ASYMPTOTIC SHAPE FOR THE CONTACT PROCESS IN RANDOM ENVIRONMENT

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Abstract. The aim of this article is to prove asymptotic shape theorems for the contact process in stationary random environment. These theorems generalize known results for the classical contact process. In particular, if H_t denotes the set of already occupied sites at time t, we show that for almost every environment, when the contact process survives, the set H_t/t almost surely converges to a compact set that only depends on the law of the environment. To this aim, we prove a new almost subadditive ergodic theorem. croissance aléatoire, processus de contact, environnement aléatoire, théorème ergodique presque sous-additif, théorème de forme asymptotique.
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1. Introduction

The aim of this paper is to obtain an asymptotic shape theorem for the contact process in random environment on $\mathbb{Z}^d$. The environment is here given by a collection $(\lambda_e)_{e\in\mathbb{E}^d}$ of random variables indexed by the set $\mathbb{E}^d$ of edges of the lattice $\mathbb{Z}^d$: the random variable λ_e gives the infection rate on edge e, while each site becomes healthy at rate 1. We assume that the law of $(\lambda_e)_{e\in\mathbb{E}^d}$ is stationary and ergodic.

Our main result is the following: if we assume that the minimal value taken by the $(\lambda_e)_{e\in\mathbb{E}^d}$ is above $\lambda_c(\mathbb{Z}^d)$ – the critical parameter for the ordinary contact process on $\mathbb{Z}^d$ – then there exists a norm μ on $\mathbb{R}^d$ such that for almost every environment $\lambda=(\lambda_e)_{e\in\mathbb{E}^d}$, the set H_t of points already infected before time t satisfies:

$$\overline{\mathbb{P}}_\lambda\left(\exists T>0\ t\geq T \Longrightarrow (1-\varepsilon)tA_\mu\subset H_t\subset(1+\varepsilon)tA_\mu\right)=1,$$

where A_μ is the unit ball for μ and $\overline{\mathbb{P}}_\lambda$ is the law of the contact process in the environment λ, conditioned to survive. The growth of the contact process in random environment is thus asymptotically linear in time, and governed by an shape theorem as in the case of the classical contact process on $\mathbb{Z}^d$.

Until now, most of the work devoted to the study of the contact process in random environment focuses on determining conditions for its survival (Liggett [29], Andjel [3], Newman and Volchan [31]) or its extinction (Klein [27]). They also mainly deal with the case of dimension $d=1$. Concerning the linear growth, Bramson, Durrett and Schonmann [6] show that a random environment can gives birth to a sublinear growth. On the contrary, they conjecture that the growth should be of linear order for $d\geq 2$ as soon as the survival is possible, and that an asymptotic shape result should hold.

For the classical contact process, the proof of the shape result mainly falls in two parts:

- The result is first proved for large values of the infection rate λ by Durrett and Griffeath [16] in 1982. They first obtain, for large λ, estimates essentially implying that the growth is of linear order, and then they get the shape result with (almost) subadditive techniques.
- Later, Bezuidenhout and Grimmett [4] show that a supercritical contact process conditioned to survive, when seen on a large scale, stochastically dominates a two-dimensional supercritical oriented percolation. They also indicate how their construction could be used to obtain a shape theorem. This last step essentially consists in proving that the estimates needed in [16] hold for the whole supercritical regime, and is done by Durrett [15] in 1988.

Similarly, in the case of a random environment, proving a shape theorem can also fall into two different parts. The first one, and undoubtedly the most hard one, would be to prove that the growth is of linear order, as soon as survival is possible: this corresponds to the Bezuidenhout-Grimmett result in random environment. The second one, which we tackle here, is to prove a shape theorem under conditions assuring that the growth is of linear order: this is the random environment analogous of the Durrett–Griffeath work. We thus chose to put conditions on the random environment that allow to obtain, with classical techniques, estimates similar to the ones needed in [16] and to focus on the proof of the shape result, which already presents serious additional difficulties when compared to the proof in the classical case.

Asymptotic shape results for random growth models are usually proved with the theory of subadditive processes, initiated by Hammersley and Welsh [20], and more precisely with Kingman's subadditive ergodic theorem [25] and its extensions. The most famous example is the shape result for first passage-percolation on $\mathbb{Z}^d$ (see also different variations of this model: Boivin [5], Garet and Marchand [18], Vahidi-Asl and Wierman [35], Howard and Newmann [23], Howard [22], Deijfen [10]).

The random growth models can be classified in two families. The first and most studied one is composed of the permanent models, in which the occupied set at time t is non-decreasing and extinction is impossible (such as the Richardson models [32], frogs model by Alves and al. [1, 2], branching random walks by Comets and Popov [9]). In these models, the main part of the work is to prove that the growth is of linear order, and the whole convergence result is then obtained by subadditivity.

The second family contains non-permanent models, in which extinction is possible. In this case, we rather look for a shape result under conditioning by the survival. Hammersley [19] himself, from the beginning of the subadditive theory, underlined the difficulties raised by the possibility of extinction. Indeed, if we want to prove that the hitting times $(t(x))_{x\in\mathbb{Z}^d}$ are such that $t(nx)/n$ converges, Kingman's theory requires and stationarity and integrability properties for the collection $t(x)$. Of course, as soon as extinction is possible, the hitting times can be infinite. On the other hand, conditioning on the survival can break independence, stationarity and even subadditivity properties. A first almost subadditive lemma is proposed by Kesten in the discussion of Kingman's paper [25], next improved by Hammersley [19] (page 674). Other sets of assumptions are later proposed (see for instance Derriennic [11], Derriennic and Hachem [12], and Schürger [33, 34]).

The contact process clearly belongs to the second family. Following Bramson and Griffeath [8, 7], it is on Kesten–Hammersley lemma that Durrett and Griffeath rely to prove their shape result. However, their proof contains a certain number of inaccuracies that have partially been corrected in [15]. Our strategy, necessarily different because of the randomness of the environment, thus offers an alternative for the proof of the asymptotic shape theorem for the classical d-dimensional contact process.

Of course, considering a random environment brings extra difficulties. On one hand, working in a given – quenched – environment, we loose all the spatial stationarity properties. On the other hand, working under the annealed probability, we loose the Markovian properties of the contact process. Thus we can not apply the Kesten–Hammersley lemma, which requires both stationarity and a kind of independence. We introduce here a new quantity $\sigma(x)$, that can be seen as a regeneration time, and that represents a time when site x is occupied and has infinitely many descendants. This σ has stationarity and almost subadditive properties that t lacks, and that allows to state the problem in the setting of (almost) subadditive ergodic theory. We then establish, with techniques inspired from Liggett, a general almost subadditive ergodic theorem that gives us an shape result for σ. Finally, by showing that the gap between t and σ is not too large, we transpose to t the shape result obtained for σ.

2. Model and results

2.1. **Environment.** In the following, we denote by $\|.\|_1$ and $\|.\|_\infty$ the norms on $\mathbb{R}^d$ respectively defined by $\|x\|_1 = \sum_{i=1}^d |x_i|$ and $\|x\|_\infty = \max_{1\le i\le d} |x_i|$. The notation $\|.\|$ will be used for an unspecified norm.

We fix $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} \le \lambda_{\max} < +\infty$, where $\lambda_c(\mathbb{Z}^d)$ stands for the critical parameter for the classical contact process in $\mathbb{Z}^d$. In the following, we restrict our study to random environments $\lambda = (\lambda_e)_{e\in\mathbb{E}^d}$ taking their value in $\Lambda = [\lambda_{\min}, \lambda_{\max}]^{\mathbb{E}^d}$. An environment is thus a collection $\lambda = (\lambda_e)_{e\in\mathbb{E}^d} \in \Lambda$.

Let $\lambda \in \Lambda$ be fixed. The contact process $(\xi_t)_{t\ge 0}$ in environment λ is a homogeneous Markov process taking its values in the set $\mathcal{P}(\mathbb{Z}^d)$ of subsets of $\mathbb{Z}^d$. For $x \in \mathbb{Z}^d$ we also use the random variable $\xi_t(z) = 1\!\!1_{\{z\in\xi_t\}}$. If $\xi_t(z) = 1$, we say that z is occupied or infected, while if $\xi_t(z) = 0$, we say that z is empty or healthy. The evolution of the process is as follows:

- an occupied site becomes empty at rate 1,
- an empty site z becomes occupied at rate $\displaystyle\sum_{\|z-z'\|_1=1} \xi_t(z')\lambda_{\{z,z'\}}$,

each of these evolutions being independent from the others. In the following, we denote by $\mathcal{D}$ the set of càdlàg functions from $\mathbb{R}_+$ to $\mathcal{P}(\mathbb{Z}^d)$: it is the set of trajectories for Markov processes with state space $\mathcal{P}(\mathbb{Z}^d)$.

To define the contact process in environment $\lambda \in \Lambda$, we use the Harris construction [21]. It allows to couple contact processes starting from distinct initial configurations by building them from a single collection of Poisson measures on $\mathbb{R}_+$.

2.2. **Construction of the Poisson measures.** We endow $\mathbb{R}_+$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$, and we denote by M the set of locally finite counting measures $m =$

$\sum_{i=0}^{+\infty} \delta_{t_i}$. We endow this set with the σ-algebra $\mathcal{M}$ generated by the applications $m \mapsto m(B)$, where B describes the set of Borel sets in $\mathbb{R}_+$.

We then define the measurable space $(\Omega, \mathcal{F})$ by setting

$$\Omega = M^{\mathbb{E}^d} \times M^{\mathbb{Z}^d} \text{ and } \mathcal{F} = \mathcal{M}^{\otimes \mathbb{E}^d} \otimes \mathcal{M}^{\otimes \mathbb{Z}^d}.$$

On this space, we consider the family $(\mathbb{P}_\lambda)_{\lambda \in \Lambda}$ of probability measures defined as follows: for every $\lambda = (\lambda_e)_{e \in \mathbb{E}^d} \in \Lambda$,

$$\mathbb{P}_\lambda = \left(\bigotimes_{e \in \mathbb{E}^d} \mathcal{P}_{\lambda_e} \right) \otimes \mathcal{P}_1^{\otimes \mathbb{Z}^d},$$

where, for every $\lambda \in \mathbb{R}_+$, $\mathcal{P}_\lambda$ is the law of a punctual Poisson process on $\mathbb{R}_+$ with intensity λ. If $\lambda \in \mathbb{R}_+$, we write $\mathbb{P}_\lambda$ (rather than $\mathbb{P}_{(\lambda)_{e \in \mathbb{E}^d}}$) for the law in deterministic environment with constant infection rate λ.

For every $t \geq 0$, we denote by $\mathcal{F}_t$ the σ-algebra generated by the applications $\omega \mapsto \omega_e(B)$ and $\omega \mapsto \omega_z(B)$, where e ranges over all edges in $\mathbb{E}^d$, z ranges over all points in $\mathbb{Z}^d$, and B ranges over the set of Borel set in $[0, t]$.

2.3. **Graphical construction of the contact process.** This construction is exposed in all details in Harris [21]; we just give here an informal description. Let $\omega = ((\omega_e)_{e \in \mathbb{E}^d}, (\omega_z)_{z \in \mathbb{Z}^d}) \in \Omega$. Above each site $z \in \mathbb{Z}^d$, we draw a time line $\mathbb{R}_+$, and we put a cross at the times given by ω_z. Above each edge $e \in \mathbb{E}^d$, we draw at the times given by ω_e an horizontal segment between the extremities of the edge.

An open path follows the time lines above sites – but crossing crosses is forbidden – and uses horizontal segments to jump from a time line to a neighboring time line: in this description, the evolution of the contact process looks like a percolation process, oriented in time but not in space. For $x, y \in \mathbb{Z}^d$ and $t \geq 0$, we say that $\xi_t^x(y) = 1$ if and only if there exists an open path from $(x, 0)$ to (y, t), then we define:

$$\begin{aligned} \xi_t^x &= \{y \in \mathbb{Z}^d : \ \xi_t^x(y) = 1\}, \\ \forall A \in \mathcal{P}(\mathbb{Z}^d) \quad \xi_t^A &= \bigcup_{x \in A} \xi_t^x. \end{aligned} \tag{1}$$

For instance, we obtain $(A \subset B) \Rightarrow (\forall t \geq 0 \quad \xi_t^A \subset \xi_t^B)$.

When $\lambda \in \mathbb{R}_+^*$, Harris shows that under $\mathbb{P}_\lambda$, the process $(\xi_t^A)_{t \geq 0}$ is the contact process with infection rate λ, starting from initial configuration A. The proof can readily be extended to a non constant $\lambda \in \Lambda$, which allows to define the contact process in environment λ starting from initial configuration A. This is a Feller process, and thus benefits from the strong Markov property.

2.4. **Time translations.** For $t \geq 0$, we define the translation operator θ_t on a locally finite counting measure $m = \sum_{i=1}^{+\infty} \delta_{t_i}$ on $\mathbb{R}_+$ by setting

$$\theta_t m = \sum_{i=1}^{+\infty} 1\!\!1_{\{t_i \geq t\}} \delta_{t_i - t}.$$

The translation θ_t induces an operator on Ω, still denoted by θ_t: for every $\omega \in \Omega$, we set

$$\theta_t \omega = ((\theta_t \omega_e)_{e \in \mathbb{E}^d}, (\theta_t \omega_z)_{z \in \mathbb{Z}^d}).$$

The Poisson point process being translation invariant, every probability measure $\mathbb{P}_\lambda$ is stationary under θ_t. The semi-group property of the contact process has here a stronger trajectorial version: for every $A \subset \mathbb{Z}^d$, for every $s,t \geq 0$, for every $\omega \in \Omega$, we have

(2) $$\xi_{t+s}^A(\omega) = \xi_s^{\xi_t^A(\omega)}(\theta_t\omega) = \xi_s^{\cdot}(\theta_t\omega) \circ \xi_t^A(\omega),$$

that can also be written in the classical markovian way:

$$\forall B \in \mathcal{B}(\mathcal{D}) \quad \mathbb{P}((\xi_{t+s}^A)_{s\geq 0} \in B|\mathcal{F}_t) = \mathbb{P}((\xi_s^{\cdot})_{s\geq 0} \in B) \circ \xi_t^A.$$

We can write in the same way the strong Markov property: if T is an $(\mathcal{F}_t)_{t\geq 0}$ stopping time, then, on the event $\{T < +\infty\}$,

$$\begin{aligned} \xi_{T+s}^A(\omega) &= \xi_s^{\xi_T^A(\omega)}(\theta_T\omega), \\ \forall B \in \mathcal{B}(\mathcal{D}) \quad \mathbb{P}((\xi_{T+s}^A)_{s\geq 0} \in B|\mathcal{F}_T) &= \mathbb{P}((\xi_s^{\cdot})_{s\geq 0} \in B) \circ \xi_T^A. \end{aligned}$$

We recall that $\mathcal{F}_T$ is defined by

$$\mathcal{F}_T = \{B \in \mathcal{F} : \quad \forall t \geq 0 \quad B \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

2.5. **Spatial translations.** The group $\mathbb{Z}^d$ can act on the process and on the environment. The action on the process changes the observer's point of view of the process: for $x \in \mathbb{Z}^d$, we define the translation operator T_x by

$$\forall \omega \in \Omega \quad T_x\omega = ((\omega_{x+e})_{e\in\mathbb{E}^d}, (\omega_{x+z})_{z\in\mathbb{Z}^d}),$$

where $x+e$ the edge e translated by vector x.

Besides, we can consider the translated environment $x.\lambda$ defined by $(x.\lambda)_e = \lambda_{x+e}$. These actions are dual in the sense that for every $\lambda \in \Lambda$, for every $x \in \mathbb{Z}^d$,

(3) $$\forall A \in \mathcal{F} \quad \mathbb{P}_\lambda(T_x\omega \in A) \quad = \quad \mathbb{P}_{x.\lambda}(\omega \in A).$$

Consequently, the law of ξ^x under $\mathbb{P}_\lambda$ coincides with the law of ξ^0 under $\mathbb{P}_{x.\lambda}$.

2.6. **Essential hitting times and associated translations.** For a set $A \subset \mathbb{Z}^d$, we define the life time τ^A of the process starting from A by

$$\tau^A = \inf\{t \geq 0 : \ \xi_t^A = \varnothing\}.$$

For $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we also define the first infection time $t^A(x)$ of site x from set A by

$$t^A(x) = \inf\{t \geq 0 : \ x \in \xi_t^A\}.$$

If $y \in \mathbb{Z}^d$, we write $t^y(x)$ instead of $t^{\{y\}}(x)$. Similarly, we simply write $t(x)$ for $t^0(x)$.

We now introduce the essential hitting time $\sigma(x)$: it is a time where the site x is infected from the origin 0 and also has an infinite life time. This essential hitting time is defined through a family of stopping times as follows: we set $u_0(x) = v_0(x) = 0$ and we define recursively two increasing sequences of stopping times $(u_n(x))_{n\geq 0}$ and $(v_n(x))_{n\geq 0}$ with $u_0(x) = v_0(x) \leq u_1(x) \leq v_1(x) \leq u_2(x) \dots$ as follows:

- Assume that $v_k(x)$ is defined. We set $u_{k+1}(x) = \inf\{t \geq v_k(x) : \ x \in \xi_t^0\}$. If $v_k(x) < +\infty$, then $u_{k+1}(x)$ is the first time after $v_k(x)$ where site x is once again infected; otherwise, $u_{k+1}(x) = +\infty$.
- Assume that $u_k(x)$ is defined, with $k \geq 1$. We set $v_k(x) = u_k(x) + \tau^x \circ \theta_{u_k(x)}$. If $u_k(x) < +\infty$, The time $\tau^x \circ \theta_{u_k(x)}$ is the life time of the contact process starting from x at time $u_k(x)$; otherwise, $v_k(x) = +\infty$.

We then set

$$K(x) = \min\{n \geq 0: \ v_n(x) = +\infty \text{ or } u_{n+1}(x) = +\infty\}. \tag{4}$$

This quantity represents the number of steps before the success of this process: either we stop because we have just found an infinite $v_n(x)$, which corresponds to a time $u_n(x)$ where x is and infected and has infinitely many descendants, or we stop because we have just found an infinite $u_{n+1}(x)$, which says that after $v_n(x)$, site x is never infected anymore.

We then set $\sigma(x) = u_{K(x)}$, and call it the essential hitting time of x. It is of course larger than the hitting time $t(x)$ and can been seen as a regeneration time. We will see that $K(x)$ is almost surely finite, so $\sigma(x)$ is well-defined. At the same time, we define the operator $\tilde{\theta}_x$ on Ω by:

$$\tilde{\theta}_x = \begin{cases} T_x \circ \theta_{\sigma(x)} & \text{if } \sigma(x) < +\infty, \\ T_x & \text{otherwise,} \end{cases}$$

or, more explicitly,

$$(\tilde{\theta}_x)(\omega) = \begin{cases} T_x(\theta_{\sigma(x)(\omega)}\omega) & \text{if } \sigma(x)(\omega) < +\infty, \\ T_x(\omega) & \text{otherwise.} \end{cases}$$

We will mainly deal with the essential hitting time $\sigma(x)$ that enjoys, unlike $t(x)$, some good invariance properties in the survival-conditioned environment. We will also control the difference between $\sigma(x)$ et $t(x)$, which will allow us to transpose to $t(x)$ the results obtained for $\sigma(x)$.

2.7. **Contact process in the survival-conditioned environment.** We now have to introduce the random environment. In the following, we fix a probability measure ν on the sets of environments $\Lambda = [\lambda_{\min}, \lambda_{\max}]^{\mathbb{E}^d}$. We assume that ν is stationary and ergodic under the action of $\mathbb{Z}^d$. This setting naturally contains the case of a deterministic environment $\lambda > \lambda_c(\mathbb{Z}^d)$: we simply take for ν the Dirac measure $(\delta_\lambda)^{\otimes \mathbb{E}^d}$.

