Asymptotic behavior of the finite-time expected time-integrated negative part of some risk processes and optimal reserve allocation

R. Biard[a], S. Loisel[a,*], C. Macci[b], N. Veraverbeke[c]

[a] *Université de Lyon, Université Claude Bernard Lyon 1, Institut de Science Financière et d'Assurances, 50 Avenue Tony Garnier, F-69007 Lyon, France*
[b] *Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, I-00133 Roma, Italy*
[c] *Center for Statistics, Hasselt University, Agoralaan, B-3590 Diepenbeek, Belgium*

Abstract

In the renewal risk model, we study the asymptotic behavior of the expected time-integrated negative part of the process. This risk measure has been introduced by [1]. Both heavy-tailed and light-tailed claim amount distributions are investigated. The time horizon may be finite or infinite. We apply the results to an optimal allocation problem with two lines of business of an insurance company. The asymptotic behavior of the two optimal initial reserves are computed.

Key words: Ruin theory, heavy-tailed and light-tailed claim size distribution, risk measure, optimal reserve allocation.

1. Introduction

The current change of regulation leads the insurance industry to address new questions regarding solvency. In Europe, insurance groups will have to comply with the new rules, namely Solvency II, by 2011. In comparison to the previous regulation system, Solvency II aims at defining solvency margins that are better adjusted to the underlying risks. Solvency requirements may either be computed thanks to a standard formula, or with internal models that companies are encouraged to develop. While a bottom-up approach is used in the standard formula (one first studies each small risk separately and then aggregates them thanks to a kind of correlation matrix), a top-down approach may be used in some internal or partial internal models: once the main risk drivers for the overall company have been identified and the global solvency capital requirement has been computed, it is necessary to split this overall buffer capital into marginal solvency capitals for each line of business, in order to avoid as far as possible that some lines of business become insolvent too often. Capital fungibility between lines of business or between entities of a large insurance group that lie in different countries is indeed limited by different entity-specific or country-specific solvency constraints.

One possible way to define optimality of the global reserve allocation is to minimize the expected sum of the penalties that each line of business would have to pay due to its temporary potential insolvency. If one neglects discounting factors, a first approximation of this penalty is given by the time-integrated expected negative part of the surplus process. In [1, 2], the author studies this penalty function in infinite time and furnishes a criterion for optimal reserve allocation with different lines of business. Closed-form formulas were available in the classical risk model for exponentially distributed claim amounts, which led to a semi-explicit optimal reserve allocation.

Unfortunately, the hypotheses used in these papers do not perfectly match real-world constraints for

*Corresponding author

Email addresses: romain.biard@univ-lyon1.fr (R. Biard), stephane.loisel@univ-lyon1.fr (S. Loisel), macci@mat.uniroma2.it (C. Macci), noel.veraverbeke@uhasselt.be (N. Veraverbeke)

practical applications. The first point is that in practice, one may have very different claim amount distributions depending on the kind of insurance risks that are covered: there may be some heavy tails, some light tails or even some very light tails for some particular risks. The second point is that insurance regulation is based on a 1-year time horizon in the standard formula, and on a finite time horizon usually comprised between 1 and 10 years in internal models. To better take those real-world constraints into account, we address the following questions in this paper: can the results obtained by [1] for exponentially distributed claim amounts be adapted to the sub-exponential case using results of [3]? For large initial global reserve and two lines of business, *with a finite time horizon,* what is the asymptotic optimal part of the initial reserve that one should allocate to each line of business to minimize the sum of the two penalty functions?
To solve these problems for regularly varying, light-tailed and super-exponential claim size distributions, we first need to compute the asymptotics of the finite-time expected time in red and of the expected time-integrated negative part of the considered risk process. In the regularly varying case, it is often said that everything behaves as if one large claim caused ruin. We wondered whether this heuristic principle could be adapted to our problem, and we show that it applies here. Our paper is organized as follows: in Section 2 we describe our model and use results of [3] to obtain analogous results to those of [1] in the regular variation case, for infinite time horizon. In Section 3 we derive the asymptotics of the expected time in red and of the expected time-integrated negative part of the considered risk process for finite time horizon and for different classes of claim size distributions. In Section 4, we use these results to obtain asymptotic optimal reserve allocation in some risk models with two lines of business.

2. The model

For a uni-dimensional risk processes $U(t) = u + X_t$ that represents the surplus of an insurance company at time t, with initial reserve u and with $X_t = ct - S(t)$, where $c > 0$ is the premium income rate, and $S(t)$ is in the most classical case a compound Poisson process, many risk measures have been considered. The finite-time probability is one and has been studied for different models of risk process. It has been investigated among others by [4], [5] and [6] for classical models. The dependent case has been studied by [7] and [8]. Sensitivity analysis has been carried out by [9], [10] and [11].
We may consider some others risk measures (see for example [12], [13] and [14]): the time to ruin $T_u = \inf\{t > 0, u + X_t < 0\}$, the severity of ruin $u + X_{T_u}$, the couple $(T_u, u + X_{T_u})$, the time in red (below 0) from the first ruin to the first time of recovery $T'_u - T_u$ where $T'_u = \inf\{t > T_u, u + X_t = 0\}$, the maximal ruin severity ($\inf_{t>0} u + X_t$), the aggregate severity of ruin until recovery $J(u) = \int_{T_u}^{T'_u} |u + X_t| dt$,... [15] studied the total time in red $\tau(u) = \int_0^{+\infty} 1_{\{u+X_t<0\}} dt$ using results of [12].
All these random variables are drawn from the infinite time ruin theory, or involve the behavior of the risk process between ruin times and recovery times. It seems interesting to consider risk measures based on some fixed time interval $[0, T]$ (T may be infinite). One other of the simplest penalty functions may be the expected value of the time-aggregated negative part of the risk process (see Figure 1):

$$E(I_T(u)) = E\left(\int_0^T 1_{\{U(t)<0\}} |U(t)| dt \right).$$

Note that the probability $P(I_T = 0)$ is the probability of non ruin within finite time T. I_T may be seen as the penalty the company will have to pay due to its insolvency until the time horizon T.
These risk measures may be differentiated with respect to the initial reserve u, which makes it possible to compute them quite easily as integrals of other functions of u such as the probability of ruin or the total time in red. Moreover, they have the advantage that the integral over t and the mathematical expectation may be permuted thanks to Fubini's Theorem. Here we recall the two main differentiation theorems (see [1]) that are going to be useful for our study:

