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SHARP QUANTITATIVE ISOPERIMETRIC INEQUALITIES IN THE L^1 MINKOWSKI PLANE

by

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An isoperimetric inequality bounds from below the perimeter of a domain in terms of its area. A *quantitative* isoperimetric inequality is a stability result: it bounds from above the distance to an isoperimetric minimizer in terms of the isoperimetric deficit. In other words, it measures how close to a minimizer an almost optimal set must be.

The euclidean quantitative isoperimetric inequality has been thoroughly studied, in particular in [**Hal92**] and [**FMP08**], but the L^1 case has drawn much less attention.

In this note we prove two quantitative isoperimetric inequality in the L^1 Minkowski plane with sharp constants and determine the extremal domains for one of them. It is usually (but not here) difficult to determine the extremal domains in a quantitative isoperimetric inequality: the only such kown result is for the Euclidean plane, due to Nitsch [**Nit08**].

1. Statement of the results

We consider the plane $\mathbb{R}^2$ endowed with the L^1 metric:

$$|(x_1, x_2) - (y_1, y_2)| = |x_1 - y_1| + |x_2 - y_2|.$$

The notation $|\cdot|$ shall be used to denote the size of an object, whatever its nature. If A is an measurable plane set then $|A|$ is its Lebesgue measure, also called its area ; if v is a vector $|v|$ is its L^1 norm ; if γ is a rectifiable curve, $|\gamma|$ is its L^1 length. We denote the boundary of a set using ∂.

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By a *domain* of the plane, we mean the closure of the bounded component of a Jordan curve. In particular, domains are compact and connected. All rectangles and squares considered are assumed to have their sides parallel to the coordinate axes. The square centered at 0 with side length 2λ is denoted by $B_\infty(\lambda)$: it is the λ ball of the L^∞ metric. Squares are known to minimize L^1 perimeter among plane domains of given area.

The measure of the distance between compact plane sets A, B we use in our main result is the L^∞ Haussdorf metric :

$$d_\infty(A,B) = \inf\{\lambda \leqslant 0 \,|\, A \subset B + B_\infty(\lambda) \text{ and } B \subset A + B_\infty(\lambda)\}.$$

Let us explain why this metric is natural here. One way to prove that almost isoperimetric domains are close to minimizers is to prove that they contain a minimizer of radius r and are included in another of radius R, with small radii difference $R-r$ and same center. In the euclidean space, such inclusions imply that the considered domain is at Haussdorf distance at most $(R-r)/2$ from some ball. However, balls and minimizers are different in the L^1 plane, so that if A is between concentric squares of radii R and r, one can only say that it is at L^1 Haussdorf distance $R-r$ from some square, while the L^∞ Hausdorf distance bound is the expected $(R-r)/2$.

It would certainly be possible to use the L^1 Haussdorf metric, and we expect that arguments of the same kind that those we use to prove Theorem 1.1, but more involved, would give a constant better than the $1/16$ obtained using the inequality $d_1 \leqslant 2d_\infty$ and Theorem 1.1.

Theorem 1.1. — *Let A be a domain of the L^1 Minkowski plane whose boundary is a rectifiable curve, and assume that*

$$|\partial A|^2 \leqslant (16+\varepsilon)|A|. \tag{1}$$

Then there is a square S such that

$$d_\infty(A,S)^2 \leqslant \frac{\varepsilon|A|}{64}. \tag{2}$$

We shall also see that Theorem 1.1 is sharp and show that up to L^1 isometry and homothety the domains that achieve the bound are the rectangles and the squares with one square deleted at a corner.

A second possible measure of the distance between domains of the same area, which present the advantage to be suitable to higher dimension as well, is simply the gap between their area and that of their intersection. In this respect we prove the following.

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Proposition 1.2. — *Let A be a domain of the L^1 Minkowski plane whose boundary is a rectifiable curve, and assume that* (1) *holds with ε sufficiently small. Then there is a square S such that $|S| = |A|$ and:*

$$|S \cap A| \geqslant (1 - \frac{\sqrt{\varepsilon}}{4} + O(\varepsilon))|A|. \tag{3}$$

In terms of Fraenkel asymmetry, this reads:

$$\frac{|S \Delta A|}{|A|} \leqslant \frac{\sqrt{\varepsilon}}{2} + O(\varepsilon) \tag{4}$$

We shall see that the 1/2 constant in (4) is sharp.

Surprisingly enough, it seems that these results are new, although similar ones can be deduced from the much more general [**FMP09**] (but with a non-optimal constant) and [**PWZ93**] (only when A is convex).

