



DISPERSION RELATIONS AND WAVE OPERATORS IN SELF-SIMILAR QUASI-CONTINUOUS LINEAR CHAINS

T.M. Michelitsch^{1*} G.A. Maugin¹ F. C. G. A. Nicolleau², A. F. Nowakowski², S. Derogar³

¹ Institut Jean le Rond d'Alembert
CNRS UMR 7190
Université Pierre et Marie Curie, Paris 6
FRANCE

² Department of Mechanical Engineering
³ Department of Civil and Structural Engineering
University of Sheffield
United Kingdom

Physical Review E **80**, 011135 (2009)

July 28, 2009

1 Abstract

We construct self-similar functions and linear operators to deduce a self-similar variant of the Laplacian operator and of the D'Alembertian wave operator. The exigence of self-similarity as a symmetry property requires the introduction of non-local particle-particle interactions. We derive a self-similar linear wave operator describing the dynamics of a quasi-continuous linear chain of infinite length with a spatially self-similar distribution of nonlocal inter-particle springs. The self-similarity of the nonlocal harmonic particle-particle interactions results in a dispersion relation of the form of a Weierstrass-Mandelbrot function which exhibits self-similar and fractal features. We also derive a continuum approximation which relates the self-similar Laplacian to fractional integrals and yields in the low-frequency regime a power law frequency-dependence of the oscillator density.

Keywords: Self-similarity, self-similar functions, affine transformations, Weierstrass-Mandelbrot function, fractal functions, fractals, power laws, fractional integrals.

PACS: 05.50.+q 81.05.Zx 63.20.D-

*Corresponding author, Email: michel@lmm.jussieu.fr

2 Introduction

In the seventies of the last century the development of the *Fractal Geometry* by Mandelbrot [1] launched a scientific revolution. However, the mathematical roots of this discipline originate much earlier in the 19th century [2]. The superior electromagnetic properties of “*fractal antennae*” have been known already for a while [3, 4]. More recently one found by means of numerical simulations that fractal gaskets such as the Sierpinski gasket reveal interesting vibrational properties [5]. Meanwhile physical problems in fractal and self-similar structures or media become more and more a subject of interest also in analytical mechanics and engineering science. This is true in statics and dynamics. However technological exploitations of effects based on self-similarity and “fractality” are still very limited due to a lack of fundamental understanding of the role of the self-similar symmetry. An improved understanding could raise an enormous new field for basic research and applications in a wide range of disciplines including fluid mechanics and the mechanics of granular media and solids. Some initial steps have been performed (see papers [5, 6, 7, 8, 9, 10] and the references therein). However a generally accepted “fractal mechanics” has yet to be developed. Therefore, it is highly desirable to develop sufficiently simple models which are on the one hand accessible to a mathematical-analytical framework and on the other hand which capture the essential features imposed by self-similar scale invariant symmetry. The goal of this demonstration is to develop such a model.

Several significant contributions of fractal and self-similar chains and lattices have been presented [13, 14, 15, 16]. In these papers problems on *discrete* lattices with fractal features are addressed. Closed form solutions for the dynamic Green’s function and the vibrational spectrum of a linear chain with spatially exponential properties are developed in a recent paper [11]. A similar fractal type of linear chain as in the present paper has been considered very recently by Tarasov [7]. Unlike in the present paper the chain considered by Tarasov in [7] is *discrete*, i.e. there remains a characteristic length scale which is given by the next-neighbor distance of the particles.

In contrast to all these works we analyze in the present paper vibrational properties in a *quasi-continuous* linear chain with (in the self-similar limiting case) infinitesimal lattice spacing with a non-local spatially self-similar distribution of power-law-scaled harmonic inter-particle interactions (springs). In this way we avoid the appearance of a characteristic length scale in our chain model. It seems there are analogue situations in turbulence [17] and other areas where the present interdisciplinary approach could be useful.

Our demonstration is organized as follows: § 3 is devoted to the construction of self-similar functions and operators where a self-similar variant of the Laplacian is deduced. This Laplacian gets his physical justification in § 4. It is further shown in § 3 that in a continuum approximation this Laplacian takes the form of fractional integrals. In § 4 we consider a self-similar quasi-continuous linear chain with self-similar harmonic interactions. The equation of motion of this chain takes the form of a self-similar wave equation containing the self-similar Laplacian defined in § 3 leading to a dispersion relation having the form of the Weierstrass-Mandelbrot function which is a self-similar and for a certain parameter range also a fractal function.

3 Construction of self-similar functions and linear operators

In this paragraph we define the term “self-similarity” with respect to functions and operators. We call a scalar function $\phi(h)$ *exact self-similar* with respect to variable h if the condition

$$\phi(Nh) = \Lambda\phi(h) \tag{1}$$

is satisfied for all values $h > 0$ of the scalar variable h . We call (1) the “affine problem”¹ where N is a fixed parameter and $\Lambda = N^\delta$ represents a continuous set of admissible eigenvalues. The band of

¹where we restrict here to affine transformations $h' = Nh + c$ with $c = 0$.

admissible $\delta = \frac{\ln \Lambda}{\ln N}$ is to be determined. A function $\phi(h)$ satisfying (1) for a certain N and admissible $\Lambda = N^\delta$ represents an unknown “solution” to the affine problem of the form $\phi_{N,\delta}(h)$ and is to be determined.