For $\lambda \in \Lambda$, we define the probability measure $\overline{\mathbb{P}}_\lambda$ on $(\Omega, \mathcal{F})$ by

$$\forall E \in \mathcal{F} \quad \overline{\mathbb{P}}_\lambda(E) = \mathbb{P}_\lambda(E|\tau^0 = +\infty).$$

It is thus the law of the family of Poisson point processes, conditioned to the survival of the contact process starting from 0. On the same space $(\Omega, \mathcal{F})$, we defined the corresponding annealed probability $\overline{\mathbb{P}}$ by setting

$$\forall E \in \mathcal{F} \quad \overline{\mathbb{P}}(E) = \int_\Lambda \overline{\mathbb{P}}_\lambda(E) \ d\nu(\lambda).$$

In other words, the environment $\lambda = (\lambda_e)_{e \in \mathbb{E}^d}$ where the contact process lives is a random variable with law ν, and it is under the probability measure $\overline{\mathbb{P}}$ that we seek the asymptotic shape theorem.

It could seem more natural to work with the following probability measure:

$$\forall E \in \mathcal{F} \quad \hat{\mathbb{P}}(E) = \mathbb{P}(E|\tau^0 = +\infty) = \frac{\int \overline{\mathbb{P}}_\lambda(E)\mathbb{P}_\lambda(\tau^0 = +\infty)d\nu(\lambda)}{\int \mathbb{P}_\lambda(\tau^0 = +\infty)d\nu(\lambda)}.$$

It appears that our proofs do not work with this probability measure. However, our restrictions on the set Λ of possible environments ensure that $\overline{\mathbb{P}}$ and $\hat{\mathbb{P}}$ are equivalent: the $\overline{\mathbb{P}}-a.s$ asymptotic shape theorem is thus also a $\hat{\mathbb{P}}-a.s$ asymptotic shape theorem.

2.8. **Organization of the paper and results.** In Section 3, we establish the invariance and ergodicity properties. In particular, we prove the following:

Theorem 1. *For every* $x \in \mathbb{Z}^d \backslash \{0\}$*, the measure-preserving dynamical system* $(\Omega, \mathcal{F}, \overline{\mathbb{P}}, \tilde{\theta}_x)$ *is ergodic.*

In Section 4, we study the integrability properties of the family $(\sigma(x))_{x\in\mathbb{Z}^d}$; we also control the discrepancy between $\sigma(x)$ and $t(x)$ and the lack of subadditivity of σ:

Theorem 2. *There exist* $A_5, B_5 > 0$ *such that for any* $\lambda \in \Lambda$*, for any* $x, y \in \mathbb{Z}^d$*,*

$$\forall t > 0 \quad \overline{\mathbb{P}}_\lambda(\sigma(x+y) - (\sigma(x) + \sigma(y) \circ \tilde{\theta}_x) \geq t) \leq A_5 \exp(-B_5\sqrt{t}). \tag{5}$$

This says that the lack of subadditivity of σ is really small: in particular, it does not depend on the considered points. Then, in the same spirit as Kingman [26] and Liggett [28], we prove in Section 5 that for every $x \in \mathbb{Z}^d$, the ratio $\frac{\sigma(nx)}{n}$ converges $\overline{\mathbb{P}} - a.s.$ to a real number $\mu(x)$. The functional $x \mapsto \mu(x)$ can be extended into a norm on $\mathbb{R}^d$, which will characterize the asymptotic shape In the following, A_μ will denote the unit ball for μ. We define the sets

$$\begin{aligned} H_t &= \{x \in \mathbb{Z}^d : \ t(x) \leq t\}, \\ G_t &= \{x \in \mathbb{Z}^d : \ \sigma(x) \leq t\}, \\ K'_t &= \{x \in \mathbb{Z}^d : \ \forall s \geq t \quad \xi^0_s(x) = \xi^{\mathbb{Z}^d}_s(x)\}, \end{aligned}$$

and we denote by $\tilde{H}_t, \tilde{G}_t, \tilde{K}'_t$ their "fattened" versions:

$$\tilde{H}_t = H_t + [0,1]^d, \ \tilde{G}_t = G_t + [0,1]^d \text{ and } \tilde{K}'_t = K'_t + [0,1]^d.$$

We can now state the asymptotic shape result:

Theorem 3 (Asymptotic shape theorem)**.** *For every* $\varepsilon > 0$, $\overline{\mathbb{P}} - a.s.$*, for every* t *large enough,*

$$(1-\varepsilon)A_\mu \subset \frac{\tilde{K}'_t \cap \tilde{G}_t}{t} \subset \frac{\tilde{G}_t}{t} \subset \frac{\tilde{H}_t}{t} \subset (1+\varepsilon)A_\mu. \tag{6}$$

The set $K'_t \cap G_t$ is the coupled zone of the process. Usually, the asymptotic shape result for the coupled zone is rather expressed in terms of $K_t \cap H_t$, where

$$K_t = \{x \in \mathbb{Z}^d : \ \xi^0_t(x) = \xi^{\mathbb{Z}^d}_t(x)\}.$$

Our result also gives the shape theorem for $K_t \cap H_t$, because $K'_t \cap G_t \subset K_t \cap H_t \subset H_t$.

Let us note that the shape result can also be formulated in the following "quenched" terms: for $\nu - a.e.$ environment, we know that on the event "the contact process survives", its growth is governed by (6) for t large enough. We can also give a complete convergence result:

Theorem 4 (Complete convergence theorem)**.** *For every* $\lambda \in \Lambda$*, the contact process in environment* Λ *admits an upper invariant measure* m_λ *defined by*

$$\forall A \subset \mathbb{Z}^d, |A| < +\infty \quad m_\lambda(\omega \supset A) = \lim_{t \to +\infty} \mathbb{P}_\lambda(\xi^{\mathbb{Z}^d}_t \supset A).$$

Then, for every finite set $A \subset \mathbb{Z}^d$ and for $\nu -$ a.e. environment λ, one has

$$\mathbb{P}^A_{\lambda,t} \Longrightarrow \mathbb{P}_\lambda(\tau^A < +\infty)\delta_\varnothing + \mathbb{P}_\lambda(\tau^A = \infty)m_\lambda,$$

where $\mathbb{P}^A_{\lambda,t}$ is the law of ξ^A_t under $\mathbb{P}_\lambda$ and $\Longrightarrow$ stands for the convergence in law.

The proof of this result does not require any new idea, and we just give a hint at the end of Section 6.

As explained in the introduction, in order to prove the asymptotic shape theorem, we need some estimates analogous to the ones needed in the proof by Durrett and Griffeath in the classical case. We set

$$B^x_r = \{y \in \mathbb{Z}^d : \ \|y - x\|_\infty \le r\},$$

and we write B_r instead of B^0_r.

Proposition 5. *There exist $A, B, M, c, \rho > 0$ such that for every $\lambda \in \Lambda$, for every $y \in \mathbb{Z}^d$, for every $t \ge 0$*

$$\mathbb{P}_\lambda(\tau^0 = +\infty) \ \ge \ \rho, \tag{7}$$

$$\mathbb{P}_\lambda(H^0_t \not\subset B_{Mt}) \ \le \ A\exp(-Bt), \tag{8}$$

$$\mathbb{P}_\lambda(t < \tau^0 < +\infty) \ \le \ A\exp(-Bt), \tag{9}$$

$$\mathbb{P}_\lambda\left(t^0(y) \ge \frac{\|y\|}{c} + t, \ \tau^0 = +\infty\right) \ \le \ A\exp(-Bt), \tag{10}$$

$$\mathbb{P}_\lambda(0 \notin K'_t, \ \tau^0 = +\infty) \ \le \ A\exp(-Bt). \tag{11}$$

All these estimates are already available for the classical contact process in the supercritical regime. For large λ, they are established by Durrett and Griffeath [16], and the extension to the entire supercritical regime is made possible thanks to Bezuidenhout and Grimmett's work [4]. For the crucial estimate (9), one can find the detailed proof in Durrett [17] or in Liggett [30]. The need for these estimates explains our restrictions on the possible range of the random environment.

We chose to focus on the stationarity and subadditivity properties of the essential hitting time σ and on the proof of the shape result. We thus admit in Sections 3, 4 and 5 the uniform controls given by Proposition 5, whose proof (via restart arguments) is postponed to Section 6. Section 6 is totally independent of the rest of the paper. Finally, in an appendix, we prove a general (almost) subadditive theorem. As we think it could also be useful in other situations, we present it in a more general form than what is needed for our aim.

3. Properties of $\tilde{\theta}_x$

3.1. **First properties.** We first check that $K(x)$ is almost surely finite and even has a sub-geometrical tail:

Lemma 6. $\forall A \subset \mathbb{Z}^d \quad \forall x \in \mathbb{Z}^d \quad \forall \lambda \in \Lambda \quad \forall n \in \mathbb{N} \quad \mathbb{P}_\lambda(K(x) > n) \le (1-\rho)^n.$

Proof. Remember that ρ is given in (7). Let $\lambda\in\Lambda$ and $n\in\mathbb{N}$. The strong Markov property applied at time $u_{n+1}(x)$ ensures that

$$\begin{aligned}\mathbb{P}_\lambda(K(x)>n+1) &= \mathbb{P}_\lambda(u_{n+2}(x)<+\infty)\\ &\le \mathbb{P}_\lambda(u_{n+1}(x)<+\infty, v_{n+1}(x)<+\infty)\\ &\le \mathbb{P}_\lambda(u_{n+1}(x)<+\infty, \tau^x\circ\theta_{u_{n+1}(x)}<+\infty)\\ &\le \mathbb{P}_\lambda(u_{n+1}(x)<+\infty)\mathbb{P}_\lambda(\tau^x<+\infty)\\ &\le \mathbb{P}_\lambda(u_{n+1}(x)<+\infty)(1-\rho)=\mathbb{P}_\lambda(K(x)>n)(1-\rho),\end{aligned}$$

which prove the lemma. □

Lemma 7. *Let $\lambda\in\Lambda$. $\mathbb{P}_\lambda-a.s.$, for every $x\in\mathbb{Z}^d$,*

$$(K(x)=k) \text{ and } (\tau^0=+\infty) \iff (u_k(x)<+\infty \text{ and } v_k(x)=+\infty), \tag{12}$$

Proof. Let $\lambda\in\Lambda$. By Lemma 6, the number $K(x)$ is $\mathbb{P}_\lambda-a.s.$ finite. Let $k\in\mathbb{N}$: the strong Markov property applied at time $v_k(x)$ ensures that

$$\begin{aligned}&\mathbb{P}_\lambda(\tau^0=+\infty, v_k(x)<+\infty, u_{k+1}(x)=+\infty|\mathcal{F}_{v_k(x)})\\ =\;& 1\!\!1_{\{v_k(x)<+\infty\}}\mathbb{P}_\lambda(\tau^\cdot=+\infty, t^\cdot(x)=+\infty)\circ\xi^0_{v_k(x)}.\end{aligned}$$

Consider now a finite non-empty set $B\subset\mathbb{Z}^d$: with (10), we get

$$\begin{aligned}\mathbb{P}_\lambda(\tau^B=+\infty, t^B(x)=+\infty) &\le \sum_{y\in B}\mathbb{P}_\lambda(\tau^y=+\infty, t^y(x)=+\infty)\\ &\le \sum_{y\in B}\mathbb{P}_{y.\lambda}(\tau^0=+\infty, t^0(x-y)=+\infty)=0.\end{aligned}$$

This gives the direct implication. The reverse one comes from (2). □

Our construction of $\sigma(x)$ is very similar to the restart process exposed in Durrett–Griffeath [16]. The essential difference is that in that paper, the aim is to find, close to x, a point that survives while we require here the point to be exactly at x. Thus, we will be able to describe precisely the law of the contact process starting from x at time $\sigma(x)$, and construct transformations under which $\overline{\mathbb{P}}$ is invariant.

Lemma 8. *Let $x\in\mathbb{Z}^d\backslash\{0\}$, A in the σ-algebra generated by $\sigma(x)$, and $B\in\mathcal{F}$. Then*

$$\forall\lambda\in\Lambda\quad \overline{\mathbb{P}}_\lambda(A\cap(\tilde\theta_x)^{-1}(B))=\overline{\mathbb{P}}_\lambda(A)\overline{\mathbb{P}}_{x.\lambda}(B).$$

Proof. We just have to check that for any $k\in\mathbb{N}^*$, one has

$$\overline{\mathbb{P}}_\lambda(A\cap(\tilde\theta_x)^{-1}(B)\cap\{K(x)=k\})=\overline{\mathbb{P}}_\lambda(A\cap\{K(x)=k\})\overline{\mathbb{P}}_{x.\lambda}(B).$$

Consider a Borel set $A'\subset\mathbb{R}$ such that $A=\{\sigma(x)\in A'\}$. The essential hitting time $\sigma(x)$ is not a stopping time, but we can use the stopping times of the construction:

$$\mathbb{P}_\lambda(\{\tau^0=+\infty\}\cap A\cap(\tilde\theta_x)^{-1}(B)\cap\{K(x)=k\})$$

$$= \mathbb{P}_\lambda(\tau^0=+\infty,\ \sigma(x)\in A',\ T_x\circ\theta_{\sigma(x)}\in B,\ u_k(x)<+\infty,\ v_k=+\infty) \tag{13}$$

$$= \mathbb{P}_\lambda(u_k(x)<+\infty, u_k(x)\in A',\ \ \tau^x\circ\theta_{u_k(x)}=+\infty,\ T_x\circ\theta_{u_k(x)}\in B) \tag{14}$$

$$= \mathbb{P}_\lambda(u_k(x)\in A',\ u_k(x)<+\infty)\mathbb{P}_\lambda(\tau^x=+\infty,\ T_x\in B) \tag{15}$$

$$= \mathbb{P}_\lambda(u_k(x)\in A',\ u_k(x)<+\infty)\mathbb{P}_{x.\lambda}(\{\tau^0=+\infty\}\cap B). \tag{16}$$

For (13), we use Equivalence (12). For (14), we notice that for any stopping time T,

$$\{T<+\infty,\ x\in\xi^0_T,\ \tau^0\circ T_x\circ\theta_T=+\infty\}\subset\{\tau^0=+\infty\}. \tag{17}$$

Equality (15) follows from the strong Markov property applied at time $u_k(x)$, while (16) comes from the spatial translation property (3). Dividing the identity by $\mathbb{P}_\lambda(\tau^0 = +\infty)$, we obtain an identity of the following form:

$$\overline{\mathbb{P}}_\lambda(A \cap (\tilde{\theta}_x)^{-1}(B) \cap \{K(x) = k\}) = \psi(x, \lambda, k, A)\overline{\mathbb{P}}_{x.\lambda}(B),$$

and the number $\psi(x, \lambda, k, A)$ is identified by taking $B = \Omega$. □

Corollary 9. *Let $x, y \in \mathbb{Z}^d$ and $\lambda \in \Lambda$. Assume that $x \neq 0$.*

- *The probability measure $\overline{\mathbb{P}}$ is invariant under the translation $\tilde{\theta}_x$.*
- *Under $\overline{\mathbb{P}}_\lambda$, $\sigma(y)\circ\tilde{\theta}_x$ and $\sigma(x)$ are independent. Moreover, the law of $\sigma(y)\circ\tilde{\theta}_x$ under $\overline{\mathbb{P}}_\lambda$ is the same as the law of $\sigma(y)$ under $\overline{\mathbb{P}}_{x.\lambda}$.*
- *The variables $(\sigma(x) \circ (\tilde{\theta}_x)^j)_{j \geq 0}$ are independent under $\overline{\mathbb{P}}_\lambda$.*

Proof. For the first point, we just apply the previous lemma with $A = \Omega$, then we integrate with respect to λ and use the stationarity of ν.

For the second point, let A', B' be two Borel sets in $\mathbb{R}$ and apply Lemma 8 with $A = \{\sigma(x) \in A'\}$ and $B = \{\sigma(y) \circ \tilde{\theta}_x \in B'\}$.

Let $n \geq 1$ and $A_0, A_1, \ldots, A_n$ be some Borel sets in $\mathbb{R}$. We have:

$$\begin{aligned} & \overline{\mathbb{P}}_\lambda(\sigma(x) \in A_0, \sigma(x) \circ \tilde{\theta}_x \in A_1, \ldots, \sigma(x) \circ (\tilde{\theta}_x)^n \in A_n) \\ = \ & \overline{\mathbb{P}}_\lambda(\sigma(x) \in A_0, (\sigma(x), \ldots, \sigma(x) \circ (\tilde{\theta}_x)^{n-1}) \circ \tilde{\theta}_x \in A_1 \times \cdots \times A_n) \\ = \ & \overline{\mathbb{P}}_\lambda(\sigma(x) \in A_0)\overline{\mathbb{P}}_{x.\lambda}(\sigma(x) \in A_1, \sigma(x) \circ \tilde{\theta}_x \in A_2, \ldots, \sigma(x) \circ (\tilde{\theta}_x)^{n-1} \in A_n), \end{aligned}$$

where the last equality comes from Lemma 8. We recursively obtain

$$\overline{\mathbb{P}}_\lambda\left(\bigcap_{0 \leq j \leq n} \{\sigma(x) \circ (\tilde{\theta}_x)^j \in A_j\}\right) = \prod_{0 \leq j \leq n} \overline{\mathbb{P}}_{jx.\lambda}\left(\sigma(x) \in A_j\right),$$

which ends the proof of the lemma. □

3.2. **Ergodicity.** To prove Theorem 1, it seems natural to estimate the evolution with m of the dependence between A and $\tilde{\theta}_x^{-m}(B)$ for some events A and B. If $m \geq 1$, the operator $\tilde{\theta}_x^m$ corresponds to a spatial translation by vector mx and to a time translation by vector $S_m(x)$:

$$\begin{aligned} \tilde{\theta}_x^m &= T_{mx} \circ \theta_{S_m(x)}, \\ \text{with } S_m(x) &= \sum_{j=0}^{m-1} \sigma(x) \circ \tilde{\theta}_x^j. \end{aligned}$$

We begin with a lemma in the same spirit as Lemma 8:

Lemma 10. *Let $t > 0$, $A \in \mathcal{F}_t$ and $B \in \mathcal{F}$.*
Then, for any $x \in \mathbb{Z}^d$, any $\lambda \in \lambda$, any $m \geq 1$,

$$\overline{\mathbb{P}}_\lambda(A \cap \{t \leq S_m(x)\} \cap (\tilde{\theta}_x^m)^{-1}(B)) = \overline{\mathbb{P}}_\lambda(A \cap \{t \leq S_m(x)\})\overline{\mathbb{P}}_{mx.\lambda}(B).$$

Proof. Set $\overline{K}_m(x) = (K(x), K(x) \circ \tilde{\theta}_x, \ldots, K(x) \circ \tilde{\theta}_x^{m-1})$. Is is sufficient to prove that for any $k = (k_0, \ldots, k_{m-1}) \in (\mathbb{N}^*)^m$, one has

$$\begin{aligned} & \overline{\mathbb{P}}_\lambda(A,\ t \leq S_m(x),\ \tilde{\theta}_x^{-m}(B),\ \overline{K}_m(x) = k) \\ = \ & \overline{\mathbb{P}}_\lambda(A,\ t \leq S_m(x),\ \overline{K}_m(x) = k)\overline{\mathbb{P}}_{mx.\lambda}(B). \end{aligned}$$

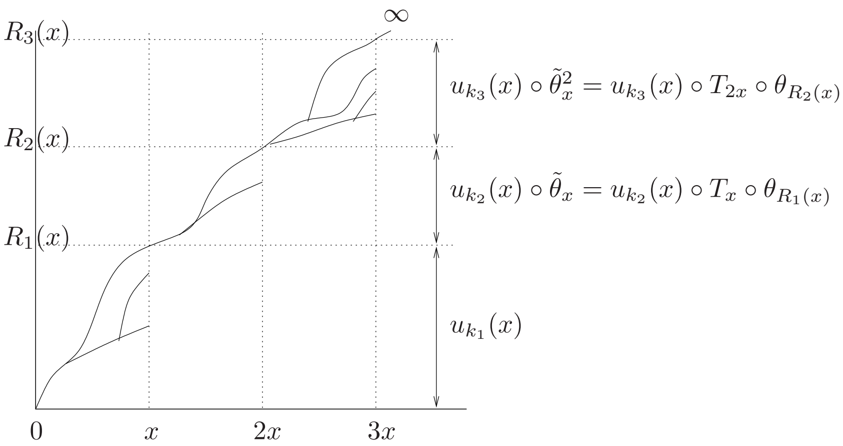

FIGURE 1. An example with $k_1 = 3$, $k_2 = 2$ and $k_3 = 4$.

