CONSTRUCTIBLE REPRESENTATIONS AND BASIC SETS IN TYPE B

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ABSTRACT. We study the parametrizations of simple modules provided by the theory of basic sets for all finite Weyl groups. In the case of type B_n, we show the existence of basic sets for the matrices of constructible representations. Then we study bijections between the various basic sets and show that they are controlled by the matrices of the constructible representations.

1. INTRODUCTION

One of the main problems in the modular representation theory of Hecke algebras of finite Weyl groups is to find "good" parametrizations of the set of simple modules. There is a natural way to solve that problem by studying the associated decomposition matrices. In fact, in characteristic zero, using these matrices, it is possible to prove the existence of certain indexing sets called "basic sets" which are in natural bijection with the set of simple modules for the Hecke algebra. Once we have the existence of these sets, it is another problem to have an explicit characterization of them. This has been achieved in type A_{n-1} by Dipper and James [7], in type B_n combining works by M. Chlouveraki, M.Geck, and the author [6], [18], in type D_n by M. Geck [13] and the author [25] and for the exceptional types by Geck, Müller and Lux, [11], [20], [34], [35]. Importantly, these results remain valid in positive characteristic under the assumptions of certain Lusztig conjectures. We refer to [15] for a survey of these results.

In the case of type B_n, the theory of basic sets provides several natural ways to label the same set of simple modules. In this paper, we are mainly interested on the connections between these various basic sets. First, we show the existence of analogues of basic sets for other types of representation introduced by Lusztig: the constructible representations (see [33]). In fact, given the matrix of the constructible characters for a choice of parameters (this includes the case where these parameters are negative), we show the existence of two different associated basic sets. As a consequence, we obtain two natural ways to parametrize the constructible characters, extending the work of Lusztig [31], [32], [33]. In addition, we describe the bijection between these two basic sets. All these results use as a crucial tool the works of Lusztig, Leclerc and Miyachi [30] and the combinatorics developed therein.

The last part of the paper is devoted to the study of the various bijections between the basic sets in type B_n. It turns out that the bipartitions labelling these sets are difficult to describe in general (we only have in principle a recursive description of them). Then, in the same spirit as in [29], we show the existence of an action of the affine extended symmetric group $\widehat{\mathfrak{S}}_2$ on these basic sets. We observe the two following remarkable facts:

- there is an easy description of the basic sets lying in a fundamental domain associated with the above action,
- the action of $\widehat{\mathfrak{S}}_2$ on the set of basic sets can be explicitly described combining our previous results with results obtained in [29].

It is then possible to describe the basic sets as orbits of the elements of the fundamental domain under this action. Finally, we remark that this action is in some sense controlled by the matrices of constructible representations. In particular the bijections between the various basic sets can be essentially read through these matrices.

We end the paper with an explanation of this phenomenon.

2. Decomposition matrices for Hecke algebras

Let (W,S) be a finite Weyl group. We assume that we have a decomposition $S=S_+\sqcup S_-$ where no elements of S_+ is conjugate to an element of S_-. Let $\phi: S\to\mathbb{Z}$ be such that

$$\phi(s)=\phi(s') \text{ if } (s,s')\in S_+^2 \text{ and } \phi(s)=\phi(s') \text{ if } (s,s')\in S_-^2 \quad (\star)$$

Let q be an indeterminate and choose $q^{1/2}$ a root of q. We then have an associated Iwahori-Hecke algebra $\mathcal{H}(W,S,\phi)$ over $A=\mathbb{Z}[q^{1/2},q^{-1/2}]$. The basis is given by $\{T_w\}_{w\in W}$ and the multiplication is determined by the following rules:

$$\begin{cases} T_wT_{w'}=T_{ww'} & \text{if } l(ww')=l(w)+l(w') \\ (T_s-q^{\phi(s)})(T_s+1)=0 & \end{cases}$$

In this section, we study the representation theory of these algebras in both the semisimple and the modular case and give extensions of some definitions and properties which were previously only known when $\phi(S)\subset\mathbb{N}$

2.1. **Decomposition matrices.** Let K be the field of fractions of A. Then by [21, §9.3.5], the algebra $\mathcal{H}_K(W,S,\phi):=K\otimes_A\mathcal{H}(W,S,\phi)$ is split semisimple and by Tits' deformation theorem, we have a canonical bijection between $\operatorname{Irr}(\mathcal{H}_K(W,S,\phi))$ and $\operatorname{Irr}(W)$. Let Λ is an indexing set for $\operatorname{Irr}(W)$:

$$\operatorname{Irr}(W)=\left\{E^{\boldsymbol{\lambda}}\mid\boldsymbol{\lambda}\in\Lambda\right\}.$$

We then have:

$$\operatorname{Irr}(\mathcal{H}_K(W,S,\phi))=\left\{V_\phi^{\boldsymbol{\lambda}}\mid\boldsymbol{\lambda}\in\Lambda\right\}$$

Let k be a field and $\xi\in k^\times$ be an element which has a square root in $k^\times$. Then there is a ring homomorphism $\theta: A\to k$ such that $\theta(q)=\xi$. Considering k as an A-module via θ, we set $\mathcal{H}_k(W,S,\phi):=k\otimes_A\mathcal{H}_A(W,S,\phi)$. As noted above, we have a canonical way to parametrize the simple modules for $\mathcal{H}_K(W,S,\phi)$. It is also desirable to obtain a "good" parametrization of the simple $\mathcal{H}_k(W,S,\phi)$-modules. As $\mathcal{H}_k(W,S,\phi)$ is not semisimple in general, Tits' deformation theorem cannot be applied. However, following [15, §4.10], one can use the associated decomposition matrix to solve that problem. Let $\boldsymbol{\lambda}\in\Lambda$ and let

$$\begin{array}{cccc}\rho^{\boldsymbol{\lambda}}: & \mathcal{H}_K(W,S,\phi) & \to & M_d(K) \\ & T_w & \mapsto & (a_{ij}(T_w))_{1\leq i,j\leq d}\end{array}$$

be a matrix representation affording the module $V_\phi^{\boldsymbol{\lambda}}\in\operatorname{Irr}(\mathcal{H}_K(W,S,\phi))$ of dimension d. The ideal $\mathfrak{p}=\ker(\theta)$ is a prime ideal in A and the localization $A_\mathfrak{p}$ is a regular local ring of Krull dimension ≤ 2. Hence, by Du–Parshall–Scott [9, §1.1.1], we can assume that $\rho^{\boldsymbol{\lambda}}$ satisfies the condition

$$\rho^{\boldsymbol{\lambda}}(T_w)\in M_d(A_\mathfrak{p}) \qquad \text{for all } w\in W.$$

Now, θ extends to a ring homomorphism $\theta_\mathfrak{p}: A_\mathfrak{p}\to k$. Applying $\theta_\mathfrak{p}$, we obtain a representation

$$\begin{array}{cccc}\rho^{\boldsymbol{\lambda}}_{k,\xi}: & \mathcal{H}_k(W,S,\phi) & \to & M_d(k), \\ & T_w & \mapsto & (\theta_\mathfrak{p}(a_{ij}(T_w)))_{1\leq i,j\leq d}.\end{array}$$

This representation may no longer be irreducible. For any $M\in\operatorname{Irr}(\mathcal{H}_k(W,S,\phi))$, let $[V_\phi^{\boldsymbol{\lambda}}:M]$ be the multiplicity of M as a composition factor of the $\mathcal{H}_k(W,S,\phi)$-module affording $\rho^{\boldsymbol{\lambda}}_{k,\xi}$. This is well defined by [9, §1.1.2]. Thus, we obtain a well-defined matrix

$$D_\theta^\phi=\left([V_\phi^{\boldsymbol{\lambda}}:M]\right)_{\boldsymbol{\lambda}\in\Lambda,M\in\operatorname{Irr}(\mathcal{H}_k(W,S,\phi))}$$

which is called the decomposition matrix associated with θ. Let $R(\mathcal{H}_K(W,S,\phi))$ (resp. $R(\mathcal{H}_k(W,S,\phi))$) be the Grothendieck group of finitely generated $\mathcal{H}_K(W,S,\phi)$-modules (resp. $\mathcal{H}_k(W,S,\phi)$-modules). It is generated by the classes $[U]$ of the simple

$\mathcal{H}_K(W,S,\phi)$-modules (resp. $\mathcal{H}_k(W,S,\phi)$-modules) U. Then we obtained a well defined decomposition map:

$$d_\theta^\phi : R(\mathcal{H}_K(W,S,\phi)) \to R(\mathcal{H}_k(W,S,\phi))$$

such that for all $\lambda \in \Lambda$ we have:

$$d_\theta^\phi([V_\phi^{\boldsymbol{\lambda}}]) = \sum_{M\in \mathrm{Irr}(\mathcal{H}_k(W,S,\phi))} [V_\phi^{\boldsymbol{\lambda}} : M][M]$$

The notion of basic sets of simple modules for Hecke algebras has first been considered by Geck in [12]. The definition depends on the decomposition matrix and has been originally given in the case where ϕ is constant and positive and then in the case where ϕ is positive in [15]. Using the above discussion, we will be able to generalize these notions to all possible ϕ.

