

# Linear combinations of orders

Emmanuel Beffara

November 3, 2009

**Abstract.** This note explores some properties of a notion of linear combination of partial order relations over a given set. A bilinear form is used to represent compatibility of orders, and we study combinations up to equivalence through this form. A precise description of the quotient space is provided according to the nature of the semiring of coefficients.

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>General setting</b>	<b>2</b>
<b>3</b>	<b>Some properties of partial orders</b>	<b>2</b>
<b>4</b>	<b>Total orders</b>	<b>4</b>
<b>5</b>	<b>Weak orders</b>	<b>5</b>
<b>6</b>	<b>Orders with equivalence</b>	<b>7</b>
<b>7</b>	<b>Further remarks</b>	<b>8</b>

## 1 Introduction

The motivation behind this note is the study interactive programs in the framework of algebraic process calculi [1, 2]. (TODO: cite event structures [6] and the notion of testing, reference the theory of scheduling, something about weak orders [3] and even Lamport [4]) In this context, programs are decomposed into linear combinations of elementary processes, called traces, which correspond to possible interaction scenarios. The interaction of two programs is in turn decomposed into the interactions of the traces that compose them. Traces are partially ordered sets of elementary actions, and the interaction of two traces consists in finding a pairing of their actions so that a common scheduling exists.

The basic notion that we study is thus compatibility of orders. Two partial order relations  $R$  and  $S$  over a given set  $X$  are compatible if there is a partial order that extends them both. We denote by  $\varphi$  the characteristic function of this relation, i.e.  $\varphi(R, S)$  is 1 if the partial orders  $R$  and  $S$  are compatible and 0 otherwise.

Considering a finite set  $X$  and an arbitrary semiring  $\mathbb{k}$ , we first consider the free  $\mathbb{k}$ -module over the partial order relations on  $X$ , in other words the set of formal linear combinations of order relations. The compatibility function extends as a bilinear form over this space, and the point of this note is to study the properties of this bilinear form. It happens that  $\varphi$  is always degenerate, and the study aims at describing the quotient of the space of formal combinations of orders by the kernel of  $\varphi$ .

In the case where  $\mathbb{k}$  is a tropical semiring (i.e. a semiring where addition is idempotent), we show that the quotient space is generated by total order relations. In the case where  $\mathbb{k}$  is a ring of characteristic 0, we show that the kernel of  $\varphi$  is generated by a single equation that relates the four non-trivial orders over three points, and that a basis of the quotient is given by the so-called weak orders, i.e. orders in which the incomparability relation is an equivalence.

## 2 General setting

For a partial order relation  $R$ , we use the notation  $x \leq_R y$  as equivalent to  $(x, y) \in R$ . The notations  $<_R, \geq_R, >_R$  are defined as expected. We denote by  $|_R$  the incomparability relation:  $x |_R y$  if and only if neither  $x \leq_R y$  nor  $y \leq_R x$ . We write  $x \parallel_R y$  if  $x = y$  or  $x |_R y$ . If there is no ambiguity, we may omit the subscript  $R$  in these notations.

The set of all partial order relations over a set  $X$  is written  $\mathcal{O}(X)$ . Two partial order relations  $R$  and  $S$  over a set  $X$  are called compatible, written  $R \sim S$ , if their union is acyclic. In this case, we call  $R \vee S$  the smallest order relation that contains them both, which is the transitive closure of  $R \cup S$ . We write  $R \not\sim S$  if  $R$  and  $S$  are not compatible.

The notation  $[a < b]$ , for  $a, b \in X$ , represents the smallest partial order for which  $a < b$ , that is  $\{(x, x) \mid x \in X\} \cup \{(a, b)\}$ . The notation extends to more complicated formulas, for instance  $[a < b < c]$  is the smallest partial order for which  $a < b$  and  $b < c$ .

Let  $\mathbb{k}$  be a unitary semiring. For an arbitrary set  $X$ , a formal linear combination over  $X$  is a function from  $X$  to  $\mathbb{k}$  with finite support. Formal linear combinations over  $X$  form the free  $\mathbb{k}$ -module over  $X$ . If  $X$  is finite, then the set of formal linear combinations is the  $\mathbb{k}$ -module  $\mathbb{k}^{\mathcal{O}(X)}$ .

**Definition 1.** Let  $\varphi$  be the symmetric bilinear form over  $\mathbb{k}^{\mathcal{O}(X)}$  such that, for all  $R, S \in \mathcal{O}(X)$ ,  $\varphi(R, S) = 1$  if  $R$  and  $S$  are compatible and 0 otherwise, that is  $\varphi(u, v) = \sum_{R \sim S} u(R)v(S)$ . Let  $\varphi_1$  be the homomorphism that maps each combination  $u$  to the linear form  $\varphi(u, \cdot)$ , which maps each  $v$  to  $\varphi(u, v)$ .

**Definition 2.** Two vectors,  $u, v \in \mathbb{k}^{\mathcal{O}(X)}$  are equivalent, written  $u \sim v$ , if  $\varphi(u, \cdot) = \varphi(v, \cdot)$ . The space of combinations of orders over  $X$  is  $E(X) = \mathbb{k}^{\mathcal{O}(X)} / \sim$ .