Let $k \in (\mathbb{N}^*)^m$. We set $R_0(x) = 0$ and, for $l \leq m-2$, $R_{l+1}(x) = R_l + u_{k_l}(x) \circ \theta_{R_l(x)}$. Thanks to Remark (17), the following sets coincide:

$$\left\{ \begin{array}{c} \tau^0 = +\infty, \\ \overline{K}_m(x) = k \end{array} \right\} = \left\{ \begin{array}{c} u_{k_1}(x) < +\infty,\ u_{k_2}(x) \circ T_x \circ \theta_{R_1(x)} < +\infty,\ \dots, \\ u_{k_m}(x) \circ T_{(m-1)x} \circ \theta_{R_{m-1}(x)} < +\infty, \\ \tau^0 \circ T_{mx} \circ \theta_{R_m(x)} = +\infty \end{array} \right\}.$$

Moreover, on this event, $S_m(x) = R_m(x)$ holds. Thus

$$\mathbb{P}_\lambda \left(\begin{array}{c} \tau^0 = +\infty,\ A, \\ t \leq S_m(x), \\ \overline{K}_m(x) = k,\ \tilde{\theta}_x^{-m}(B) \end{array} \right) = \mathbb{P}_\lambda \left(\begin{array}{c} A,\ u_{k_1}(x) < +\infty, \\ u_{k_2}(x) \circ T_x \circ \theta_{R_1(x)} < +\infty, \dots, \\ u_{k_m}(x) \circ T_{(m-1)x} \circ \theta_{R_{m-1}(x)} < +\infty, \\ t \leq R_m(x),\ \tau^0 \circ T_{mx} \circ \theta_{R_m(x)} = +\infty, \\ T_{mx} \circ \theta_{R_m(x)} \in B \end{array} \right).$$

By construction, $R_m(x)$ is a stopping time and the event

$$A \cap \{u_{k_1}(x) < +\infty\} \cap \dots \cap \{u_{k_m}(x) \circ T_{(m-1)x} \circ \theta_{R_{m-1}(x)} < +\infty\} \cap \{t \leq R_m(x)\}$$

is measurable with respect to $\mathcal{F}_{R_m(x)}$. Using the strong Markov property and the spatial translation property (3), we get:

$$\mathbb{P}_\lambda \left(\begin{array}{c} \tau^0 = +\infty,\ A, \\ t \leq S_m(x), \\ \overline{K}_m(x) = k,\ \tilde{\theta}_x^{-m}(B) \end{array} \right) = \mathbb{P}_\lambda \left(\begin{array}{c} A,\ u_{k_1}(x) < +\infty, \\ u_{k_2}(x) \circ T_x \circ \theta_{u_{k_1}(x)} < +\infty, \dots \\ u_{k_m}(x) \circ T_{(m-1)x} \circ \theta_{R_{m-1}(x)} < +\infty, \\ t \leq R_m(x) \end{array} \right) \times \mathbb{P}_{mx.\lambda}(\{\tau = +\infty\} \cap B).$$

Dividing the identity by $\mathbb{P}_\lambda(\tau = +\infty)$, we obtain an identity of the form

$$\overline{\mathbb{P}}_\lambda(A,\ t \leq S_m(x),\ \tilde{\theta}_x^{-m}(B),\ \overline{K}_m(x) = k) = \psi(x, \lambda, k, m, A) \overline{\mathbb{P}}_{mx.\lambda}(B),$$

and we identify the value of $\psi(x, \lambda, k, m, A)$ by taking $B = \Omega$. □

We can now state a mixing property:

Lemma 11. *Let $t>0$ and $q>1$ be fixed. There exists a constant $A(t,q)$ such that for any $x\in\mathbb{Z}^d\backslash\{0\}$, for any $A\in\mathcal{F}_t$, for any $B\in\mathcal{F}$, $\lambda\in\Lambda$ and every $\ell\ge 1$,*

$$|\overline{\mathbb{P}}_\lambda(A\cap(\tilde{\theta}_x^\ell)^{-1}(B))-\overline{\mathbb{P}}_\lambda(A)\overline{\mathbb{P}}_{\ell x.\lambda}(B)|\le A(t,q)q^{-\ell}.$$

Proof. Let $\ell\ge 1$. With Lemma 10, we get

$$\begin{aligned}
&|\overline{\mathbb{P}}_\lambda(A\cap\tilde{\theta}_x^{-\ell}(B))-\overline{\mathbb{P}}_\lambda(A)\overline{\mathbb{P}}_\lambda(\tilde{\theta}_x^{-\ell}(B))|\\
\le\ &|\overline{\mathbb{P}}_\lambda(t\le S_\ell(x),A\cap\tilde{\theta}_x^{-\ell}(B))-\overline{\mathbb{P}}_\lambda(t\le S_\ell(x),A)\overline{\mathbb{P}}_\lambda(\tilde{\theta}_x^{-\ell}(B))|\\
&\quad+2\overline{\mathbb{P}}_\lambda(t>S_\ell(x))\\
=\ &2\overline{\mathbb{P}}_\lambda(t>S_\ell(x)).
\end{aligned}$$

Let us now fix $\alpha>0$.
With the Markov inequality, $\overline{\mathbb{P}}_\lambda(S_\ell(x)\le t)\le\exp(\alpha t)\overline{\mathbb{E}}_\lambda(\exp(-\alpha S_\ell(x)))$. Using the two last points of Corollary 9, one has

$$\begin{aligned}
\overline{\mathbb{E}}_\lambda(\exp(-\alpha S_\ell(x)))&\le\overline{\mathbb{E}}_\lambda\left(\exp\left(-\alpha\sum_{j=0}^{\ell-1}\sigma(x)\circ\tilde{\theta}_x^j\right)\right)\\
&\le\prod_{j=0}^{\ell-1}\overline{\mathbb{E}}_\lambda\left(\exp(-\alpha\sigma(x)\circ\tilde{\theta}_x^j)\right)=\prod_{j=0}^{\ell-1}\overline{\mathbb{E}}_{jx.\lambda}(\exp(-\alpha\sigma(x)).
\end{aligned}$$

Now we just have to prove the existence of some $\alpha>0$ such that for every $\lambda\in\Lambda$,

$$\overline{\mathbb{E}}_\lambda(\exp(-\alpha\sigma(x)))\le q^{-1}.$$

Let ρ be the constant given in (7).

$$\overline{\mathbb{E}}_\lambda(\exp(-\alpha\sigma(x)))\le\frac{1}{\rho}\mathbb{E}_\lambda(\exp(-\alpha\sigma(x)))\le\frac{1}{\rho}\mathbb{E}_{\lambda_{\max}}(\exp(-\alpha\sigma(x)))\le\frac{1}{\rho}\frac{2d\lambda_{max}}{\alpha+2d\lambda_{\max}},$$

because $\sigma(x)\ge t(x)$, and $t(x)$ stochastically dominates an exponential random variable with parameter $2d\lambda_{\max}$. This gives the desired inequality if α is large enough. □

We can now move forward to the proof of the ergodicity properties of the systems $(\Omega,\mathcal{F},\overline{\mathbb{P}},\tilde{\theta}_x)$.

Proof of Theorem 1. We have already seen in Corollary 9 that for any $x\in\mathbb{Z}^d$, the probability measure $\overline{\mathbb{P}}$ is invariant under the action of $\tilde{\theta}_x$. To prove ergodicity, we use an embedding in a larger space to consider simultaneously a random environment and a random contact process.

We thus set $\tilde{\Omega}=\Lambda\times\Omega$, equipped with the σ-algebra $\tilde{\mathcal{F}}=\mathcal{B}(\Lambda)\otimes\mathcal{F}$, and we define a probability measure $\overline{\mathbb{Q}}$ on $\tilde{\mathcal{F}}$ by

$$\forall(A,B)\in\mathcal{B}(\Lambda)\times\mathcal{F}\quad\overline{\mathbb{Q}}(A\times B)=\int_\Lambda 1\!\!1_A(\lambda)\overline{\mathbb{P}}_\lambda(B)\ d\nu(\lambda).$$

We define the transformation $\tilde{\Theta}_x$ on $\tilde{\Omega}$ by setting $\tilde{\Theta}_x(\lambda,\omega)=(x.\lambda,\tilde{\theta}_x(\omega))$. It is easy to see that $\overline{\mathbb{Q}}$ is invariant under $\tilde{\Theta}_x$. Indeed, for $(A,B)\in\mathcal{B}(\Lambda)\times\mathcal{F}$, using

Lemma 8, one has

$$\begin{aligned}\overline{\mathbb{Q}}(\tilde{\Theta}_x(\lambda,\omega)\in A\times B) &= \overline{\mathbb{Q}}(x.\lambda\in A,\ \tilde{\theta}_x(\omega)\in B)\\ &= \int_\Lambda 1\!\!1_A(x.\lambda)\overline{\mathbb{P}}_\lambda(\tilde{\theta}_x(\omega)\in B)\ d\nu(\lambda)\\ &= \int_\Lambda 1\!\!1_A(x.\lambda)\overline{\mathbb{P}}_{x.\lambda}(B)\ d\nu(\lambda)\\ &= \int_\Lambda 1\!\!1_A(\lambda)\overline{\mathbb{P}}_\lambda(B)\ d\nu(\lambda)=\overline{\mathbb{Q}}(A\times B).\end{aligned}$$

Note that if $g(\lambda,\omega)=f(\lambda)$, then $\int g\ d\overline{\mathbb{Q}}=\int f\ d\nu$.
Similarly, if $g(\lambda,\omega)=f(\omega)$, then $\int g\ d\overline{\mathbb{Q}}=\int f\ d\overline{\mathbb{P}}$.

Note that $\mathcal{A}=\underset{t\geq 0}{\cup}\mathcal{F}_t$ is an algebra that generates $\mathcal{F}$. To prove that $\tilde{\theta}_x$ is ergodic, it is then sufficient to show that for every $A\in\mathcal{A}$,

$$\frac{1}{n}\sum_{k=0}^{n-1}1\!\!1_A(\tilde{\theta}_x^k)\text{ converges in }L^2(\overline{\mathbb{P}})\text{ to }\overline{\mathbb{P}}(A). \tag{18}$$

The quantity above can be seen as a function of the two variables (λ,ω). Thus, it is equivalent to prove that the sequence of functions $(\lambda,\omega)\mapsto\frac{1}{n}\sum_{k=0}^{n-1}1\!\!1_A(\tilde{\theta}_x^k\omega)$ converges to $\overline{\mathbb{P}}(A)$ in $L^2(\overline{\mathbb{Q}})$. Let $A\in\mathcal{A}$ and $t>0$ be such that $A\in\mathcal{F}_t$. For every $(\omega,\lambda)\in\tilde{\Omega}$, we split the sum in two terms:

$$\frac{1}{n}\sum_{k=0}^{n-1}1\!\!1_A(\tilde{\theta}_x^k\omega) = \frac{1}{n}\sum_{k=0}^{n-1}\Big(1\!\!1_A(\tilde{\theta}_x^k\omega)-\overline{\mathbb{P}}_{kx.\lambda}(A)\Big)+\frac{1}{n}\sum_{k=0}^{n-1}\overline{\mathbb{P}}_{kx.\lambda}(A)$$

If we set $f(\lambda)=\overline{\mathbb{P}}_\lambda(A)$, the second term can be written

$$\frac{1}{n}\sum_{k=0}^{n-1}\overline{\mathbb{P}}_{kx.\lambda}(A)=\frac{1}{n}\sum_{k=0}^{n-1}f(kx.\lambda).$$

Since ν is ergodic, the Von Neumann ergodic theorem says that this quantity converges in $L^2(\nu)$ to $\int f d\nu=\overline{\mathbb{P}}(A)$. Seen as a function of (λ,ω), it also converges in $L^2(\overline{\mathbb{Q}})$ to $\overline{\mathbb{P}}(A)$. Set, $k\geq 0$,

$$Y_k=1\!\!1_A(\tilde{\theta}_x^k\omega)-\overline{\mathbb{P}}_\lambda(\tilde{\theta}_x^{-k}(A))=1\!\!1_A(\tilde{\theta}_x^k\omega)-\overline{\mathbb{P}}_{kx.\lambda}(A)$$

and $L_n = Y_0 + Y_1 + \cdots + Y_{n-1}$. It only remains to prove that L_n/n converges to 0 in $L^2(\overline{\mathbb{Q}})$. As $Y_k = Y_0 \circ \tilde{\Theta}_x^k$, the field $(Y_k)_{k\geq 0}$ is stationary. We thus have

$$\begin{aligned}
\int L_n^2 \, d\overline{\mathbb{Q}} &= \sum_{0\leq i,j\leq n-1} \int Y_i Y_j \, d\overline{\mathbb{Q}} \\
&= \sum_{i=0}^{n-1} \int Y_i^2 \, d\overline{\mathbb{Q}} + 2\sum_{\ell=1}^{n-1} (n-\ell) \int Y_0 Y_\ell \, d\overline{\mathbb{Q}} \\
&\leq 2n \left(\sum_{\ell=0}^{+\infty} \left| \int Y_0 Y_\ell \, d\overline{\mathbb{Q}} \right| \right) \quad \leq \quad 2n \left(\sum_{\ell=0}^{+\infty} \int_\Lambda |\overline{\mathbb{E}}_\lambda(Y_0 Y_\ell)| \, d\nu(\lambda) \right) \\
&\leq 2n \left(\sum_{\ell=0}^{+\infty} \int_\Lambda |\overline{\mathbb{P}}_\lambda(A \cap \tilde{\theta}_x^{-\ell}(A)) - \overline{\mathbb{P}}_\lambda(A)\overline{\mathbb{P}}_\lambda(\tilde{\theta}_x^{-\ell}(A))| \, d\nu(\lambda) \right) \\
&\leq 2n \left(\sum_{\ell=0}^{+\infty} A(t,2)2^{-\ell} \right) = 4A(t,2)n,
\end{aligned}$$

thanks to Lemma 11. This ends the proof of (18), hence the proof of Theorem 1. □

4. Control of the subadditivity flaw

In this section, we are going to bound quantities such as $\sigma(x+y) - [\sigma(x) + \sigma(y) \circ \tilde{\theta}_x]$ and $\sigma(x) - t(x)$.

We will use these results in the application of a (almost) subadditive ergodic theorem in Section 5. In both case, we use a kind of restart argument. Considering the definition of the essential hitting time σ, we will have to deal with two types of sums of random variables, that are quite different: sums of $v_i - u_i$ on one hand, and sums of $u_{i+1} - v_i$ on the other hand:

- The life time $v_i(x) - u_i(x)$ of the contact process starting from x at time $u_i(x)$ can be bounded independently of the precise configuration of the process at time $u_i(x)$. So the control is quite simple.
- On the contrary, $u_{i+1}(x) - v_i(x)$, which represents the amount of time needed to reinfect site x after time $v_i(x)$, clearly depends on the whole configuration of the process at time $v_i(x)$, which is not easy to control precisely and uniformly in x. This explains why the restart argument we use is more complex and the estimates we obtain less accurate than in more classical situations (For instance, in Section 6, we obtain the exponential estimates of Proposition 5 by standard restart arguments.)

As an illustration of the first point, we easily obtain:

Lemma 12. *There exist $A, B > 0$ such that for every $\lambda \in \Lambda$,*

$$(19) \qquad \forall x \in \mathbb{Z}^d \quad \forall t > 0 \quad \overline{\mathbb{P}}_\lambda(\exists i < K(x) : \; v_i(x) - u_i(x) > t) \leq A \exp(-Bt).$$

Proof. Let $F : \Omega \to \mathbb{R}_+$ be a measurable function and $x \in \mathbb{Z}^d$. We set

$$\mathcal{L}_x(F) = \sum_{i=0}^{+\infty} \mathbb{1}_{\{u_i(x)<+\infty\}} F \circ \theta_{u_i(x)}.$$

With the Markov property and the definition of $K(x)$, we have

$$\begin{aligned}\mathbb{E}_\lambda[\mathcal{L}_x(F)] &= \sum_{i=0}^{+\infty}\mathbb{E}_\lambda[\mathbb{1}_{\{u_i(x)<+\infty\}}]\mathbb{E}_\lambda[F] = \left(1+\sum_{i=0}^{+\infty}\mathbb{P}_\lambda(K(x)>i)\right)\mathbb{E}_\lambda[F]\\ &= (1+\mathbb{E}_\lambda[K(x)])\mathbb{E}_\lambda[F] \le \left(1+\frac{1}{\rho}\right)\mathbb{E}_\lambda[F],\end{aligned}$$

where the last equality comes from Lemma 6. We choose $F=\mathbb{1}_{\{t<u_i(x)-v_i(x)<+\infty\}}$, and with Estimate (7), we obtain:

$$\begin{aligned}\overline{\mathbb{P}}_\lambda(\exists i<K(x):\ v_i(x)-u_i(x)>t) &\le \frac{1}{\rho}\mathbb{P}_\lambda(\exists i<K(x):\ v_i(x)-u_i(x)>t)\\ &\le \frac{1}{\rho}\mathbb{P}_\lambda(\mathcal{L}_x(F)\ge 1)\le\frac{1}{\rho}\mathbb{E}_\lambda[\mathcal{L}_x(F)]\\ &\le \frac{1}{\rho}\left(1+\frac{1}{\rho}\right)\mathbb{P}_\lambda(t<\tau^x<+\infty).\end{aligned}$$

We can then conclude with Inequality (9). □

To deal with the reinfection times $u_{i+1}(x)-v_i(x)$, the idea is to look for a point (y,t) – in space-time coordinates – close to $(x,u_i(x))$, infected from $(0,0)$ and with infinite life time: The at-least-linear-growth estimate (10) will then ensure it does not take too long to reinfect x after time t, just by looking infection starting from the new source point (y,t). The difficulty lies in the control of the distance between $(x,u_i(x))$ and a source point (y,t): if the configuration around $(x,u_i(x))$ is "reasonable", this point will not be too not far from $(x,u_i(x))$, and we will obtain a good control of $u_{i+1}(x)$ and $u_i(x)$.