2.2. **Basic sets.** In this section, we adopt the following notations. Let $\phi : S \to \mathbb{Z}$ be a map satisfying

$$\phi(s) = \phi(s') \text{ if } (s,s') \in S_+^2 \text{ and } \phi(s) = \phi(s') \text{ if } (s,s') \in S_-^2 \qquad (\star).$$

We denote by $|\phi| : S \to \mathbb{Z}$ the map such that $|\phi|(s) = |\phi(s)|$ for all $s \in S$. Note that this map satisfy $(\star)$. Set:

$$S^- := \{s \in S \mid \phi(s) < 0\}.$$

Let $\varepsilon : W \to \mathbb{Q}^\times$ be the one dimensional representation of W such that $\varepsilon(s) = 1$ if $s \in S^+$ and $\varepsilon(s) = -1$ if $s \in S^-$. For $\boldsymbol{\lambda} \in \Lambda$, the module $(E^{\boldsymbol{\lambda}})^\varepsilon$ remains simple and we define $\boldsymbol{\lambda}^\varepsilon \in \Lambda$ such that $E^{\boldsymbol{\lambda}^\varepsilon} \simeq (E^{\boldsymbol{\lambda}})^\varepsilon$.

We turn to the definition of basic sets associated with a specialization of the Hecke algebra.

Definition 2.1. We say that $\mathcal{H}(W,S,\phi)$ admits a basic set $\mathcal{B}(\phi) \subset \Lambda$ with respect to $\theta : A \to k$ and to a map $\alpha^\phi : \Lambda \to \mathbb{Q}$ if and only if:

(1) For all $M \in \mathrm{Irr}(\mathcal{H}_k(W,S,\phi))$ there exists $\lambda_M \in \mathcal{B}(\phi)$ such that

$$[V_\phi^{\boldsymbol{\lambda}_M}, M] = 1 \text{ and } \alpha^\phi(\boldsymbol{\mu}) > \alpha^\phi(\boldsymbol{\lambda}_M) \text{ if } [V_\phi^{\boldsymbol{\mu}}, M] \neq 0$$

(2) The map

$$\begin{array}{ccc} \mathrm{Irr}(\mathcal{H}_k(W,S,\phi)) & \to & \mathcal{B}(\phi) \\ M & \mapsto & \lambda_M \end{array}$$

is a bijection

Assume that $\mathcal{H}(W,S,\phi)$ admits a basic set $\mathcal{B}(\phi) \subset \Lambda$ with respect to θ and to a map $\alpha^\phi : \Lambda \to \mathbb{Q}$. This implies that the associated decomposition matrix has a lower triangular shape with one along the diagonal for a “good” order on Λ induced by the map α^ϕ. Hence, it gives a way to label $\mathrm{Irr}(\mathcal{H}_k(W,S,\phi))$.

It is now natural to ask if these basic sets always exist. The question has been considered in [23], [12], [15] (see [19] for a complete survey on this theory) and in [6] (where the question of existence of basic sets in characteristic 0 and for any weight function is complete),

The first step is to define the canonical map $a^\phi : \Lambda \to \mathbb{N}$ which will play the role of α^ϕ. This can be done using the symmetric algebra structure of $\mathcal{H}(W,S,\phi)$ We define a linear map $\tau \colon \mathcal{H}(W,S,\phi) \to A$ by

$$\tau(T_1) = 1 \qquad \text{and} \qquad \tau(T_w) = 0 \quad \text{for } w \neq 1.$$

Then one can show that τ is a trace function and we have

$$\tau(T_w T_{w'}) = \begin{cases} \pi(w) & \text{if } w' = w^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $\mathcal{H}(W,S,\phi)$ is a symmetric algebra (see [21, Ch. 7] for a study of the representation theory of this type of algebras). The above trace form extends to a trace form $\tau_K\colon \mathcal{H}_K(W,S,\phi) \to K$. Now since $\mathcal{H}_K(W,S,\phi)$ is split semisimple, we have

$$\tau_K(T_w) = \sum_{\boldsymbol{\lambda}\in\Lambda} \frac{1}{c_{\boldsymbol{\lambda}}} \mathrm{trace}(T_w, V_\phi^{\boldsymbol{\lambda}}), \qquad \text{for all } w \in W,$$

where $c_{\boldsymbol{\lambda}} \in A$ is called the Schur element associated to $\boldsymbol{\lambda} \in \Lambda$. For all $\boldsymbol{\lambda} \in \Lambda$, we now have

$$c_{\boldsymbol{\lambda}} = f_{\boldsymbol{\lambda}} q^{-a^\phi_{\boldsymbol{\lambda}}} + \text{ combination of higher powers of } q,$$

where $f_{\boldsymbol{\lambda}}$ and $a^\phi_{\boldsymbol{\lambda}}$ are both integers such that $f_{\boldsymbol{\lambda}} > 0$ and $a^\phi_{\boldsymbol{\lambda}} \geq 0$ (see [21, Ch. 20] for details.) The map

$$\begin{array}{rccc} a^\phi: & \Lambda & \to & \mathbb{N} \\ & \boldsymbol{\lambda} & \mapsto & a^\phi_{\boldsymbol{\lambda}} \end{array}$$

is called the Lusztig a-function.

In the next theorem, we need the following definition: we say that k is good with respect to $\mathcal{H}(W,S,\phi)$ if $f_{\boldsymbol{\lambda}} 1_k \neq 0$ for all $\boldsymbol{\lambda} \in \Lambda$. The proof of the existence of basic set for Hecke algebras (in characretistic 0) with respect to the a-function has been given Geck and Geck-Rouquier (see [15]) whenΦ is positive. The following proposition obtained in [6, Prop 2.5] allows the extension of the result for arbitrary Φ. This result will also be crucial in the rest of the paper.

Proposition 2.2. *For all $\boldsymbol{\lambda} \in \Lambda$, we have:*

$$a^\phi(V_\phi^{\boldsymbol{\lambda}}) = a^{|\phi|}(V_{|\phi|}^{\boldsymbol{\lambda}^\varepsilon})$$

The theorem of existence becomes then the following:

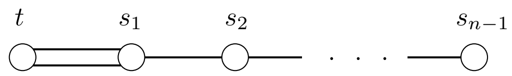
Theorem 2.3. *We keep the above notations. Assume in addition that Lusztig's conjectures* **P1-P15** *in* [33, §14.2] *hold and that k is good with respect to $\mathcal{H}(W,S,\phi)$. Then $\mathcal{H}(W,S,\phi)$ admits a basic set $\mathcal{B}(\phi) \subset \Lambda$ with respect to any specialization and to the map a^ϕ, the Lusztig a-function. This basic set is called the canonical basic set and it only depends on e and ϕ.*

Remark 2.4. Lusztig's conjectures are known to hold in the following cases:

(1) For all finite Weyl group, in the so called "equal parameter case", that is when there exists $a \in \mathbb{N}$ such that $\phi(s) = a$ for all $s \in S$ by Lusztig (see [33, Ch. 15].)
(2) In type B_n, in the so called "asymptotic case" by the works of Bonnafé-Iancu [4], Bonnafé [2], Geck [16] and Geck-Iancu [17].

In fact, the results in [6] show that the canonical basic set $\mathcal{B}(\phi) \subset \Lambda$ can be deduced from $\mathcal{B}(|\phi|)$. We illustrate this fact in type B_n which is our main centre of interest in this paper. Hence, let W be a Weyl group of type B_n.