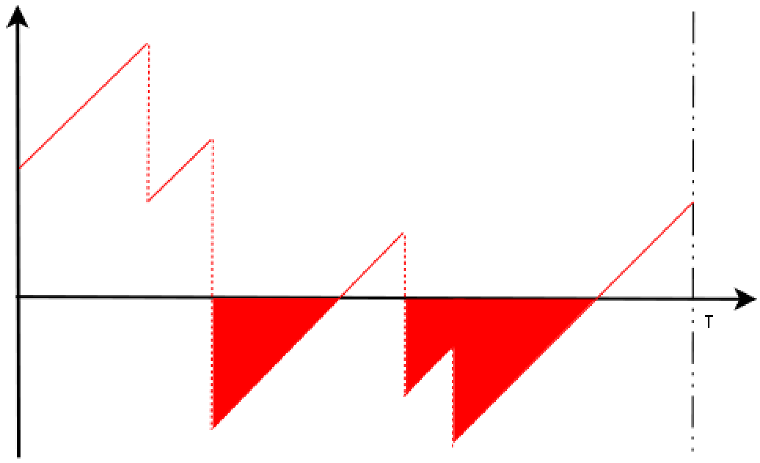

Figure 1: Example of a time-aggregated negative part of a risk process.

Theorem 2.1. *Assume $T \in \mathbb{R}^+$. Let $(X_t)_{t\in[0,T)}$ be a renewal risk process (possibly modulated by an environment process with finite state space) with time-integrable sample paths. For $u \in \mathbb{R}$, denote by $\tau(u,T)$ the random variable corresponding to the time spent under zero by the process $u + X_t$ between the fixed times 0 and T:*

$$\tau(u,T) = \int_0^T 1_{\{u+X_t<0\}} dt,$$

Let $\tau_0(u,T)$ correspond to the time spent in zero by the process $u + X_t$:

$$\tau_0(u,T) = \int_0^T 1_{\{u+X_t=0\}} dt.$$

Let $I_T(u)$ represent the time-integrated negative part of the process $u + X_t$ between 0 and T:

$$I_T(u) = \int_0^T 1_{\{u+X_t<0\}} |u + X_t| dt$$

and $f(u) = E(I_T(u))$.
For $u \in \mathbb{R}$, if $E(\tau_0(u,T)) = 0$, then f is differentiable at u, and $f'(u) = -E(\tau(u,T))$.

Theorem 2.2. *Let $X_t = ct - S(t)$, where $S(t)$ is a compound Poisson process. Consider $T < +\infty$ and define h by $h(u) = E(\tau(u))$ for $u \in \mathbb{R}$. h is differentiable on $\mathbb{R}^+_* = (0,\infty)$, and for $u > 0$,*

$$h'(u) = -\frac{1}{c} E\left(N^0(u,T)\right),$$

where $N^0(u,T) = Card(\{t \in [0,T], \quad u + ct - S(t) = 0\})$.

We introduce here more notations in the classical compound Poisson model:
An insurance company has an initial surplus $u \geq 0$ and receives premiums continuously at a constant rate $c > 0$. Claims arise according to a homogeneous Poisson process $\{N(t)\}$ with mean λ per unit of time, and, independently of this process, the successive claim amounts $\{W_i\}$ are non-negative independent and identically distributed random variables, with common distribution function $F_W(x)$ and mean μ. So, the

aggregate claims constitute a compound Poisson process $\{S(t)\}$ where $S(t) = \sum_{i=1}^{N(t)} W_i$. The surplus at time t is then given by

$$U(t) = u + ct - S(t), \tag{2.1}$$

and ruin occurs as soon as the surplus becomes negative. One assumes that the net profit condition holds:

$$c > \lambda\mu.$$

Let $\phi(u, T)$ be the probability of non-ruin until time T:

$$\phi(u, T) = P[U(t) = u + ct - S(t) > 0 \ \text{ for } \ 0 < t \le T], \tag{2.2}$$

and let $\psi(u, T) = 1 - \phi(u, T)$ be the probability of ruin before time T. As $T \to \infty$, (2.2) becomes the ultimate non-ruin probability $\phi(u)$, the ultimate ruin probability being $\psi(u) = 1 - \phi(u)$.
For $T = \infty$, from [1], if $\tau(u)$ is integrable for all $u > 0$, we have

$$E\left(N^0(u, \infty)\right) = \frac{\psi(u)}{1 - \psi(0)}, \tag{2.3}$$

for the compound Poisson case and $u > 0$.

3. Asymptotics of $E(I_T(u))$ and $E(\tau_T(u))$

This Section gives some results on asymptotics of risk measures we have introduced before. Several cases for the claim size distribution are studied.

3.1. A heuristic result with Pareto claim amounts

In the Pareto case, with very large initial reserve u one would expect that one large claim would be responsible for ruin and for the main contribution to the penalty function

$$E(I_T(u)).$$

This is a well-known heuristic result for ruin probabilities, but does it remain true for the expected time-integrated negative part of the risk process? Denote by T_u the time to ruin. Using the decomposition

$$E(I_T(u)) = E\left(I_T(u) \mid T_u \le T\right) \psi(u, T),$$

the result we expect is that one large claim is likely to cause ruin. Given that this claim occurs, the conditional distribution of this large claim instant is uniform on the interval $[0, T]$ (with average $T/2$), and the average severity at ruin is of the same order as

$$e(u) \sim \frac{1}{\alpha - 1} u.$$

Consequently, with this approach, it is tempting to say that at the first order, given that ruin occurs before T the risk process stays below zero during an average time $T/2$ at a level equivalent to $-\frac{1}{\alpha-1}u$, which correspond to an average surface in red

$$\frac{T}{2}\frac{1}{\alpha - 1}u.$$

This would lead to the following equivalent:

$$E(I_T(u)) = E\left(I_T(u) \mid T_u \le T\right) \psi(u, T) \sim \left[\frac{T}{2}\frac{1}{\alpha - 1}u\right]\left[\lambda T u^{-\alpha}\right],$$

which may be rewritten as

$$E(I_T(u)) \sim \frac{\lambda T^2}{2(\alpha - 1)} u^{-\alpha+1} \tag{3.4}$$

as $u \to +\infty$.
A similar heuristic approach would lead us to guess that the average time spent below zero by the risk process up to time T is

$$E(\tau(u,T)) \sim \frac{\lambda T^2}{2} u^{-\alpha} \tag{3.5}$$

as $u \to +\infty$, as the risk process would remain below zero in average during a time $T/2$ in case of ruin: if ruin occurs, the large claim causing ruin occurs in average at time $T/2$ and the expected severity at ruin is $e(u) = u/(\alpha-1)$, so that recovery is almost impossible before time u if u is large enough.