We recall that for every $x\in\mathbb{Z}^d$, ω_x is the Poisson point process giving the possible death times at site x, and that M and c are respectively given in (8) et (10). Note that we can assume that $M>1$. We note

$$\gamma = 3M(1+1/c) > 3. \tag{20}$$

For $x,y\in\mathbb{Z}^d$ and $t>0$, we say that the growth from $(y,0)$ is bad at scale t with respect to x if the following event occur:

$$\begin{aligned}E^y(x,t) &= \{\omega_y[0,t/2]=0\}\cup\{H_t^y\not\subset y+B_{Mt}\}\\ &\cup\ \{t/2<\tau^y<+\infty\}\cup\{\tau^y=+\infty,\ \inf\{s\ge 2t:\ x\in\xi_s^y\}>\gamma t\},\end{aligned}$$

We want to check that with a high probability, there is no such bad growth point in a box around x: for every $x\in\mathbb{Z}^d$, every $L>0$ and every $t>0$. So we define

$$N_L(x,t)=\sum_{y\in x+B_{Mt+2}}\int_0^L \mathbb{1}_{E^y(x,t)}\circ\theta_s\ d\left(\omega_y+\sum_{e\in\mathbb{E}^d:y\in e}\omega_e+\delta_0\right)(s).$$

In other words, we count the number of points (y,s) in the space-time box $(x+B_{Mt+1})\times[0,L]$ such that there happens something for site y at time t, either a possible death, or a possible infection, and at this time the bad event $E^y(x,t)\circ\theta_s$ occurs. We first check that if the space-time box has no bad points and if $u_i(x)$ is in the time window, then we can control the delay before the next infection :

Lemma 13. *If $N_L(x,t)\circ\theta_s=0$ and $s+t\le u_i(x)\le s+L$, then $v_i(x)=+\infty$ or $u_{i+1}(x)-u_i(x)\le\gamma t$.*

Proof. By definition of $u_i(x)$, site x is infected from $(0,0)$ at time $u_i(x)$. Since $s+t \le u_i(x) \le s+L$ and $u_i(x)$ is a possible infection time for x, the non-occurrence of $E^x(x,t) \circ \theta_{u_i(x)}$ ensures that $\tau^x \circ \theta_{u_i(x)} = +\infty$ or that $\tau^x \circ \theta_{u_i(x)} \le t/2$. If $\tau^x \circ \theta_{u_i(x)} = +\infty$, we are done because then $v_i(x) = +\infty$. Otherwise, note that $v_i(x) - u_i(x) \le t/2$.

By definition, there exists an infection path $\gamma_i : [0, u_i(x)] \to \mathbb{Z}^d$ from $(0,0)$ to $(x, u_i(x))$, *i.e.* such that $\gamma_i(0) = 0$ and $\gamma_i(u_i(x)) = x$. Consider the portion of γ_i between time $u_i(x) - t$ and time $u_i(x)$. Denote by $x_0 = \gamma_i(u_i(x) - t)$ and let us see that $x_0 \in x + B_{Mt+2}$. Indeed, if $x_0 \notin x + B_{Mt+2}$, we seek the first time t_1 after time $u_i(x) - t$ when γ_i enters in $x + B_{Mt+2}$ at a site we call x_1 (note that since x_1 is in the inside boundary of $x + B_{Mt+2}$, we have $\|x - x_1\|_\infty \ge Mt+1$): time t_1 is a possible infection time for x_1, and the non-occurrence of $E^{x_1}(x,t) \circ \theta_{t_1}$ ensures that the infection of x from (x_1, t_1) will at least require a delay t, which contradicts $u_i(x) - t \ge 0$.

So $x_0 \in x + B_{Mt+2}$: since $N_L(x,t) \circ \theta_s = 0$, the first possible death at site x_0 after time $u_i(x) - t$ can not occur after a delay of $t/2$; thus the first time t_2 when the path γ_i jumps to a different point x_2 satisfies $t_2 \le u_i(x) - t + t/2 = u_i(x) - t/2$. Consequently, when (x_2, t_2) infects $(x, u_i(x))$, it is at least $t/2$ aged, and the non-occurrence of $E^{x_2}(x,t) \circ \theta_{t_2}$ ensure it lives forever and

$$\inf\{u \ge 2t : \; x \in \xi_u^{x_2}\} \circ \theta_{t_2} \le \gamma t.$$

So there exists $t_3 \in [t_2 + 2t, t_2 + \gamma t]$, with $x \in \xi^0_{t_3}$. Since $v_i(x) - u_i(x) \le t/2$, one has

$$t_3 \ge t_2 + 2t \ge (u_i(x) - t) + 2t = u_i(x) + t \ge v_i(x).$$

Finally, $u_{i+1}(x) - u_i(x) \le t_3 - u_i(x) \le t_2 - u_i(x) + \gamma t \le \gamma t$. □

Now we estimate the probability that a space-time box contains no bad points:

Lemma 14. *There exist $A_{21}, B_{21} > 0$ such that for every $\lambda \in \Lambda$,*

(21) $$\forall L > 0 \quad \forall x \in \mathbb{Z}^d \quad \forall t > 0 \quad \mathbb{P}_\lambda(N_L(x,t) \ge 1) \le A_{21}(1+L)\exp(-B_{21}t).$$

Proof. Let us first prove there exist $A, B > 0$ such that for every $\lambda \in \Lambda$,

(22) $$\forall x \in \mathbb{Z}^d \quad \forall t > 0 \quad \forall y \in x + B_{Mt+2} \quad \mathbb{P}_\lambda(E^y(x,t)) \le A\exp(-Bt).$$

Let $x \in \mathbb{Z}^d$, $t > 0$ and $y \in x + B_{Mt+2}$. If $\tau^y = +\infty$, there exists $z \in \xi^y_{2t}$ with $\tau^z \circ \theta_{2t} = +\infty$. Thus the definition (20) of γ implies that

$$\begin{aligned}
& \{\tau^y = +\infty, \; \inf\{s \ge 2t : \; x \in \xi^y_s\} > \gamma t\} \\
\subset \; & \{\xi^y_{2t} \not\subset y + B_{2Mt}\} \cup \bigcup_{z \in y + B_{2Mt}} \{t^z(x) \circ \theta_{2t} > (\gamma - 2M)t\} \\
\subset \; & \{\xi^y_{2t} \not\subset y + B_{2Mt}\} \cup \bigcup_{z \in y + B_{2Mt}} \left\{ t^z(x) \circ \theta_{2t} > \frac{\|x - z\|}{c} + Mt - \frac{3}{c} \right\}.
\end{aligned}$$

Hence, with (8) and (10),

$$\begin{aligned}
& \mathbb{P}_\lambda(\tau^y = +\infty, \; \inf\{s \ge 2t : \; x \in \xi^y_s\} > \gamma t) \\
\le \; & A\exp(-2BMt) + (1 + 4Mt)^d A \exp(-B(Mt - 3/c)).
\end{aligned}$$

The distribution of the number $\omega_y([0, t/2])$ of possible deaths on site y between time 0 and time $t/2$ is a Poisson law with parameter $t/2$, so

$$\mathbb{P}_\lambda(\omega_y([0, t/2]) = 0) = \exp(-t/2).$$

The two remaining terms are controlled with (8) and (9); this gives (22).

Fix now $y \in x + B_{Mt+2}$ and note $\beta_y = \omega_y + \sum_{e \in \mathbb{E}^d : y \in e} \omega_e$. Under $\mathbb{P}_\lambda$, β_y is a Poisson point process with intensity $2d\lambda_e$. Let $S_0 = 0$ and $(S_n)_{n \geq 1}$ be the increasing sequence of the times given by this process.

$$\int_0^L 1\!\!1_{E^y(x,t)} \circ \theta_s \; d(\beta_y + \delta_0)(s) = \sum_{n=0}^{+\infty} 1\!\!1_{\{S_n \leq L\}} 1\!\!1_{E^y(x,t)} \circ \theta_{S_n};$$

so, with the Markov property,

$$\begin{aligned}
& \mathbb{E}_\lambda \left(\int_0^L 1\!\!1_{E^y(x,t)} \circ \theta_s \; d(\beta_y + \delta_0)(s) \right) \\
= & \sum_{n=0}^{+\infty} \mathbb{E}_\lambda \left(1\!\!1_{\{S_n \leq L\}} 1\!\!1_{E^y(x,t)} \circ \theta_{S_n}] \right) = \sum_{n=0}^{+\infty} \mathbb{E}_\lambda \left(1\!\!1_{\{S_n \leq L\}} \right) \mathbb{P}_\lambda(E^y(x,t)) \\
= & (1 + \mathbb{E}_\lambda[\beta_y([0,L])]) \, \mathbb{P}_\lambda(E^y(x,t)) = (1 + L(2d\lambda_e + 1)) \mathbb{P}_\lambda(E^y(x,t)).
\end{aligned}$$

So (21) follows from (22), from the remark that $\mathbb{P}_\lambda(N_L(x,t) \geq 1) \leq \mathbb{E}_\lambda[N_L(x,t)]$, and from on obvious bound on the cardinality of B_{Mt+2}. □

Once the process is initiated, Lemma 13 can be used recursively to control $u_{i+1}(x) - u_i(x)$. To initiate the process, we assume that there exists a point (u,s), reached from $(0,0)$, living infinitely and close to x in space:

Lemma 15. *For any $t, s > 0$, for every $x \in \mathbb{Z}^d$, the following inclusion holds:*

$$\{\tau^0 = +\infty\} \cap \left\{ \exists u \in x + B_{Mt+2}, \; \tau_u \circ \theta_s = +\infty, \; u \in \xi_s^0 \right\}$$

$$\cap \left\{ N_{K(x)\gamma t}(x,t) \circ \theta_s = 0 \right\} \tag{23}$$

$$\cap \bigcap_{1 \leq i < K(x)} \{v_i(x) - u_i(x) < t\} \tag{24}$$

$$\subset \{\tau^0 = +\infty\} \cap \{\sigma(x) \leq s + K(x)\gamma t\}. \tag{25}$$

Proof. If every finite $u_i(x)$ is smaller than $s+t$, we are done because $\sigma(x) \leq s+t \leq s + K(x)\gamma t$. So set

$$i_0 = \max\{i : u_i(x) \leq s + t\}.$$

Since $v_{i_0}(x) < +\infty$, the event (24) ensures that $v_{i_0}(x) - u_{i_0}(x) < t$, and so $v_{i_0}(x) \leq s + 2t$. Now, since $\tau^u = +\infty$, the non-occurrence of $E^u(x,t) \circ \theta_s$ implied by (23) says that

$$\inf\{s \geq 2t : \; x \in \xi_s^u\} \circ \theta_s \leq \gamma t,$$

which leads to $u_{i_0+1}(x) \leq s + \gamma t$. Noting that for any $j \geq 1$, $u_{i_0+j}(x+y) \geq s + t$, we prove by a recursive use of Lemma 13 with the event $\{N_{K(x)\gamma t}(x,t) \circ \theta_s = 0\}$ that

$$\forall j \in \{1, \ldots, K(x) - i_0\} \quad u_{i_0+j} \leq s + j\gamma t.$$

For $j = K(x) - i_0$, we get $\sigma(x) = u_{i_0+j}(x) \leq s + (K(x) - i_0)\gamma t \leq s + K(x)\gamma t$, which proves (25). □

4.1. **Bound for the lack of subadditivity.** To bound $\sigma(x+y)-[\sigma(x)+\sigma(y)\circ\tilde{\theta}_x]$, we apply the strategy we have just explained around site $x+y$. To initiate the recursive process, one can benefit here from the existence of an infinite start at the precise point $(x+y,\sigma(x)+\sigma(y)\circ\tilde{\theta}_y)$.

Proof of Theorem 2. Let $x,y\in\mathbb{Z}^d$, $\lambda\in\Lambda$ and $t>0$. We set $s=\sigma(x)+\sigma(y)\circ\tilde{\theta}_x$.

$$\begin{aligned}&\overline{\mathbb{P}}_\lambda(\sigma(x+y)>\sigma(x)+\sigma(y)\circ\tilde{\theta}_x+t)\\ \le\ &\overline{\mathbb{P}}_\lambda\left(K(x+y)>\frac{\sqrt{t}}{\gamma}\right)+\overline{\mathbb{P}}_\lambda\left(\begin{array}{c}\tau^0=+\infty,\ K(x+y)\le\frac{\sqrt{t}}{\gamma}\\ \sigma(x+y)\ge s+K(x+y)\gamma\sqrt{t}\end{array}\right).\end{aligned}$$

With the sub-geometrical behavior of the tail of K given in Lemma 6 and the uniform control (7), we can control the first term. Note that if $K(x+y)\le\frac{\sqrt{t}}{\gamma}$, then $K(x+y)\gamma\sqrt{t}\le t$, and so that

$$\{N_{K(x+y)\gamma\sqrt{t}}(x+y,\sqrt{t})\ge 1\}\subset\{N_t(x+y,\sqrt{t})\ge 1\}.$$

We apply Lemma 15 around $x+y$, on a scale $\sqrt{t}$, an initial time $s=\sigma(x)+\sigma(y)\circ\tilde{\theta}_x$ and a source point $u=x+y$:

$$\begin{aligned}&\overline{\mathbb{P}}_\lambda\left(\tau^0=+\infty,\ K(x+y)\le\frac{\sqrt{t}}{\gamma},\ \sigma(x+y)\ge s+K(x+y)\gamma\sqrt{t}\right)\\ \le\ &\overline{\mathbb{P}}_\lambda(N_t(x+y,\sqrt{t})\circ\theta_s\ge 1)\\ &+\overline{\mathbb{P}}_\lambda(\exists i<K(x+y):\ v_i(x+y)-u_i(x+y)>\sqrt{t}).\end{aligned}\tag{26}$$

Since $N_t(x+y,\sqrt{t})=N_t(0,\sqrt{t})\circ T_x\circ T_y$ and $s=\sigma(x)+\sigma(y)\circ\tilde{\theta}_x$, we have

$$N_t(x+y,\sqrt{t})\circ\theta_s=N_t(0,\sqrt{t})\circ\tilde{\theta}_y\circ\tilde{\theta}_x.$$

Thus $\overline{\mathbb{P}}_\lambda(N_t(x+y,\sqrt{t})\circ\theta_s\ge 1)=\overline{\mathbb{P}}_{(x+y).\lambda}(N_t(0,\sqrt{t})\ge 1)$, which is controlled by Lemma 14 and estimate (7). Finally, (26) is bounded with Lemma 12. □

Corollary 16. *For $x,y\in\mathbb{Z}^d$, set $r(x,y)=(\sigma(x+y)-(\sigma(x)+\sigma(y)\circ\tilde{\theta}_x))^+$.*
For any $p\ge 1$, there exists $M_p>0$ such that

$$\forall\lambda\in\Lambda\quad\forall x,y\in\mathbb{Z}^d\quad\overline{\mathbb{E}}_\lambda[r(x,y)^p]\le M_p.\tag{27}$$

Proof. We write $\overline{\mathbb{E}}_\lambda[r(x,y)^p]=\int_0^{+\infty}pu^{p-1}\overline{\mathbb{P}}_\lambda(r(x,y)>u)\,du$ and use Theorem 2. □

4.2. **Control of the discrepancy between hitting times and essential hitting times.** To bound $\sigma(x)-t(x)$, we would like to apply the same strategy starting from $(x,t(x))$ but we do not have any natural candidate for an infinite start close to this point. We are going to look for such a point along the infection path between $(0,0)$ and $(x,t(x))$, which requires controls on a space-time box whose height – in time – of order $t(x)$, *i.e* of order $\|x\|$. So we will lose in the precision of the estimates and in their uniformity.

Proposition 17. *There exist $A_{28},B_{28},\alpha_{28}>0$ such that for every $z>0$, every $x\in\mathbb{Z}^d$, every $\lambda\in\Lambda$:*

$$\overline{\mathbb{P}}_\lambda\left(\sigma(x)\ge t(x)+K(x)(\alpha_{28}\log(1+\|x\|)+z)\right)\le A_{28}\exp(-B_{28}z).\tag{28}$$

Proof. For $x,y\in\mathbb{Z}^d$ and $t,L>0$, we define

$$\begin{aligned}\tilde{E}^y(t) &= \{\tau_y<+\infty,\ \bigcup_{s\ge 0} H_s^x \not\subset B_{Mt}\},\\ \tilde{N}_L(x,t) &= \sum_{y\in x+B_{Mt+1}} \int_0^L \mathbb{1}_{\tilde{E}^y(t)}\circ\theta_s\ d\left(\sum_{e\in\mathbb{E}^d: y\in e}\omega_e\right)(s).\end{aligned}$$

With (7), (8) and (9), it is easy to get the existence of $A,B>0$ such that

$$\forall\lambda\in\Lambda\quad \forall x\in\mathbb{Z}^d\quad \forall t>0\quad \overline{\mathbb{P}}_\lambda(\tilde{N}_L(x,t)\ge 1)\le A(1+L)\exp(-Bt). \tag{29}$$

Now, we choose the last point (u,s) on the infection path between $(0,0)$ and $(x,t(x))$ such that $\tau^u\circ\theta_s=+\infty$. Note that on $\{\tau^0=+\infty\}$, such an s always exists.

Let us see that if $\tilde{N}_{t(x)}(x,t)=0$, then $u\in x+B_{Mt+2}$. Indeed, if $\|u-x\|>Mt+2$, we consider the first point (u',s') on the infection path after (u,s) to be in $x+B_{Mt}$: The definition of s ensures that the contact process starting from (u',s') does not survive, but, since it contains $(x,t(x))$, its diameter must be larger than Mt, which implies that $\tilde{N}_{t(x)}(x,t)\ge 1$, and gives the desired implication.

On event $\{\tilde{N}_{t(x)}(x,t)=0\}$, we are going to apply Lemma 15 around point $(x,0)$, at scale

$$t=\frac{\alpha\log(1+\|x\|)+z}{\gamma}\ge\frac{z}{\gamma},$$

with source point (u,s) and a time length $L=K(x)\gamma t$. Here and in the following, $\alpha>0$ is a large constant that will be chosen later. Since $s\le t(x)$,

$$\begin{aligned}&\overline{\mathbb{P}}_\lambda\left(\sigma(x)\ge t(x)+K(x)(\alpha\log(1+\|x\|)+z)\right)=\overline{\mathbb{P}}_\lambda\left(\sigma(x)\ge t(x)+K(x)\gamma t\right)\\ \le\ &\overline{\mathbb{P}}_\lambda\left(\sigma(x)\ge s+K(x)\gamma t\right)\\ \le\ &\overline{\mathbb{P}}_\lambda\left(\sigma(x)\ge s+K(x)\gamma t,\ \tilde{N}_{t(x)}(x,t)=0\right)+\overline{\mathbb{P}}_\lambda\left(\tilde{N}_{t(x)}(x,t)\ge 1\right)\\ \le\ &\overline{\mathbb{P}}_\lambda(N_{K(x)\gamma t}(x,t)\circ\theta_s\ge 1)+\overline{\mathbb{P}}_\lambda(\exists i<K(x):\ v_i(x)-u_i(x)>t)\\ &+\overline{\mathbb{P}}_\lambda(\tilde{N}_{t(x)}(x,t)\ge 1).\end{aligned}\tag{30}$$

The second term in (30) is bounded with Lemma 12. For the last term, we write:

$$\overline{\mathbb{P}}_\lambda\left(\tilde{N}_{t(x)}(x,t)\ge 1\right)\ \le\ \overline{\mathbb{P}}_\lambda\left(\tilde{N}_{\frac{\|x\|}{c}+z}(x,t)\ge 1\right)+\overline{\mathbb{P}}_\lambda\left(t(x)>\frac{\|x\|}{c}+z\right).$$

The second term is controlled with (7) and (10), and (29) ensures that

$$\begin{aligned}\overline{\mathbb{P}}_\lambda\left(\tilde{N}_{\frac{\|x\|}{c}+t}(x,t)\ge 1\right) &\le A\left(1+\frac{\|x\|}{c}+z\right)\exp(-Bt)\\ &\le A\left(1+\frac{\|x\|}{c}+z\right)\exp\left(-\frac{B(\alpha\log(1+\|x\|)+z)}{\gamma}\right)\\ &\le A'\exp(-B'z),\end{aligned}$$

as soon as α is large enough.
For the first term of (30), we note that $N_{K(x)\gamma t}(x,t)\circ\theta_s\le N_{t(x)+K(x)\gamma t}(x,t)$. Thus

$$\overline{\mathbb{P}}_\lambda(N_{K(x)\gamma t}(x,t)\circ\theta_s\ge 1)\le\overline{\mathbb{P}}_\lambda\left(N_{\frac{\|x\|}{c}+z+K(x)\gamma t}(x,t)\ge 1\right)+\overline{\mathbb{P}}_\lambda\left(t(x)\ge\frac{\|x\|}{c}+z\right).$$

As previously, the second term is bounded with (7) and (10), while using (21), we get:

$$\begin{aligned}
&\overline{\mathbb{P}}_\lambda\left(N_{\frac{\|x\|}{c}+z+K(x)\gamma t}(x,t)\geq 1\right)\\
\leq\ &\sum_{k=1}^{+\infty}\sqrt{\overline{\mathbb{P}}_\lambda(K(x)=k)}\sqrt{\overline{\mathbb{P}}_\lambda\left(N_{k\gamma t+\frac{\|x\|}{c}+z}(x,t)\geq 1\right)}\\
\leq\ &\sum_{k=1}^{+\infty}\sqrt{\overline{\mathbb{P}}_\lambda(K(x)=k)}\sqrt{A_{21}\left(k\gamma t+\frac{\|x\|}{c}+z\right)\exp(-B_{21}t)}\\
\leq\ &\sqrt{A_{21}\left(1+\frac{\|x\|}{c}\right)(1+z)(1+\gamma t)\exp\left(-\frac{B_{21}}{2}t\right)}\sum_{k=1}^{+\infty}\sqrt{(1+k)\overline{\mathbb{P}}_\lambda(K(x)=k)}.
\end{aligned}$$

The sub-geometrical behavior of the tail of $K(x)$ given by Lemma (6) ensures that the sum is finite, and we end the proof by increasing α if necessary. □

Lemma 18. *For every $p\geq 1$, there exists $C_{31}(p)>0$ such that for every $x\in\mathbb{Z}^d$*

$$\forall\lambda\in\Lambda\quad \overline{\mathbb{E}}_\lambda(|\sigma(x)-t(x)|^p)\leq C_{31}(p)(\log(1+\|x\|))^p. \tag{31}$$

Proof. Set $V_x=\frac{\sigma(x)-t(x)}{K(x)}-\alpha_{28}\log(1+\|x\|)$. By Proposition 17, there exists a random variable W with exponential moments that stochastically dominates V_x under $\overline{\mathbb{P}}_\lambda$ for every x and every λ. Moreover, Lemma 6 ensures that $K(x)$ is stochastically dominated by a geometrical random variable K' .