B_n



In this case, Λ can be defined to be the set Π_n^2 of bipartitions of rank n. In the following, it will be useful to introduce a "more generic" Hecke algebra than the one defined in the introduction of this section.

Let V and v be indeterminates and consider the generic Hecke algebra $\mathcal{H}(\{V,v\})$ of type B_n over $\mathbb{Z}[V^{\pm 1/2}, v^{\pm 1/2}]$ with presentation as follows:

$$\begin{array}{l} (T_0 - V)(T_0 + 1) = 0 \\ (T_i - v)(T_i + 1) = 0 \quad \text{if } i = 1, ..., n-1 \end{array}$$

Let K be the field of fractions of A. We set

$$\mathrm{Irr}(\mathcal{H}_K(\{V,v\})) = \{V^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \Pi_n^2\}.$$

and $\Lambda = \Pi_n^2$.

We assume that we have a specialization $\theta : \mathbb{Z}[V^{\pm 1/2}, v^{\pm 1/2}] \to k$ where k is a field. By results of Dipper and James [8, Th. 4.17], one can restrict ourselves to the following case. We assume that $\theta(V)=-q^{da}$ and $\theta(v)=q^a$ for $(a,b)\in\mathbb{N}^2$ and for $q\in k^\times$. The resulting algebra $\mathcal{H}_k(\{q^a,-q^{ad}\})$ has a presentation as follows:

$$\begin{aligned}(T_0+q^{da})(T_0+1)&=0\\ (T_i-q^a)(T_i+1)&=0 \quad \text{if } i=1,...,n-1.\end{aligned}$$

It is in general a non semisimple algebra and as in §2.1, we have an associated decomposition matrix.

$$D=([V^{\boldsymbol{\lambda}}:M])_{\boldsymbol{\lambda}\in\Pi_n^2, M\in\mathrm{Irr}(\mathcal{H}_k(\{q^a,-q^{ad}\}))}$$

By a deep theorem of Ariki [1, Thm 14.49], this matrix is nothing but the (evaluation at $v=1$ of the) matrix of the canonical basis for the irreducible highest weight $\mathcal{U}_v(\widehat{\mathfrak{sl}_e})$-module with weight $\Lambda_d+\Lambda_0$ (where the Λ_i with $i\in\mathbb{Z}/e\mathbb{Z}$ denote the fundamental weights).

Applying [6] to type B_n leads the following proposition.

Proposition 2.5. *Let (W,S) be the Weyl group of type B_n. We assume that we have a map $\phi:S=\{t,s_1,...,s_n\}\to\mathbb{Z}$ satisfying $(\star)$. We set $|\phi|:S=\{t,s_1,...,s_n\}\to\mathbb{Z}$ such that $|\phi|(s)=|\phi(s)|$ for all $s\in S$. Then, for all specialization θ, we have*

$$\mathcal{B}(\phi)=\left\{(\lambda^0,\lambda^1)^\varepsilon \mid (\lambda^0,\lambda^1)\in\mathcal{B}(|\phi|)\right\}$$

where

(1) $(\lambda^0,\lambda^1)^\varepsilon=(\lambda^1,\lambda^0)$ *if* $S^+=\{s_1,...,s_{n-1}\}$ *and* $S^-=\{t\}$,
(2) $(\lambda^0,\lambda^1)^\varepsilon=(\lambda^{1\prime},\lambda^{0\prime})$ *if* $S^+=\emptyset$ *and* $S^-=\{s_1,...,s_{n-1},t\}$,
(3) $(\lambda^0,\lambda^1)^\varepsilon=(\lambda^{0\prime},\lambda^{1\prime})$ *if* $S^+=\{t\}$ *and* $S^-=\{s_1,...,s_{n-1}\}$,

From now, we will denote $\kappa(\lambda^0,\lambda^1):=(\lambda^1,\lambda^0)$.

2.3. **Constructible representations.** From the above definitions and the results in [3], it will be easy to extend known results of constructible representations as defined by Lusztig [33, §22.1]. Let (W,S) be a Weyl group and let ϕ be a map $\phi:S\to\mathbb{Z}$ satisfying $(\star)$. Let $I\subset S$ and let (W_I,I) be the corresponding parabolic subgroup. Let ϕ_I be the restriction of ϕ to I. Each simple $\mathbb{C}[W]$-module $E^{\boldsymbol{\lambda}}$ can be seen as a specialization of a simple $\mathcal{H}(W,S,\phi)$-module $V_\phi^{\boldsymbol{\lambda}}$. By §2.2, each simple $\mathbb{C}[W]$-module U comes equipped with an associated invariant $a^\phi(U)$ depending on the choice of ϕ. Let U be a simple $\mathbb{C}[W_I]$-module. Then we can uniquely write:

$$\mathrm{Ind}_I^S(U)=U_{(0)}\oplus U_{(1)}\oplus ....$$

where for any integer i

$$U_{(i)}=\bigoplus_V[\mathrm{Ind}_I^S(U):V]V \text{ (sum over all } V\in\mathrm{Irr}(\mathbb{C}[W]) \text{ such that } a^\phi(V)=i).$$

Then the **J**-induction of a simple $\mathbb{C}[W_I]$-module U is

$$\mathbf{J}_I^S(U)=U_{(a^{\phi_I}(U))}$$

Using this "truncated" induction, the constructible representations with respect to ϕ are defined inductively in the following way:

(1) If $W=\{1\}$, only the trivial module is constructible.
(2) If $W\neq\{1\}$, the set of constructible $\mathbb{C}[W]$-modules consists of the $\mathbb{C}[W]$-modules of the form

$$\mathbf{J}_I^S(V) \quad \text{or} \quad \mathrm{sgn}\otimes\mathbf{J}_I^S(V),$$

where sgn is the sign representation of W, and I some proper subset of S.

One can define an analogue of the decomposition matrix for this type of representations: the constructible matrix D_{cons}^ϕ which is defined as follows.

- The rows are labelled by Λ,
- the coefficients in a fixed column give the expansion of the corresponding constructible representation in terms of the irreducible ones.

Proposition 2.6. *Let $\phi : S \to \mathbb{Z}$ be a map satisfying $(\star)$. Keeping the above notations $\bigoplus_{\boldsymbol{\lambda}\in\Lambda} \alpha_{\boldsymbol{\lambda}} V_{\phi}^{\boldsymbol{\lambda}}$ is a constructible $\mathbb{C}[W]$-module with respect to ϕ if and only if $\bigoplus_{\boldsymbol{\lambda}\in\Lambda} \alpha_{\boldsymbol{\lambda}} V_{|\phi|}^{\boldsymbol{\lambda}^{\varepsilon}}$ is a constructible $\mathbb{C}[W]$-module with respect to $|\phi|$*

Proof. The result follows directly from Prop. 2.2 and the definition of the constructible representation. □

Remark 2.7. If ϕ is positive, the blocks of the matrix D_{cons}^{ϕ} are known as the families of characters. This notion plays an important role in the theory of reductive group. In fact, it is possible to generalize the definition of families of characters to any map ϕ (ie. not necessary positive) and to a wider class of algebras : the cyclotomic Hecke algebras, using the notion of Rouquier blocks. The associated families have been studied by Chlouveraki [5]. When W is a Weyl group, one can easily see that for any map ϕ, the blocks of the matrix D_{cons}^{ϕ} corresponds to the families of characters as given in [5].

3. Bijections of basic sets

We now focus on the case of the Hecke algebras of type B_n. Before the study of the representation theory, we first give the formula for the Lusztig a-function in the case where $\phi(t) = b \geq 0$ and $\phi(s_i) = a \geq 0$ for $i = 1, ..., n-1$.