Note that from differentiation theorems in [1], Equation (3.5) holds as long as (3.4) holds. We shall now prove that our intuition is correct and that (3.4) holds.

3.2. Sub-exponential case

In this Section, we give the asymptotics of $E(I_T(u))$ when u tends to infinity for claim amount distributions that belong to two sub-classes of the subexponential class.

Definition 3.1. *A cdf F with support* $(0,\infty)$ *is subexponential, if for all* $n \geq 2$,

$$\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n.$$

The class of subexponential cdfs will be denoted by $\mathcal{S}$.

3.2.1. Regular variation case

Definition 3.2. *A function l on* $(0,\infty)$ *is slowing varying at* ∞ *(we write* $l \in \mathcal{R}_0$*) if*

$$\lim_{x\to\infty} \frac{l(tx)}{l(x)} = 1, \quad t > 0.$$

The convergence is uniform on each compact subset of $t \in (0,\infty)$.

Definition 3.3. *A cdf F with support* $(0,\infty)$ *belong to the regular variation class if for some* $\alpha > 0$

$$\lim_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = y^{-\alpha}, \quad for\ y > 0.$$

or equivalently if,

$$\overline{F}(x) = x^{-\alpha} l(x),$$

with $l \in \mathcal{R}_0$. *We note* $F \in \mathcal{R}_{-\alpha}$.
The convergence is uniform on each subset $y \in [y_0,\infty)$ $(0 < y_0 < \infty)$.

Theorem 3.4 (Karamata's Theorem). *Let* $l \in \mathcal{R}_0$ *be locally bounded in* $[x_0,\infty]$ *for some* $x_0 \geq 0$. *Then*

- *for* $0 < \alpha < 1$,

$$\int_{x_0}^{x} t^{-\alpha} l(t)dt \sim (1-\alpha)^{-1} x^{-\alpha+1} l(x), \quad x \to \infty,$$

- *for* $\alpha > 1$,

$$\int_{x}^{\infty} t^{-\alpha} l(t)dt \sim (\alpha-1)^{-1} x^{-\alpha+1} l(x), \quad x \to \infty.$$

We first investigate the infinite-time case.
In the sub-exponential case, [3] have shown that

$$\psi(u) \sim \frac{\lambda}{c-\lambda\mu}\int_u^{+\infty}(1-F_W(x))dx.$$

In the α-regularly varying case with $\alpha > 1$ (this means that

$$1-F_W(x) \sim x^{-\alpha}l(x) \text{ as } x \to +\infty,$$

where l is a slowly varying function), this corresponds to

$$\psi(u) \sim \frac{\lambda}{c-\lambda\mu}\frac{1}{\alpha-1}u^{-\alpha+1}l(u).$$

From Theorems 2.2 and 2.1 and (2.3), we get that

Proposition 3.5.

$$E\left[\tau(u)\right] \sim \frac{1}{c}\frac{1}{1-\psi(0)}\frac{\lambda}{c-\lambda\mu}\frac{1}{(\alpha-1)(\alpha-2)}u^{-\alpha+2}l(u)$$

for $\alpha > 2$ *and*

$$E\left[I_\infty(u)\right] \sim \frac{1}{c}\frac{1}{1-\psi(0)}\frac{\lambda}{c-\lambda\mu}\frac{1}{(\alpha-1)(\alpha-2)(\alpha-3)}u^{-\alpha+3}l(u)$$

for $\alpha > 3$.

For real-world applications, finite-time horizon is preferred to infinite-time horizon. This is the reason why we consider a finite-time ruin horizon in the sequel.

Definition 3.6 (Mean excess function). *For a random variable X, the mean excess function* $e_X(u)$ *is*

$$e_X(u) = E(X-u|X>u).$$

Remark 3.7. *A continuous c.d.f. is uniquely determined by its mean excess function since we have*

$$\begin{aligned} e_X(u) &= \int_0^\infty (x-u)dF_X(x)/\overline{F_X}(u) \\ &= \frac{1}{\overline{F_X}(u)}\int_u^\infty \overline{F_X}(x)dx, \quad 0<u<\infty, \end{aligned}$$

and

$$\overline{F_X}(x) = \frac{e_X(0)}{e_X(x)}\exp\left\{-\int_0^x \frac{1}{e_X(u)}du\right\}, \quad x>0.$$

Proposition 3.8. *For a random variable X with c.d.f.* $F_X \in \mathcal{R}_{-\alpha}$ *for some* $\alpha > 1$, *we have for large* u

$$e_X(u) \sim \frac{u}{\alpha-1}.$$

Proof. For the proof, see for example [16], p 162.