Set $v(x)=\sigma(x)-t(x)=K(x)(\alpha\log(1+\|x\|)+V_x)$ and let $p\geq 1$. With the Minkowski inequality, we have

$$\begin{aligned}
(\overline{\mathbb{E}}_\lambda v(x)^p)^{1/p} &\leq \alpha\log(1+\|x\|)(\overline{\mathbb{E}}_\lambda K(x)^p)^{1/p}+\left(\overline{\mathbb{E}}_\lambda[K(x)^pV_x^p]\right)^{1/p}\\
&\leq \alpha\log(1+\|x\|)(\overline{\mathbb{E}}_\lambda K(x)^p)^{1/p}+\left(\overline{\mathbb{E}}_\lambda K(x)^{2p}\overline{\mathbb{E}}_\lambda V_x^{2p}\right)^{\frac{1}{2p}}\\
&\leq \alpha\log(1+\|x\|)(\mathbb{E}K'^p)^{1/p}+\left(\mathbb{E}K'^{2p}\mathbb{E}W^{2p}\right)^{\frac{1}{2p}},
\end{aligned}$$

which ends the proof. □

Corollary 19. $\overline{\mathbb{P}}-a.s.,\ \lim_{\|x\|\to+\infty}\dfrac{|\sigma(x)-t(x)|}{\|x\|}=0.$

Proof. Let $p>d$: Equation (31) gives

$$\sum_{x\in\mathbb{Z}^d}\overline{\mathbb{E}}\frac{|\sigma(x)-t(x)|^p}{(1+\|x\|)^p}\quad\leq\quad C_{31}(p)\sum_{x\in\mathbb{Z}^d}\frac{(\log(1+\|x\|))^p}{(1+\|x\|)^p}<+\infty.$$

So $\left(\frac{|\sigma(x)-t(x)|}{(1+\|x\|)}\right)_{x\in\mathbb{Z}^d}$ is almost surely in $\ell^p(\mathbb{Z}^d)$, and thus goes to 0. □

Corollary 20. *There exist $A_{32},B_{32},C_{32}>0$ such that for every $\lambda\in\Lambda$,*

$$\forall x\in\mathbb{Z}^d\quad\forall t>0\quad \overline{\mathbb{P}}_\lambda\left(\sigma(x)\geq C_{32}\|x\|+t\right)\leq A_{32}\exp(-B_{32}\sqrt{t}). \tag{32}$$

Proof. Let $\alpha=\alpha_{28}$ be given in Proposition 17, and note that if $K(x)\leq\frac{1}{2\alpha}\sqrt{\|x\|+t/2}$ and $z=\alpha\sqrt{\|x\|+t/2}$, then, since $\log(1+u)\leq\sqrt{u}$, we get:

$$K(x)[\alpha\log(1+\|x\|)+z]\leq 2zK(x)\leq\|x\|+t/2.$$

Thus with (10) and (7),

$$\begin{aligned}
&\overline{\mathbb{P}}_\lambda\left(\sigma(x) > \left(\frac{1}{c}+1\right)\|x\| + t\right)\\
\le\ &\overline{\mathbb{P}}_\lambda\left(t(x) \ge \frac{\|x\|}{c} + t/2\right) + \overline{\mathbb{P}}_\lambda\left(K(x) > \frac{1}{2\alpha}\sqrt{\|x\|+t/2}\right)\\
&\quad +\overline{\mathbb{P}}_\lambda\left(\sigma(x) > t(x) + K(x)(\alpha\log(1+\|x\|) + \alpha\sqrt{\|x\|+t/2})\right).
\end{aligned}$$

The first term is controlled with (10), the second one with Lemma 6, and the last one by Proposition 17. □

Corollary 21. *For any $p \ge 1$, there exists $C_{33}(p) > 0$ such that*

$$\forall \lambda \in \Lambda \quad \forall x \in \mathbb{Z}^d \quad \overline{\mathbb{E}}_\lambda[\sigma(x)^p] \le C_{33}(p)(1+\|x\|)^p. \tag{33}$$

Proof. With the Minkowski inequality, one has

$$(\overline{\mathbb{E}}_\lambda[\sigma(x)^p])^{1/p} \le C_{32}\|x\| + (\overline{\mathbb{E}}_\lambda[((\sigma(x)-C_{32}\|x\|)^+)^p])^{1/p}.$$

Moreover

$$\overline{\mathbb{E}}_\lambda[((\sigma(x)-C_{32}\|x\|)^+)^p] = \int_0^{+\infty} pu^{p-1}\overline{\mathbb{P}}_\lambda(\sigma(x)-C_{32}\|x\| > u)\,du < +\infty$$

by Corollary 20. □

Remarks. In classical restart arguments, the existence of exponential moments for a random variable usually comes from the following argument: if $(X_n)_{n\in\mathbb{N}}$ are independent identically distributed random variables with exponential moments, if K is independent of the $(X_n)_{n\in\mathbb{N}}$'s and also has exponential moments, then $\sum_{0\le n\le K} X_n$ has exponential moments. Here, our difficulties to precisely bound the reinfection times $u_{i+1}-v_i$ prevents us to use this scheme: we thus have to use *ad hoc* arguments, which lead to weaker estimates.

5. Asymptotic shape theorems

We can now move forward to the proof of Theorem 3. The first step consists in proving convergence for ratios of the type $\frac{\sigma(nx)}{n}$. With Corollary 16, we know that for every $n,p \ge 0$:

$$\overline{\mathbb{E}}[\sigma((n+p)x)] \le \overline{\mathbb{E}}[\sigma(nx)] + \overline{\mathbb{E}}[\sigma(px)] + M_1.$$

Thus the Fekete lemma says that $\frac{1}{n}\overline{\mathbb{E}}[\sigma(nx)]$ has a finite limit when n goes to $+\infty$ and the natural candidate for the limit of $\frac{\sigma(nx)}{n}$ is thus

$$\mu(x) = \lim_{n\to+\infty}\frac{\overline{\mathbb{E}}(\sigma(nx))}{n}.$$

Theorem 22. $\overline{\mathbb{P}} - a.s.\quad \forall x \in \mathbb{Z}^d \quad \lim_{n\to+\infty}\frac{\sigma(nx)}{n} = \lim_{n\to+\infty}\frac{\overline{\mathbb{E}}\sigma(nx)}{n} = \mu(x)$.
This convergence also holds in any $L^p(\overline{\mathbb{P}})$, $p \ge 1$.

To prove this result, we need the two following (almost) subadditive ergodic theorems, whose proof will be given in the appendix.

Theorem 23. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $(\theta_n)_{n\ge 1}$ a collection of transformations leaving the probability measure $\mathbb{P}$ invariant. On this space, we consider a collection $(f_n)_{n\ge 1}$ of integrable functions, a collection $(g_n)_{n\ge 1}$ of nonnegative functions, and a collection $(r_{n,p})_{n,p\ge 1}$ of real functions such that*

$$\forall n,p \ge 1 \quad f_{n+p} \le f_n + f_p\circ\theta_n + g_p\circ\theta_n + r_{n,p}. \tag{34}$$

We assume that

- $c = \inf_{n\ge 1} \frac{\mathbb{E}f_n}{n} > -\infty$.
- g_1 *is integrable,* g_n/n *almost surely converges to* 0 *and* $\frac{\mathbb{E}g_n}{n}$ *converges to* 0.
- *There exists* $\alpha > 1$ *and a sequence of positive numbers* $(C_p)_{p\ge 1}$ *such that* $\mathbb{E}[(r^+_{n,p})^\alpha] \le C_p$ *for every* n,p *and*

$$\sum_{p=1}^{+\infty} \frac{C_p}{p^\alpha} < +\infty.$$

Then $\frac{1}{n}\mathbb{E}f_n$ *converges; if* μ *denotes its limit, one has*

$$\mathbb{E}\left[\underline{\lim}_{n\to+\infty} \frac{f_n}{n}\right] \ge \mu.$$

If we set $\underline{f} = \underline{\lim}_{n\to+\infty} \frac{f_n}{n}$, *then* $\underline{f}$ *is invariant under the action of each* θ_n.

Theorem 24. *We keep the setting and assumptions of Theorem 23. We assume moreover that for every* k,

$$\frac{1}{n}\left(f_{nk} - \sum_{i=0}^{n-1} f_k\circ(\theta_k)^i\right)^+ \to 0 \quad a.s.$$

Then f_n/n *converges a.s. to* $\underline{f}$.

Proof of Theorem 22. We apply Theorem 23 with the choices $f_n = \sigma(nx)$, $\theta_n = \tilde{\theta}_{nx}$, $g_p = 0$, $r_{n,p} = r(nx, px)$ and the probability measure $\mathbb{P} = \overline{\mathbb{P}}$. We take $\alpha > 1$. Corollary 21 gives the integrability of $\sigma(x)$ under $\overline{\mathbb{P}}$ and Corollary 16 gives the necessary controls on its moments.

We now check the extra assumption of Theorem 24: it is easy to see that

$$t(nkx) \le \sum_{i=0}^{n-1} \sigma(kx)\circ(\tilde{\theta}_{kx})^i,$$

which implies that $\left(\sigma(nkx) - \sum_{i=0}^{n-1}\sigma(kx)\circ(\tilde{\theta}_{kx})^i\right)^+ \le \sigma(nkx) - t(nkx)$; Corollary 19 ensures that this quantity is $o(n)$. Thus $\sigma(nx)/n$ converges to a random variable $\mu(x)$, which is invariant under the action of $\tilde{\theta}_x$. But Theorem 1 says that this $\mu(x)$ is in fact a constant, which ends the proof of the a.s. convergence.

To prove that a sequence converges in L^p, it suffices to show that it converges a.s. and that it is bounded in L^q for some $q > p$. Since Corollary 21 says that f_n/n is bounded in any L^p, the proof is over. □

The next step is to prove the asymptotic shape result, namely Theorem 3. We start by proving the shape result for the essential hitting time σ, by following the classical strategy:

- We extend μ to a norm on $\mathbb{R}^d$ in Lemma 25.
- We prove that the directional convergence given by Theorem 22 is in fact uniform in the direction in Lemma 27.
- We easily deduce the shape result from this lemma in Lemma 28.

To transpose this shape result for the classical hitting time t (Lemma 29), we just need to control the discrepancy between σ and t: this was done in Lemma 19. Finally, the shape result for the coupled zone is proved in Lemma 30, by introducing a coupling time t' and by bounding the difference between this time t' and the essential hitting time σ.

Lemma 25. *The functional μ can be extended to a norm on $\mathbb{R}^d$.*

Proof. Homogeneity in integers. We know that $\mu(x) = \lim \overline{\mathbb{E}}\frac{\sigma(nx)}{n}$, and that $\sigma(nx)$ and $\sigma(-nx)$ have the same law under $\overline{\mathbb{P}}$, so $\mu(x) = \mu(-x)$. By extracting subsequences, we prove the homogeneity in integers:

$$\forall k \in \mathbb{Z} \quad \forall x \in \mathbb{Z}^d \quad \mu(kx) = |k|\mu(x).$$

Subadditivity. One has $\sigma(nx+ny) \le \sigma(nx) + \sigma(ny) \circ \tilde{\theta}_{nx} + r(nx, ny)$.
Since $\overline{\mathbb{P}}$ is invariant under the action of $\tilde{\theta}_{nx}$, we get with Corollary 16:

$$\overline{\mathbb{E}}\sigma(nx+ny) \le \overline{\mathbb{E}}\sigma(nx) + \overline{\mathbb{E}}\sigma(ny) + \overline{\mathbb{E}}r(nx, ny) \le \overline{\mathbb{E}}\sigma(nx) + \overline{\mathbb{E}}\sigma(ny) + M_1.$$

We deduce that $\forall x \in \mathbb{Z}^d \quad \forall y \in \mathbb{Z}^d \quad \mu(x+y) \le \mu(x) + \mu(y)$.

Extension to $\mathbb{R}^d$. The Fekete Lemma ensures that

$$\mu(x) + M_1 = \inf_{n\ge 1} \frac{\overline{\mathbb{E}}\sigma(nx) + M_1}{n},$$

so $\mu(x) \le \overline{\mathbb{E}}\sigma(x)$. Corollary 21 gives some $L > 0$ such that $\overline{\mathbb{E}}\sigma(x) \le L\|x\|$ for any x. Finally, $\mu(x) \le L\|x\|$ for every $x \in \mathbb{Z}^d$, which leads to $|\mu(x) - \mu(y)| \le L\|x - y\|$: we can then extend μ to $\mathbb{Q}^d$ par homogeneity, then to $\mathbb{R}^d$ by uniform continuity.

Positivity. Let M be given by Proposition 5. With (8), we obtain

$$\begin{aligned}
\overline{\mathbb{P}}\left(\sigma(nx) < \frac{n\|x\|}{2M}\right) &\le \overline{\mathbb{P}}\left(t(nx) < \frac{n\|x\|}{2M}\right) \le \overline{\mathbb{P}}\left(\xi_{\frac{n\|x\|}{M}} \not\subset B_{\frac{n\|x\|}{2}}\right) \\
&\le \int \frac{\mathbb{P}_\lambda\left(\tau^0 = +\infty,\ \xi^0_{\frac{n\|x\|}{2M}} \not\subset B_{\frac{n\|x\|}{2}}\right)}{\mathbb{P}_\lambda(\tau^0 = +\infty)}\, d\nu(\lambda) \\
&\le \frac{A}{\rho}\exp\left(-B\frac{n\|x\|}{2M}\right).
\end{aligned}$$

With the Borel–Cantelli Lemma, we deduce that $\mu(x) \ge \frac{1}{2M}\|x\|$. This inequality, once established for every $x \in \mathbb{Z}^d$, can be extended by homogeneity and continuity to $\mathbb{R}^d$. So μ is a norm. □

In the following, we set $C = 2C_{32}$, where C_{32} in given in Corollary 20.

Lemma 26. *For every $\varepsilon > 0$, $\overline{\mathbb{P}} - a.s.$, there exists $R > 0$ such that*

$$\forall x, y \in \mathbb{Z}^d \quad (\|x\| \ge R \text{ and } \|x - y\| \le \varepsilon\|x\|) \Longrightarrow (|\sigma(x) - \sigma(y)| \le C\varepsilon\|x\|).$$

Proof. For $m \in \mathbb{N}$ and $\varepsilon > 0$, we define the event

$$A_m(\varepsilon) = \left\{\exists x, y \in \mathbb{Z}^d : \ \|x\| = m, \ \|x-y\| \le \varepsilon m \text{ et } |\sigma(x) - \sigma(y)| > C\varepsilon m\right\}.$$

Noting that

$$A_m(\varepsilon) \subset \bigcup_{\substack{(1-\varepsilon)m \le \|x\| \le (1+\varepsilon)m \\ \|x-y\| \le \varepsilon m}} \left\{\sigma(y-x) \circ \tilde{\theta}_x + r(x, y-x) > C\varepsilon m\right\},$$

we see, with Corollaries 20 and 16, that

$$\begin{aligned}
\overline{\mathbb{P}}_\lambda(A_m(\varepsilon)) &\le \sum_{\substack{(1-\varepsilon)m \le \|x\| \le (1+\varepsilon)m \\ \|z\| \le \varepsilon m}} \overline{\mathbb{P}}_\lambda(\sigma(z) \circ \tilde{\theta}_x + r(x,z) > C\varepsilon m) \\
&\le \sum_{\substack{(1-\varepsilon)m \le \|x\| \le (1+\varepsilon)m \\ \|z\| \le \varepsilon m}} \overline{\mathbb{P}}_{x.\lambda}(\sigma(z) > 2C\varepsilon m/3) \\
&\qquad + \overline{\mathbb{P}}_\lambda(r(x, y-x) > C\varepsilon m/3) \\
&\le (1+2\varepsilon m)^d (1+2(1+\varepsilon)m)^d A_{32} \exp(-B_{32}\sqrt{C\varepsilon m/3}) \\
&\qquad + A_{27} \exp(-B_{27}\sqrt{C'\varepsilon m/3}))
\end{aligned}$$

by Corollary 20 and Theorem 2. Integrating then with respect to λ, we conclude the proof with the Borel–Cantelli Lemma. □

Lemma 27. $\overline{\mathbb{P}} - a.s.$ $\displaystyle\lim_{\|x\| \to +\infty} \frac{|\sigma(x) - \mu(x)|}{\|x\|} = 0.$

Proof. Assume by contradition that there exists $\varepsilon > 0$ such that the event "$|\sigma(x) - \mu(x)| > \varepsilon\|x\|$ for infinitely many values of x" has a positive probability. We focus on this event. There exists a random sequence $(y_n)_{n\ge 0}$ of sites in $\mathbb{Z}^d$ such that $\|y_n\|_1 \to +\infty$ and, for every n, $|\sigma(y_n) - \mu(y_n)| \ge \varepsilon\|y_n\|_1$. By extracting a subsequence, we can assume that

$$\frac{y_n}{\|y_n\|_1} \to z.$$

Fix $\varepsilon_1 > 0$ (to be chosen later); we can find $z' \in \mathbb{Z}^d$ such that

$$\left\| \frac{z'}{\|z'\|_1} - z \right\|_1 \le \varepsilon_1.$$

For each y_n, we can find an integer point on $\mathbb{R}z'$ close to y_n. Let h_n be the integer part of $\frac{\|y_n\|_1}{\|z'\|_1}$; we have

$$\begin{aligned}
\|y_n - h_n.z'\|_1 &\le \left\| y_n - \frac{\|y_n\|_1}{\|z'\|_1} z' \right\|_1 + \left| \frac{\|y_n\|_1}{\|z'\|_1} - h_n \right| \|z'\|_1 \\
&\le \|y_n\|_1 \left\| \frac{y_n}{\|y_n\|_1} - \frac{z'}{\|z'\|_1} \right\|_1 + \|z'\|_1
\end{aligned}$$

Take $N > 0$ large enough to have $\big(n \ge N\big) \Rightarrow \big(\|\frac{y_n}{\|y_n\|_1} - z\|_1 \le \varepsilon_1\big)$. By our choice for z', one has

$$\big(n \ge N\big) \Rightarrow \left(\left\| \frac{y_n}{\|y_n\|_1} - \frac{z'}{\|z'\|_1} \right\|_1 \le 2\varepsilon_1 \right),$$

and, consequently, $\|y_n - h_n.z'\|_1 \le 2\varepsilon_1\|y_n\|_1 + \|z'\|_1$. Thus, increasing N if necessary, one has, for every $n \ge N$, $\|y_n - h_n.z'\|_1 \le 3\varepsilon_1\|y_n\|_1$. But if N is large enough, Lemma 26 ensures that

$$\forall n \ge N \quad |\sigma(y_n) - \sigma(h_n.z')| \le 3C\varepsilon_1\|y_n\|_1.$$

Finally, for every large n we have

$$\begin{aligned}|\sigma(y_n) - \mu(y_n)| &\le |\sigma(y_n) - \sigma(h_n.z')| + |\sigma(h_n.z') - \mu(h_n.z')| + |\mu(h_n.z') - \mu(y_n)| \\ &\le 3C\varepsilon_1\|y_n\|_1 + h_n\left|\frac{\sigma(h_n.z')}{h_n} - \mu(z')\right| + L\|h_n.z' - y_n\|_1 \\ &\le 3C\varepsilon_1\|y_n\|_1 + (1+\varepsilon_1)\frac{\|y_n\|_1}{\|z'\|_1}\left|\frac{\sigma(h_n.z')}{h_n} - \mu(z')\right| + 3\varepsilon_1 L\|y_n\|_1.\end{aligned}$$

But the a.s. convergence in the z' direction ensures that for every large n,

$$\left|\frac{\sigma(h_n.z')}{h_n} - \mu(z')\right| \le \varepsilon_1.$$

Now if $\varepsilon_1 > 0$ is small, we obtain, for every large n, $|\sigma(y_n) - \mu(y_n)| < \varepsilon\|y_n\|_1$; this brings contradiction and ends the proof. □

We can now prove the shape result for the "fattened" version $\tilde{G}_t$ of $G_t = \{x \in \mathbb{Z}^d : \sigma(x) \le t\}$; we recall that A_μ is the unit ball for μ.