2. Proof of the inequalities

Assume A satisfies (1) for some ε and let R be the smallest rectangle containing A. This rectangle plays the role of a convex hull.

Lemma 2.1. — *We have $|\partial A| \geqslant |\partial R|$.*

Proof. — Since R is minimal, each of its sides contains a point of the boundary of A. Denote r_1, r_2, r_3, r_4 such points so that r_i and r_{i+1} lie on two adjacent sides of R for all i (modulo 4). It is possible that some $r_i = r_{i+1}$, but this does not affect what follows.

There are four curves γ_i in ∂A that connect r_i to r_{i+1} and meet only at their endpoints (see figure 1). Similarly, the boundary of R is made of four curves η_i connecting r_i to r_{i+1}. Since R is a rectangle, the η_i are L^1 geodesics. The length of γ_i is at least $|r_i - r_{i+1}| = |\eta_i|$, so that

$$|\partial A| = |\gamma_1| + |\gamma_2| + |\gamma_3| + |\gamma_4| \geqslant |\eta_1| + |\eta_2| + |\eta_3| + |\eta_4| = |\partial R|.$$

□

Let ℓ and α be such that $\ell - 2\alpha$ and $\ell + 2\alpha$ are the side lengths of R.

Lemma 2.2. — *We have*

$$|A| \leqslant \ell^2 \leqslant \frac{16 + \varepsilon}{16}|A| \tag{5}$$

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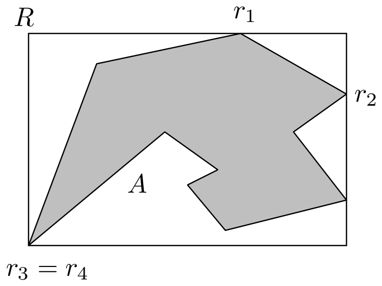

FIGURE 1. The L^1 perimeter of A is at least that of R

and

$$\alpha^2 \leqslant \frac{\varepsilon|A|}{64} \tag{6}$$

Proof. — From previous lemma we have $|\partial A| \geqslant 4\ell$, so that using (1) we get $16\ell^2 \leqslant (16+\varepsilon)|A|$. Since $A \subset R$ we have $|A| \leqslant |R| = \ell^2 - 4\alpha^2$ and (5) follows.

Next we have

$$\begin{aligned} 16\ell^2 &\leqslant (16+\varepsilon)(\ell^2 - 4\alpha^2) \\ 0 &\leqslant \varepsilon\ell^2 - 4(16+\varepsilon)\alpha^2 \\ \alpha^2 &\leqslant \frac{\varepsilon\ell^2}{4(16+\varepsilon)} \\ \alpha^2 &\leqslant \frac{\varepsilon}{64}|A| \end{aligned}$$

and we are done. □

Note that this lemma is sufficient to deduce the L^1 isoperimetric inequality and its equality case: if $\varepsilon = 0$, then $\alpha = 0$ and $|A| = \ell^2$.

2.1. Proof of Theorem 1.1. — We have $\min_S d_\infty(A,S) \geqslant \alpha$ (where S runs over all squares, see figure 2) and if there is equality, Lemma 2.2 is sufficient to conclude. We therefore assume $\delta := \min_S d_\infty(A,S) > \alpha$.

The following is the main step of the proof.

Lemma 2.3. — *We have either*

$$|A| \leqslant \ell^2 - 4\alpha^2 - 8\delta(\delta - \alpha)$$

or

$$|A| \leqslant \ell^2 - 4\alpha^2 - 4\delta^2.$$

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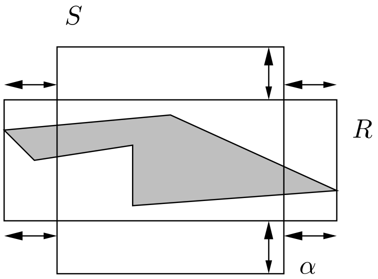

Figure 2. The closest square to R.

Proof. — Choose the origin so that R has its bottom side at height 0. Let S_η be the square that is at distance δ from each short side of R (so that it has side length $\ell + 2\alpha - 2\delta$) and whose bottom side is at height η.

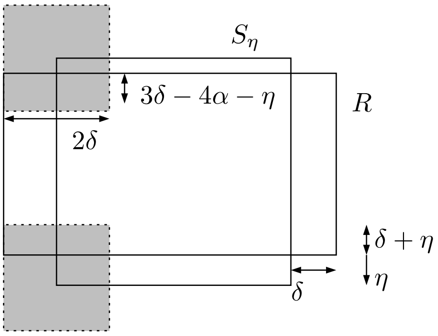

Figure 3. The domain A avoids one of the grey squares.