It is clear that addition and scalar multiplication are compatible with the relation  $\sim$ , hence  $E(X)$  is a module over the semiring  $\mathbb{k}$  and the function that maps each combination to its equivalence class is linear. When  $\mathbb{k}$  is a ring,  $E(X)$  is obviously the quotient of  $\mathbb{k}^{\mathcal{O}(X)}$  by the kernel of  $\varphi_1$ .

As an abuse of notation, an order relation  $R \in \mathcal{O}(X)$  is identified with its characteristic function  $\chi_R \in \mathbb{k}^{\mathcal{O}(X)}$  and also with the equivalence class of this function in  $E(X)$ .

Our aim is to describe the  $\mathbb{k}$ -module  $E(X)$ , in particular by providing a basis. However, the notion of basis is not obvious in this context, because linear independence does not have a unique definition. The one we use is the following:

**Definition 3.** Let  $\mathbb{k}$  be a semiring and  $E$  a module over  $\mathbb{k}$ . A family  $u_1, \dots, u_n$  is linearly independent in  $\mathbb{k}$  if, for any two families  $\lambda_1, \dots, \lambda_n \in \mathbb{k}$  and  $\mu_1, \dots, \mu_n \in \mathbb{k}$ , if  $\sum_{i=1}^n \lambda_i u_i = \sum_{i=1}^n \mu_i u_i$  then for all  $i$ ,  $\lambda_i = \mu_i$ . A basis of  $E$  is a linearly independent generating family.

## 3 Some properties of partial orders

**Lemma 1.** Let  $X$  be an arbitrary set. For all  $R, S \in \mathcal{O}(X)$ ,  $R \sim S$  if and only if there is a total order  $T$  such that  $R \subseteq T$  and  $S \subseteq T$ .

*Proof.* If  $R \sim S$ , then by definition  $R \cup S$  is acyclic so  $R \vee S$  is a partial order over  $X$  with  $R \subseteq R \vee S$  and  $S \subseteq R \vee S$ . Since  $R \vee S$  is a partial order, it has a topological ordering  $T$ , that is a total order that contains it, and clearly  $R \subseteq T$  and  $S \subseteq T$ . Reciprocally, if  $R \subseteq T$  and  $S \subseteq T$  for some total order  $T$ , then obviously  $R \cup S$  is acyclic since  $T$  is acyclic.  $\square$

**Lemma 2.** Let  $X$  be an arbitrary set. For all  $R \in \mathcal{O}(X)$  and  $a, b \in X$  with  $a \neq b$ ,  $a |_R b$  if and only if  $R \sim [a < b]$  and  $R \sim [b < a]$ .

*Proof.* If  $R$  is incompatible with  $[a < b]$ , then  $R \cup [a < b]$  has a cycle, so there exists a sequence  $a_0, \dots, a_n$  such that  $a_n = a_0$  and for each  $i < n$ ,  $a_i <_R a_{i+1}$  or  $(a_i, a_{i+1}) = (a, b)$ . Since  $R$  is acyclic, the couple  $(a, b)$  must occur at least once in the sequence. We may assume that it occurs at least twice, for instance by repeating the cycle twice. Between two successive occurrences of  $(a, b)$  we have a sequence  $b = a_i <_R a_{i+1} <_R \dots <_R a_{i+j} = a$ , hence  $b <_R a$ . Therefore, by contraposition,  $a \mid_R b$  implies  $R \sim [a < b]$ . It implies  $R \sim [b < a]$  by the same argument.

Reciprocally, assume  $R \sim [a < b]$  and  $R \sim [b < a]$ . Then there are two total orders  $T$  and  $U$  that contain  $R$  and with  $[a < b] \subseteq T$  and  $[b < a] \subseteq U$ . Therefore we have  $R \subseteq T \cap U$ . By  $a <_T b$  and  $b <_U a$  we deduce  $a \mid_{T \cap U} b$  hence  $a \mid_R b$ .  $\square$

**Proposition 1.** *Let  $(X, \leq)$  be a partially ordered set. The following conditions are equivalent:*

- *For all  $x, y, z \in X$ , if  $x < y$  then  $x < z$  or  $z < y$ .*
- *The relation  $\parallel$  is an equivalence.*
- *There is a totally ordered set  $(Y, \leq)$  and a function  $f : X \rightarrow Y$  such that, for all  $x, y \in X$ ,  $x < y$  if and only if  $f(x) < f(y)$ .*

*A partial order that satisfies these conditions is called a weak order. Let  $\mathcal{W}(X)$  be the set of weak orders over  $X$ .*

*Proof.* Firstly, assume that for all  $x, y, z \in X$ , if  $x < y$  then  $x < z$  or  $z < y$ . It is clear that  $\parallel$  is always reflexive and symmetric. Let  $x, y, z \in X$  such that  $x \parallel z$  and  $z \parallel y$ . If  $x < y$ , then by hypothesis we must have  $x < z$  or  $z < y$ , which contradicts the hypothesis on  $x, y, z$ . Similarly we cannot have  $y < x$ , so  $x \parallel y$ . Therefore  $\parallel$  is transitive and it is an equivalence relation.