As we will see below for a given N solutions $\phi(h)$ exist only in a certain range of admissible Λ . From the definition of the problem follows that if $\phi(h)$ is a solution of (1) it is also a solution of $\phi(N^s h) = \Lambda^s \phi(h)$ where $s \in \mathbf{Z}$ is discrete and can take all positive and negative integers including zero. We emphasize that non-integer s are not admitted. The discrete set of pairs Λ^s, N^s are for all $s \in \mathbf{Z}$ related by a power law with the same power δ , i.e. $\Lambda = N^\delta$ hence $\Lambda^s = (N^s)^\delta$. By replacing Λ and N by Λ^{-1} and N^{-1} in (1) defines the identical problem. Hence we can restrict our considerations on fixed values of $N > 1$.

We can consider the affine problem (1) as the eigenvalue problem for a linear operator \hat{A}_N with a certain given fixed parameter N and eigenfunctions $\phi(h)$ to be determined which correspond to an *admissible* range of eigenvalues $\Lambda = N^\delta$ (or equivalently to an admissible range of exponent $\delta = \ln \Lambda / \ln N$). For a function $f(x, h)$ we denote by $\hat{A}_N(h)f(x, h) =: f(x, Nh)$ when the affine transformation is only performed with respect to variable h .

We assume $\Lambda, N \in \mathbb{R}$ for physical reasons without too much loss of generality to be real and positive. For our convenience we define the “affine” operator \hat{A}_N as follows

$$\hat{A}_N f(h) =: f(Nh) \quad (2)$$

It is easily verified that the affine operator \hat{A}_N is *linear*, i.e.

$$\hat{A}_N (c_1 f_1(h) + c_2 f_2(h)) = c_1 f_1(Nh) + c_2 f_2(Nh) \quad (3)$$

and

$$\hat{A}_N^s f(h) = f(N^s h), \quad s = 0 \pm 1, \pm 2, \dots \pm \infty \quad (4)$$

We can define affine operator functions for any smooth function $g(\tau)$ that can be expanded into a Maclaurin series as

$$g(\tau) = \sum_{s=0}^{\infty} a_s \tau^s \quad (5)$$

We define an affine operator function in the form

$$g(\xi \hat{A}_N) = \sum_{s=0}^{\infty} a_s \xi^s \hat{A}_N^s \quad (6)$$

where ξ denotes a scalar parameter. The operator function which is defined by (6) acts on a function $f(h)$ as follows

$$g(\xi \hat{A}_N) f(h) = \sum_{s=0}^{\infty} a_s \xi^s f(N^s h) \quad (7)$$

where relation (4) with expansion (6) has been used. The convergence of series (7) has to be verified for a function $f(h)$ to be admissible. An explicit representation of the affine operator \hat{A}_N can be obtained when we write $f(h) = f(e^{\ln h}) = \bar{f}(\ln h)$ to arrive at

$$\hat{A}_N(h) = e^{\ln N \frac{d}{d(\ln h)}} \quad (8)$$

This relation is immediately verified in view of

$$\hat{A}_N(h) f(h) = e^{\ln N \frac{d}{d(\ln h)}} f(e^{\ln h}) = f(e^{\ln h + \ln N}) = f(Nh) \quad (9)$$

With this machinery we are now able to construct self-similar functions and operators.

3.1 Construction of self-similar functions

A self-similar function solving problem (1) is formally given by the series

$$\phi(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} \hat{A}_N^s f(h) = \sum_{s=-\infty}^{\infty} \Lambda^{-s} f(N^s h) \quad (10)$$

for any function $f(h)$ for which the series (10) is uniformly convergent for all h . We introduce the self-similar operator

$$\hat{T}_N = \sum_{s=-\infty}^{\infty} \Lambda^{-s} \hat{A}_N^s \quad (11)$$

that fulfils formally the condition of self-similarity $\hat{A}_N \hat{T}_N = \Lambda \hat{T}_N$ and hence (10) solves the affine problem (1). In view of the symmetry with respect to inversion of the sign of s in (10) and (11) we can restrict ourselves to $N > 1$ ($N, \Lambda \in \mathbb{R}$) without any loss of generality²: Let us look for admissible functions $f(t)$ for which (10) is convergent. To this end we have to demand simultaneous convergence of the partial sums over positive and negative s . Let us assume that (where we can confine ourselves to $t > 0$)

$$\lim_{t \rightarrow 0} f(t) = a_0 t^\alpha \quad (12)$$

For $t \rightarrow \infty$ we have to demand that $|f(t)|$ increases not stronger than a power of t , i.e.

$$\lim_{t \rightarrow \infty} f(t) = c_\infty t^\beta \quad (13)$$

with a_0, c_∞ denoting constants. Both exponents $\alpha, \beta \in \mathbb{R}$ are allowed to take positive or negative values which do not need to be integers. A brief consideration of partial sums yields the following requirements for $\Lambda = N^\delta$, namely: Summation over $s < 0$ in (10) requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda N^{-\alpha} < 1$. That is we have to demand $\delta < \alpha$. The partial sum over $s > 0$ requires absolute convergence of a geometrical series leading to the condition for its argument $\Lambda^{-1} N^\beta < 1$ which corresponds to $\delta > \beta$. Both conditions are simultaneously met if

$$N^\beta < \Lambda = N^\delta < N^\alpha \quad (14)$$

or equivalently

$$\beta < \delta = \frac{\ln \Lambda}{\ln N} < \alpha \quad (15)$$

Relations (14) and (15) require additionally $\beta < \alpha$, that is only functions $f(t)$ with the behaviour (12) and (13) with $\beta < \alpha$ are *admissible* in (10). The case $\beta = 0$ includes for instance certain bounded functions $|f(t)| < M$ such as some periodic functions.