Lemma 28. *For every $\varepsilon > 0$, $\overline{\mathbb{P}}$ − a.s., for every large t,*

$$(1-\varepsilon)A_\mu \subset \frac{\tilde{G}_t}{t} \subset (1+\varepsilon)A_\mu.$$

Proof. Let us prove by contradiction that if t is large enough, $\frac{G_t}{t} \subset (1+\varepsilon)A_\mu$. Thus assume that there exists a increasing sequence$(t_n)_{n\ge1}$, with $t_n \to +\infty$ and $\frac{G_{t_n}}{t_n} \not\subset (1+\varepsilon)A_\mu$: so there exists x_n with $\sigma(x_n) \le t_n$ and $\mu(x_n)/t_n > 1+\varepsilon$. So $\mu(x_n)/\sigma(x_n) > 1+\varepsilon$, which contradicts the uniform convergence of Lemma 27: since $\mu(x_n) > t_n(1+\varepsilon)$, the sequence $(\|x_n\|)_{n\ge1}$ goes to infinity.

For the inverse inclusion, we still assume by contradiction that there exists a increasing sequence $(t_n)_{n\ge1}$, with $t_n \to +\infty$ and $(1-\varepsilon)A_\mu \not\subset \frac{\tilde{G}_{t_n}}{t_n}$; this means we can find x_n with $\mu(x_n) \le (1-\varepsilon)t_n$, but $\sigma(x_n) > t_n$. Since t_n goes to $+\infty$, the sequence $(x_n)_{n\ge1}$ is not bounded and satisfies $\frac{\mu(x_n)}{\sigma(x_n)} < 1-\varepsilon$; this contradicts once again the uniform convergence of Lemma 27 and ends the proof. □

Then we immediately recover the uniform convergence result for the hitting time t via Lemma 19, and, by an argument similar to the one used in Lemma 28, the asymptotic shape result for the "fattened" version $\tilde{H}_t$ of $H_t = \{x \in \mathbb{Z}^d : t(x) \le t\}$.

Lemma 29. $\overline{\mathbb{P}}$ − *a.s.*, $\displaystyle\lim_{\|x\|\to+\infty} \frac{t(x) - \mu(x)}{\|x\|} = 0,$

and for every $\varepsilon > 0$, $\overline{\mathbb{P}}$ − a.s., for every large t, $(1-\varepsilon)A_\mu \subset \dfrac{\tilde{H}_t}{t} \subset (1+\varepsilon)A_\mu$.

It only remains now to prove the shape result for the coupled zone $\tilde{K}'_t$, which is the "fattened" version of $K'_t = \{x \in \mathbb{Z}^d : \ \forall s \ge t \quad \xi^0_s(x) = \xi^{\mathbb{Z}^d}_s(x)\}$:

Lemma 30. *For every $\varepsilon > 0$, $\overline{\mathbb{P}}$ − a.s., for every large t, $(1-\varepsilon)A_\mu \subset \dfrac{\tilde{K}'_t \cap \tilde{G}_t}{t}$.*

Proof. Since $t \mapsto K'_t \cap G_t$ is non-decreasing, we use the same scheme of proof as for Lemma 28. We set, for $x \in \mathbb{Z}^d$,

$$t'(x) = \inf\{t \geq 0 : \ x \in K'_t \cap G_t\}.$$

It is then sufficient to prove that $\overline{\mathbb{P}} - a.s.$, $\lim_{\|x\| \to +\infty} \frac{|t'(x)-\sigma(x)|}{\|x\|} = 0$.
By definition, $t'(x) \geq \sigma(x)$; thus it is sufficient to prove the existence of constants $A', B' > 0$ such that

$$\forall x \in \mathbb{Z}^d \quad \forall s \geq 0 \quad \overline{\mathbb{P}}(t'(x) - \sigma(x) \geq s) \leq A' e^{-B's}. \tag{35}$$

• Note first that for every $t \geq 0$, $K_{\sigma(x)+t} \supset x + K_t \circ \tilde{\theta}_x$.
Indeed, let $z \in x + K_t \circ \tilde{\theta}_x$. Consider first the case $z \notin \xi^{\mathbb{Z}^d}_{\sigma(x)+t}$. Since, by additivity (1), $\xi^0_{\sigma(x)+t} \subset \xi^{\mathbb{Z}^d}_{\sigma(x)+t}$, we have $z \notin \xi^0_{\sigma(x)+t}$, and so that $z \in K_{\sigma(x)+t}$.

Consider now the case $z \in \xi^{\mathbb{Z}^d}_{\sigma(x)+t}$. Since, by additivity, $\xi^{\mathbb{Z}^d}_{\sigma(x)} \subset \xi^{\mathbb{Z}^d}_0 \circ \tilde{\theta}_x$, we have $y = z - x \in \xi^{\mathbb{Z}^d}_t \circ \tilde{\theta}_x$. But since $y \in K_t \circ \tilde{\theta}_x$, the definition of K_t implies that $\xi^0_t(y) \circ \tilde{\theta}_x = \xi^{\mathbb{Z}^d}_t(y) \circ \tilde{\theta}_x = 1$. Since $x \in \xi^0_{\sigma(x)}$ and $y \in \xi^0_t \circ \tilde{\theta}_x$, we obtain $z = x + y \in \xi^0_{\sigma(x)+t}$, and so $z \in K_{\sigma(x)+t}$.

• Fix $s \geq 0$. The previous point says that

$$\left(\bigcap_{t \geq s} K_{\sigma(x)+t}\right) \supset \left(x + \bigcap_{t \geq s} (K_t \circ \tilde{\theta}_x)\right), \text{ and so } K'_{\sigma(x)+s} \supset \Big(x + (K'_s \circ \tilde{\theta}_x)\Big).$$

Since $\overline{\mathbb{P}}$ is invariant under $\tilde{\theta}_x$, we get:

$$\begin{aligned}
\overline{\mathbb{P}}(t'(x) > \sigma(x) + s) &= \overline{\mathbb{P}}(x \notin K'_{\sigma(x)+s} \cap G_{\sigma(x)+s}) \\
&= \overline{\mathbb{P}}(x \notin K'_{\sigma(x)+s}) \\
&\leq \overline{\mathbb{P}}\Big(x \notin (x + K'_s \circ \tilde{\theta}_x)\Big) \\
&\leq \overline{\mathbb{P}}(0 \notin K'_s \circ \tilde{\theta}_x) = \overline{\mathbb{P}}(0 \notin K'_s).
\end{aligned}$$

We conclude with (11). □

6. Uniform controls of the growth

The aim of this section is to establish some of the uniform controls announced in Proposition 5. To control the growth of the contact process, we need some lemmas on the Richardson model.

6.1. **Some lemmas on the Richardson model.** We call Richardon model with parameter λ the time-homogeneous, $\mathcal{P}(\mathbb{Z}^d)$-valued Markov process $(\eta_t)_{t \geq 0}$, whose evolution is defined as follows: an empty site z becomes infected at rate $\lambda \sum_{\|z-z'\|_1=1} \eta_t(z')$, the different evolutions being independent. Thanks to the graphical construction, we can, for each $\lambda \in \Lambda$, build a coupling of the contact process in environment λ with the Richardson model with parameter $\lambda_{\max}$, in the following way: at any time t, the space occupied by the contact process is contained in the space occupied by the Richardson model.

The first lemma, whose proof is omitted, easily follows from the representation of the Richardson model in terms of first passage percolation, together with a path counting argument.

Lemma 31. *For every $\lambda > 0$, there exist constants $A, B > 0$ such that*

$$\forall t \geq 0 \quad \mathbb{P}(\eta_1 \not\subset B_t) \leq A \exp(-Bt).$$

Lemma 32. *For every $\lambda > 0$, there exist constant $A, B, M > 0$ such that*

$$\forall s \geq 0 \quad \mathbb{P}(\exists t \geq 0 : \ \eta_t \not\subset B_{Mt+s}) \leq A \exp(-Bs).$$

Proof. The representation of the Richardson model in terms of first passage percolation ensures the existence of $A', B', M' > 0$ such that for each $t \geq 0$,

$$\mathbb{P}(\eta_t \not\subset B_{M't}) \leq A' \exp(-B't). \tag{36}$$

For more details, one can refer to Kesten [24].

We first control the process in integer times thanks to the following estimate:

$$\begin{aligned}
\mathbb{P}(\exists k \in \mathbb{N} : \ \eta_k \not\subset B_{M'k+s/2}) &\leq \mathbb{P}(\exists k \in \mathbb{N} : \ \eta_{k+s/(2M')} \not\subset B_{M'k+s/2}) \\
&\leq \sum_{k=0}^{+\infty} \mathbb{P}(\eta_{k+s/(2M')} \not\subset B_{M'k+s/2}) \\
&\leq \frac{A'}{1-\exp(-B')} \exp\left(-\frac{B's}{2M'}\right). \qquad (37)
\end{aligned}$$

Let us now control the fluctuations between integer times. Let $M > M'$:

$$\begin{aligned}
&\mathbb{P}(\{\exists t \geq 0 : \ \eta_t \not\subset B_{Mt+s}\} \cap \{\forall k \in \mathbb{N}, \ \eta_k \subset B_{M'k+s/2}\}) \\
\leq & \sum_{k=0}^{+\infty} \mathbb{P}(\exists t \in [k, k+1] : \ \eta_k \subset B_{M'k+s/2} \text{ et } \eta_t \not\subset B_{Mt+s}). \qquad (38)
\end{aligned}$$

Then, denoting by $C' > 0$ a constant such that $|B_t| \leq C'(1+t)^d$ and by A, B the constants appearing in Lemma 31,

$$\begin{aligned}
&\mathbb{P}(\exists t \in [k, k+1] : \ \eta_k \subset B_{M'k+s/2} \text{ et } \eta_t \not\subset B_{Mt+s}) \\
\leq\ & \mathbb{P}(\eta_k \subset B_{M'k+s/2} \text{ et } \eta_{k+1} \not\subset B_{Mk+s}) \\
\leq\ & |B_{M'k+s/2}| \mathbb{P}(\eta_1 \not\subset B_{k(M-M')+s/2}) \qquad (39) \\
\leq\ & C'(1+M'k+s/2)^d A \exp(-B(k(M-M')+s/2)) \\
\leq\ & AC'(1+s/2)^d \exp(-Bs/2)(1+M'k)^d \exp(-B(k(M-M')).
\end{aligned}$$

Inequality (39) comes from the Markov property and from the subadditivity of the contact process. Since the series $((1+M'k)^d \exp(-B(k(M-M')))_{k \geq 1}$ converges, the desired result follows from (37) and (38). □ □

6.2. **A restart procedure.** We will use here a so-called restart argument, which can be summed up as follows. We couple the system that we want to study (the strong system) with a system that it stochastically dominates (the weak system), and that is best understood. Then, we can transport some of the properties of the known system to the one we study: we let the processes simultaneously evolve and, each time the weaker dies and the stronger remains alive, we restart a copy of the weakest, coupled with the strongest again. Thus, either both processes die before we found any weak process surviving; in this case the control of large finite lifetimes for the weak can be transposed to the strongest one, or the strongest indefinitely survives and is finally coupled with a weak surviving one. In that case, a bound for the time that is necessary to find a successful restart permits to transfer properties of the weak surviving process to the strong one.

This technique is already old; that can be found for example in Durrett [14], section 12, in a very pure form. It is also used by Durrett and Griffeath [16], in order to transfer some controls for the one-dimensional contact process to the contact process in a larger dimension. We will use it here by coupling the contact process in inhomogeneous environment $\lambda \in \Lambda$ with the contact process with a constant birth rate $\lambda_{\min}$. Here, the assumption $\lambda_{\min} > \lambda_c(\mathbb{Z}^d)$ matters.

To this end, we will couple collections of Poisson point processes. Fix $\lambda \in \Lambda$. We can build a probability measure $\tilde{\mathbb{P}}_\lambda$ on $\Omega \times \Omega$ under which

- The first coordinate ω is a collection $((\omega_e)_{e\in\mathbb{E}^d}, (\omega_z)_{z\in\mathbb{Z}^d})$ of Poisson point processes, with respective intensities $(\lambda_e)_{e\in\mathbb{E}^d}$ for the bond-indexed processes, and intensity 1 for the site-indexed processes.
- The second coordinate η is a collection $((\eta_e)_{e\in\mathbb{E}^d}, (\eta_z)_{z\in\mathbb{Z}^d})$ of Poisson point processes, with intensity $\lambda_{\min}$ for the bond-indexed processes, and intensity 1 for the site-indexed processes.
- Site-indexed Poisson point processed (death times) coincide: for every $z \in \mathbb{Z}^d$, $\eta_z = \omega_z$.
- Bond-indexed Poisson point processed (birth-times candidates) are coupled: for each $e \in \mathbb{E}^d$, the support of η_e is included in the support of ω_e.

We denote by $\xi^A = \xi^A(\omega, \eta)$ the contact process in environment λ starting from A and built from the Poisson process collection ω, and $\zeta^B = \zeta^B(\omega, \eta)$ the contact process in environment $\lambda_{\min}$ starting from B and built from the Poisson process collection η. If $B \subset A$, then $\tilde{\mathbb{P}}_\lambda$ almost surely, $\zeta_t^B \subset \xi_t^A$ holds for each $t \geq 0$. We can note that the process (ξ^A, ζ^B) is a Markov process.

We introduce the lifetimes of both processes:

$$\tau = \inf\{t \geq 0 : \ \xi_t^0 = \varnothing\} \text{ and, for } x \in \mathbb{Z}^d, \ \tau'_x = \inf\{t \geq 0 : \ \zeta_t^x = \varnothing\}.$$

Note that the law of τ'_x under $\tilde{\mathbb{P}}_\lambda$ is the law of τ_x under $\mathbb{P}_{\lambda_{\min}}$; it does actually not depend on the process starting point, because the model with constant birth rate is translation invariant.

We recursively define a sequence of stopping times $(u_k)_{k\geq 0}$ and a sequence of points $(z_k)_{k\geq 0}$, letting $u_0 = 0$, $z_0 = 0$, and for each $k \geq 0$:

- if $u_k < +\infty$ and $\xi_{u_k} \neq \varnothing$, then $u_{k+1} = \tau'_{z_k} \circ \theta_{u_k}$;
- if $u_k = +\infty$ or if $\xi_{u_k} = \varnothing$, then $u_{k+1} = +\infty$;
- if $u_{k+1} < +\infty$ and $\xi_{u_{k+1}} \neq \varnothing$, then z_{k+1} is the smallest point of $\xi_{u_{k+1}}$ for the lexicographic order;
- if $u_{k+1} = +\infty$ or if $\xi_{u_{k+1}} = \varnothing$, then $z_{k+1} = +\infty$.

In other words, until $u_k < +\infty$ and $\xi_{u_k} \neq \varnothing$, we take in ξ_{u_k} the smallest point z_k for the lexicographic order, and look at the lifetime of the weakest process, namely ζ, starting from z_k at time u_k. The restart procedure can stop in two ways: either we find k such that $u_k < +\infty$ and $\xi_{u_k} = \varnothing$, which implies that that the strongest process (which contains the weak) precisely dies at time u_k; or we find k such that $u_k < +\infty$, $\xi_{u_k} \neq \varnothing$, and $u_{k+1} = +\infty$. In this case, we have found a point z_k such that the weak process which starts from z_k at time u_k survives; particularly this implies that the strongest also survives. We then define

$$K=\inf\{n\ge 0:\ u_{n+1}=+\infty\}.$$

The name of the K variable is chosen by analogy with Section 3. The current section being independent from the rest of the article, confusion should not be possible. It comes from the preceding discussion that

(40) $$(\tau=+\infty \Longleftrightarrow \xi^0_{u_K}\neq\varnothing) \quad \text{and if } \tau<+\infty, \text{ then } u_K=\tau.$$

We regroup in the next lemma some estimates on the restart procedure that are necessary to prove Proposition 5. Recall that ρ is introduced in Equation (7).

Lemma 33. *We work in the preceding frame. Then,*

- $\forall\lambda\in\Lambda\quad \forall n\in\mathbb{N}\quad \tilde{\mathbb{P}}_\lambda(K>n)\le(1-\rho)^n$.
- $\forall B\in\mathcal{B}(\mathcal{D})\quad \tilde{\mathbb{P}}_\lambda(\tau=+\infty,\zeta^{z_K}\circ\theta_{u_K}\in B)=\mathbb{P}_\lambda(\tau=+\infty)\overline{\mathbb{P}}_{\lambda_{\min}}(\xi^0\in B)$.
- *There exist* $\alpha,\beta>0$ *such that for every* $\lambda\in\Lambda$, $\tilde{\mathbb{E}}_\lambda(\exp(\alpha u_K))<\beta$.