Secondly, assume  $\parallel$  is an equivalence relation. Let  $Y$  be the set of equivalence classes of  $\parallel$ . Define the relation  $\sqsubseteq$  on  $Y$  as  $A \sqsubseteq B$  if  $a \leq b$  for some  $a \in A$  and  $b \in B$ . The relation  $\sqsubseteq$  is reflexive since for all  $A \in Y$ , for any  $a \in A$  we have  $a \leq a$  so  $A \sqsubseteq A$ . Assume  $A \sqsubseteq B$  and  $B \sqsubseteq A$  for some  $A, B \in Y$ , then there are  $a, a' \in A$  and  $b, b' \in B$  such that  $a \leq b$  and  $b' \leq a'$ ; if  $a < b'$  then  $a < a'$  which contradicts  $a \parallel a'$ , similarly if  $b' < a$  then  $b' < b$  which contradicts  $b' \parallel b$ , so  $a \parallel b'$ , which implies that  $A$  and  $B$  are the same class, therefore  $\sqsubseteq$  is antisymmetric. Assume  $A \sqsubseteq B$  and  $B \sqsubseteq C$  for some  $A, B, C \in Y$ , then there are  $a \in A$ ,  $b, b' \in B$  and  $c \in C$  such that  $a \leq b$  and  $b' \leq c$ ; if  $a \parallel c$  then  $A = C$  hence  $A \sqsubseteq C$ , otherwise we must have  $a < c$  or  $c < a$ , but the second case implies  $b' \leq c < a \leq b$  which contradicts  $b \parallel b'$ , so  $a < c$  and  $A \sqsubseteq C$ , hence  $\sqsubseteq$  is transitive. Totality is immediate: if  $A$  and  $B$  are two distinct classes, then every pair  $(a, b) \in A \times B$  is comparable. Let  $f$  be the function that maps each element of  $X$  to its class. If  $x < y$  then  $f(x) \sqsubset f(y)$  by definition. Reciprocally, if  $f(x) \sqsubset f(y)$ , then  $x$  and  $y$  must be comparable (since they are in distinct classes), and  $y < x$  would imply  $f(y) \sqsubset f(x)$ , so  $x < y$ .

Finally, assume there is  $f : X \rightarrow Y$  where  $Y$  is totally ordered such that  $x < y$  if and only if  $f(x) < f(y)$ . Let  $x, y, z$  be such that  $x < y$ , then  $f(x) < f(y)$ . Since the order on  $Y$  is total, we must have either  $f(x) < f(z)$  or  $f(z) < f(y)$  (or both), hence  $x < z$  or  $z < y$ .  $\square$

**Definition 4.** Let  $R \in \mathcal{O}(X)$ . Two elements  $a, b \in X$  are equivalent in  $R$ , written  $a \sim_R b$ , if for all  $c \in X \setminus \{a, b\}$ ,  $a <_R c$  if and only if  $b <_R c$ , and  $c <_R a$  if and only if  $c <_R b$ . For a pair  $a \sim_R b$  with  $a \neq b$ , let  $R/(a \sim b)$  be the order  $R \cap (X \setminus \{b\})^2$  over  $X \setminus \{b\}$ .

**Lemma 3.** *For all  $R \in \mathcal{W}(X)$ , for all  $a, b \in X$ , if  $a \mid_R b$  then  $a \sim_R b$ .*

*Proof.* Assume that  $a \mid b$  and let  $c$  be an element of  $X$  distinct from  $a$  and  $b$ . If  $a < c$  then, by definition, we must have  $a < b$  or  $b < c$ , but  $a \mid b$  so  $b < c$ . The same argument proves that  $b < c$  implies  $a < c$ , and that  $c < a$  if and only if  $c < b$ , hence  $a \sim b$ .  $\square$

**Definition 5.** Let  $a, b \in X$  with  $a \neq b$ . For each  $R \in \mathcal{O}(X \setminus \{b\})$ , define the relations

$$\begin{aligned} R_{a|b} &:= R \cup \{(x, b) \mid (x, a) \in R\} \cup \{(b, x) \mid (a, x) \in R\}, \\ R_{a < b} &:= R_{a|b} \cup \{(a, b)\}, \\ R_{a > b} &:= R_{a|b} \cup \{(b, a)\}. \end{aligned}$$

Clearly,  $R_{a|b}$ ,  $R_{a < b}$  and  $R_{a > b}$  are partial orders over  $X$  in which  $a$  and  $b$  are equivalent.

**Lemma 4.** Let  $R \in \mathcal{O}(X)$  and let  $a, b$  be two distinct elements of  $X$  such that  $a \sim_R b$ . Let  $S$  be a partial order over  $X \setminus \{b\}$ . Then  $R \sim S_{a|b}$  if and only if  $R/(a \sim b) \sim S$ .