⋄

Theorem 3.9. *For a risk process with claim amounts distribution in the regular variation class for some $\alpha > 1$ and c.d.f F_W, we have, for $T > 0$ and large u,*

$$E(I_T(u)) \sim \frac{\lambda T^2}{2(\alpha-1)} u\overline{F_W}(u).$$

Proof. With Proposition 3.8 and Remark 3.7, we can express $E(I_T(u))$ with the mean-excess function of the compound process $S(t) = \sum_{i=1}^{N(t)} W_i$ which has a c.d.f which belong to the regular variation class with the same parameter as F_W. Hence, we have

$$\begin{aligned}
E(I_T(u)) &= E\left(\int_{t=0}^{T} \mathbb{1}_{u+X_t<0} \left|u + X_t\right| dt\right) \\
&= \int_{t=0}^{T} E\left(\mathbb{1}_{u+X_t<0} \left|u + X_t\right|\right) dt && \text{using Fubini's Theorem} \\
&= \int_{t=0}^{T}\int_{x=0}^{\infty} P\left(S(t) > u + ct + x\right) dxdt \\
&= \int_{t=0}^{T}\int_{y=u+ct}^{\infty} P\left(S(t) > y\right) dydt \\
&= \int_{t=0}^{T} \overline{F}_{S(t)}(u+ct) e_{S(t)}(u+ct) dt \\
&\sim \lambda\overline{F}_W(u) \int_{t=0}^{T} t e_{S(t)}(u+ct) dt && \textit{as} \quad u \to \infty \\
&\sim \frac{\lambda T^2}{2(\alpha-1)} u\overline{F_W}(u).
\end{aligned}$$

⋄

3.2.2. *An other subclass of the subexponential class*

Theorem 3.10. *For a risk process with claim amounts distribution function F_W we have for $T > 0$: if*

$$\frac{1}{\mu}\int_0^x \overline{F}_W(y)dy \in \mathcal{S},$$

then for large u,

$$E(I_T(u)) \sim \frac{\lambda T^2}{2}\left(\int_u^{\infty} \overline{F_W}(v)dv\right).$$

Proof. We have (cf Theorem 3.9)

$$E(I_T(u)) = \int_0^T \overline{F_{S(T)}}(u+ct) e_{S(t)}(u+ct) dt.$$

Since Condition (3.10) implies that $\overline{F_W} \in \mathcal{S}$, we have from Theorem 3 in [17] that $F_{S(t)} \in \mathcal{S}$ and that

$$\overline{F_{S(t)}}(x) \sim \lambda t \overline{F_W}(x) \text{ for } x \to \infty.$$

It follows that

$$e_{S(t)}(x) = \frac{\int_x^{\infty} \overline{F_{S(t)}}(y)dy}{\overline{F_{S(t)}}(x)} \sim \frac{\int_x^{\infty} \overline{F_W}(y)dy}{\overline{F_W}(x)} = e_W(x).$$

Hence, as $u \to \infty$,

$$E(I_T(u)) \sim \frac{\lambda T^2}{2}\overline{F_W}(u)e_W(u) = \frac{\lambda T^2}{2}\int_u^{\infty}\overline{F_W}(x)dx.$$

$\diamond$

Remark 3.11. *If F_W is regularly varying for some $\alpha > 1$, then Condition (3.10) is satisfied and we retrieve $E(I_T(u)) \sim \frac{\lambda T^2}{2}\frac{u\overline{F_W}(u)}{\alpha-1}$.*

3.3. Case where the Cramer-Lundberg coefficient exists

In this Subsection, we assume that the Cramer-Lundberg coefficient of the risk process $(U_t)_{t\geq 0}$ exists and is equal to R.

With these assumptions and in the infinite-time case, we have the following well-known result.

Theorem 3.12 (The Cramer-Lundberg Approximation). *If we denote $\hat{F_W}$ the m.g.f. of F_W, we have*

$$\psi(u) \sim Ce^{-Ru} \quad \text{as } u \to \infty\ ,$$

where

$$C = \frac{1-\lambda\mu}{\lambda\hat{F_W}'(R) - 1}.$$

From Theorems 3.12, 2.2 and 2.1 and (2.3), we get that

Proposition 3.13.

$$E\left[\tau(u)\right] \sim \frac{1}{c}\frac{1}{1-\psi(0)}\frac{C}{R}e^{-Ru}$$

and

$$E\left[I_\infty(u)\right] \sim \frac{1}{c}\frac{1}{1-\psi(0)}\frac{C}{R^2}e^{-Ru}.$$

In the finite-time case, a convexity argument enables us to show that:

$$C'e^{-Ru}\left(1-e^{-R(c-\lambda\mu)T}\right) \sim E\left[I_{+\infty}(u)\right] - E\left[I_{+\infty}\left(E\left[U(T)\right]\right)\right] \leq E\left[I_T(u)\right] \leq E\left[I_{+\infty}(u)\right] \sim C'e^{-Ru},$$

with $C' = \frac{1}{c}\frac{1}{1-\psi(0)}\frac{C}{R^2}$.

3.4. Super-exponential case

In this Section we consider the super-exponential case, i.e. we assume that $E[e^{\theta W_1}] < \infty$ for all $\theta > 0$. The aim is to present a large deviation principle (LDP) based on the results in [18]; see [19] for the definition of LDP. We start introducing the function $\Lambda : \mathbb{R} \to \mathbb{R}$ defined by $\Lambda(\theta) = c\theta + \lambda(E[e^{-\theta W_1}] - 1)$; moreover let Λ^* be Fenchel-Legendre transform of Λ, i.e. the function $\Lambda^*(x) = \sup_{\theta\in\mathbb{R}}\{\theta x - \Lambda(\theta)\}$. We recall that $\Lambda'(0) = c - \lambda E[W_1]$, and the net profit condition is $\Lambda'(0) > 0$.

Proposition 3.14. *Assume $\Lambda'(0) \geq -1/T$. Then $\left\{\frac{1}{u^2}I_{Tu}(u) : u > 0\right\}$ satisfies the LDP with good rate function J defined by*

$$J(z) = \begin{cases} T\Lambda^*\left(\frac{1}{T}\left(-\frac{z}{T} - \sqrt{\left(\frac{z}{T}\right)^2 + \frac{2z}{T}} - 1\right)\right) & if\ z > 0 \\ 0 & if\ z = 0 \\ \infty & if\ z < 0. \end{cases}$$

This means that

$$-\inf_{z\in E^\circ} J(z) \leq \liminf_{u\to\infty}\frac{1}{u}\log P\left(\frac{1}{u^2}I_{Tu}(u) \in E\right) \leq \limsup_{u\to\infty}\frac{1}{u}\log P\left(\frac{1}{u^2}I_{Tu}(u) \in E\right) \leq -\inf_{z\in\overline{E}} J(z)$$

for all measurable sets E (E° is the interior of E and $\overline{E}$ is the closure of E).