3.2 A self-similar analogue to the Laplace operator

In the sprit of (10) and (11) we construct an exactly self-similar function from the second difference according to

$$\phi(x, h) = \hat{T}_N(h) (u(x+h) + u(x-h) - 2u(x)) \quad (16)$$

²We also can exclude the trivial case $N = 1$.

where $u(\dots)$ denotes an arbitrary smooth continuous field variable and $\hat{T}_N(h)$ expresses that the affine operator $\hat{A}_N(h)$ acts only on the dependence on h , that is $\hat{A}_N(h)v(x, h) = v(x, Nh)$. We have with $\xi = \Lambda^{-1}$ the expression

$$\phi(x, h) = \sum_{s=-\infty}^{\infty} \xi^s \{u(x + N^s h) + u(x - N^s h) - 2u(x)\} \quad (17)$$

which is a self-similar function with respect to its dependence on h with $\hat{A}_N(h)\phi(x, h) = \phi(x, Nh) = \xi^{-1}\phi(x, h)$ but a regular function with respect to x . The function $\phi(x, h)$ exists if the series (17) is convergent. Let us assume that $u(x)$ is a smooth function with a convergent Taylor series for any h . Then we have with $u(x \pm h) = e^{\pm h \frac{d}{dx}} u(x)$ and $u(x + h) + u(x - h) - 2u(x) = (e^{h \frac{d}{dx}} + e^{-h \frac{d}{dx}} - 2) u(x)$ which can be written as

$$u(x + h) + u(x - h) - 2u(x) = 4 \sinh^2 \left(\frac{h}{2} \frac{d}{dx} \right) u(x) = h^2 \frac{d^2}{dx^2} u(x) + \text{orders } h^{\geq 4} \quad (18)$$

thus $\alpha = 2$ in criteria (12) is met. If we demand $u(x)$ being Fourier transformable we have as necessary condition that

$$\int_{-\infty}^{\infty} |u(x)| dx < \infty \quad (19)$$

exists. This is true if $|u(t)|$ tends to zero as $t \rightarrow \pm\infty$ as $|t|^\beta$ where $\beta < -1$. We have then the condition that

$$\beta < 0 < \delta = -\frac{\ln \xi}{\ln N} < \alpha = 2 \quad (20)$$

We will see below that only $\delta > 0$ is *physically admissible*, i.e. compatible with harmonic particle-particle interactions which decrease with increasing particle-particle distance.

The 1D Laplacian Δ_1 is defined by

$$\Delta_1 u(x) = \frac{d^2}{dx^2} u(x) = \lim_{\tau \rightarrow 0} \frac{(u(x + \tau) + u(x - \tau) - 2u(x))}{\tau^2} \quad (21)$$

Let us now define a self-similar analogue to the 1D Laplacian (21) where we put with $\xi = N^{-\delta}$

$$\Delta_{(\delta, N, \tau)} u(x) =: \text{const} \lim_{\tau \rightarrow 0} \tau^{-\lambda} \phi(x, \tau) \quad (22)$$

$$= \text{const} \lim_{\tau \rightarrow 0} \tau^{-\lambda} \sum_{s=-\infty}^{\infty} \xi^s (u(x + N^s \tau) + u(x - N^s \tau) - 2u(x)) \quad (23)$$

where we have introduced a renormalisation-multiplier $\tau^{-\lambda}$ with the unknown power λ to be determined such that the limiting case is finite. The constant factor *const* indicates that there is a certain arbitrariness in this definition and will be chosen conveniently. Let us consider the limit $\tau \rightarrow 0$ by the special sequence $\tau_n = N^{-n} h$ with $n \rightarrow \infty$ and h being constant. Unlike in the 1D case (21), the result of this limiting process depends crucially on the choice of the sequence τ_n . We see here that the self-similar Laplacian cannot be defined uniquely as in the 1D case. We have (by putting in (22) $\text{const} = h^\lambda$)

$$\Delta_{(\delta, N, h)} u(x) = \lim_{n \rightarrow \infty} N^{\lambda n} \xi^n \sum_{s=-\infty}^{\infty} \xi^{s-n} (u(x + N^{s-n} h) + u(x - N^{s-n} h) - 2u(x)) \quad (24)$$

which assumes by replacing $s - n \rightarrow s$ the form

$$\Delta_{(\delta, N, h)} u(x) = \phi(x, h) \lim_{n \rightarrow \infty} N^{-(\delta-\lambda)n} \quad (25)$$

which is only finite and nonzero if $\lambda = \delta$. The ‘‘Laplacian’’ can then be defined simply by

$$\Delta_{(\delta,N,h)}u(x) =: \lim_{n \rightarrow \infty} N^{\delta n} \phi(x, N^{-n}h) = \phi(x, h) \quad (26)$$

or by using (16) and (18) we can simply write³

$$\Delta_{(\delta,N,h)} = 4\hat{T}_N(h) \sinh^2 \left(\frac{h}{2} \frac{\partial}{\partial x} \right) = 4 \sum_{s=-\infty}^{\infty} N^{-\delta s} \sinh^2 \left(\frac{N^s h}{2} \frac{\partial}{\partial x} \right) \quad (27)$$