*Proof.* First suppose that  $R \sim S_{a|b}$ . Then there is a total order  $T$  over  $X$  such that  $R \subseteq T$  and  $S_{a|b} \subseteq T$ . Then clearly  $T \cap (X \setminus \{b\})^2$  is a total order over  $X \setminus \{b\}$  that contains  $R/(a \sim b)$  and  $S$ , so  $R/(a \sim b) \sim S$ . Reciprocally, suppose  $R/(a \sim b) \sim S$ , then there is a total order  $T$  over  $X \setminus \{b\}$  that contains  $R/(a \sim b)$  and  $S$ . It is clear that  $T_{a|b}$  contains  $S_{a|b}$  and  $(R/(a \sim b))_{a|b}$ , but the latter is actually equal to  $R$ , by the hypothesis  $a \sim_R b$ , so  $R \sim S_{a|b}$ .  $\square$

**Lemma 5.** Let  $a, b$  be two distinct elements of  $X$ . For all  $R \in \mathcal{W}(X)$  and  $S \in \mathcal{O}(X \setminus \{b\})$ ,

- if  $a <_R b$  then  $R \sim S_{a|b}$  if and only if  $R \sim S_{a < b}$ , moreover  $R \not\sim S_{a > b}$ ,
- if  $a >_R b$  then  $R \sim S_{a|b}$  if and only if  $R \sim S_{a > b}$ , moreover  $R \not\sim S_{a < b}$ ,
- if  $a \sim_R b$  then  $R \sim S_{a|b}$  if and only if  $R \sim S_{a < b}$  if and only if  $R \sim S_{a > b}$ .

*Proof.* If  $a <_R b$ , we have  $R \cup S_{a|b} = R \cup S_{a < b}$ , since  $S_{a|b}$  and  $S_{a < b}$  only differ on  $(a, b)$ , so the compatibility of the two pairs is equivalent to this union being acyclic. The same argument applies to the case  $a >_R b$ . For the case  $a \sim_R b$ , first assume  $R \sim S_{a|b}$  and let  $T = R \vee S_{a|b}$ . By lemma 3 we have  $a \sim_R b$ . If  $a <_T b$  then there exists a sequence  $a = a_0, \dots, a_n = b$  such that for each  $i < n$ ,  $a_i <_R a_{i+1}$  or  $a_i <_{S_{a|b}} a_{i+1}$ , but since  $a$  and  $b$  are equivalent in both  $R$  and  $S_{a|b}$ , we can replace  $b$  with  $a$  in this sequence, which leads to the contradiction  $a <_T a$ . By the same argument we cannot have  $b <_T a$ , so  $a \sim_T b$ . By lemma 2, we thus have  $T \sim [a < b]$  hence  $S_{a < b} = S_{a|b} \vee [a < b] \sim R$ , and similarly  $R \sim S_{a > b}$ . The reverse implications are immediate since  $S_{a|b}$  is included in both  $S_{a < b}$  and  $S_{a > b}$ .  $\square$

## 4 Total orders

Let  $\mathcal{T}(X)$  be the set of total orders over  $X$ .

**Proposition 2.** Total orders form a linearly independent family in  $E(X)$ .

*Proof.* Let  $n$  be the cardinal of  $X$  and let  $\{T_i \mid 1 \leq i \leq n!\}$  be an enumeration of  $\mathcal{T}(X)$ . Let  $u = \sum_{i=1}^{n!} u_i T_i$  and  $v = \sum_{i=1}^{n!} v_i T_i$  be two combinations such that  $u \sim v$ . If  $R$  and  $S$  be two distinct total orders over  $X$ , there exists a pair  $(a, b) \in X^2$  such that  $a <_R b$  and  $b <_S a$ , hence  $R$  and  $S$  are not compatible, so  $\varphi(R, S) = 0$ . As a consequence, for all  $i$ ,  $\varphi(u, T_i) = u_i$  and  $\varphi(v, T_i) = v_i$ , so  $u \sim v$  implies  $u_i = v_i$  and  $u = v$ .  $\square$

However,  $\mathcal{T}(X)$  is not a generating family for  $E(X)$ . The simplest counter-example can be found if  $X$  has two points. Write  $X = \{a, b\}$ , then  $\mathcal{O}(X)$  has three elements:

$$\mathcal{O}(\{a, b\}) = \{[a \mid b], [a < b], [a > b]\}.$$

$$\left[ \begin{array}{c} b \bullet \\ | \\ a \bullet \end{array} \right] \cdot c + \left[ \begin{array}{c} b \bullet \\ | \\ c \bullet \\ | \\ a \bullet \end{array} \right] = \left[ \begin{array}{c} b \bullet \\ | \\ a \bullet \end{array} \right] \cdot c + \left[ \begin{array}{c} b \bullet \\ | \\ a \bullet \\ \diagdown \\ c \bullet \end{array} \right]$$

Figure 1: The basic equation.