Proof. We start noting that

$$\begin{aligned}\frac{1}{u^2} I_{Tu}(u) &= \frac{1}{u^2}\int_0^{Tu} 1_{\{u+ct-\sum_{k=1}^{N(t)} W_k<0\}} \left|u+ct-\sum_{k=1}^{N(t)} W_k\right| dt \\ &= \frac{1}{u^2}\int_0^{T} 1_{\{u+cus-\sum_{k=1}^{N(us)} W_k<0\}} \left|u+cus-\sum_{k=1}^{N(us)} W_k\right| uds \\ &= \int_0^{T} 1_{\{1+cs-\frac{1}{u}\sum_{k=1}^{N(us)} W_k<0\}} \left|1+cs-\frac{1}{u}\sum_{k=1}^{N(us)} W_k\right| ds.\end{aligned}$$

Then the LDP holds by Proposition 2.1 in [18] with $u = 1$; indeed here we have $\frac{1}{u}$ in place of ε in [18]. The expression of the rate function is provided by equation (7) in [18] with $u = 1$.
◇

We remark that we could have $\lim_{z\to 0+} J(z) > 0 = J(0)$; see the discussion in Remark 5.1 in [18].

4. Optimal reserve allocation strategy for large initial reserve

In this Section, we consider an insurance company with two lines of business. Two main kinds of phenomena may generate dependence between the two processes.

- Firstly, in some cases, claims for the two lines of business may come from a common event : for example, a car accident may cause a claim for driving insurance, liability and disablement insurance. Hurricanes might cause losses in different countries. This should correspond to simultaneous jumps for the two processes. The most common tool to take this into account is the Poisson common shock model.
- Secondly, there exist other sources of dependence, for example the influence of the weather on health insurance and on agriculture insurance. In this case, claims seem to outcome independently for each line of business, depending on the weather. This seems to correspond rather to models with modulation by a Markov process which describes the evolution of the state of the environment.

The environment state process, denoted by $(J(t))_{t\geq 0}$ is a Markov process with state space $\mathcal{S} = \{1, ..., J\}$, initial distribution μ and intensity matrix A.
For $i \in \{1, 2\}$, let us define the J independent processes

$$Y_i^j = c_i^j t - \sum_{n=1}^{N_i^j(t)} W_{i,n}^j , \quad j = 1, ..., J,$$

- where $c_i^j > 0$,
- $(W_{i,n}^j)_{n\geq 1}$ is a i.i.d. sequence with common c.d.f. $F_{W_i^j}$ and mean μ_i^j,
- and independent from a Poisson process $(N_i^j(t))_{t\geq 0}$ described below.

Let T_p be the instant of the *pth* jump of the process $(J(t))_{\tau\geq 0}$, and define $(U_i(t))_{t\geq 0}$, for $i \in \{1, 2\}$ by

$$U_i(t) = u + \sum_{p\geq 1}\sum_{1\leq j\leq J}\left[Y_i^j(T_p) - Y_i^j(T_{p-1})\right]\mathbb{1}_{\{J_{T_{p-1}}=j, T_p\leq t\}} + \sum_{p\geq 1}\sum_{1\leq j\leq J}\left[Y_i^j(t) - Y_i^j(T_{p-1})\right]\mathbb{1}_{\{J_{T_{p-1}}=j, T_{p-1}\leq t\leq T_p\}}.$$

Thus, we have built the two processes modulated by a common process. For an illustration of a single modulated process see Figure 2.

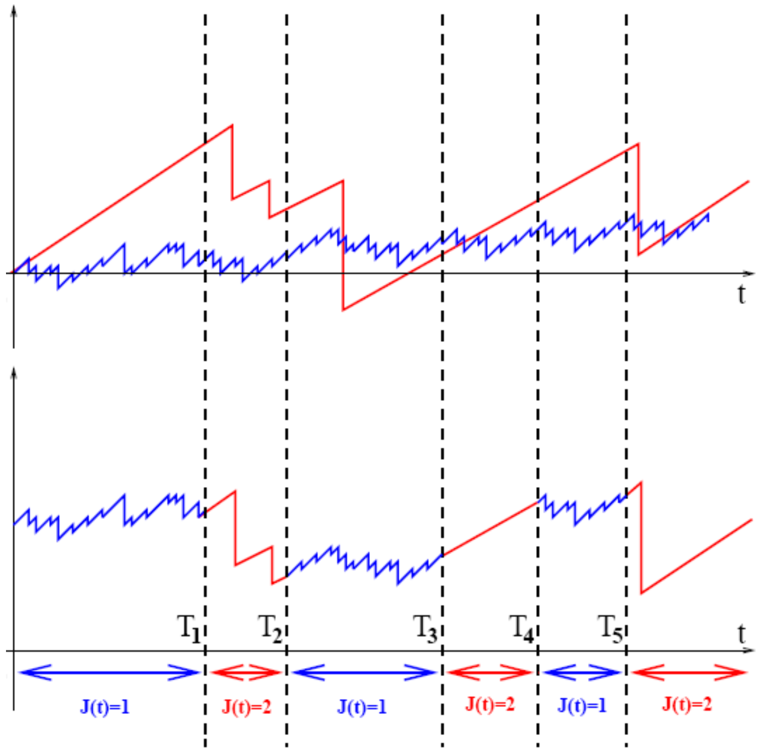

Figure 2: A typical modulated risk process with two states (red and blue).

To model common shocks, we decompose, for all $j \in \{1, .., J\}$, $(N_1(t))_{t\geq 0}$ and $(N_2(t))_{t\geq 0}$ as follow

$$N_1^j(t) = M_1^j(t) + M^j(t)$$
$$N_2^j(t) = M_2^j(t) + M^j(t)$$

with $(M_1^j(t))_{t\geq 0}$, $(M_2^j(t))_{t\geq 0}$ and $(M^j(t))_{t\geq 0}$ three independent processes with parameter λ_1^j, λ_2^j and λ_j respectively.
For $i = 1, 2$ and $u > 0$, we note $\psi_i(u) = P[U_i(t) < 0 \text{ for some } t \geq 0 | U_i(0) = u]$.

The allocation problem is to minimize the risk measure

$$I_T(u_1, u_2) = E\left[I_T^1(u_1)\right] + E\left[I_T^2(u_2)\right],$$

under the constraint $u_1 + u_2 = u$ for large u where

$$I_T^i(u_i) = \int_0^T 1_{\{U_i(t)<0\}} |U_i(t)| dt \quad i = 1, 2.$$

For an illustration, see Figure 3.