3.1. **a-function in type B_n.** We first introduce a combinatorial object which will be useful in the following: the shifted symbol of a bipartition. Let $\beta = (\beta_1, ..., \beta_k)$ be a sequence of strictly increasing integers and let s be a rational nonnegative number. We denote by $[s]$ the integer part of s. We set

$$\beta(s) := (s - [s], s - [s] + 1, ..., s - 1, \beta_1 + s, ..., \beta_k + s).$$

Let $r \in \mathbb{Q}$ and let $\boldsymbol{\lambda} = (\lambda^0, \lambda^1)$ be a bipartition of rank n. Let h^0 and h^1 be the heights of the partitions λ^0 and λ^1 and let h be a positive integer such that $h \geq \max(h^0, h^1) + 1$. We say that h is an admissible size for $\boldsymbol{\lambda}$. We define two sequences of strictly decreasing integers

$$\beta^0 = (\lambda_h^0 - h + h, ..., \lambda_j^0 - j + h, ..., \lambda_1^0 - 1 + h)$$

and

$$\beta^1 = (\lambda_h^1 - h + h, ..., \lambda_j^1 - j + h, ..., \lambda_1^1 - 1 + h)$$

The *shifted r-symbol* of $\boldsymbol{\lambda}$ of size h is then the family of sequence

$$\mathbf{B}_r(\boldsymbol{\lambda}) := (B^0, B^1)$$

such that

$$B^0 = \begin{cases} \beta^0(r) & \text{if } r \geq 0 \\ \beta^1(-r) & \text{otherwise.} \end{cases} \quad \text{and} \quad B^1 = \begin{cases} \beta^0 & \text{if } r \leq 0 \\ \beta^1 & \text{otherwise.} \end{cases}$$

The shifted symbol of size h is usually written as a two row tableaux as follows:

$$\mathbf{B}_r(\boldsymbol{\lambda}) := \begin{pmatrix} B^0 \\ B^1 \end{pmatrix}$$

Note that $\mathbf{B}_r(\boldsymbol{\lambda}) = \mathbf{B}_{-r}(\kappa(\boldsymbol{\lambda}))$.

Example 3.1. Let $r = 1/2$ and $\boldsymbol{\lambda} = (2.1, 3.3.2)$. We have $h^0 = 2$ and $h^1 = 3$. Then the associated shifted r-symbol of size 4 is

$$\begin{pmatrix} 1/2 & 3/2 & 7/2 & 11/2 \\ 0 & 3 & 5 & 6 \end{pmatrix}$$

Let $r = -5/2$ and $\boldsymbol{\lambda} = (2.1, 3.3.2)$. We have $h^0 = 2$ and $h^1 = 3$ then the associated shifted r-symbol of size 4 is

$$\begin{pmatrix} 1/2 & 3/2 & 1/2 & 7/2 & 11/2 & 13/2 \\ 0 & 1 & 3 & 5 & & \end{pmatrix}$$

Assume now that $\phi(t)=b\geq 0$ and $\phi(s_i)=a>0$ for $i=1,...,n-1$. Let $\boldsymbol{\lambda}\in\Lambda$ and let $\mathbf{B}_{b/a}(\boldsymbol{\lambda})$ be the shifted b/a symbol of $\boldsymbol{\lambda}$ of size h. Let

$$\gamma_1\geq\gamma_2\geq ....\geq\gamma_t,$$

be the elements of this symbol written in decreasing order (with repetition). Then write

$$a_1^{\phi,h}(\boldsymbol{\lambda})=\sum_{i=1}^{t}(i-1)\gamma_i.$$

Then by Lusztig [33, §22.14], we have:

$$a^{\phi}(\boldsymbol{\lambda})=a_1^{\phi,h}(\boldsymbol{\lambda})-a_1^{\phi,h}(\boldsymbol{\emptyset}).$$

Regarding the above definition and the connection between Lusztig a-function when $b<0$ and $b>0$ given by Prop. 2.2, we deduce:

Proposition 3.2. *Assume that $\phi(t)=b$ and $\phi(s_i)=a\geq 0$ for $i=1,...,n-1$. Let $\boldsymbol{\lambda}\in\Lambda$ and let $\mathbf{B}_{b/a}(\boldsymbol{\lambda})=(B^0,B^1)$ be the shifted b/a-symbol of $\boldsymbol{\lambda}$. Let*

$$\gamma_1\geq\gamma_2\geq ....\geq\gamma_t$$

be the elements of this symbol written in decreasing order (with repetition). We denote $\gamma^h_{b/a}(\boldsymbol{\lambda})=(\gamma_1,...,\gamma_t)$ the associated partition. We then write

$$a_1^{\phi}(\boldsymbol{\lambda})=\sum_{i=1}^{t}(i-1)\gamma_i$$

Then we have:

$$a^{b/a}(\boldsymbol{\lambda}):=a^{\phi}(\boldsymbol{\lambda})=a_1^{\phi,h}(\boldsymbol{\lambda})-a_1^{\phi,h}(\boldsymbol{\emptyset})$$

It does not depend on the size of the symbol.

Proof. When $b\geq 0$, this is a result of Lusztig. Assume that $b<0$, then we have by Prop. 2.2,

$$a^{b/a}(\boldsymbol{\lambda}):=a^{-b/a}(\kappa(\boldsymbol{\lambda}))$$

and $\mathbf{B}_{b/a}(\boldsymbol{\lambda})=\mathbf{B}_{-b/a}(\kappa(\boldsymbol{\lambda}))$ so the result follows $\square$

The following result is a direct consequence of the formula of the a-function.

Corollary 3.3. *Assume that $\phi(t)=b$ and $\phi(s_i)=a\geq 0$ for $i=1,...,n-1$. Let $\boldsymbol{\lambda}\in\Lambda$ and $\boldsymbol{\mu}\in\Lambda$. Let h be an admissible size for $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. Assume that $\gamma^h_{b/a}(\boldsymbol{\lambda})\rhd\gamma^h_{b/a}(\boldsymbol{\mu})$ then we have $a^{b/a}(\boldsymbol{\lambda})<a^{b/a}(\boldsymbol{\mu})$.*

3.2. **Constructible representations in type B_n.** Let V and v be indeterminates and consider the Hecke algebra $\mathcal{H}(\{V,v\})$ of type B_n over $\mathbb{Z}[V^{\pm 1/2},v^{\pm 1/2}]$ as in §2.2. We have

$$\operatorname{Irr}(\mathcal{H}_K(\{V,v\}))=\{V^{\boldsymbol{\lambda}}\mid\boldsymbol{\lambda}\in\Pi^2_n\}.$$

We assume that we have a specialization $\theta:\mathbb{Z}[V^{\pm 1/2},v^{\pm 1/2}]\to\mathbb{Q}(q^{\frac{1}{2}})$ where q is an indeterminate such that $\theta(V)=-q^{da}$ and $\theta(v)=q^a$ for $(a,b)\in\mathbb{N}^2$. For $d\in\mathbb{Z}$. Then we have an associated decomposition matrix.

$$D_\theta=([V^{\boldsymbol{\lambda}}:M])_{\boldsymbol{\lambda}\in\Pi^2_n,M\in\operatorname{Irr}(\mathcal{H}_{\mathbb{Q}(q^{1/2})}(q^a,-q^{da}))}$$

By Ariki's theorem, this decomposition matrix is the matrix of the canonical basis of the irreducible highest weight $\mathcal{U}_v(\mathfrak{sl}_\infty)$-module with weight $\Lambda_d+\Lambda_0$. Assume that $d\geq 0$ then by [30] it has another interpretation in terms of Kazhdan-Lusztig theory: this is the matrix D^{ϕ}_{cons} of the constructible representations for the algebra $\mathcal{H}(W,S,\phi)$ where $\phi(t)=d$ and $\phi(s_i)=1$ for all $i=1,...,n-1$. Thus, we have $D^{\phi}_{\text{cons}}=D_\theta$. In other words, the columns of this decomposition matrix give the expansion of the constructible representations associated to the map ϕ in terms of the simple $\mathcal{H}(W,S,\phi)$-modules $V^{\boldsymbol{\mu}}_\phi$.

It is natural to ask if a basic set as defined in Def. 2.1 can be found in this situation that is if one can order the rows and columns of the constructible matrix such that it has a unitriangular shape. The explicit determination of the matrix D_θ has been given by Lusztig and by Leclerc-Miyachi using different technics when $d \geq 0$. We here follows the latter exposition [30] and extend it to the case $d \in \mathbb{Z}$.

Hence we assume from now that ϕ is such that $\phi(t) = d \in \mathbb{Z}$ and $\phi(s_i) = 1$ for all $i = 1, ..., n-1$. First, we need some additional combinatorial definition. Let $\boldsymbol{\lambda}$ be a bipartition and consider its d-symbol $\mathbf{B}_d(\boldsymbol{\lambda}) = (B^0, B^1)$. We say that $\boldsymbol{\lambda}$ is standard, or equivalently that $\mathbf{B}_d(\boldsymbol{\lambda})$ is standard if

$$B_i^1 \geq B_i^0 \text{ for all } i \geq 1.$$

The set of standard bipartitions is denoted by $\mathrm{Std}(d)$.