Then in the canonical basis  $([a | b], [a < b], [a > b])$ , the matrix of  $\varphi$  is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

If  $\mathbb{k}$  is the field of reals, for instance, then this matrix is invertible, which means that all three orders are linearly independent, hence  $E(X)$  is isomorphic to  $\mathbb{k}^{\mathcal{O}(X)}$  (this isomorphism holds if and only if the cardinal of  $X$  is at most 2, as we shall see below).

There is one case where  $[a < b]$  and  $[b < a]$  do generate  $E(\{a, b\})$ , namely when addition in  $\mathbb{k}$  is idempotent i.e.  $1 + 1 = 1$ . A semiring that has this property is sometimes called *tropical* [5], and the canonical example is that of min-plus semirings.

**Theorem 1.**  $\mathbb{k}$  is tropical if and only if  $\mathcal{T}(X)$  is a basis of  $E(X)$  for all  $X$ .

*Proof.* By proposition 2, we know that  $\mathcal{T}(X)$  is always a linearly independent family, so all we have to prove is that it generates  $E(X)$  if and only if  $\mathbb{k}$  is tropical.

Firstly, assume that  $\mathcal{T}(X)$  generates  $E(X)$ . Then, for  $X = \{a, b\}$ , there are two scalars  $\lambda, \mu \in \mathbb{k}$  such that  $[a | b] \sim \lambda[a < b] + \mu[a > b]$ . Then we have

$$\varphi([a | b], [a < b]) = \lambda\varphi([a < b], [a < b]) + \mu\varphi([a > b], [a < b]) = \lambda$$

but by definition of  $\varphi$  we have  $\varphi([a | b], [a < b]) = 1$ , so  $\lambda = 1$ . Similarly, we get  $\mu = 1$ , and  $[a | b] = [a < b] + [a > b]$ . As a consequence, we have

$$1 = \varphi([a | b], [a | b]) = \varphi([a < b], [a | b]) + \varphi([a > b], [a | b]) = 1 + 1.$$

Hence  $\mathbb{k}$  is tropical.

Reciprocally, assume  $\mathbb{k}$  is tropical. Let  $X$  be an arbitrary finite set and let  $R \in \mathcal{O}(X)$ . Let  $u = \sum_{i=1}^k T_i$  be the sum of all total orders that are compatible with  $R$ . Consider an arbitrary order  $S \in \mathcal{O}(X)$ . Then we have  $\varphi(u, S) = \sum_{i=1}^k \varphi(T_i, S)$  and each term of this sum is 0 or 1. If  $S$  is compatible with  $R$ , then there is a total order  $T$  that extends both  $R$  and  $S$ , so  $T$  is one of the  $T_i$ ; since  $S$  and  $T$  are compatible, the sum contains at least one 1 so  $\varphi(u, S) = 1 = \varphi(R, S)$ . If  $S$  is incompatible with  $R$ , then it is incompatible with any order that contains  $R$ , and in particular it is incompatible with all the  $T_i$ , so  $\varphi(u, S) = 0 = \varphi(R, S)$ . As a consequence we have  $R \sim u$ , which proves that  $\mathcal{T}(X)$  generates  $E(X)$ .  $\square$

## 5 Weak orders

**Definition 6.** Let  $E(X)$  be the quotient of  $\mathbb{k}^{\mathcal{O}(X)}$  by the submodule  $Z(X) := \{f \mid \forall g, \varphi(f, g) = 0\}$ . If  $R$  is a partial order over  $X$ , we denote by  $R$  the class of  $\chi_R$  in  $E$ .

**Proposition 3.** Let  $R$  be a partial order over a set  $X$ . Let  $a, b, c \in X$  such that  $a <_R b$ ,  $a |_R c$  and  $b |_R c$ . Then  $R + (R \vee [a < c < b]) \sim (R \vee [a < c]) + (R \vee [c < b])$ .

*Proof.* We use the following notations:  $R_{\vee} := R \vee [a < c]$ ,  $R_{\wedge} := R \vee [c < b]$ ,  $R_{\perp} := R \vee [a < c < b]$ .

Let  $S$  be a partial order over  $X$ . First remark that  $S \sim R$  if and only if  $S \sim R_{\vee}$  or  $S \sim R_{\wedge}$ . Indeed, assume  $S \sim R$ . By lemma 1, there is a total order  $T$  that contains  $R$  and  $S$ . If  $a <_T c$  then  $[a < c] \subseteq T$  so  $R \vee [a < c] \subseteq T$ , hence  $S \sim R \vee [a < c] = R_{\vee}$ . Otherwise  $c <_T a <_T b$  so  $c <_T b$  then  $S \sim R_{\wedge}$ . Reciprocally, if  $S \sim R_{\vee}$  or  $S \sim R_{\wedge}$  then  $S \sim R$  since  $R$  is included in  $R_{\vee}$  and  $R_{\wedge}$ .