4.1. Infinite-time regular variation case

In the Subsection, we assume that the dependence between the two lines of business is only generated by common shocks. There is no environment process. We also assume that the claim amount distribution

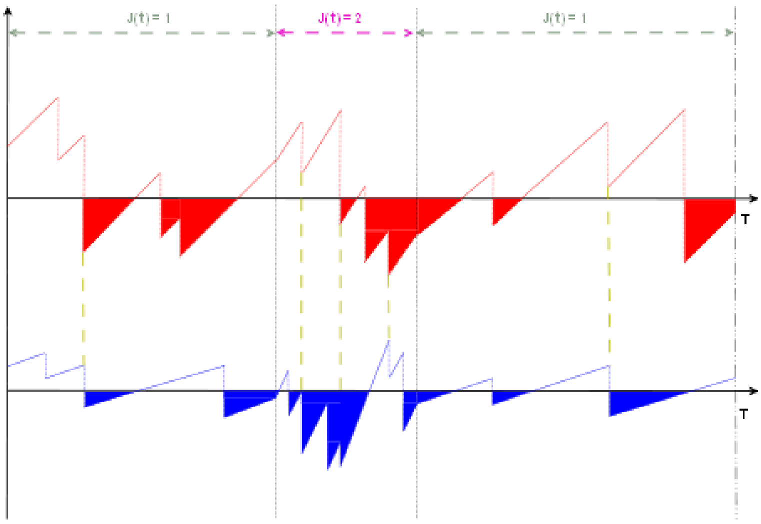

Figure 3: Two modulated risk processes with common shocks.

of the first (resp. second) line of business belongs to the regular variation class with parameter α_1 (resp. α_2) with $\alpha_1 < \alpha_2$. Thus, the second line of business is safer than the first one.
As there are no environment process, the notation in this Subsection is the same but without the state exponent j.

From Proposition 3.5, we have for large u and $i = 1, 2$,

$$E\left[I^i_\infty(u)\right] \sim D_i u^3 \overline{F_{W_i}}(u),$$

with

$$D_i = \frac{1}{c_i}\frac{1}{1-\psi_i(0)}\frac{\lambda_i + \lambda}{c - (\lambda_i + \lambda)\mu_i}\frac{1}{(\alpha_i - 1)(\alpha_i - 2)(\alpha_i - 3)}.$$

Lemma 4.1. *The couples $(u, 0)$ and $(0, u)$ do not solve our optimization problem for u large enough.*

Proof. Let us choose for example $u/2$ for each line of business. We have

$$I_T(u/2, u/2) \xrightarrow[u\to\infty]{} 0.$$

Since $I_T(0, u) = E\left[I^1_T(0)\right] + E\left[I^2_T(u)\right] \geq E\left[I^1_T(0)\right] > 0$ and $I_T(u, 0) = E\left[I^1_T(u)\right] + E\left[I^2_T(0)\right] \geq E\left[I^2_T(0)\right] > 0$ for all $u \in \mathbb{R}$, we have the result.
◇

Theorem 4.2. *Under the assumptions of this Subsection, the couple (u_1, u_2) which minimizes $I_\infty(u_1, u_2)$ satisfies*

$$\begin{cases} \frac{\partial E\left[I^1_\infty(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I^1_\infty(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

Moreover, if we denote $u_1 = (1 - \beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ we have for large u

$$\beta(u) \sim \left(\frac{D'_2}{D'_1}\frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)}\right)^{1/(\alpha_2 - 2)},$$

where

$$D_i^{'} = (\alpha_i - 3)^{-1} D_i \quad i = 1, 2.$$

Note that $\beta(u)$ represents the proportion of the global reserve we allocate to the safer line of business.

Proof. From Lemma 4.1, u_1 and u_2 are not equal to zero, we know from the Lagrange multiplier method, (see [1]), that the solution of our problem satisfies

$$\begin{cases} \frac{\partial E\left[I_\infty^1(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I_\infty^1(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

We know from Proposition 3.5 that for $i = 1, 2$ and large u,

$$E\left[I_\infty^i(u)\right] \sim D_i u^3 \overline{F_{W_i}}(u),$$

with

$$D_i = \frac{1}{c_i} \frac{1}{1 - \psi_i(0)} \frac{\lambda_i + \lambda}{c - (\lambda_i + \lambda)\mu} \frac{1}{(\alpha_i - 1)(\alpha_i - 2)(\alpha_i - 3)}.$$

For $i = 1, 2$, since $u \mapsto E\left[I_\infty^i(u_i)\right]$ is regularly varying with index $\alpha_i - 3$, we have for large u,

$$\frac{\partial E\left[I_\infty^i(u_i)\right]}{\partial u_i} \sim D_i^{'} u_i^2 \overline{F_{W_i}}(u_i) \quad i = 1, 2,$$

with $D_i^{'} = (\alpha_i - 3)^{-1} D_i$.

Let us denote $u_1 = (1 - \beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ ($\beta(u)$ represents the proportion of the global reserve u we allocate to the line of business 2). With this notation, we are able to give the asymptotic behavior of u_1 and u_2.

Indeed, we have this following equation to solve, with large u,

$$D_1^{'} \left((1 - \beta(u))u\right)^2 \overline{F_{W_1}}((1 - \beta(u))u) = D_2^{'} \left(\beta(u)u\right)^2 \overline{F_{W_2}}(\beta(u)u),$$

or equivalently, since $\overline{F_{W_i}}$ is regularly varying with index α_i for $i = 1, 2$ and using the uniform convergence property (cf Definition 3.3),

$$D_1^{'}(1 - \beta(u))^{-\alpha_1 + 2} u^2 \overline{F_{W_1}}(u) = D_2^{'} \beta(u)^{-\alpha_2 + 2} u^2 \overline{F_{W_2}}(u).$$

Thus we have

$$\beta(u)^{\alpha_2 - 2} = \frac{D_2^{'}}{D_1^{'}} \frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)} (1 - \beta(u))^{\alpha_1 - 2} \to 0,$$

since $\alpha_2 > \alpha_1 > 1$ and $1 - \beta(u) \in (0, 1)$.