where $\hat{T}_N(h)$ is the self-similar operator defined in (11). The self-similar analogue of Laplace operator defined by (27) depends on the parameters δ, N, h . We furthermore observe the self-similarity of Laplacian (27) with respect to its dependence on h , namely

$$\Delta_{(\delta,N,Nh)} = N^\delta \Delta_{(\delta,N,h)} \quad (28)$$

3.3 Continuum approximation - link to fractional integrals

For numerical evaluations it may be convenient to utilize a continuum approximation of the self-similar Laplacian (27). To this end we put $N = 1 + \epsilon$ (with $0 < \epsilon \ll 1$ thus $\epsilon \approx \ln N$) where ϵ is assumed to be ‘‘small’’ and $s\epsilon = v$ such that $dv \approx \epsilon$ and $N^s = (1 + \epsilon)^{\frac{v}{\epsilon}} \approx e^v$. In this approximation $N^s \approx e^v$ becomes a (quasi)-continuous variable when s runs through $s \in \mathbf{Z}$. Then we can write (10) in the form

$$\phi(h) = \sum_{s=-\infty}^{\infty} N^{-s\delta} f(N^s h) \approx \frac{1}{\epsilon} \int_{-\infty}^{\infty} e^{-\delta v} f(he^v) dv \quad (29)$$

which can be further written with $he^v = \tau$ ($h > 0$) and $\frac{d\tau}{\tau} = dv$ and $\tau(v \rightarrow -\infty) = 0$ and $\tau(v \rightarrow \infty) = \infty$ as

$$\phi(h) \approx \frac{h^\delta}{\epsilon} \int_0^\infty \frac{f(\tau)}{\tau^{1+\delta}} d\tau \quad (30)$$

In this continuum approximation the function $\phi(h)$ obeys the same scaling behaviour as (10), namely $\phi(h\lambda) = \lambda^\delta \phi(h)$ but in contrast to (10) λ can assume any continuous positive value. This is due to the fact that (30) is holding for $N = 1 + \epsilon$ with sufficiently small $\epsilon > 0$ since in this limiting case there exists for any continuous value $\lambda > 0$ an $m \in \mathbf{Z}$ such that $N^m \approx \lambda$. The existence requirement for integral (30) leads to the same requirements for $f(t)$ as in (10), namely inequality (15). Application of the approximate relation (30) to Laplacian (27) yields

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \frac{h^\delta}{\epsilon} \int_0^\infty \frac{(u(x-\tau) + u(x+\tau) - 2u(x))}{\tau^{1+\delta}} d\tau \quad (31)$$

where this integral exists for $\beta < 0 < \delta < 2$ and $\beta < -1$ because the required existence of integral (19) and relation (18). By performing two partial integrations and by taking into account the vanishing boundary terms at $\tau = 0$ and $\tau = \infty$ for $0 < \delta < 2$, we can re-write (31) in the form of a convolution of the conventional 1D Laplacian $\frac{d^2 u}{dx^2}(x)$, namely

$$\Delta_{(\delta,\epsilon,h)}u(x) \approx \int_{-\infty}^{\infty} g(|x-\tau|) \frac{d^2 u}{d\tau^2}(\tau) d\tau \quad (32)$$

with the kernel

³We have to replace $\frac{d}{dx} \rightarrow \frac{\partial}{\partial x}$ if the Laplacian acts on a field $u(x, t)$ as in Sec. 4.

$$g(|x|) = \frac{h^\delta}{\delta(\delta-1)\epsilon} |x|^{1-\delta}, \quad \delta \neq 1 \quad (33)$$

where $0 < \delta < 2$ and $g(|x|) = -\frac{h}{\epsilon} \ln |x|$ for $\delta = 1$. We can further write for $\delta \neq 1$ (32) in terms of *fractional integrals*

$$\Delta_{(\delta=2-D,\epsilon,h)} u(x) \approx \frac{h^{2-D}}{\epsilon} \frac{\Gamma(D)}{(D-1)(D-2)} \left(\mathcal{D}_{-\infty,x}^{-D} + (-1)^D \mathcal{D}_{\infty,x}^{-D} \right) \Delta_1 u(x) \quad (34)$$

where $\Delta_1 u(x) = \frac{d^2}{dx^2} u(x)$ denotes the conventional 1D-Laplacian and $D = 2 - \delta > 0$ which is positive in the admissible range of $0 < \delta < 2$. For $0 < \delta < 1$ the quantity D can be identified with the estimated fractal dimension of the fractal dispersion relation of the Laplacian [18] which is deduced in the next section. In (34) we have introduced the Riemann-Liouville fractional integral $\mathcal{D}_{a,x}^{-D}$ which is defined by (e.g. [19, 20])

$$\mathcal{D}_{a,x}^{-D} v(x) = \frac{1}{\Gamma(D)} \int_a^x (x-\tau)^{D-1} v(\tau) d\tau \quad (35)$$

where $\Gamma(D)$ denotes the Γ -function which represents the generalization of the factorial function to non-integer $D > 0$. The Γ -function is defined as

$$\Gamma(D) = \int_0^\infty \tau^{D-1} e^{-\tau} d\tau, \quad D > 0 \quad (36)$$

For positive integers $D > 0$ the Γ -function reproduces the factorial-function $\Gamma(D) = (D-1)!$ with $D = 1, 2, \dots, \infty$.