When $\eta \in [-\delta, 3\delta - 4\alpha]$, R and A are contained in the L^∞ neighborhood of size δ around S_η, thus there is some point $p_\eta \in S_\eta$ that is at L^∞ distance at least δ from A.

This excludes A from a square centered at p_η; the worst case (with respect to our goal of bounding $|A|$ from above) is when this excluded squares intersect only small parts of R and have maximal overlap. This is achieved when p_η is a corner of S_η for all η and the short side of R closest to p_η is constant.

In this case, for each η, if p_η is a lower corner then there is a $2\delta \times (\delta + \eta)$ sub-rectangle of R excluded, else p_η is a upper corner and there is a $2\delta \times (3\delta - 4\alpha - \eta)$ sub-rectangle of R excluded. These values assume

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that $\eta < \delta$ and $3\delta - 4\alpha - \eta < \delta$ respectively, otherwise there is simply an excluded square of area $4\delta^2$.

Let x be the supremum of the η such that p_η is a lower corner. There is an excluded sub-rectangle of area $2\delta \times (\delta + x)$ and for all $\eta > x$ the point p_η must be a higher corner, so that there is another excluded sub-rectangle of area $2\delta \times (3\delta - 4\alpha - x)$.

Summing up, either there are excluded sub-rectangles of total area at least $2\delta \times 4(\delta - \alpha)$, or there is an excluded sub-rectangle of area at least $4\delta^2$, and we get the desired bounds on $|A|$. □

We can now conclude the proof of Theorem 1.1. First, if $|A| \leqslant (\ell^2 - 4\alpha^2) - 8\delta(\delta - \alpha)$ then we have

$$\begin{aligned}
|\partial A|^2 &\leqslant (16+\varepsilon)|A| \\
(4\ell)^2 &\leqslant (16+\varepsilon)\big(\ell^2 - 4\alpha^2 - 8\delta(\delta-\alpha)\big) \\
0 &\leqslant \varepsilon\ell^2 - 4(16+\varepsilon)(\alpha^2 + 2\delta(\delta-\alpha)) \\
\alpha^2 + 2\delta(\delta-\alpha) &\leqslant \frac{\varepsilon\ell^2}{4(16+\varepsilon)} \\
2\delta^2 - 2\alpha\delta + \alpha^2 &\leqslant \frac{\varepsilon}{64}|A|
\end{aligned}$$

the last inequality coming from (5).

Since the function $x \mapsto 2\delta^2 - 2x\delta + x^2$ is minimal when $x = \delta$, we have $2\delta^2 - 2\alpha\delta + \alpha^2 \geqslant 2\delta^2 - 2\delta^2 + \delta^2 = \delta^2$, so that $\delta^2 \leqslant \frac{\varepsilon}{64}|A|$.

In the case when $|A| \leqslant \ell^2 - 4\alpha^2 - 4\delta^2$, we get:

$$\begin{aligned}
|\partial A|^2 &\leqslant (16+\varepsilon)(\ell^2 - 4\alpha^2 - 4\delta^2) \\
16\ell^2 &\leqslant 16\ell^2 + \varepsilon\ell^2 - 4(16+\varepsilon)(\alpha^2 + \delta^2) \\
\alpha^2 + \delta^2 &\leqslant \frac{\varepsilon\ell^2}{4(16+\varepsilon)} \\
\delta^2 &\leqslant \frac{\varepsilon}{64}|A|
\end{aligned}$$

2.2. Proof of Proposition 1.2.

— Let $\mu = \max_S |S \cap A|/|A|$ where S runs over the squares having same area than A.

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Lemma 2.4. — *One of the following holds:*

$$\begin{aligned}\mu &\geqslant 2 - \frac{\ell^2 - 4\alpha^2}{|A|}\\ \mu &\geqslant 2 - \frac{\ell + 2\alpha}{\sqrt{|A|}}.\end{aligned}$$

Proof. — Define S_0 to be a square that shares a corner of R and intersects its interior, and that have the same area than A (see figure 4). The definition of μ implies that $|A \cap S_0| \leqslant \mu|A|$.