Consequently, $\beta(u) \xrightarrow[u \to \infty]{} 0$ and for large u,

$$\frac{\beta(u)^{\alpha_2 - 2}}{\frac{D_2^{'}}{D_1^{'}} \frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)}} = (1 - \beta(u))^{\alpha_1 - 2} \xrightarrow[u \to \infty]{} 1.$$

So, we have

$$\beta(u) \sim \left(\frac{D_2^{'}}{D_1^{'}} \frac{\overline{F_{W_2}}(u)}{\overline{F_{W_1}}(u)} \right)^{1/(\alpha_2 - 2)}.$$

⋄

4.2. *Finite-time regular variation case*

We assume here that claim size distribution is regularly varying with parameter α_i^j for state j and process i with $j \in \{1, ..., J\}$ and $i \in \{1, 2\}$. We also assume that $\alpha_1^1 < \alpha_1^j$ for all $j \in \{2, ..., J\}$ and $\alpha_2^1 < \alpha_2^j$ for all $j \in \{2, ..., J\}$ and $1<\alpha_1^1 < \alpha_2^2$. That is to say, state 1 corresponds to a crisis environment with more severe claims and the first line of business is also riskier that the second one.

Proposition 4.3. *Under the assumptions of this Section, we have for large u and $i \in \{1, 2\}$*

$$E\left[I_T^i(u)\right] \sim \left(\sum_{j=1}^{J} \mu(j) \left[\int_0^T E(N_i^1(V_j^1(t)))dt\right]\right) \frac{u\overline{F_{W_i^1}}(u)}{\alpha_i^1 - 1},$$

where $V_j^1(t)$ is the time spent by the environment process in state 1 during $[0, t]$ given $J(0) = j$.

Proof. First, rewrite, for $i = 1, 2$, $U_i(t)$ as follows,

$$U_i(t) = u + C_i(t) - S_i(t),$$

where
$$C_i(t) = \sum_{p\geq 1} \sum_{1\leq j\leq J} \left(c_i^j(T_p - T_{p-1})\right) \mathbb{1}_{\{J_{T_{p-1}}=j, T_p\leq t\}} + \sum_{p\geq 1} \sum_{1\leq j\leq J} \left(c_i^j(t - T_{p-1})\right) \mathbb{1}_{\{J_{T_{p-1}}=j, T_{p-1}\leq t\leq T_p\}},$$

and where

$$S_i(t) = \sum_{p\geq 1} \sum_{1\leq j\leq J} \left(\sum_{n=1}^{N_i^j(T_p)} W_{i,n}^j - \sum_{n=1}^{N_i^j(T_{p-1})} W_{i,n}^j\right) \mathbb{1}_{\{J_{T_{p-1}}=j, T_p\leq t\}} + \sum_{p\geq 1} \sum_{1\leq j\leq J} \left(\sum_{n=1}^{N_i^j(t)} W_{i,n}^j - \sum_{n=1}^{N_i^j(T_{p-1})} W_{i,n}^j\right) \mathbb{1}_{\{J_{T_{p-1}}=j, T_{p-1}\leq t\leq T_p\}}.$$

Then, notice that, for all $t > 0$, $S_i(t)$ has the same distribution as

$$\tilde{S}_i(t) = \sum_{j=1}^{J} \sum_{n=1}^{N^j(V^j(t))} W_{i,n}^j,$$

where for $j = 1, ..., J$ and $t > 0$, $V^j(t)$ is the time spent by the environment process in state j during $[0, t]$.
In [7], we have the following result :

$$\begin{aligned} P(U_i(t) < 0) &= \sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > u + C_i(t) | J(0) = i), \\ &\sim \left(\sum_{j=1}^{J} \mu(j) E(N_i^1(V_j^1(t)))\right) \overline{F_{W^1,i}}(u) \quad \text{as } u \to \infty, \end{aligned}$$

for $i = 1, 2$.
Thus, $u \mapsto \sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > u + C_i(t) | J(0) = i)$ is regularly varying with parameter α_i^1 and from Karamata's

Theorem we have for large u,

$$\begin{aligned}
E(I_T^i(u)) &= E\left(\int_{t=0}^{T} \mathbb{1}_{U_i(t)<0} \, |U_i(t)| \, dt\right) \\
&= \int_{t=0}^{T} E\left(\mathbb{1}_{U_i(t)<0} \, |U_i(t)|\right) dt \qquad \text{using Fubini's Theorem} \\
&= \int_{t=0}^{T} \int_{x=0}^{\infty} \sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > u + C_i(t) + x | J(0) = j) dx dt \\
&= \int_{t=0}^{T} \int_{y=u}^{\infty} \sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > y + C_i(t) | J(0) = j) dy dt \\
&\sim \int_{t=0}^{T} \frac{u}{\alpha_i^1 - 1} \sum_{j=1}^{J} \mu(j) P(\tilde{S}_i(t) > u | J(0) = j) dt \\
&\sim \left(\sum_{j=1}^{J} \mu(j) \left[\int_0^T E(N_i^1(V_j^1(t))) dt\right]\right) \frac{u \overline{F_{W_i^1}}(u)}{\alpha_i^1 - 1}
\end{aligned}$$

⋄

Lemma 4.4. *The couples $(u, 0)$ and $(0, u)$ do not solve our optimization problem for u large enough.*

Proof. The proof is the same as in Lemma 4.1.
⋄

Theorem 4.5. *Under the assumptions of this Subsection, the couple (u_1, u_2) which minimizes $I_T(u_1, u_2)$ satisfies*

$$\begin{cases} \frac{\partial E[I_T^1(u_1)]}{\partial u_1} = \frac{\partial E[I_T^1(u_2)]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

Moreover, if we denote $u_1 = (1 - \beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ we have for large u

$$\beta(u) \sim \left(\frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)}\right)^{1/\alpha_2},$$

where

$$K_i = \left(\sum_{j=1}^{J} \mu(j) \left[\int_0^T E(N_i^1(V_j^1(t))) dt\right]\right) \quad i = 1, 2.$$

Note that $\beta(u)$ represents the proportion of the global reserve we allocate to the safer line of business.