Let $\mathbf{B}_d(\boldsymbol{\lambda})$ be a standard symbol. We define an injection $\Psi : B^1 \to B^0$ such that $\Psi(j) \leq j$ for all $j \in B^1$. It is obtained by describing the subsets

$$B_l^1 := \{j \in B^1 \mid \Psi(j) = j - l\}.$$

We set $B_0^1 = B^1 \cap B^0$ and for $l \geq 1$, we put:

$$B_l^1 = \{j \in B^1 \setminus \{B_0^1, ..., B_{l-1}^1\} \mid j - l \in B^0 \setminus \Psi(B_0^1 \cup ... \cup B_{l-1}^1)\}$$

the pairs $(j, \Psi(j))$ with $\Psi(j) \neq j$ are called the pairs of the symbols $\mathbf{B}_d(\boldsymbol{\lambda})$. Let $\mathcal{C}(\boldsymbol{\lambda})$ be the set of all bipartitions $\boldsymbol{\mu}$ such that the symbol of $\boldsymbol{\mu}$ is obtained from $\mathbf{B}_d(\boldsymbol{\lambda})$ by permuting some pairs in $\mathbf{B}_d(\boldsymbol{\lambda})$ and reordering the rows. We also define $\mathrm{Inv}_d(\boldsymbol{\lambda})$ to be the bipartition in $\mathcal{C}(\boldsymbol{\lambda})$ whose symbols is obtained from the symbol of $\boldsymbol{\lambda}$ after all possible permutations of the pairs.

The following is a result by Lusztig and Leclerc-Miyachi when $d \geq 0$ which is easily extend in the case where $d \in \mathbb{Z}$ by Prop. 2.6 and by the definition of symbols.

Proposition 3.4. *Assume that ϕ is such that $\phi(t) = d \in \mathbb{Z}$ and $\phi(s_i) = 1$ for all $i = 1, ..., n-1$. The constructible representations with respect to ϕ are labelled by the standard bipartitions. Moreover, if $\boldsymbol{\lambda}$ is a standard bipartition, the associated constructible representation with respect to ϕ is*

$$\bigoplus_{\boldsymbol{\mu} \in \mathcal{C}(\boldsymbol{\lambda})} V_\phi^{\boldsymbol{\mu}}.$$

Example 3.5. We consider the Weyl group of type B_3. Let ϕ be such that $\phi(t) = 2 \in \mathbb{Z}$ and $\phi(s_i) = 1$ for all $i = 1, 2$. From above, one can check that the only non trivial constructible representations are:

$$V_\phi^{(\emptyset,3)} \oplus V_\phi^{(1,2)},\ V_\phi^{(1,2)} \oplus V_\phi^{(1.1,1)}, V_\phi^{(1.1,1)} \oplus V_\phi^{(1.1.1,\emptyset)}$$

In other words, we have:

$$D_{\mathrm{cons}}^\phi = \begin{pmatrix} 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 1&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&1&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&1&0&0 \end{pmatrix} \begin{matrix} (\emptyset,3) \\ (\emptyset,2.1) \\ (\emptyset,1.1.1) \\ (1,2) \\ (1,1.1) \\ (2,1) \\ (1.1,1) \\ (3,\emptyset) \\ (2.1,\emptyset) \\ (1.1.1,\emptyset) \end{matrix}$$

Note that the constructible representations are labelled by the set of standard bipartitions which is in this case $\Pi_3^2 \setminus \{(1.1.1, \emptyset)\}$. Note also that the unique non trivial block of the above matrix is given by the following set of bipartitions

$$\{(\emptyset, 3), (1, 2), (1.1, 2), (1.1.1, \emptyset)\}$$

which corresponds to the unique non trivial family of characters.

Now, if we set ϕ' be such that $\phi'(t) = -2 \in \mathbb{Z}$ and $\phi'(s_i) = 1$ for all $i = 1, 2$. From above, one can check that the only non trivial constructible representations are:

$$V_{\phi'}^{(3,\emptyset)} \oplus V_{\phi'}^{(2,1)},\ V_{\phi'}^{(2,1)} \oplus V_{\phi'}^{(1,1.1)}, V_{\phi'}^{(1,1.1)} \oplus V_{\phi'}^{(\emptyset,1.1.1)}$$

The unique non trivial block of the above matrix is given by the following set of bipartitions

$$\{(3,\emptyset),(2,1),(1,1.1),(\emptyset,1.1.1)\}$$

which corresponds to the unique non trivial family of characters.

3.3. **Basic sets of constructible representations.** The aim of this section is to show the existence of basic sets for the matrix affording the constructible representations. To do this, we have to study the matrix D^{ϕ}_{cons} of the constructible representations for the algebra $\mathcal{H}(W,S,\phi)$ where $\phi(t)=d\in\mathbb{Z}$ and $\phi(s_i)=1$ for all $i=1,...,n-1$. We denote by $\text{Cons}(d)$ the set of constructible $\mathcal{H}(W,S,\phi)$-modules. The main result of this section is the following one.

Theorem 3.6. *We keep the above notations and we put $r\in\mathbb{Q}$ such that $r\neq d$.*

(1) *For all $U\in\text{Cons}(d)$ there exists $\lambda_U\in\Lambda$ such that*

$$[V^{\boldsymbol{\lambda}_U},U]=1 \textit{ and } a^r(\boldsymbol{\mu})>a^r(\boldsymbol{\lambda}_U) \textit{ if } [V^{\boldsymbol{\mu}},U]\neq 0$$

(2) *Let*

$$\mathcal{B}^r_\infty:=\{\lambda_U\mid U\in\text{Cons}(d)\}$$

Then the map

$$\begin{array}{ccc}\text{Cons}(d) & \to & \mathcal{B}^r_\infty\\ U & \mapsto & \lambda_U\end{array}$$

is a bijection

(3) *We have*

$$\mathcal{B}^r_\infty=\begin{cases}\text{Std}(d) & \textit{if } r<d\\ \text{Inv}_d(\text{Std}(d)) & \textit{if } r>d\end{cases}$$

Proof. Let us first assume that $d\geq 0$. Let $\boldsymbol{\lambda}$ be a standard bipartition then by Prop 3.4, we have an associated constructible representation labelled by this bipartition. Assume that $V^{\boldsymbol{\mu}}_\phi$ appears as an irreducible constituent in the expansion of this constructible representation. Then $\boldsymbol{\mu}$ can be constructed from $\boldsymbol{\lambda}$ by permuting some pairs in the shifted d-symbol of $\boldsymbol{\lambda}$. Let h be an admissible size for $\boldsymbol{\lambda}$. By construction, it implies that this size is admissible for $\boldsymbol{\mu}$. Let $\mathbf{B}_d(\boldsymbol{\mu})=(B^0,B^1)$ be the shifted d-symbol of $\boldsymbol{\mu}$ of size h. By §3.1, we have $B^0=\beta^0(d)$ and $a=\beta^1$.

- If $r>d$ the shifted r-symbol of $\boldsymbol{\mu}$ is $\mathbf{B}_r(\boldsymbol{\mu})=(B^0(r-d),B^1)$. By the definition of $\mathcal{C}(\boldsymbol{\lambda})$, It is easy to see that
$$\gamma^h_r(\text{Inv}_d(\boldsymbol{\lambda}))\trianglerighteq\gamma^h_r(\boldsymbol{\mu}),$$
for all $\boldsymbol{\mu}\in\mathcal{C}(\boldsymbol{\lambda})$. Hence by Cor 3.3, we obtain:
$$a^r(\text{Inv}_d(\boldsymbol{\lambda}))\leq a^r(\boldsymbol{\mu})$$
- if $r<d$, the shifted r-symbol of $\boldsymbol{\mu}$ of size h is $\mathbf{B}_r(\boldsymbol{\mu})=(B^0(d-r),B^1)$ if r is positive. In this case, by the definition of $\mathcal{C}(\boldsymbol{\lambda})$ It is easy to see that
$$\gamma^h_r(\boldsymbol{\lambda})\trianglerighteq\gamma^h_r(\boldsymbol{\mu}),$$
for all $\boldsymbol{\mu}\in\mathcal{C}(\boldsymbol{\lambda})$ such that $\boldsymbol{\mu}\neq\boldsymbol{\lambda}$. Hence we obtain:
$$a^r(\boldsymbol{\lambda})\leq a^r(\boldsymbol{\mu}).$$
If r is negative, the shifted r-symbol of $\boldsymbol{\mu}$ of size $h+d$ is $\mathbf{B}_r(\boldsymbol{\mu})=(B^1(d-r),B^0)$. In this case, by the definition of $\mathcal{C}(\boldsymbol{\lambda})$ It is easy to see that
$$\gamma^{h+d}_r(\boldsymbol{\lambda})\triangleright\gamma^{h+d}_r(\boldsymbol{\mu}),$$
and we can conclude as above.