4 The physical chain model

We consider an infinitely long quasi-continuous linear chain of identical particles. Any space-point x corresponds to a “material point” or particle. The mass density of particles is assumed to be spatially homogeneous and equal to one for any space point x . Any particle is associated with one degree of freedom which is represented by the displacement field $u(x, t)$ where x is its spatial (Lagrangian) coordinate and t indicates time. In this sense we consider a quasi continuous spatial distribution of particles. Any particle at space-point x is non-locally connected by harmonic springs of strength ξ^s to particles located at $x \pm N^s h$, where $N > 1$ and $N \in \mathbb{R}$ is not necessarily integer, $h > 0$, and $s = 0, \pm 1, \pm 2, \dots, \pm \infty$. The requirement of decreasing spring constants with increasing particle-particle distance leads to the requirement that $\xi = N^{-\delta} < 1$ ($N > 1$) i.e. only chains with $\delta > 0$ are physically admissible. In order to get exact self-similarity we avoid the notion of “next-neighbour particles” in our chain which would be equivalent to the introduction of an internal length scale (the next neighbour distance). To admit particle interactions over arbitrarily close distances $N^s h \rightarrow 0$ ($s \rightarrow -\infty$, $h = \text{const}$) our chain has to be *quasi-continuous*. This is the principal difference to the *discrete* chain considered recently by Tarasov [7] which is discrete and not self-similar.

The Hamiltonian which describes our chain can be written as

$$H = \frac{1}{2} \int_{-\infty}^{\infty} \left(\dot{u}^2(x, t) + \mathcal{V}(x, t, h) \right) dx \quad (37)$$

In the spirit of (10) the elastic energy density $\mathcal{V}(x, t, h)$ is assumed to be constructed self-similarly, namely⁴

$$\mathcal{V}(x, t, h) = \frac{1}{2} \hat{T}_N(h) \left[(u(x, t) - u(x + h, t))^2 + (u(x, t) - u(x - h, t))^2 \right] \quad (38)$$

where $\hat{T}_N(h)$ is the self-similar operator (11) with $\xi = \Lambda^{-1} = N^{-\delta}$ to arrive at

$$\mathcal{V}(x, t, h) = \frac{1}{2} \sum_{s=-\infty}^{\infty} \xi^s \left[(u(x, t) - u(x + hN^s, t))^2 + (u(x, t) - u(x - hN^s, t))^2 \right] \quad (39)$$

The elastic energy density $\mathcal{V}(x, t, h)$ fulfills the condition of self-similarity with respect to h , namely

$$\hat{A}_N(h) \mathcal{V}(x, t, h) = \mathcal{V}(x, t, Nh) = \xi^{-1} \mathcal{V}(x, t, h) \quad (40)$$

We have to demand in our physical model that the energy is finite, i.e. (39) needs to be convergent which yields $\alpha = 2$ as for the Laplacian (17). To determine β we have to demand that $u(x, t)$ is a Fourier transformable field⁵. Thus we have to have an asymptotic behaviour of $|u(x \pm \tau, t)| \rightarrow 0$ as τ^β where $\beta < -1$ as $\tau \rightarrow \infty$. From this follows $|u(x, t) - u(x \pm \tau, t)|^2$ behaves then as $|u(x, t)|^2$. Hence, the elastic energy density (39) is finite if

$$0 < \delta < 2 \quad (41)$$

where $\beta < -1$.

This inequality determines the range of the admissible values of δ in order to achieve convergence. The equation of motion is obtained from

$$\frac{\partial^2 u}{\partial t^2} = -\frac{\delta H}{\delta u} \quad (42)$$

(where $\delta./\delta u$ stands for a functional derivative) to arrive at

$$\frac{\partial^2 u}{\partial t^2} = - \sum_{s=-\infty}^{\infty} \xi^s (2u(x, t) - u(x + hN^s, t) - u(x - hN^s, t)) \quad (43)$$

$$\frac{\partial^2 u}{\partial t^2} = \Delta_{(\delta, N, h)} u(x, t) \quad (44)$$

with the self-similar Laplacian $\Delta_{(\delta, N, h)}$ of equation (27). As the elastic energy density (39) the equation of motion is convergent for δ being in the interval (41) where $\beta < -1$. We can re-write (44) in the compact form of a wave equation

$$\square_{(\delta, N, h)} u(x, t) = 0 \quad (45)$$

where $\square_{(\delta, N, h)}$ is the *self-similar analogue of the d'Alembertian wave operator* having the form

$$\square_{(\delta, N, h)} = \Delta_{(\delta, N, h)} - \frac{\partial^2}{\partial t^2} \quad (46)$$

The d'Alembertian (46) with the Laplacian (27) describes the wave propagation in the self-similar chain with Hamiltonian (37). The present approach seems to be useful as a point of departure to establish a generalized theory of wave propagation in self-similar media.

Now we determine the dispersion relation, which is constituted by the (negative) eigenvalues of the (semi-)negative definite Laplacian (27). To this end we make use of the fact that the displacement

⁴The additional factor 1/2 in the elastic energy avoids double counting.