If $\sqrt{|A|} \leqslant \ell - 2\alpha$ we have:

$$\begin{aligned}|A| &\leqslant \mu|A| + (\ell + 2\alpha)(\ell - 2\alpha - \sqrt{|A|}) + \sqrt{|A|}(\ell + 2\alpha - \sqrt{|A|})\\ &\leqslant \mu|A| + \ell^2 - 4\alpha^2 - \sqrt{|A|}(\ell + 2\alpha) + \sqrt{|A|}(\ell + 2\alpha) - |A|\\ &\leqslant \ell^2 - 4\alpha^2 + (\mu - 1)|A|\\ \mu|A| &\geqslant 2|A| - (\ell^2 - 4\alpha^2).\end{aligned}$$

Otherwise, we get

$$\begin{aligned}|A| &\leqslant \mu|A| + (\ell - 2\alpha)(\ell + 2\alpha - \sqrt{|A|})\\ &\leqslant \mu|A| + \sqrt{|A|}(\ell + 2\alpha - \sqrt{|A|})\\ \mu|A| &\geqslant 2|A| - (\ell + 2\alpha)\sqrt{|A|}\end{aligned}$$

□

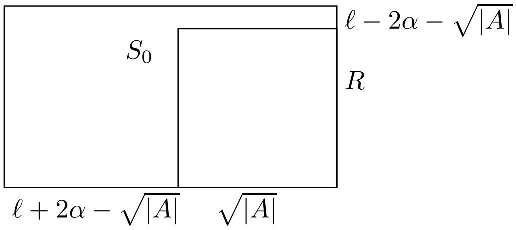

Figure 4. The domain A is included in R and cannot meet a too large proportion of S_0

If the first conclusion holds, using Lemma 2.2 it comes

$$\begin{aligned}\mu &\geqslant 2 - \frac{(\ell^2 - 4\alpha^2)(16 + \varepsilon)}{16\ell^2}\\ &\geqslant 2 - \frac{16 + \varepsilon}{16} = 1 - \frac{\varepsilon}{16}\end{aligned}$$

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If the second conclusion holds, using Lemma 2.2 we get

$$\begin{aligned}\mu &\geqslant 2-\frac{\ell}{\sqrt{|A|}}-\frac{2\alpha}{\sqrt{|A|}}\\ &\geqslant 2-\sqrt{1+\frac{\varepsilon}{16}}-\frac{\sqrt{\varepsilon}}{4}\\ &\geqslant 1-\frac{\sqrt{\varepsilon}}{4}+O(\varepsilon)\end{aligned}$$

But for all sufficiently small ε, this second expression is smaller than $1-\varepsilon/16$, and Proposition 1.2 is proved.

3. Sharpness

Two examples showing sharpness of Theorem 1.1 steam out from its proof.

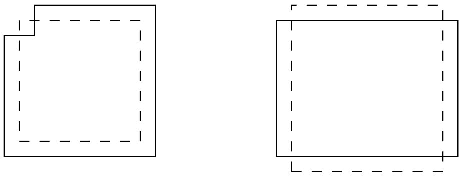

Figure 5. Two domain that are almost isopermetric and as far as possible from squares: a square with a small corner deleted and a rectangle with sides of almost the same lengths.

The first one is the domain S'_δ obtained from the unit square by deleting a $2\delta\times 2\delta$ square at one corner ($\delta<1/2$)). We have $|S'_\delta|=1-4\delta^2$ and $|\partial S'_\delta|=4$, so that (1) holds with

$$\varepsilon=\frac{64\delta^2}{1-4\delta^2}$$

and $\inf_S d_\infty(S,S'_\delta)=\delta$ so that equality holds in (2).

The second one is the rectangle R_α whose side length are $1-2\alpha$ and $1+2\alpha$ (where $\alpha<1/2$)). We have $|R_\alpha|=1-4\alpha^2$, $|\partial R_\alpha|=4$ and $\inf_S d_\infty(S,R_\alpha)=\alpha$ so that (2) is an equality once again.

Let us show that S'_δ and R_α are the only possible (up to homothety and L^1 isometry) exemple realizing equality in both (1) and (2) for the same ε. In the first case of Lemma 2.3, for $2\delta^2-2\alpha\delta+\alpha^2\leqslant\delta^2$ to be an equality it is necessary that $\alpha=\delta$, so that A must be equal to

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R (otherwise R would have smaller isoperimetric inequality and same distance to squares). In the second case of the lemma, one is lead to $\alpha = 0$ in the last lines of the proof of Theorem 1.1, so that R is a square and according to the proof of Lemma 2.3, A is contained in a S'_δ having the same isoperimetric deficit and the same minimal rectangle. They must therefore be equal.

At last, R_α shows asymptotic sharpness of Proposition 1.2:

$$\sup_{|S|=|R_\alpha|} |S \cap R_\alpha| = (1-2\alpha)\sqrt{1-4\alpha^2} = 1 - 2\alpha + o(\alpha)$$

and

$$1 - \frac{1}{4}\sqrt{\varepsilon} = 1 - 2\alpha + o(\alpha)$$

when ε takes the extremal value $64\alpha^2/(1-4\alpha^2)$.

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