Secondly, remark that  $S \sim R_{\perp}$  if and only if  $S \sim R_{\vee}$  and  $S \sim R_{\wedge}$ . Indeed, assume that  $S \sim R_{\vee} = R \vee [a < c]$  and  $S \sim R_{\wedge} = R \vee [c < b]$ . Let  $S' = S \vee R_{\vee} = S \vee R \vee [a < c]$ . Suppose  $S' \not\sim [c < b]$ , then  $b <_{S'} c$ . By hypothesis we cannot have  $b <_{R \vee S} c$ , so  $(b, c)$  occurs in  $S'$  but not in  $(S \vee R) \cup [a < c]$ , which implies  $b <_{S \vee R} a$ . This contradicts the hypothesis  $a <_R b$ , hence  $S' \sim [c < b]$ , so  $S \sim R \vee [a < c] \vee [c < b] = R_{\perp}$ . The reciprocal implication is immediate since  $R_{\vee} \subseteq R_{\perp}$  and  $R_{\wedge} \subseteq R_{\perp}$ .

As a consequence of the two remarks above, we have  $\varphi(R, S) = 1$  if and only if  $\varphi(R_{\vee}, S) = 1$  or  $\varphi(R_{\wedge}, S) = 1$ , which is equivalent to  $\varphi(R_{\vee} + R_{\wedge}, S) \in \{1, 2\}$ . Moreover,  $\varphi(R_{\vee} + R_{\wedge}, S) = 2$  if and only if  $\varphi(R_{\vee}, S) = 1$  and  $\varphi(R_{\wedge}, S) = 1$ , which is equivalent to  $\varphi(R_{\perp}, S) = 1$ . Therefore  $\varphi(R + R_{\perp}, S) = \varphi(R_{\vee} + R_{\wedge}, S)$ .  $\square$

This property holds for any semiring  $\mathbb{k}$ . In the case where  $\mathbb{k}$  is actually a ring, it allows us to express each pattern from the basic equation in figure 1 as a combination of the others with coefficients 1 and  $-1$ . That means that for each of these patterns:



the set of all orders that do not contain said pattern generates  $E(X)$ . In each case, the forbidden pattern defines a particular class of orders, respectively weak orders (as of proposition 1), orders of height at most 2, forests and reversed forests. It turns out that weak orders form a basis.

**Theorem 2.** *For all finite set  $X$ ,  $\mathcal{W}(X)$  is a basis of  $E(X)$ .*

*Proof.* Let  $Z(X)$  be the kernel of  $\varphi$ , that is the set of all combinations  $u$  such that for all  $R \in \mathcal{O}(X)$ ,  $\varphi(u, R) = 0$ . We actually prove the fact that  $\mathbb{k}^{\mathcal{O}(X)}$  is isomorphic to the direct sum  $\mathbb{k}^{\mathcal{W}(X)} \oplus Z(X)$ , which is equivalent since  $E(X)$  is  $\mathbb{k}^{\mathcal{O}(X)}/Z(X)$  by definition.

We first prove that for all  $R \in \mathcal{O}(X)$  there is an  $R' \in \mathbb{k}^{\mathcal{W}(X)}$  such that  $R - R' \in Z(X)$ . Let  $N(R) = \{(a, b, c) \in X^3 \mid a <_R b, a \mid_R c, b \mid_R c\}$ , we proceed by induction on  $\sharp N(R)$ . If  $R = \emptyset$ , then by proposition 1 we have  $R \in \mathcal{W}(X)$ , so we can set  $R' = R$ . Otherwise, consider a triple  $(a, b, c) \in N(R)$ . Define the orders  $R_1 := R \vee [a < c]$ ,  $R_2 := R \vee [c < b]$  and  $R_3 := R \vee [a < c < b]$ . By proposition 3, we have  $R_1 + R_2 - R_3 - R \in Z(X)$ . Besides, for each  $i \in \{1, 2, 3\}$ , clearly  $N(R_i) \subseteq N(R)$  and  $(a, b, c) \in N(R) \setminus N(R_i)$ , so  $\sharp N(R_i) < \sharp N(R)$ . We can then apply the induction hypothesis to get an  $R'_i \in \mathbb{k}^{\mathcal{W}(X)}$  such that  $R_i - R'_i \in Z(X)$ . We can then conclude by setting  $R' := R'_1 + R'_2 - R'_3$ .

As a consequence we have  $\mathbb{k}^{\mathcal{O}(X)} = \mathbb{k}^{\mathcal{W}(X)} + Z(X)$ , and we now prove that this sum is a direct sum, i.e. that  $\mathbb{k}^{\mathcal{W}(X)} \cap Z(X) = \{0\}$ . We proceed by recurrence on the size of  $X$ . If  $X$  has 0 or 1 element, then  $\mathcal{O}(X)$  only contains the trivial order  $T$ , and  $\varphi(T, T) = 1 \neq 0$ , so  $Z(X) = \{0\}$  and the result trivially holds. Now let  $n \geq 2$ , suppose the result holds for all  $X$  with at most  $n - 1$  points, and let  $u \in \mathbb{k}^{\mathcal{W}(X)} \cap Z(X)$ . We now prove that  $u$  is the zero function.