⁵This assumption defines the (function) space of eigenmodes and corresponds to infinite body boundary conditions.

field $u(x, t)$ is Fourier transformable (guaranteed by choosing $\beta < -1$) and that the exponentials e^{ikx} are eigenfunctions of the self-similar Laplacian (27). We hence write the Fourier integral

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{u}(k, t) e^{ikx} dk \quad (47)$$

to re-write (44) for the Fourier amplitudes $\tilde{u}(k, t)$ in the form

$$\frac{\partial^2 \tilde{u}}{\partial t^2}(k, t) = -\bar{\omega}^2(k) \tilde{u}(k, t) \quad (48)$$

and obtain

$$\omega^2(kh) = 4 \sum_{s=-\infty}^{\infty} N^{-\delta s} \sin^2\left(\frac{khN^s}{2}\right) \quad (49)$$

The series (49) describes a *Weierstrass-Mandelbrot function* which is a continuous and for $0 < \delta \leq 1$ a nowhere differentiable function [1, 18]. The Weierstrass-Mandelbrot function (49) fulfills the condition of self-similar symmetry, namely

$$\omega^2(Nkh) = N^\delta \omega^2(kh) \quad (50)$$

where the interval of convergence of the series of the *Weierstrass-Mandelbrot function* (49) is also given by (41). We emphasize that indeed *only* exponents δ in the interval (41) are *admissible* in Hamiltonian (37) with the elastic energy density (39) in order to have a “well-posed” problem.

It was shown by Hardy [18] that for $\xi N > 1$ and $\xi = N^{-\delta} < 1$ or equivalently for

$$0 < \delta < 1 \quad (51)$$

the Weierstrass-Mandelbrot function of the form (49) is not only self-similar but also a *fractal curve* of (estimated) non-integer fractal (Hausdorff) dimension $D = 2 - \delta > 1$. Figs. 2-4 show dispersion curves $\omega^2(kh)$ for different decreasing values of admissible $0 < \delta < 1$ and increasing fractal dimension D . Fig. 1 corresponds to the non-fractal case ($\delta = 1.2 > 1$). The increase of the fractal dimension from Figs. 2-4 is indicated by the increasingly irregular harsh behaviour of the curves. In Fig. 4 the fractal dimension of the dispersion curve is with $D = 1.9$ already close to the plane-filling dimension 2.

To evaluate (49) approximately it is convenient to replace the series by an integral utilizing a similar substitution as in Sec. 3.3 ($\epsilon \approx \ln N$). By doing so we smoothen the Weierstrass-Mandelbrot function (49). It is important to notice that the resulting approximate dispersion relation is hence differentiable and has not any more a fractal dimension $D > 1$ in the interval (51). For sufficiently “small” $|k|h$ ($h > 0$), i.e. in the long-wave regime we arrive at

$$\omega^2(kh) \approx \frac{(h|k|)^\delta}{\epsilon} C \quad (52)$$

which is only finite if $(|k|h)^\delta$ is in the order of magnitude of ϵ or smaller. This regime which includes the long-wave limit $k \rightarrow 0$ is hence characterized by a power law behaviour $\bar{\omega}(k) \approx \text{Const} |k|^{\delta/2}$ of the dispersion relation. The constant C introduced in (52) is given by the integral

$$C = 2 \int_0^\infty \frac{(1 - \cos \tau)}{\tau^{1+\delta}} d\tau \quad (53)$$

which exists for δ being within interval (41).

This approximation holds for “small” $\epsilon \approx \ln N \neq 0$ ($0 < \epsilon \ll 1$)⁶ which corresponds to the limiting case that $N^s = e^v$ is continuous. In this limiting case we obtain the oscillator density from [11]⁷

$$\rho(\omega) = 2 \frac{1}{2\pi} \frac{d|k|}{d\omega} \quad (54)$$

which is normalized such that $\rho(\omega)d\omega$ counts the number (per unit length) of normal oscillators having frequencies within the interval $[\omega, \omega + d\omega]$. We obtain then a power law of the form

$$\rho(\omega) = \frac{2}{\pi\delta h} \left(\frac{\epsilon}{C} \right)^{\frac{1}{\delta}} \omega^{\frac{2}{\delta}-1} \quad (55)$$

where δ is restricted within interval (41). We observe hence that the power $\frac{2}{\delta}-1$ is restricted within the range $0 < \frac{2}{\delta}-1 < \infty$ for $0 < \delta < 2$, especially with always vanishing oscillator density $\rho(\omega \rightarrow 0) = 0$.

We emphasize that neither is the dependence on k of the Weierstrass-Mandelbrot function (49) represented by a *continuous* $|k|^\delta$ -dependence nor is this function differentiable with respect to k . Application of (54) is hence only justified to be applied to the approximative representation (52) if $0 < \epsilon \ll 1$ thus $N = 1 + \epsilon$ is sufficiently close to 1 so that N^s is a quasi-continuous function when s runs through $s \in \mathbf{Z}$. Hence (54) is not generally applicable to (49) for any arbitrary $N > 1$. We can consider (55) as the low-frequency regime $\omega \rightarrow 0$ of the oscillator density holding *only* in the quasi-continuous case $N = 1 + \epsilon$ with $0 < \epsilon \ll 1$.

5 Conclusions

We have depicted how self-similar functions and linear operators can be constructed in a simple manner by utilizing a certain category of conventional “admissible” functions. This approach enables us to construct non-local self-similar analogues to the Laplacian and d’Alembert wave operator. The linear self-similar equation of motion describes the propagation of waves in a quasi-continuous linear chain with harmonic non-local self-similar particle-interactions. The complexity which comes into play by the self-similarity of the non-local interactions is completely captured by the dispersion relations which assume the forms of Weierstrass-Mandelbrot functions (49) exhibiting exact self-similarity and for certain parameter combinations (relation (51)) fractal features. In a continuum approximation the self-similar Laplacian is expressed in terms of fractional integrals (eq. (34)) leading for small k (long-wave limit) to a power-law dispersion relation (eq. (52)) and to a power-law oscillator density (eq. (55)) in the low-frequency regime.