To summarize, We know that an arbitrary column of the constructible matrix D_θ is naturally labelled by a standard bipartition $\boldsymbol{\lambda}_U$ with $U\in\text{Cons}(d)$. The above discussion shows that the minimal bipartition $\boldsymbol{\lambda}$ with respect to a^r such that $[V^{\boldsymbol{\lambda}}:M]\neq 0$ is $\boldsymbol{\lambda}=\boldsymbol{\lambda}_U$ if $r<d$, $\boldsymbol{\lambda}=\text{Inv}_d(\boldsymbol{\lambda}_U)$ otherwise. In addition by Prop 3.4, if $[V^{\boldsymbol{\lambda}}:U]\neq 0$ then $[V^{\boldsymbol{\lambda}}:U]=1$. This proves the Theorem if $d>0$.

The case $d<0$ is deduced from the above one using Prop. 2.2 and the facts that $\kappa(\mathrm{Std}(d))=\mathrm{Std}(-d)$, $\kappa(\mathrm{Inv}_d(\mathrm{Std}(d)))=\mathrm{Inv}_{-d}(\mathrm{Std}(-d))$

□

In the expansion of a constructible character, all the simple modules have the same value with respect a^d. This follows from the definition of constructible representations and can also be easily seen in the formula above. Theorem 3.6 shows that modifying the a-function by adding an integer s to d leads to a natural way to order the rows and columns of the constructible matrix so that it is unitriangular. This induces the existence of a canonical basic set which only depends on the sign of s.

Let us now describe the consequences on the parametrisations of the simple modules for Hecke algebras of type B_n in the modular case.

4. Basic sets in type B_n

4.1. **Explicit determination in a special case.** Recall that $A:=\mathbb{Z}[q^{1/2},q^{-1/2}]$. Let ϕ such that $\phi(t)=b\geq 0$ and $\phi(s_i)=a>0$ for $i=1,...,n-1$, Let

$$\theta : A\to \mathbb{Q}(q_0^{1/2})$$

be a specialization such that $\theta(q)=q_0\in\mathbb{C}^*$. Let $e\geq 2$ be the multiplicative order of $\eta_e:=q_0^a$. Assume that $q_0^b=-q_0^{ad}$ for some $d\in\mathbb{Z}$. We have an associated decomposition matrix D_θ and a canonical basic set $\mathcal{B}(\phi)$ which will, from now, be rather denoted by $\mathcal{B}_e^{b/a}$. Consider now ϕ^1 such that $\phi^1(t)=b+ae$ and $\phi^1(s_i)=a$ for $i=1,...,n-1$. Applying the specialization θ, we obtain again the algebra $\mathcal{H}_k(q_0^a,-q_0^{da})$. Hence we have another basic set denoted by $\mathcal{B}_e^{b/a+e}$ which has the same cardinality as $\mathcal{B}_e^{b/a}$ but is computed with respect to the a-function $a^{b/a+e}$. Continuing in this way we obtain several basic sets

$$\mathcal{B}^{b/a},\mathcal{B}_e^{b/a+e},...,\mathcal{B}_e^{b/a+te},...$$

and one can assume that $0\leq b/a<e$.

These basic sets have been computed in [18] without the assumptions of **P1-P15** in characteristic 0. It has been shown that the bipartitions labelling these sets are given by the so called Uglov bipartitions. These bipartitions appear as natural labelling of the Kashiwara's crystal basis for irreducible highest weight-modules of level two. As several combinatorial definitions are necessary to introduce them, we have chosen to omit the definition of these bipartitions here. We refer to [19] or [28] for details on them.

Theorem 4.1 (Geck-Jacon)**.** *We keep the above notations. Let $e\geq 2$ be the multiplicative order of q^a and let $p_0\in\mathbb{Z}$ be such that*

$$d+p_0e<\frac{b}{a}<d+(p_0+1)e.$$

(Note that the above conditions imply that $b/a\not\equiv d \bmod e$.) Then for all $t\geq 0$, we have

$$\mathcal{B}_e^{b/a+te}=\Phi_{e,n}^{(d+(p_0+t)e,0)}$$

where $\Phi_{e,n}^{(d+(p_0+t)e,0)}$ is defined in [18, Def. 4.4].

Without loss of generality, one can (and do) assume that $p_0=0$. By the results in the second section, the datum of ϕ' such that $\phi'(t)=\varepsilon_1 b$ and $\phi'(s_i)=\varepsilon_2 a$ (for $i=1,...,n-1$) with $(\varepsilon_1,\varepsilon_2)\in\{\pm 1\}^2$ also yields the existence of a basic set $\mathcal{B}(\phi')$. By Prop. 2.5, they can be easily computed using the above theorem. In particular, the case $\varepsilon_2=1$ and $\varepsilon_1=-1$ implies the existence of a basic set $\mathcal{B}^{-b/a}$. In fact, considering all the basic sets we have already obtained:

$$\mathcal{B}^{b/a},\mathcal{B}_e^{b/a+e},...,\mathcal{B}_e^{b/a+te},...$$

we obtain several other basic sets

$$\mathcal{B}^{-b/a},\mathcal{B}_e^{-b/a-e},...,\mathcal{B}_e^{-b/a-te},...$$

Now recall the specialization

$$\theta : A \to \mathbb{Q}(q_0^{1/2})$$

such that $\theta(q) = q_0$. Looking at the Hecke algebra $\mathcal{H}(\{q^{-b}, q^a\})$ and applying the specisalization θ, we obtain a decomposition matrix which can be identify with D^θ by [6, §3.1] . Keeping the above notations, we have: $q_0^{-b+ae} = -q_0^{-ad}$. Note that $-b+ae$ and a are both positives. We then obtain a basic set $\mathcal{B}^{-b/a+e}$. Actually, using the same arguments as above, one obtain several basic sets:

$$\mathcal{B}^{-b/a+e}, \mathcal{B}_e^{-b/a+2e}, ..., \mathcal{B}_e^{-b/a+te}, ...$$

Keeping the notation of thm 4.1 (recall that $p_0 = 0$), note that we have

$$-d < -\frac{b}{a} + e < -d + e.$$

The above theorem gives also the explicit determination of these basic sets:

Corollary 4.2. *Keeping the above notations, for all $t > 0$, we have*

$$\mathcal{B}_e^{-b/a+te} = \Phi_{e,n}^{(-d+(t-1)e,0)}$$

where $\Phi_{e,n}^{(-d+(t-1)e,0)}$ is defined in [18, Def. 4.4].

Finally, as above, the existence of basic sets

$$\mathcal{B}^{-b/a+e}, \mathcal{B}_e^{-b/a+2e}, ..., \mathcal{B}_e^{-b/a+te}, ...$$

yields the existence of basic sets:

$$\mathcal{B}^{b/a-e}, \mathcal{B}_e^{b/a-2e}, ..., \mathcal{B}_e^{b/a-te}, ...$$

by Prop 3.6.

4.2. **An Action of the affine Weyl group $\widehat{\mathfrak{S}}_2$.** Let us summarize the different basic sets we have obtained and the parametrizations by the Uglov l-partitions. The results in [28] allow to change the parametrization of the sets. First by [28, Prop. 3.1 (2)] For all $(s_0, s_1) \in \mathbb{Z}^2$, we have

$$\kappa(\Phi_{e,n}^{(s_0,s_1)}) = \Phi_{e,n}^{(s_1,s_0+e)}$$

Moreover, by [28, Prop 3.1 (1)], for all $m \in \mathbb{Z}$, we have:

$$\Phi_{e,n}^{(s_0,s_1)} = \Phi_{e,n}^{(s_0+m,s_1+m)}$$

To summarize, the following tabular gives the basic sets and the parametrizations by the Uglov bipartitions.