Proof. By the strong Markov property, we have

$$\begin{aligned}\tilde{\mathbb{P}}_\lambda(K\ge n+1) &= \tilde{\mathbb{P}}_\lambda(u_{n+1}<+\infty)\\ &= \tilde{\mathbb{P}}_\lambda(u_n<+\infty,\ \xi_{u_n}\neq\varnothing,\ \tau'_{z_n}\circ\theta_{u_n})<+\infty)\\ &\le \tilde{\mathbb{P}}_\lambda(u_n<+\infty)(1-\rho)=\tilde{\mathbb{P}}_\lambda(K\ge n)(1-\rho).\end{aligned}$$

Thus, K has a subexponential tail, which proves the first point. Particularly, K is almost surely finite.

Using (40) and the strong Markov property, we have also

$$\begin{aligned}&\tilde{\mathbb{P}}_\lambda(\tau=+\infty,\zeta^{z_K}\circ\theta_{u_K}\in B)=\tilde{\mathbb{P}}_\lambda(\xi_{u_K}\neq\varnothing,\zeta^{z_K}\circ\theta_{u_K}\in B)\\ =&\sum_{k=0}^{+\infty}\sum_{z\in\mathbb{Z}^d}\tilde{\mathbb{P}}_\lambda(K=k,\ \xi^0_{u_k}\neq\varnothing,\ z_k=z,\ \zeta^{z_K}\circ\theta_{u_K}\in B)\\ =&\sum_{k=0}^{+\infty}\sum_{z\in\mathbb{Z}^d}\tilde{\mathbb{P}}_\lambda(u_k<+\infty,\ \xi^0_{u_k}\neq\varnothing,\ z_k=z,\ \tau'_{z_K}\circ\theta_{u_k}=+\infty,\ \zeta^{z_K}\circ\theta_{u_K}\in B)\\ =&\sum_{k=0}^{+\infty}\sum_{z\in\mathbb{Z}^d}\tilde{\mathbb{P}}_\lambda(u_k<+\infty,\ \xi^0_{u_k}\neq\varnothing,\ z_k=z)\mathbb{P}_{\lambda_{\min}}(\tau=+\infty,\ \xi^0\in B)\\ =&\ \mathbb{P}_{\lambda_{\min}}(\tau=+\infty,\ \xi^0\in B)\sum_{k=0}^{+\infty}\tilde{\mathbb{P}}_\lambda(u_k<+\infty,\ \xi^0_{u_K}\neq\varnothing).\end{aligned}$$

Taking for B the whole set of trajectories, we can identify:

$$\tilde{\mathbb{P}}(\tau=+\infty)=\mathbb{P}_\lambda(\tau=+\infty)=\mathbb{P}_{\lambda_{\min}}(\tau=+\infty)\sum_{k=0}^{+\infty}\tilde{\mathbb{P}}_\lambda(u_k<+\infty,\xi^0_{u_K}\neq\varnothing),$$

which gives us the second point.

Since $\lambda_{\min}>\lambda_c(\mathbb{Z}^d)$, The results by Durrett and Griffeath [16] for large λ, extended to the whole supercritical regime by Bezuidenhout and Grimmett [4], ensures the existence of $A,B>0$ such that

$$\forall t\ge 0\quad \mathbb{P}_{\lambda_{\min}}(t\le\tau<+\infty)\le A\exp(-Bt),$$

which gives the existence of exponential moments for $\tau 1\!\!1_{\{\tau<+\infty\}}$. Since $\mathbb{P}_{\lambda_{\min}}(\tau=+\infty)>0$, we can chose (e.g. by dominated convergence) some $\alpha>0$ such that

$\mathbb{E}_{\lambda_{\min}}(\exp(\alpha\tau)1\!\!1_{\{\tau<+\infty\}}) = r < 1$.
For $k \geq 0$, we note

$$S_k = \exp\left(\alpha\sum_{i=0}^{k-1}\tau'_{z_i}\circ\theta_{u_i}\right)1\!\!1_{\{u_k<+\infty\}}.$$

We note that S_k is $\mathcal{F}_{u_k}$-measurable. Let $k \geq 0$. We have

$$\exp(\alpha u_K)1\!\!1_{\{K=k\}} \leq S_k.$$

Thus, applying the strong Markov property at time $u_{k-1} < +\infty$, we get, for $k \geq 1$

$$\begin{aligned}\tilde{\mathbb{E}}_\lambda[\exp(\alpha u_K)1\!\!1_{\{K=k\}}] &\leq \tilde{\mathbb{E}}_\lambda(S_k) = \tilde{\mathbb{E}}_\lambda(S_{k-1})\mathbb{E}_{\lambda_{\min}}(\exp(\alpha\tau)1\!\!1_{\{\tau<+\infty\}})\\ &\leq r\tilde{\mathbb{E}}_\lambda(S_{k-1}).\end{aligned}$$

Since $r < 1$, it comes that $\tilde{\mathbb{E}}_\lambda[\exp(\alpha u_K)] \leq \frac{r}{1-r} < +\infty$. □

6.3. **Proof of Proposition 5.** Estimates (8) and (7) follow from a simple stochastic comparizon:

Proof of (7). It suffices to note that for every environment $\lambda \in \Lambda$ and each $z \in \mathbb{Z}^d$, we have

$$\mathbb{P}_\lambda(\tau^z = +\infty) \geq \mathbb{P}_{\lambda_{\min}}(\tau^z = +\infty) = \mathbb{P}_{\lambda_{\min}}(\tau^0 = +\infty) > 0.$$

□

Proof of (8). We use the stochastic domination of the contact process in environment λ by the Richardson model with parameter $\lambda_{\max}$. For this model, (36) ensures a growth which is at least linear. □

Then, it remains to prove (9), (10) and (11) with a restart procedure.

Proof of (9). Let $\alpha, \beta > 0$ as given in the third point of Lemma 33. Recall that $u_K = \tau$ on $\{\tau < +\infty\}$. For each $\lambda \in \Lambda$ and each $t > 0$, we have

$$\begin{aligned}\mathbb{P}_\lambda(t < \tau < +\infty) &= \mathbb{P}_\lambda(e^{\alpha t} < e^{\alpha\tau}, \tau < +\infty) = \tilde{\mathbb{P}}_\lambda(e^{\alpha t} < e^{\alpha u_K}, \tau < +\infty)\\ &\leq \tilde{\mathbb{P}}_\lambda(e^{\alpha t} < e^{\alpha u_K}) \leq e^{-\alpha t}\tilde{\mathbb{E}}_\lambda e^{\alpha u_K} \leq \beta e^{-\alpha t},\end{aligned}$$

which concludes the proof. □

Proof of (10). Since $\lambda_{\min} > \lambda_c(\mathbb{Z}^d)$, Durrett and Griffeath's results [16] for largeλ, extended to the whole supercritical regime by Bezuidenhout and Grimmett [4], ensure the existence of constants $A, B, c > 0$ such that, for each $y \in \mathbb{Z}^d$, for each $t \geq 0$,

$$\overline{\mathbb{P}}_{\lambda_{\min}}\left(t(y) \geq \frac{\|y\|}{c} + t\right) \leq A\exp(-Bt). \tag{41}$$

Besides, the domination by the Richardson model with parameter $\lambda_{\max}$ and Lemma 32 ensure the existence of $A, B, M > 0$ such that for every $\lambda \in \Lambda$, for each $s \geq 0$,

$$\mathbb{P}_\lambda(\exists t \geq 0,\ \xi_t^0 \not\subset B_{Mt+s}) \leq A\exp(-Bs). \tag{42}$$

By decreasing c or increasing M if necessary, we can also assume that $\frac{c}{M} \leq 1$. Now,

$$\begin{aligned}
&\tilde{\mathbb{P}}_\lambda\left(t(y) \geq \frac{\|y\|}{c} + t,\ \tau = +\infty\right) \\
\leq\ &\tilde{\mathbb{P}}_\lambda\left(u_K \geq \frac{tc}{6M}\right) + \tilde{\mathbb{P}}_\lambda\left(u_K \leq \frac{tc}{6M},\ \xi^0_{u_K} \not\subset B_{tc/3}\right) \\
&+\tilde{\mathbb{P}}_\lambda\left(\tau = +\infty,\ u_K \leq \frac{tc}{6M},\ \xi^0_{u_K} \subset B_{tc/3},\ t(y) \geq \frac{\|y\|}{c} + t\right).
\end{aligned}$$

The existence of exponential moments for $u_K$given by Lemma 33 enables to bound the first term: there exist $C, \alpha > 0$ such that for each $\lambda \in \Lambda$, for each $t \geq 0$,

$$\tilde{\mathbb{P}}_\lambda\left(u_K \geq \frac{tc}{6M}\right) \leq C \exp\left(-\frac{\alpha ct}{6M}\right).$$

The second term is controlled with the help of (42):

$$\tilde{\mathbb{P}}_\lambda\left(u_K \leq \frac{tc}{6M},\ \xi^0_{u_K} \not\subset B_{tc/3}\right) \leq \mathbb{P}_\lambda(\exists t \geq 0,\ \xi^0_t \not\subset B_{Mt+\frac{tc}{6}}) \leq A \exp\left(-B\frac{tc}{6}\right).$$

It remains to bound the last term. We note here

$$t'(y) = \inf\{t \geq 0 :\ y \in \zeta^0_t\}.$$

Recall that if $\tau = +\infty$, then $\xi_{u_K} \neq \varnothing$ and z_K is well-defined. Since $t(y)$ is the hitting time of y and $\xi^0_t \supset \zeta^0_t$ for each t, we have, on $\{\tau = +\infty\}$,

$$t(y) \leq u_K + t'(y - z_K) \circ T_{z_K} \circ \theta_{u_K}.$$

If $u_K \leq \frac{tc}{6M} \leq \frac{t}{6}$, then $t(y) \leq \frac{t}{6} + t'(y - z_K) \circ T_{z_K} \circ \theta_{u_K}$. If, moreover, $\xi^0_{u_K} \subset B_{tc/3}$, we have $\|y\| \geq \|y - z_K\| - \frac{tc}{3}$, which gives, with the second point in Lemma 33,

$$\begin{aligned}
&\tilde{\mathbb{P}}_\lambda\left(\tau = +\infty,\ u_K \leq \frac{ct}{6M},\ \xi^0_{u_K} \subset B_{ct/3},\ t(y) \geq \frac{\|y\|}{c} + t\right) \\
\leq\ &\tilde{\mathbb{P}}_\lambda\left(\tau = +\infty,\ t'(y - z_K) \circ T_{z_K} \circ \theta_{u_K} \geq \frac{\|y - z_K\|}{c} + \frac{t}{2}\right) \\
\leq\ &\mathbb{P}_\lambda(\tau = +\infty) \sup_{z \in \mathbb{Z}^d} \overline{\mathbb{P}}_{\lambda_{\min}}\left(t(y - z) \geq \frac{\|y - z\|}{c} + \frac{t}{2}\right) \leq A \exp(-Bt/2),
\end{aligned}$$

where the last inequality follows from (41). This concludes the proof. □

Proof of (11). Let $s \geq 0$, and denote by n the integer part of s. Let $\gamma > 0$ be a fixed number, whose precise value will be specified later.

$$\begin{aligned}
&\overline{\mathbb{P}}(0 \notin K'_s) = \overline{\mathbb{P}}(\exists t \geq s :\ 0 \notin K_t) \\
\leq\ &\sum_{k=n}^{+\infty} \overline{\mathbb{P}}(B_{\gamma k} \not\subset K_k) + \sum_{k=n}^{+\infty} \overline{\mathbb{P}}(B_{\gamma k} \subset K_k,\ \exists t \in [k, k+1) \text{ such that } 0 \notin K_t).
\end{aligned}$$

Let us first bound the second sum. Fix $k \geq n$. Assume that $B_{\gamma k} \subset K_k$ and consider $t \in [k, k+1)$ such that $0 \notin K_t$. Then, there exists $x \in \mathbb{Z}^d$ such that $0 \in \xi^x_t \backslash \xi^0_t$. Since $0 \in \xi^x_t$ and $t \geq k$, there exists $y \in \mathbb{Z}^d$ such that $y \in \xi^x_k$ and $0 \in \xi^y_{t-k} \circ \theta_k$. If $y \in B_{\gamma k} \subset K_k$, then $\xi^0_k(y) = \xi^{\mathbb{Z}^d}_k(y) = 1$, which implies that $y \in \xi^0_k$.

Now, since $0 \in \xi^y_{t-k} \circ \theta_k$, we obtain $0 \in \xi^0_t$, which contradicts the assumption $0 \notin \xi^0_t$. Thus, we necessary have $y \notin B_{\gamma k}$, so:

$$\begin{aligned}
& \overline{\mathbb{P}}_\lambda(B_{\gamma k} \subset K_k,\ \exists t \in [k, k+1) \text{ tel que } 0 \notin K_t) \\
\le\ & \frac{1}{\mathbb{P}_\lambda(\tau = +\infty)} \mathbb{P}_\lambda\left(\theta_k^{-1}\left(0 \in \underset{s\in[0,1]}{\cup} \xi_s^{\mathbb{Z}^d \backslash B_{\gamma k}}\right)\right) \\
\le\ & \frac{1}{\rho}\mathbb{P}_\lambda\left(0 \in \underset{s\in[0,1]}{\cup} \xi_s^{\mathbb{Z}^d \backslash B_{\gamma k}}\right) \\
\le\ & \frac{1}{\rho}\mathbb{P}_\lambda\left(\underset{s\in[0,1]}{\cup} \xi_s^0 \not\subset B_{\gamma k}\right) = \frac{1}{\rho}\mathbb{P}_\lambda(H_1^0 \not\subset B_{\gamma k}).
\end{aligned}$$

Since the Richardson model with parameter $\lambda_{\max}$ stochastically dominates the contact process in environment λ, we control the last term thanks to Lemma 31.

To control the first sum, it is sufficient to prove that there exist positive constants A, B, γ – and this will fix the precise value of γ – such that for each $\lambda \in \Lambda$ and each $t \ge 0$

$$\mathbb{P}_\lambda(B_{\gamma t} \not\subset K_t,\ \tau^0 = +\infty) \le A \exp(-Bt). \tag{43}$$

The number of integer points in a ball being polynomial with respect to the radius, it is sufficient to prove that there exist some constants $A, B, c' > 0$ such that for each $t \ge 0$, for each $x \in \mathbb{Z}^d$,

$$\|x\| \le c't \Longrightarrow \tilde{\mathbb{P}}_\lambda(\xi^0_t \ne \varnothing, x \in \xi_t^{\mathbb{Z}^d} \backslash \xi^0_t) \le A \exp(-Bt). \tag{44}$$

To prove (44), we will use the following result, that has been obtained by Durrett [17] as a consequence of the Bezuidenhout–Grimmett construction [4]: if ξ^0 and $\tilde{\xi}^x$ are two independent contact processes with parameter $\lambda > \lambda_c(\mathbb{Z}^d)$, respectively starting from 0 and from x, then there exist positive constants A, B, α such that for each $t \ge 0$ and each $x \in \mathbb{Z}^d$,

$$\|x\| \le \alpha t \Longrightarrow \quad \mathbb{P}(\xi^0_t \cap \tilde{\xi}^x_t = \varnothing, \tilde{\xi}^x_t \ne \varnothing, \xi^0_t \ne \varnothing) \le A \exp(-Bt). \tag{45}$$

Let α and M be the constants respectively given by Equations (45) and (8). We put $c' = \alpha/2$ and choose $\varepsilon > 0$ such that $c' + 2\varepsilon M \le \alpha$.

Let $a \in B^0_{\alpha t/4}$ and $b \in B^x_{\alpha t/4}$. We set

$$\alpha_{a,s} = \zeta^a_s \circ \theta_{\varepsilon t/2} \text{ and } \beta_{b,s} = \{y \in \mathbb{Z}^d :\ b \in \zeta^y_s \circ \theta_{t(1-\varepsilon/2)-s}\}.$$

Then, $(\alpha_{a,s})_{0\le s\le t/2(1-\varepsilon)}$ and $(\beta_{a,s})_{0\le s\le t/2(1-\varepsilon)}$ are independent contact processes with constant birth rate $\lambda_{\min}$, respectively starting from a and from b. The process $(\beta_{a,s})_{0\le s\le t/2(1-\varepsilon)}$ is a contact process, but for which the time axis has been reverted. In the same way, we set

$$\hat{\xi}^x_s = \{y \in \mathbb{Z}^d :\ x \in \xi^y_s \circ \theta_{t-s}\}.$$

Note that $(\hat{\xi}^x_s)_{0\le s\le t/2}$ has the same law as $(\xi^x_s)_{0\le s\le t/2}$. Note that:

- Assuming that $a \in \xi^0_{\varepsilon t/2}$, $\alpha_{a,(1-\varepsilon)t/2} \cap \beta_{b,(1-\varepsilon)t/2} \ne \varnothing$ and $b \in \hat{\xi}^x_{\varepsilon t/2}$, then $x \in \xi^0_t$.
- If $x \in \xi^{\mathbb{Z}^d}_t$, then $\hat{\xi}^x_{t/2}$ is non-empty.
- If ξ^0_t is non-empty, then $\xi^0_{t/2}$ is non-empty.

Thus, letting

$$\begin{aligned} E^0 &= \left\{\xi^0_{t/2} \neq \varnothing\right\} \setminus \left\{\exists a \in B^0_{\alpha t/4} \cap \xi^0_{\varepsilon t/2} :\ \alpha_{a,(1-\varepsilon)t/2} \neq \varnothing\right\} \\ \text{and} \quad \hat{E}^x &= \left\{\hat{\xi}^x_{t/2} \neq \varnothing\right\} \setminus \left\{\exists b \in B^x_{\alpha t/4} \cap \hat{\xi}^x_{\varepsilon t/2} :\ \beta_{b,(1-\varepsilon)t/2} \neq \varnothing\right\}, \end{aligned}$$

we get

$$\begin{aligned} (46) \quad \tilde{\mathbb{P}}_\lambda(\xi^0_t \neq \varnothing,\ x \in \xi^{\mathbb{Z}^d}_t \backslash \xi^0_t) &\leq \tilde{\mathbb{P}}_\lambda(\xi^0_{t/2} \neq \varnothing,\ \hat{\xi}^x_{t/2} \neq \varnothing,\ \xi^0_{t/2} \cap \hat{\xi}^x_{t/2} = \varnothing) \\ &\leq \tilde{\mathbb{P}}_\lambda(E^0) + \tilde{\mathbb{P}}_\lambda(\hat{E}^x) + S, \end{aligned}$$

where $S = \sum_{\substack{a \in B^0_{\alpha t/4} \\ b \in B^x_{\alpha t/4}}} \tilde{\mathbb{P}}_\lambda \left(\alpha_{a,\frac{(1-\varepsilon)t}{2}} \neq \varnothing,\ \beta_{b,\frac{(1-\varepsilon)t}{2}} \neq \varnothing,\ \alpha_{a,\frac{(1-\varepsilon)t}{2}} \cap \beta_{b,\frac{(1-\varepsilon)t}{2}} = \varnothing\right)$.