Proof. From Lemma 4.4, u_1 and u_2 are not equal to zero, we know from the Lagrange multiplier method, (see [1]), that the solution of our problem satisfies

$$\begin{cases} \frac{\partial E[I_T^1(u_1)]}{\partial u_1} = \frac{\partial E[I_T^1(u_2)]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

We know from Proposition 4.3 that for large u,

$$E\left[I_T^i(u_i)\right] \sim K_i u_i \frac{\overline{F_{W_i^1}}(u_i)}{\alpha_i^1 - 1} \quad i = 1, 2,$$

with

$$K_i = \left(\sum_{j=1}^{J} \mu(j) \left[\int_0^T E(N_i^1(V_j^1(t)))dt\right]\right) \quad i = 1, 2.$$

For $i = 1, 2$, since $\overline{F_{W^1,i}}$ is regularly varying with index α_i^1, we have for large u,

$$\frac{\partial E\left[I_T^i(u_i)\right]}{\partial u_i} \sim K_i \overline{F_{W_i^1}}(u_i) \quad i = 1, 2.$$

Let us denote $u_1 = (1 - \beta(u))u$ and $u_2 = \beta(u)u$ with $\beta(u) \in (0, 1)$ ($\beta(u)$ represents the proportion of the global reserve u we allocate to the line of business 2). With this notation, we are able to give the asymptotic behavior of u_1 and u_2.
Indeed, we have this following equation to solve, with large u,

$$K_1 \overline{F_{W_1^1}}((1 - \beta(u))u) = K_2 \overline{F_{W_2^1}}(\beta(u)u).$$

or equivalently, since $\overline{F_{W_i^1}}$ is regularly varying with index α_i^1 for $i = 1, 2$ and using the uniform convergence property (cf Definition 3.3),

$$K_1(1 - \beta(u))^{-\alpha_1^1} \overline{F_{W_1^1}}(u) = K_2 \beta(u)^{-\alpha_2^1} \overline{F_{W_2^1}}(u).$$

Thus we have

$$\beta(u)^{\alpha_2^1} = \frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)} (1 - \beta(u))^{\alpha_1^1} \to 0,$$

since $\alpha_2 > \alpha_1 > 1$ and $1 - \beta(u) \in (0, 1)$.
Consequently, $\beta(u) \xrightarrow[u\to\infty]{} 0$ and for large u,

$$\frac{\beta(u)^{\alpha_2^1}}{\frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)}} = (1 - \beta(u))^{\alpha_1^1} \xrightarrow[u\to\infty]{} 1.$$

So,

$$\beta(u) \sim \left(\frac{K_2}{K_1} \frac{\overline{F_{W_2^1}}(u)}{\overline{F_{W_1^1}}(u)}\right)^{1/\alpha_2^1}.$$

⋄
Note that K_1 and K_2 may be computed from an adaptation of Proposition 5.2 in [7]. For example, if we consider only one state (e.g. state 1), we have $K_1 = \frac{\lambda_1^1 T^2}{2}$ and $K_2 = \frac{\lambda_2^1 T^2}{2}$ and for large u,

$$\beta(u) \sim \left(\frac{\lambda_2^1 \overline{F_{W_2^1}}(u)}{\lambda_1^1 \overline{F_{W_1^1}}(u)}\right)^{1/\alpha_2^1}.$$

4.3. Infinite time case where Cramer-Lundberg coefficient exists

In the Subsection, we assume that the dependence between the two lines of business is only generated by common shocks. There is no environment process. We also assume that the Cramer-Lundberg exponent of the risk process $(U_1(t))_{t\geq 0}$ (resp. $(U_2(t))_{t\geq 0}$) exists and is equal to R_1 (resp. R_2). Finally, we assume that $R_1 < R_2$, that is to say that the second line of business is safer than the first one.
As there is no environment process, the notations in this Subsection are the same but without the state exponent j.

From Proposition 3.13, we have for large u and $i = 1, 2$,

$$E\left[I^i_\infty(u)\right] \sim M_i e^{-R_i u},$$

with

$$M_i = \frac{1}{c_i}\frac{1}{1-\psi_i(0)}\frac{1-(\lambda_i+\lambda)\mu_i}{R_i^2((\lambda_i+\lambda)\hat{F_{W_i}}'(R_i)-1)}.$$

Lemma 4.6. *The couples $(u, 0)$ and $(0, u)$ do not solve our optimization problem for u large enough.*

Proof. The proof is the same as in Lemma 4.1.
⋄

Theorem 4.7. *Under the assumptions of this Subsection, the couple (u_1, u_2) which minimizes $I_\infty(u_1, u_2)$ satisfies*

$$\begin{cases} \frac{\partial E\left[I^1_\infty(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I^1_\infty(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

For large u, the solution is given by

$$u_1 = \frac{R_2}{R_1+R_2}u + \frac{1}{R_1+R_2}\log\left(\frac{M_2'}{M_1'}\right) + o(1),$$
$$u_2 = u - u_1 + o(1),$$

where

$$M_i' = -R_i M_i \quad i = 1, 2.$$

Proof. From Lemma 4.6, u_1 and u_2 are not equal to zero, we know from the Lagrange multiplier method, see [1], that the solution of our problem satisfies

$$\begin{cases} \frac{\partial E\left[I^1_\infty(u_1)\right]}{\partial u_1} = \frac{\partial E\left[I^1_\infty(u_2)\right]}{\partial u_2}, \\ u_1 + u_2 = u. \end{cases}$$

We know from Proposition 3.13 that for $i = 1, 2$ and large u,

$$E\left[I^i_\infty(u)\right] \sim M_i e^{-Ru},$$

with

$$M_i = \frac{1}{c_i}\frac{1}{1-\psi_i(0)}\frac{1-(\lambda_i+\lambda)\mu_i}{R_i^2((\lambda_i+\lambda)\hat{F_{W_i}}'(R_i)-1)}.$$

For $i = 1, 2$, we have for large u,

$$\frac{\partial E\left[I^i_\infty(u_i)\right]}{\partial u_i} \sim M_i' e^{-R_i u_i} \quad i = 1, 2,$$

with $M_i' = -R_i M_i$.
We have this following equation to solve, with large u,

$$M_1' e^{-R_1 u_1} = M_2' e^{-R_2(u-u_1)}.$$

The solution is as in the statement of the theorem.
◇

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