The self-similar wave operator (46) with the Laplacian (27) can be generalized to describe wave propagation in fractal and self-similar structures which are fractal subspaces embedded in Euclidean spaces of 1-3 dimensions. The development of such an approach could be a crucial step towards a better understanding of the dynamics in materials with scale hierarchies of internal structures (“multiscale materials”) which may be idealized as fractal and self-similar materials.

We hope to inspire further work and collaborations in this direction to develop appropriate approaches useful for the modelling of static and dynamic problems in self-similar and fractal structures in a wider interdisciplinary context.

⁶ $\epsilon = 0$ has to be excluded since it corresponds to $N = 1$.

⁷The additional prefactor “2” takes into account the two branches of the dispersion relation (49) (one for $k < 0$ and one for $k > 0$).

6 Acknowledgements

Fruitful discussions with J.-M. Conoir, D. Queiros-Conde and A. Wunderlin are gratefully acknowledged.

References

- [1] B. B. Mandelbrot, *Fractals, Form, Chance, and Dimension* (Springer, New York, 1978).
- [2] E.E. Kummer: *J. Reine Angew. Math.* **44**, 93 (1852).
- [3] N. Cohen, *Commun. Q.* **5**(3), 7 (1995).
- [4] S. Hohlfeld, N. Cohen, *Fractals*, **7**(1), 79 (1999).
- [5] A.N. Bondarenko, V.A. Levin, The 9th Russian-Korean International Symposium 2005 (unpublished) pp. 33-35.
- [6] J. Kigami, *Japan J. Appl. Math.* **6**(2), 259 (1989).
- [7] V.E. Tarasov, *J. Phys. A: Math. Theor.* **41**, 035101 (2008).
- [8] M. Ostoja-Starzewski, *ZAMP* **58**, 1085 (2007).
- [9] J.C. Claussen, J. Nagler, H.G. Schuster, *Phys. Rev. E* **70**, 032101 (2004).
- [10] M. Epstein, S.M. Adeeb, *Int. J. Solids Struct.* **45**(11-12), 3238 (2008).
- [11] T.M. Michelitsch, G.A. Maugin, A.F. Nowakowski, F.C.G.A Nicolleau, *Int. J. Eng. Sci.* **47**(2), 209 (2009).
- [12] K. Ghosh, R. Fuchs, *Phys. Rev. B* **44**, 7330 (1991).
- [13] K. Li, M.I. Stockman, D.J. Bergman, *Phys. Rev. Lett.* **91**(22), 227402 (2003).
- [14] S. Raghavachari, J.A. Glazier, *Phys. Rev. Lett.* **74**(16), 3297 (1995).
- [15] R. Kopelman, M. Shortreed, Z.-Y. Shi, W. Tan, Z. Xu, J.S. Moore, A. Bar-Haim, J. Klafter, *Phys. Rev. Lett.* **78**(7), 1239 (1997).
- [16] E. Domany, S. Alexander, D. Bensimon, L.P. Kadanoff, *Phys. Rev. B* **28**(6), 3110 (1983).
- [17] J.A.C. Humphrey, C.A. Schuler, B. Rubinsky, *Fluid dynamics research* **9** (1-3), 81, ISSN 0169-5983 (1992).
- [18] G.H. Hardy, *Trans - Amer. Math. Soc.*, **17**, 301 (1916).
- [19] K.S. Miller, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, by Kenneth S. Miller, Bertram Ross (Editor), John Wiley & Sons; 1 edition (May 19, 1993). ISBN 0-471-58884-9.
- [20] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Application of Fractional Differential Equations*, Mathematical Studies 204, Jan von Mill (Editor) (Elsevier, Amsterdam, 2006). ISBN-10: 0444518320.