Let  $R$  be a weak order that is not total, let  $a, b \in X$  such that  $a \mid_R b$ . Let  $X' = X \setminus \{b\}$ . Define  $u' \in \mathbb{k}^{\mathcal{W}(X')}$  by  $u'(T) = u(T_{a|b})$ , so that  $u(R) = u'(R/(a \sim b))$ . For any orders  $S \in \mathcal{W}(X)$  and  $T \in \mathcal{O}(X')$ , by lemma 5 we have that  $\varphi(S, T_{a < b} + T_{a > b} - T_{a|b})$  is 0 if  $a$  and  $b$  are comparable in

$S$ , otherwise it is equal to  $\varphi(S, T_{a|b})$ , which is itself equal to  $\varphi(S/(a \sim b), T)$  by lemma 4. Let  $S' = S/(a \sim b)$ , we have

$$\varphi(u, T_{a < b} + T_{a > b} - T_{a|b}) = \sum_{S \in \mathcal{W}(X), a|_S b} u(S) \varphi(S, T_{a|b}) = \sum_{S \in \mathcal{W}(X), a|_S b} u'(S') \varphi(S', T)$$

The mapping  $S \mapsto S/(a \sim b)$  is a bijection from weak orders over  $X$  such that  $a | b$  to weak orders over  $X'$ , so the latter sum is equal to  $\sum_{S' \in \mathcal{W}(X')} u'(S') \varphi(S', T) = \varphi(u', T)$ . Besides,  $u$  is in  $Z(X)$  so  $\varphi(u, T_{a < b} + T_{a > b} - T_{a|b}) = 0$ , which implies  $\varphi(u', T) = 0$ . This holds for all  $T$ , so  $u' \in Z(X')$ . By construction we have  $u' \in \mathbb{k}^{\mathcal{W}(X')}$  so  $u'$  is in  $\mathbb{k}^{\mathcal{W}(X')} \cap Z(X')$ . By the induction hypothesis this is  $\{0\}$ , so  $u' = 0$  and as a consequence we have  $u(R) = u'(R/(a \sim b)) = 0$ .

By the argument above, we thus know that  $u(R) = 0$  as soon as  $R$  is not a total order. In other words,  $u$  is a linear combination of total orders. From proposition 2 we know that total orders are linearly independent, so we can conclude that  $u = 0$ .  $\square$

## 6 Orders with equivalence

Let  $X$  be a finite set and let  $\equiv$  be an equivalence relation on  $X$ . Let  $\mathfrak{S}(X, \equiv)$  be the set of bijections  $\sigma : X \rightarrow X$  such that for all  $a \in X$ ,  $a \equiv \sigma(a)$ . For  $\sigma \in \mathfrak{S}(X, \equiv)$  and  $R \in \mathcal{O}(X)$ , define  $\sigma(R) = \{(\sigma(a), \sigma(b)) \mid (a, b) \in R\}$ .

Two orders  $R, S \in \mathcal{O}(X)$  are equivalent, written  $R \equiv S$ , if there is a permutation  $\sigma \in \mathfrak{S}(X, \equiv)$  such that  $\sigma(R) = S$ . Obviously, any permutation of a weak order is a weak order.

**Definition 7.** The bilinear form  $\varphi_{\equiv}$  over  $\mathbb{k}^{\mathcal{O}(X)}$  is defined as

$$\varphi_{\equiv}(u, v) := \sum_{R, S \in \mathcal{O}(X)} u(R) \cdot v(S) \cdot \#\{\sigma \in \mathfrak{S}(X, \equiv), \sigma(R) \sim S\}$$

The module  $E(X, \equiv)$  is the quotient of  $\mathbb{k}^{\mathcal{O}(X)}$  by the relation  $\sim$  such that  $u \sim v$  when  $\varphi_{\equiv}(u, \cdot) = \varphi_{\equiv}(v, \cdot)$ .

It is clear that, for any equivalent orders  $R \sim S$  we have  $\varphi_{\equiv}(R, \cdot) = \varphi_{\equiv}(S, \cdot)$ , since the sum in the definition of  $\varphi_{\equiv}$  ranges over the whole group of permutations. Therefore we can identify each order with the equivalence class of its characteristic function in  $E(X, \equiv)$ .

**Theorem 3.** *If  $\mathbb{k}$  is an integral domain of characteristic zero, then for any finite set  $X$  and equivalence relation  $\equiv$  over  $X$ ,  $\mathcal{W}(X)/\equiv$  is a basis of  $E(X, \equiv)$ .*

*Proof.* Let  $Z(X, \equiv) = \{u \mid \forall v, \varphi_{\equiv}(u, v) = 0\}$ . For all  $u \in \mathbb{k}^{\mathcal{O}(X)}$  define  $p(u) = \sum_{\sigma \in \mathfrak{S}(X, \equiv)} u \circ \sigma$ . Then by definition of  $\varphi_{\equiv}$ , for all  $u, v \in \mathbb{k}^{\mathcal{O}(X)}$  we have  $\varphi_{\equiv}(u, v) = \varphi(p(u), v) = \varphi(u, p(v))$ . From this we can deduce that  $Z(X) \subseteq Z(X, \equiv)$ , indeed for any  $u \in Z(X)$  and any  $v \in \mathbb{k}^{\mathcal{O}(X)}$  we have  $\varphi_{\equiv}(u, v) = \varphi(u, p(v)) = 0$  by definition of  $Z(X)$ .