For every couple (a,b) that appears in S, we have $\|a-b\| \leq \|a\| + \|b-x\| + \|x\| \leq \alpha t/4 + \alpha t/4 + \alpha t/2 = \alpha t$, which allows to use (45), and gives the existence of constants $A, B, C' > 0$ such that

$$S \leq C'(1+\alpha t/4)^{2d} A \exp(-B(1-\varepsilon)t/2).$$

By another time reversal, we see that $\tilde{\mathbb{P}}_\lambda(\hat{E}^x) = \tilde{\mathbb{P}}_{x.\lambda}(E^0)$; then it suffices to control $\tilde{\mathbb{P}}_\lambda(E^0)$ uniformly in λ. Let

$$E_1 = \{\xi^0_{t/2} \neq \varnothing\} \backslash \{\exists a \in \mathbb{Z}^d :\ a \in \xi^0_{\varepsilon t/2},\ \alpha_{a,(1-\varepsilon)t/2} \neq \varnothing\}.$$

We have $\tilde{\mathbb{P}}_\lambda(E^0) \leq \tilde{\mathbb{P}}_\lambda(E_1) + \tilde{\mathbb{P}}_\lambda(\xi^0_{\varepsilon t/2} \not\subset B^0_{\alpha t/4})$. By the choice we made for ε and Inequality (8), we have

$$\forall \lambda \in \Lambda \quad \forall t \geq 0 \quad \tilde{\mathbb{P}}_\lambda(\xi^0_{\varepsilon t/2} \not\subset B(0, \alpha t/4)) \leq A \exp(-B\varepsilon t/2).$$

Thanks to the restart Lemma 33, we can see that

$$\tilde{\mathbb{P}}_\lambda(u_K > \varepsilon t/2) \leq \beta \exp(-\alpha \varepsilon t/2).$$

Suppose then that $u_K \leq \varepsilon t/2$ and $\xi^0_{t/2} \neq \varnothing$: z_K is thus well defined and we have $\tau'_{z_K} \circ \theta_{u_K} = +\infty$. Then, there exists an infinite infection branch in the coupled process in environment $\lambda_{\min}$ starting from $\xi^0_{u_K}$. This branch contains at least one point $a \in \xi^0_{(1-\varepsilon)t/2}$. By construction $a \in \xi^0_{(1-\varepsilon)t/2}$ and $\alpha_{a,(1-\varepsilon)t/2} \neq \varnothing$, which completes the proof of (43). □

Remark On our way, we proved that for each $\lambda \in \Lambda$,

$$\lim_{t \to +\infty} \tilde{\mathbb{P}}_\lambda(\xi^0_t \neq \varnothing,\ \hat{\xi}^x_t \neq \varnothing,\ \xi^0_t \cap \hat{\xi}^x_t = \varnothing) = 0,$$

which is the essential ingredient in the proof of the complete convergence Theorem 4. One can refer to the article by Durrett [17] for the details in the case of the classical contact process.

Appendix: proof of almost subadditive ergodic Theorems 23 and 24

Proof of Theorem 23. Let $a_p = C_p^{1/\alpha}$ and $u_n = \mathbb{E}[f_n]$: for every $n, p \in \mathbb{N}$, we have $\mathbb{E}[r^+_{n,p}] \leq (\mathbb{E}[(r^+_{n,p})^\alpha])^{1/\alpha} \leq C_p^{1/\alpha} = a_p$, hence

$$u_{n+p} \leq u_n + u_p + \mathbb{E}[g_p] + \mathbb{E}[r_{n,p}] \leq u_n + u_p + \mathbb{E}[g_p] + a_p.$$

The general term of a convergent series tends to 0, so $C_p = o(p^\alpha)$, or $a_p = o(p)$. Since $\frac{a_n+\mathbb{E}g_n}{n}$ tends to 0, the convergence of u_n/n is classical (see Derriennic [11] for instance). The limit μ is finite because $u_n \ge cn$ holds for each n.

We are going to show that $\underline{f} = \varliminf_{n\to+\infty} \frac{f_n}{n}$ stochastically dominates a random variable whose mean value is not less than μ.

For every random variable X, let us denote by $\mathcal{L}(X)$ its law under $\mathbb{P}$. We denote by $\mathcal{K}$ the set of probability measures on $\mathbb{R}_+^{\mathbb{N}^*}$ whose marginals m satisfy:

$$\forall t>0 \quad m(]t,+\infty[) \le \mathbb{P}(f_1+g_1>t/2) + C_1(2/t)^\alpha.$$

Define, for $k \ge 1$,

$$\Delta_k = f_{k+1} - f_k,$$

and denote by Δ the process $\Delta = (\Delta_k)_{k\ge1}$. For $k \in \mathbb{N}$, subadditivity ensures that $\Delta_k \le (f_1+g_1)\circ\theta_k + r_{k,1}$, hence, for each $t>0$,

$$\begin{aligned}\mathbb{P}(\Delta_k > t) &\le \mathbb{P}((f_1+g_1)\circ\theta_k > t/2) + \mathbb{P}(r_{k,1}^+ > t/2)\\ &\le \mathbb{P}(f_1+g_1>t/2) + C_1(2/t)^\alpha.\end{aligned}$$

This ensures that $\Delta \in \mathcal{K}$.

We denote by s the shift operator: $s((u_k)_{k\ge0}) = (u_k)_{k\ge1}$, and consider the sequence of probability measures on $\mathbb{R}^{\mathbb{N}^*}$:

$$(L_n)_{n\ge1} = \left(\frac{1}{n}\sum_{j=1}^n \mathcal{L}(s^j\circ\Delta)\right)_{n\ge1}.$$

Since $\mathcal{K}$ is convex and invariant by s, the sequence $(L_n)_{n\ge1}$ is $\mathcal{K}$-valued. Let $n,k\ge1$.

$$\begin{aligned}\int \pi_k(x)\ dL_n(x) &= \frac{1}{n}\sum_{j=1}^n \mathbb{E}(\pi_k(s^j\circ\Delta))\\ &= \frac{1}{n}\sum_{j=1}^n \mathbb{E}(f_{k+j+1}-f_{k+j}) = \frac{1}{n}(\mathbb{E}[f_{n+k+1}] - \mathbb{E}[f_{k+1}]).\end{aligned}$$

Let $M_k = \sup_{n\ge1} \frac{1}{n}|\mathbb{E}[f_{n+k+1}] - \mathbb{E}[f_{k+1}]|$. The convergence of u_n/n implies that M_k is finite. Similarly, the subadditivity gives

$$\begin{aligned}\int \pi_k^+(x)\ dL_n(x) &= \frac{1}{n}\sum_{j=1}^n \mathbb{E}(\pi_k^+(s^j\circ\Delta))\\ &= \frac{1}{n}\sum_{j=1}^n \mathbb{E}[(f_{k+j+1}-f_{k+j})^+] \le \mathbb{E}[f_1^+] + \mathbb{E}[g_1] + a_1.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int |\pi_k(x)|\ dL_n(x) &\le \int 2\pi_k^+(x)\ dL_n(x) + |\int \pi_k(x)\ dL_n(x)|\\ &\le M_k + 2\mathbb{E}[f_1^+] + 2\mathbb{E}[g_1^+] + 2a_1.\end{aligned}$$

Let $\mathcal{K}'$ be the family of laws m on $\mathbb{R}^{\mathbb{N}^*}$ such that for each k, $\int |\pi_k|\ dm \le 2M_k + \mathbb{E}[f_1^+] + \mathbb{E}[g_1^+] + a_1$. $\mathcal{K}'$ is compact for the topology of the convergence in law and the sequence $(L_n)_{n\ge1}$ is $\mathcal{K}'$-valued. So, let γ be a limit point of $(L_n)_{n\ge1}$ and $(n_k)_{k\ge1}$

a sequence of indexes such that $L_{n_k} \Longrightarrow \gamma$. By construction, γ is invariant under the shift s.

Now, the sequence of the laws of the first coordinate $\pi_1(x)$ under $(L_{n_k})_{k\geq 0}$ weakly converges to the law of the first coordinate under γ. Also, by definition of $\mathcal{K}$, the positive parts of these elements form an uniformly integrable collection, so $\int \pi_1^+ \, d\gamma = \lim \int \pi_1^+ \, dL_{n_k}$. However the Fatou Lemma tells us that $\int \pi_1^- \, d\gamma \leq \underline{\lim}_{k\to+\infty} \int \pi_1^- \, dL_{n_k}$, hence finally

$$\int \pi_1 \, d\gamma \geq \underline{\lim}_{k\to+\infty} \int \pi_1 \, dL_{n_k} = \mu.$$

Let $Y = (Y_k)_{k\geq 1}$ be a process whose law is γ. Since γ is invariant under the shift s, the Birkhoff Theorem tells us that the sequence $(\frac{1}{n}\sum_{k=1}^n Y_k)_{n\geq 1}$ a.s. converges to a random variable Y_∞, which satisfies then $\mathbb{E}(Y_\infty) = \int \pi_1 \, d\gamma \geq \mu$.

It remains to see that the law of Y_∞ is stochastically dominated by the law of $\underline{f} = \underline{\lim}_{n\to+\infty} \frac{1}{n} f_n$. We will show that for each $a \in \mathbb{R}$, $\mathbb{P}(Y_\infty > a) \leq \mathbb{P}(\underline{f} > a)$. By left-continuity, it is sufficient to prove the inequality in a dense subset of $\mathbb{R}$. Thus, we can assume that a is not an atom for the law of $\underline{f}$.

$$\{Y_\infty > a\} = \left\{ \underline{\lim}_{n\to+\infty} \frac{Y_1 + \cdots + Y_n}{n} > a \right\} = \bigcup_{k\geq 1} \left\{ \inf_{n\geq k} \frac{Y_1 + \cdots + Y_n}{n} > a \right\}.$$

Hence

$$\begin{aligned}
&\mathbb{P}(Y_\infty > a) \\
= \;& \overline{\lim}_{k\to+\infty} \; \mathbb{P}_Y \left(\inf_{n\geq k} \frac{\pi_1 + \cdots + \pi_n}{n} > a \right) \\
= \;& \overline{\lim}_{k\to+\infty} \; \inf_{n\geq k} \; \mathbb{P}_Y \left(\inf_{k\leq i\leq n} \frac{\pi_1 + \cdots + \pi_i}{i} > a \right) \\
\leq \;& \overline{\lim}_{k\to+\infty} \; \inf_{n\geq k} \; \underline{\lim}_{K\to+\infty} \frac{1}{n_K} \sum_{j=1}^{n_K} \mathbb{P} \left(\inf_{k\leq i\leq n} \frac{\pi_1 + \cdots + \pi_i}{i} \circ s^j \circ \Delta > a \right).
\end{aligned}$$

Let $\varepsilon > 0$. We have, for fixed k, n, j:

$$\begin{aligned}
&\mathbb{P} \left(\inf_{k\leq i\leq n} \frac{\pi_1 + \cdots + \pi_i}{i} \circ s^j \circ \Delta > a \right) \\
= \;& \mathbb{P} \left(\inf_{k\leq i\leq n} \frac{f_{i+j+1} - f_{j+1}}{i} > a \right) \\
\leq \;& \mathbb{P} \left(\inf_{k\leq i\leq n} \frac{(f_i + g_i) \circ \theta_{j+1} + r_{j+1,i}}{i} > a \right) \\
\leq \;& \mathbb{P} \left(\inf_{k\leq i\leq n} \frac{(f_i + g_i) \circ \theta_{j+1}}{i} > a - \varepsilon \right) + \mathbb{P} \left(\sup_{i\geq k} \frac{r_{j+1,i}}{i} > \varepsilon \right)
\end{aligned}$$

On one hand, we have

$$\begin{aligned}\mathbb{P}\left(\sup_{i\ge k}\frac{r_{j+1,i}}{i}>\varepsilon\right) &\le \mathbb{P}\left(\sum_{i\ge k}\left(\frac{r^+_{j+1,i}}{i}\right)^\alpha>\varepsilon^\alpha\right)\le\varepsilon^{-\alpha}\sum_{i\ge k}\frac{1}{i^\alpha}\mathbb{E}[(r^+_{j+1,i})^\alpha]\\ &\le \varepsilon^{-\alpha}\sum_{i\ge k}\frac{C_i}{i^\alpha}.\end{aligned}$$

We can note that this term does not depend on j nor on n. On the other hand,

$$\mathbb{P}\left(\inf_{k\le i\le n}\frac{(f_i+g_i)\circ\theta_{j+1}}{i}>a-\varepsilon\right) = \mathbb{P}\left(\inf_{k\le i\le n}\frac{f_i+g_i}{i}>a-\varepsilon\right),$$

which does not depend on j. Then, for each $\varepsilon>0$, we have for every n,k, with $n\ge k$:

$$\begin{aligned}&\overline{\lim_{K\to+\infty}}\frac{1}{n_K}\sum_{j=1}^{n_K}\mathbb{P}\left(\inf_{k\le i\le n}\frac{\pi_1+\dots+\pi_i}{i}\circ s^j\circ\Delta>a\right)\\ \le\ &\varepsilon^{-\alpha}\sum_{i\ge k}\frac{C_i}{i^\alpha}+\mathbb{P}\left(\inf_{k\le i\le n}\frac{f_i+g_i}{i}>a-\varepsilon\right),\end{aligned}$$

next

$$\begin{aligned}&\inf_{n\ge k}\ \overline{\lim_{K\to+\infty}}\frac{1}{n_K}\sum_{j=1}^{n_K}\mathbb{P}\left(\inf_{k\le i\le n}\frac{\pi_1+\dots+\pi_i}{i}\circ s^j\circ\Delta>a\right)\\ \le\ &\varepsilon^{-\alpha}\sum_{i\ge k}\frac{C_i}{i^\alpha}+\inf_{n\ge k}\mathbb{P}\left(\inf_{k\le i\le n}\frac{f_i+g_i}{i}>a-\varepsilon\right).\end{aligned}$$

Finally,

$$\begin{aligned}\mathbb{P}(Y_\infty>a) &\le \overline{\lim_{k\to+\infty}}\ \inf_{n\ge k}\mathbb{P}\left(\inf_{k\le i\le n}\frac{f_i+g_i}{i}>a-\varepsilon\right)+\overline{\lim_{k\to+\infty}}\ \varepsilon^{-\alpha}\sum_{i\ge k}\frac{C_i}{i^\alpha}\\ &\le \overline{\lim_{k\to+\infty}}\ \mathbb{P}\left(\inf_{i\ge k}\frac{f_i+g_i}{i}>a-\varepsilon\right)\\ &\le \mathbb{P}\left(\underline{\lim}_{i\to+\infty}\frac{f_i+g_i}{i}>a-\varepsilon\right)=\mathbb{P}\left(\underline{\lim}_{i\to+\infty}\frac{f_i}{i}>a-\varepsilon\right),\end{aligned}$$

considering that g_i/i almost surely converges to 0. Letting ε tend to zero, we obtain

$$\mathbb{P}(Y_\infty>a)\le\mathbb{P}\left(\underline{\lim}_{i\to+\infty}\frac{f_i}{i}\ge a\right)=\mathbb{P}(\underline{f}>a).$$

It remains to see that $\underline{f}$ is invariant under the θ_n's. Fix $n\ge1$. We have

$$\mathbb{E}\left[\sum_{p=1}^{+\infty}\left(\frac{r^+_{n,p}}{p}\right)^\alpha\right]=\sum_{p=1}^{+\infty}\mathbb{E}\left[\left(\frac{r^+_{n,p}}{p}\right)^\alpha\right]\le\sum_{p=1}^{+\infty}\frac{C_p}{p^\alpha}<+\infty.$$

Particularly, $\frac{r^+_{n,p}}{p}$ almost surely converges to 0 when p tends to infinity. Since $f_{n+p}\le f_n+f_p\circ\theta_n+g_p\circ\theta_n+r^+_{n,p}$, dividing by $n+p$ and letting p tend to $+\infty$, it comes that

$$\underline{f}\le\underline{f}\circ\theta_n\quad\text{a.s.}$$

Since $\mathbb{P}$ is invariant under θ_n, we classically conclude that $\underline{f}$ is invariant under θ_n. □

Remarks. In the present article, we made no use of the possibility to take a non-zero g_p. In the case where the (g_p) are not zero, but the $r_{n,p}$'s are, we obtain a result which sounds much like Theorem 3 in Schürger [34]. Like Schürger [33], we use the idea of a coupling with a stationarized process. This idea is due to Durrett [13] and has been popularized by Liggett [28]. However, there is here a refinement, because we directly establish a stochastic comparison with the random variable Y, whereas previous papers establish a stochastic comparisons with the whole process $(Y_n)_{n\ge 1}$, that admits Y as its infimum limit.

In most almost subadditive ergodic theorems, almost sure convergence requires strong conditions on the lack of subadditivity (stationarity for instance). We obtain here an almost sure behavior by only considering a condition on the moments (of order greater than 1) of the lack of subadditivity. Besides, we know that bounding the first moment of the lack of subadditivity is not sufficient to get an almost sure behavior (see the remark by Derriennic [11] and the counter-example by Derriennic and Hachem [12]).

Proof of Theorem 24. It remains to prove that $\mathbb{E}\left(\overline{\lim_{n\to+\infty}}\frac{f_n}{n}\right)\le \mu$.
We fix $k\ge 1$. By subadditivity, we have for each $n\ge 0$ and every $0\le r\le k-1$:

$$\begin{aligned} f_{nk+r} &\le f_{nk}+(f_r+g_r)\circ\theta_{nk}+r^+_{nk,r}\\ &\le \left(\sum_{i=0}^{n-1} f_k\circ(\theta_k)^i\right)+\left(f_{nk}-\sum_{i=0}^{n-1} f_k\circ(\theta_k)^i\right)^+ +(f_r+g_r)\circ\theta_{nk}+r^+_{nk,r}\end{aligned}$$

Since $\mathbb{P}$ is invariant under θ_k, the Birkhoff Theorem gives the L^1 and almost-sure convergence:

$$\lim_{n\to+\infty}\frac{1}{n}\sum_{j=0}^{n-1}\frac{f_k\circ(\theta_k)^j}{k}=\frac{\mathbb{E}(f_k|\mathcal{I}_k)}{k},$$

where $\mathcal{I}_k$ is the σ-algebra of the θ_k-invariant events. Let us now control the residual terms. Since the finite collection $(f_r+g_r)_{0\le r\le k-1}$ is equi-integrable and $\mathbb{P}$ is invariant under θ_k, the collection $(\sup_{0\le r\le k-1}(f_r+g_r)\circ\theta_k^n)_{n\ge 1}$ is equi-integrable, which ensures the almost sure and L^1 convergence:

$$\lim_{n\to+\infty}\frac{1}{n}\sup_{0\le r\le k-1}(f_r+g_r)\circ(\theta_k)^n=0.$$

We have $\sum_{n=1}^{+\infty}\mathbb{E}\left[(\frac{r^+_{nk,r}}{n})^\alpha\right]\le\sum_{n=1}^{+\infty}\frac{C_r}{n^\alpha}<+\infty$, which implies, as previously, that $r^+_{nk,r}/n$ almost surely converges to 0. Finally,

$$\forall r\in\{0,\dots,k-1\}\quad \overline{\lim_{n\to+\infty}}\frac{f_{nk+r}}{nk+r}\le\frac{\mathbb{E}[f_k|\mathcal{I}_k]}{k},$$

hence $\mathbb{E}\left[\overline{\lim_{n\to+\infty}}\frac{f_n}{n}\right]\le\frac{\mathbb{E}[f_k]}{k}$. We complete the proof by letting k tend to $+\infty$. □

Remark. When there is no lack of subadditivity, the assumptions of Theorem 24 obviously hold; we obtain thus a subadditive ergodic theorem which sounds very much like Liggett's [28]. However, these theorems are not strictly comparable, in the following sense that no one implies the other one.

Indeed, extending a remark made by Kingman in his Saint-Flour's course [26] (page 178), we can note that the assumption of Kingman's original article – the stationarity of the doubly indexed process $(X_{s,t})_{s\geq 0,t\geq 0}$ – can be weakened in two different ways:

- Either assuming that for each k, the process $(X_{(r-1)k,rk})_{r\geq 1}$ is stationary – this assumption will be used by Liggett [28].
- Or assuming that the law of $X_{n,n+p}$ does not depend on p. That assumption, suggested by Hammersley and Welsh, is the one that we use here, also used by Schürger in [34].

Note however that the special assumption of stationarity is used in Liggett's proof [28] only in the so-called easy part, *i.e.* the bound for the supremum limit.

Kingman thought that the first set of assumptions surpassed the second one, in view of possible applications. More than 30 years later, the progresses of subadditive ergodic theorems, particularly about bounding the infimum limit, lead to moderate this affirmation.

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