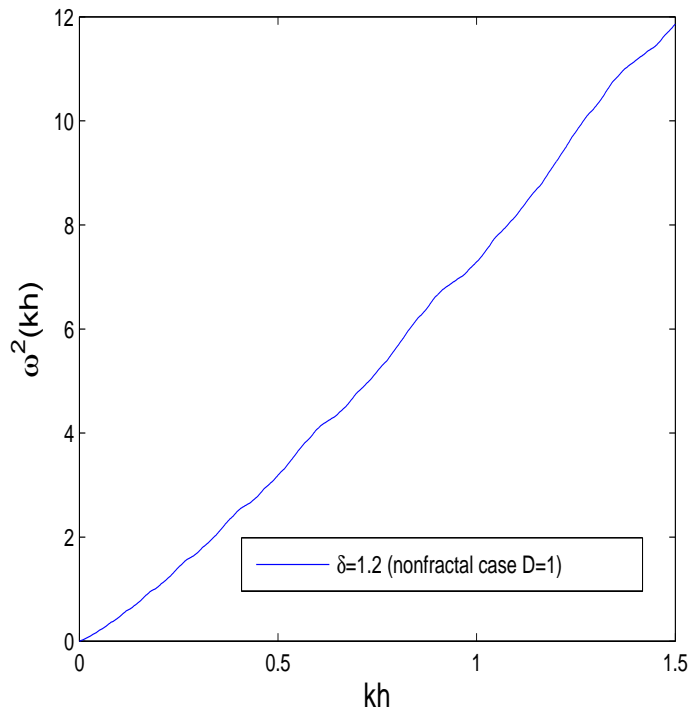


Figure 1: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 1.2$

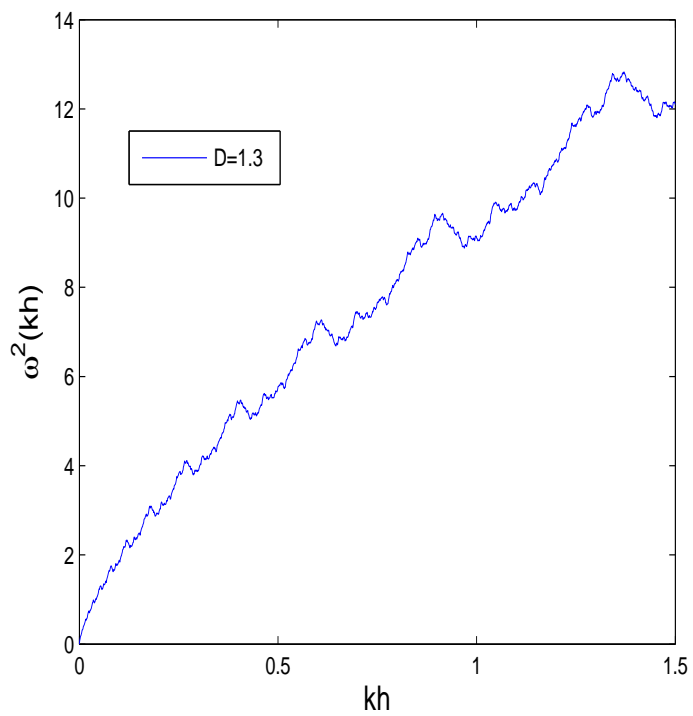


Figure 2: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 0.7$

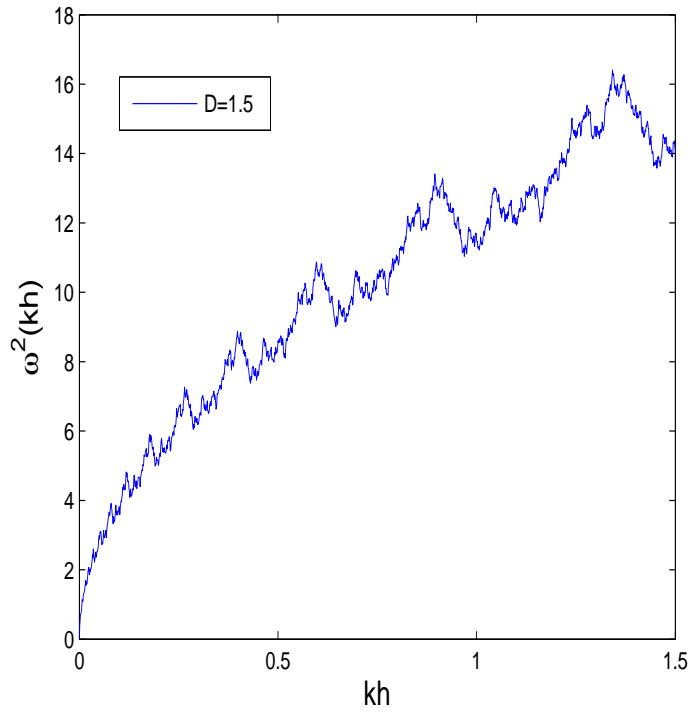


Figure 3: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 0.5$

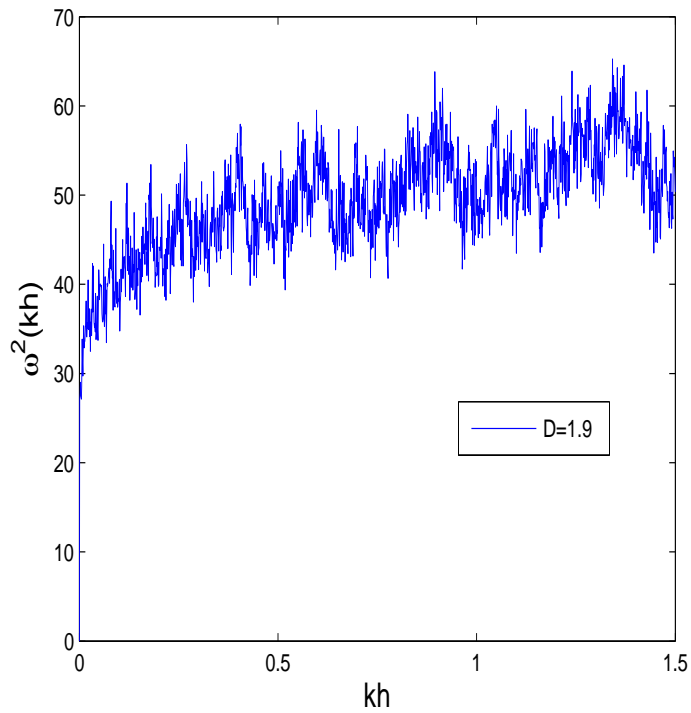


Figure 4: Dispersion relation $\omega^2(kh)$ in arbitrary units for $N = 1.5$ and $\delta = 0.1$