Basic set with positive parameters	Associated set of bipartitions	Basic set with negative parameters	Associated set of bipartitions
...	...	...	...
$\mathcal{B}_e^{-b/a+te}$	$\Phi_{e,n}^{(0,d-(t-1)e)}$	$\mathcal{B}_e^{b/a-te}$	$\Phi_{e,n}^{(d-te,0)}$
...	...	...	...
$\mathcal{B}_e^{-b/a+2e}$	$\Phi_{e,n}^{(0,d-e)}$	$\mathcal{B}_e^{b/a-2e}$	$\Phi_{e,n}^{(d-2e,0)}$
$\mathcal{B}_e^{-b/a+e}$	$\Phi_{e,n}^{(0,d)}$	$\mathcal{B}_e^{b/a-e}$	$\Phi_{e,n}^{(d-e,0)}$
$\mathcal{B}_e^{b/a}$	$\Phi_{e,n}^{(d,0)}$	$\mathcal{B}_e^{-b/a}$	$\Phi_{e,n}^{(0,d+e)}$
$\mathcal{B}_e^{b/a+e}$	$\Phi_{e,n}^{(d+e,0)}$	$\mathcal{B}_e^{-b/a-e}$	$\Phi_{e,n}^{(0,d+2e)}$
$\mathcal{B}_e^{b/a+2e}$	$\Phi_{e,n}^{(d+2e,0)}$	$\mathcal{B}_e^{-b/a-2e}$	$\Phi_{e,n}^{(0,d+3e)}$
...	...	...	...
$\mathcal{B}_e^{b/a+te}$	$\Phi_{e,n}^{(d+te,0)}$	$\mathcal{B}_e^{-b/a-te}$	$\Phi_{e,n}^{(0,d+(t+1)e)}$
...	...	...	...

Following [29], we set

$$\mathcal{F} = \{\pm b/a + te \mid t \in \mathbb{Z}\}$$

Let $\widehat{\mathfrak{S}}_2$ be the extended affine symmetric group with generators σ and y_0, y_1 and relations

$$y_0 y_1 = y_1 y_0, \quad \sigma^2 = 1, \quad y_0 = \sigma y_1 \sigma.$$

One can define an action of $\widehat{\mathfrak{S}}_2$ on $\mathcal{F}$ determined by the following identities. For all $t \in \mathbb{Z}$, we set:

$$\sigma.(b/a + te) = -b/a - te, \quad y_0.(b/a + te) = b/a + (t+1)e,$$

$$y_0.(-b/a + te) = -b/a + (t+1)e$$

One can easily check that this action is well defined. The above dicussion yields the existence of an action on the basic sets by setting for all $w \in \widehat{\mathfrak{S}}_2$ and $\gamma \in \mathcal{F}$:

$$w.\mathcal{B}_e^{\gamma} = \mathcal{B}_e^{w.\gamma}$$

By Prop 3.6, we have:

$$\begin{aligned} \sigma.\mathcal{B}_e^{\gamma} &= \kappa(\mathcal{B}_e^{\gamma}) \\ &= \left\{(\lambda^0, \lambda^1) \mid (\lambda^1, \lambda^0) \in \mathcal{B}_e^{\gamma}\right\} \end{aligned}$$

Hence, to describe the action of $\widehat{\mathfrak{S}}_2$ on $\mathcal{F}$, it suffices to understand the action of y_0 on an arbitrary basic set. By the results in [28] (see also the generalizations in [29]) together with Thm 4.1, the action of y_0 corresponds to a crystal isomorphism. Using the combinatorial study of this isomorphism in this paper, It can be expressed using the map Inv_D defined in the previous section. This is given by the following theorem which uses the combinatorics developed in [28].

Theorem 4.3. *Assume that $\gamma \in \mathcal{F}$. Then there exists $D \in \mathbb{Z}$ such that:*

$$\mathcal{B}_e^{\gamma} = \Phi_{e,n}^{(D-e,0)}$$

To describe the action of y_0, one can assume that $D \geq 0$, then we have

$$y_0.\mathcal{B}_e^{\gamma} = \mathrm{Inv}_D(\mathcal{B}_e^{\gamma})$$

Proof. Let $\gamma \in \mathcal{F}$. By Thm 4.1, there exists $D \in \mathbb{Z}$ such that $\mathcal{B}_e^{\gamma} = \Phi_{e,n}^{(D-e,0)}$. Since we know that $\sigma.\mathcal{B}_e^{\delta} = \kappa(\mathcal{B}_e^{\delta})$ for all $\delta \in \mathcal{F}$, one can assume that $D \geq 0$.

Let $(\lambda^0, \lambda^1) \in \mathcal{B}_e^{\gamma} = \Phi_{e,n}^{(D-e,0)}$. Then by [28, Prop. 3.1], we have $(\lambda^1, \lambda^0) \in \Phi_{e,n}^{(0,D)}$. Using [28, Prop. 4.1], we deduce that $(\lambda^0, \lambda^1) \in \mathrm{Std}(D)$. Again, by [28, Prop. 4.1], we have $\kappa(\mathrm{Inv}_D(\lambda^0, \lambda^1)) \in \Phi_{e,n}^{(0,D+e)}$ which implies that $\mathrm{Inv}_D(\lambda^0, \lambda^1) \in \Phi_{e,n}^{(D,0)}$. The map sending $(\lambda^0, \lambda^1) \in \Phi_{e,n}^{(D-e,0)}$ to $\mathrm{Inv}_D(\lambda^0, \lambda^1) \in \Phi_{e,n}^{(D,0)}$ is a bijection. □

Hence, remarkably, this action does not depend on e but only on D ! This will be developed in the following section.

Remark 4.4. Assume that $b/a = -b/a + e$ then $2b = ae$ and we have $q_0^{2b} = 1$. Then we have two cases to consider

- if $q_0^b = 1$ then $q_0^{ad} = -1$ implies that e is even and $d = e/2$ which is impossible because then $q_0^b = q_0^{ae/2} = -q_0^{ae/2}$.
- if $q_0^b = -1$ then $q_0^{ad} = 1$ implies $d = e$ and $d = 0$.

In this case, note that $\Phi_{e,n}^{(d,0)} = \Phi_{e,n}^{(0,d)}$ and then $\mathcal{B}_e^{b/a} = \mathcal{B}_e^{-b/a+e}$. Hence the above result is coherent with this case.

4.3. **Factorization of the decomposition map.** The aim of this section is to give an interpretation of Prop. 4.3 in terms of constructible representations. We here keep the notations of this proposition.

Let $\mathcal{H}(W,S,\{Q,q^a\})$ be the generic Hecke algebra with parameters Q and q^a (where Q and q are indeterminates). We consider a first specialization:

$$\theta_q : \mathbb{Z}[Q^{\pm 1/2}, q^{\pm 1/2}] \to \mathbb{Q}(q^{\frac{1}{2}})$$

We obtain a well defined decomposition map:

$$d_{\theta_q} : R(\mathcal{H}_{\mathbb{Q}(Q^{1/2},q^{1/2})}(W,S,\{Q,q^a\})) \to R(\mathcal{H}_{\mathbb{Q}(q^{1/2})}(W,S,\{-q^{D.a},q^a\}))$$

and an associated decomposition matrix D_{θ_q}. We also have a specialization

$$\theta : A \to k$$

We obtain a well defined decomposition map:

$$d_\theta : R(\mathcal{H}_{\mathbb{Q}(q^{1/2})}(W,S,\{-q^{D.a},q^a\})) \to R(\mathcal{H}_{\mathbb{Q}(q_0^{1/2})}(W,S,\{-q_0^{D.a},q_0^a\}))$$

and an associated decomposition matrix D_θ. On the other hand, one can also define a specialization

$$\theta_1 : \mathbb{Z}[Q^{\pm 1/2}, q^{\pm 1/2}] \to \mathbb{Q}(q_0^{1/2})$$

We obtain a well defined decomposition map:

$$d_{\theta_1} : R(\mathcal{H}_{Q^{\pm 1/2},q^{\pm 1/2}}(W,S,\{Q,q^a\})) \to R(\mathcal{H}_{\mathbb{Q}(q_0^{1/2})}(W,S,\{-q_0^{D.a},q_0^a\}))$$

and an associated decomposition matrix D_{θ_1}.