Let  $Y$  be the submodule of  $\mathbb{k}^{\mathcal{O}(X)}$  generated by the  $R - S$  for  $R \equiv S$ . For such a pair, we have  $p(R) = p(S)$  hence  $p(R - S) = 0$ , so  $R - S \in Z(X, \equiv)$ , and subsequently  $Y \subseteq Z(X, \equiv)$ . Let  $\psi : \mathcal{W}(X) \rightarrow \mathcal{W}(X)$  be a function such that for all  $R, S \in \mathcal{W}(X)$ , if  $R \equiv S$  then  $\psi(R) = \psi(S)$ , so that  $\psi$  chooses one element in each equivalence class in  $\mathcal{W}(X)$ , and let  $W$  be the image of  $\psi$ . The function  $\psi$  extends naturally as a morphism from  $\mathbb{k}^{\mathcal{W}(X)}$  to  $\mathbb{k}^W$  as  $\psi(u)(R) = \sum_{S \equiv R} u(S)$ , and by construction we have  $\psi(u) - u \in Y$  for all  $u \in \mathbb{k}^{\mathcal{W}(X)}$ . As a consequence we have  $\mathbb{k}^{\mathcal{W}(X)} = \mathbb{k}^W + Y$  and  $\mathbb{k}^{\mathcal{O}(X)} = \mathbb{k}^W + Z(X, \equiv)$ .

Now let  $u \in \mathbb{k}^W \cap Z(X, \equiv)$ . By the remark above,  $u \in Z(X, \equiv)$  is equivalent to  $p(u) \in Z(X)$ . Any permutation of a weak order is a weak order, so  $u \in \mathbb{k}^W$  implies  $p(u) \in \mathbb{k}^{W(X)}$ , and by theorem 2 we get  $p(u) = 0$ . Besides, for any  $R \in \mathcal{W}(X)$ ,

$$p(u)(R) = \sum_{\sigma \in \mathfrak{S}(X, \equiv)} u(\sigma(R)) = \#\{\sigma \mid \sigma(R) = R\} \cdot u(R)$$

since for each  $S \in \mathcal{W}(X)$ ,  $u(S) = 0$  if  $R \equiv S$  and  $R \neq S$ . The coefficient in front of  $u(R)$  is a strictly positive integer (since the set of permutations always contains the identity), so if  $\mathbb{k}$  has characteristic zero and no zero divisors, we get  $u(R) = 0$ . As a consequence  $u = 0$ , so  $\mathbb{k}^W \cap Z(X, \equiv) = \{0\}$ , which implies  $\mathbb{k}^{\mathcal{O}(X)} = \mathbb{k}^W \oplus Z(X, \equiv)$ .

Equivalent orders are mapped to the same class in the quotient  $E(X, \equiv)$  and  $W$  contains one representant for each class in  $\mathcal{W}(X)/\equiv$ , so  $\mathcal{W}(X)/\equiv$  is a basis  $E(X, \equiv)$ .  $\square$

## 7 Further remarks

The bilinear form  $\varphi$  is not a scalar product, because it is not positive. Consider the case  $X = \{a, b\}$ , and call its elements

$$x = [a \mid b] \qquad y = [a < b] \qquad z = [a > b]$$

and let  $x' = x - y - z$ . From the remarks in section 4, we get that

$$\begin{aligned} \varphi(x', x') &= \varphi(x, x) + \varphi(y, y) + \varphi(z, z) - 2\varphi(x, y) - 2\varphi(x, z) + 2\varphi(y, z) \\ &= 1 + 1 + 1 - 2 - 2 + 2 \times 0 = -1 \end{aligned}$$

## References

- [1] Emmanuel BEFFARA. *An algebraic process calculus*. In Proceedings of the twenty-third annual IEEE symposium on logic in computer science (LICS), pages 130–141, 2008.
- [2] Emmanuel BEFFARA. *Quantitative testing semantics for non-interleaving*. Technical report hal-00397551, Institut de Mathématiques de Luminy, April 2009.
- [3] Karel BERTET, Jens GUSTEDT and Michel MORVAN. *Weak-order extensions of an order*. Theoretical Computer Science, 304:249–268, 2003.
- [4] Leslie LAMPORT. *Time, clocks and the ordering of events in a distributed system*. Communications of the ACM, 21(7):558–565, July 1978.
- [5] Jean-Éric PIN. *Tropical semirings*. In Jeremy GUNAWARDENA, editor, Idempotency (Bristol, 1994), Publications of the Newton Institute, volume 11, pages 50–69. Cambridge University Press, 1998.
- [6] Glynn WINSKEL. *Event structures*. In Advances in Petri nets: applications and relationships to other models of concurrency, pages 325–392. Springer Verlag, 1987.