to this remark and with $u_m < m + \sum_{i=1}^{m} u_{f_m(i)}$, (the sequence of sizes is increasing), we obtain $u_{2^n} < 2^n + 2^{n-1} + \sum_{i=2}^{n-1} 2^{n-i}u_{2^i} - \sum_{i=2}^{n-1} u_{2^i} + u_{2^{n-1}} + u_{2^{n-2}}$. Thus, in evaluating the sums and reorganizing the terms, we obtain :

$$u_{2^n} < \alpha_n + 2^{nc}\beta$$

with

$$\alpha_n = 2^{n-1} + 1 + 2^c + \frac{1}{2^c - 1} - \left(\frac{2^n}{2^{(c-1)} - 1} + 2^{n(c-1)}\right)$$

$$\beta = \frac{1}{2^c} + \frac{1}{2^{2c}} + \frac{1}{2^{(c-1)} - 1} - \frac{1}{2^c - 1}$$

Now $\alpha_n < 0$ when $m > 1$ and $\beta \leq 1$ by definition of c.
So $u_{2^n} < 2^{nc}$, hence $u_m < (2m)^c$. □

Remark 2. We observe that $c = \frac{\ln(x)}{\ln(2)}$, where x is the positive root of $X^4 - 5X^3 + 4X^2 + X - 2 = 0$.

Theorem 2 then follows since any rooted tree which is strongly m-universal for the rooted trees is also clearly m-universal for the class of trees.

6 Conclusion and Related Questions

When using the sequence $\Phi = (\varphi_1, \varphi_2, ...)$ of lemma 7, the induction step leads to involved expressions that do not allow us to find the asymptotic behavior of the corresponding term u_m. A computer simulation gives that such a m-universal tree for the trees has less than $m^{1.88}$ edges. In any case, the constructive approach we proposed here, seems to be hopeless to reach the asymptotic best size of a m-universal tree for the trees.

Conjecture 1. The minimal size of a m-universal tree for the trees is $m^{1+o(1)}$.

As a possible way to prove such a conjecture, it would be interesting to obtain an explicit effective coding of a tree of size m using a list of contracted edges taken in a m-universal tree for the trees.

A variant of our problem consists in determining a minimum tree which contains as a subtree every tree of size m. This is closely related to a well known still open conjecture due to Erdös and Sös (see [5]).

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