Theorem 4.5 (Geck [10], [14] Geck-Rouquier [22])**.** *The following diagram is commutative*

$$\begin{array}{ccc} R(\mathcal{H}_{\mathbb{Q}(q^{1/2},Q^{1/2})}(W,S,\{Q,q^a\})) & \xrightarrow{\quad d_{\theta_1}\quad} & R(\mathcal{H}_{\mathbb{Q}(q_0^{1/2})}(W,S,\{-q_0^{Da},q_0^a\})) \\ & {\scriptstyle d_{\theta_q}}\searrow \qquad \nearrow{\scriptstyle d_\theta} & \\ & R(\mathcal{H}_{\mathbb{Q}(q^{1/2})}(W,S,\{-q^{Da},q^a\})) & \end{array}$$

In other words, we have:

$$D_{\theta_1} = D_{\theta_q} D_\theta$$

The following gives a first consequence of this result which can be also easily checked using the results of the previous sections and the properties of Uglov biartitions.

Corollary 4.6. *We have:*

$$\mathcal{B}_e^\gamma \subset \mathcal{B}_\infty^\gamma.$$

Proof. This follows directly from the above theorem. □

Proposition 4.3 and the above result show that the bijection between $y_0.\mathcal{B}_e^\gamma$ and $\mathcal{B}_e^\gamma$ is "controlled" by the matrix of the constructible representations through the above factorization in the following sense.

Note that $\gamma < D < \gamma + e$. We have a bijection

$$\Psi : \mathcal{B}_\infty^\gamma \to \mathcal{B}_\infty^{\gamma+e}$$

which is naturally defined using D_{θ_q}. Consider a constructible character and the expansion of it in the standard basis. This is given by a column of D_{θ_q}. Let $\boldsymbol{\lambda}$ be the element appearing in this column with non zero coefficient and with minimal value with respect to a^γ. Then $\boldsymbol{\lambda} \in \mathcal{B}_\infty^\gamma$. Let $\boldsymbol{\mu}$ be the element appearing in this column with non zero coefficient and with minimal value with respect to $a^{\gamma+e}$. Then $\boldsymbol{\mu} \in \mathcal{B}_\infty^{\gamma+e}$. We then set $\Psi(\boldsymbol{\lambda}) = \boldsymbol{\mu}$. Combining this with the above result we get

$$\Psi(\mathcal{B}_e^\gamma) = \mathcal{B}_e^{\gamma+e}.$$

References

[1] S.Ariki, Representations of quantum algebras and combinatorics of Young tableaux. University Lecture Series, 26. American Mathematical Society, Providence, RI, 2002.

[2] C.Bonnafé, Two-sided cells in type B (asymptotic case), J. Algebra 304 (2006), 216-236.

[3] C.Bonnafé, Semicontinuity properties of Kazhdan-Lusztig cells, preprint 2008 **arXiv:0808.3522**.

[4] C.Bonnafé and L.Iancu, Left cells in type B_n with unequal parameters, Represent. Theory 7 (2003), 587-609.

[5] M.Chlouveraki, Blocks and families for cyclotomic Hecke algebras, to appear in Springer Lecture Notes in Mathematics.

[6] M.Chlouveraki and N.Jacon, Schur elements and basic sets for cyclotomic Hecke algebras, preprint 2009 **arXiv:0910.4931**.

[7] R.Dipper and G. D.James, Representations of Hecke algebras of general linear groups, Proc. London Math. Soc. **52** (1986), 20–52.

[8] R.Dipper and G. D.James, Representations of Hecke algebras of type B_n . J. Algebra 146, 454-481 (1992)

[9] J.Du, B.Parshall and L.Scott, Stratifying endomorphism algebras associated to Hecke algebras, J. Algebra **203** (1998), 169–210.

[10] M.Geck, Brauer trees of Hecke algebras. Comm. Algebra (1992), 2937-2973.

[11] M.Geck, The decomposition numbers of the Hecke algebra of type E_6. Comp. Math., 61, no. 204: 889-899, 1993.

[12] M.Geck, Kazhdan-Lusztig cells and decomposition numbers. Represent. Theory **2** (1998), 264–277 (electronic).

[13] M.Geck, On the representation theory of Iwahori-Hecke algebras of extended finite Weyl groups. Representation theory (electronic journal), 4 : 370-397, 2000.

[14] M.Geck, On the number of simple modules of Iwahori-Hecke algebras of finite Weyl groups. Bul. Stiit. Univ. Baia Mare, Ser. B 16 (2000), 235-246.

[15] M.Geck, Modular representations of Hecke algebras. Group representation theory, 301–353, EPFL Press, Lausanne, 2007.

[16] M.Geck, Relative Kazhdan-Lusztig cells. Represent. Theory 10 (2006), 481–524

[17] M.Geck and L.Iancu, Lusztig's a-function in type B_n in the asymptotic case. Nagoya Math. J. 182 (2006), 199–240.

[18] M.Geck and N.Jacon, Canonical basic sets in type B_n. J. Algebra 306 (2006), no. 1, 104–127.

[19] M.Geck and N.Jacon, Representations of Hecke algebras at roots of unity, monograph, in preparation.

[20] M.Geck and K.Lux, The decomposition numbers of the Hecke algebra of type F_4. Manuscripta Math., 70, no. 3: 285-306, 1991.

[21] M.Geck and G.Pfeiffer, Characters of finite Coxeter groups and Iwahori-Hecke algebras. London Mathematical Society Monographs. New Series, 21. The Clarendon Press, Oxford University Press, New York, 2000. .

[22] M.Geck and R.Rouquier, Centers and simple modules for Iwahori-Hecke algebras. In: Finite reductive groups, related structures and representations (M. Cabanes, ed.), Progress in Math. 141, pp. 251-272. Birkhauser, Boston (1997).

[23] M.Geck and R.Rouquier, Filtrations on projective modules for Iwahori–Hecke algebras. *In*: Modular Representation Theory of Finite Groups (Charlottesville, VA, 1998; eds. M. J. Collins, B. J. Parshall and L. L. Scott), p. 211–221, Walter de Gruyter, Berlin 2001.

[24] N.Jacon, On the parametrization of the simple modules for Ariki-Koike algebras at roots of unity. J. Math. Kyoto Univ. 44 (2004), no. 4, 729–767.

[25] N.Jacon, Sur les représentations modulaires des algèbres de Hecke de type D_n. J. Algebra 274 (2004), no. 2, 607–628.

[26] N.Jacon, Canonical basic sets for Hecke algebras, Infinite-dimensional aspects of representation theory and applications, 33–41, Contemp. Math., 392, Amer. Math. Soc., Providence, RI, 2005

[27] N.Jacon, Crystal graphs of higher level q-deformed Fock spaces, Lusztig a-values and Ariki-Koike algebras. Algebr. Represent. Theory 10 (2007), no. 6, 565–591.

[28] N.Jacon, Crystal graphs of irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of level two and Uglov bipartitions. J. Algebraic Combin. 27 (2008), no. 2, 143–162.

[29] N.Jacon and C.Lecouvey, Crystal isomorphisms for irreducible $\mathcal{U}_v(\widehat{\mathfrak{sl}}_e)$-modules of higher level. to appear in Algebras and Rep Theory

[30] B.Leclerc and H.Miyachi, Constructible characters and canonical bases. J. Algebra 277 (2004), no. 1, 298–317.

[31] G.Lusztig, A class of irreducible representations of a Weyl group, Indag. Math., 41 (1979), 323-335.

[32] G.Lusztig, Characters of reductive groups over a finite field, Annals of Math. Studies, Princeton Univ. Press 1984.

[33] G.Lusztig, Hecke algebras with unequal parameters. CRM Monograph Series, 18. American Mathematical Society, Providence, RI, 2003.

[34] J.Müller, Zerlegungszahlen für generische Iwahori-Hecke Algebren von exzeptionellem Typ. PhD Thesis, Aachen, 1995.

[35] J.Müller, Decomposition numbers for generic Iwahori-Hecke algebras of non-cristallographic type. J. Algebra, 125-149, 189, 1997.

[36] D.Uglov, Denis Canonical bases of higher-level q-deformed Fock spaces and Kazhdan-Lusztig polynomials. Physical combinatorics (Kyoto, 1999), 249–299, Progr. Math., 191, Birkhäuser Boston, Boston, MA, 2000.