ALMOST INDISCERNIBLE SEQUENCES AND CONVERGENCE OF CANONICAL BASES

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Abstract. Nous étudions et comparons trois notions de convergence de types dans une théorie stable : le convergence logique, c.à.d., formule par formule, la convergence métrique (toutes deux déjà bien étudiées) et la convergence des bases canoniques.

(i) Nous caractérisons les suites qui admettent des sous-suites presque indiscernables.
(ii) Nous étudions les théories pour lesquelles la convergence métrique coïncide avec la convergence des bases canoniques (*a priori* plus faible). Pour les théories $\aleph_0$-catégoriques nous caractérisons cette propriété par la $\aleph_0$-catégoricité de la théorie des belles paires associée. En particulier nous montrons que c'est le cas pour la théorie des espaces des variables aléatoires.
(iii) Utilisant ces outils nous donnons des preuves modèle théoriques à des résultats sur les suites des variables aléatoires figurant dans Berkes & Rosenthal [BR85].

We study and compare three notions of convergence of types in a stable theory: logic convergence, i.e., formula by formula, metric convergence (both already well studied) and convergence of canonical bases.

(i) We characterise sequences which admit almost indiscernible sub-sequences.
(ii) We study theories for which metric converge coincides with canonical base convergence (*a priori* weaker). For $\aleph_0$-categorical theories we characterise this property by the $\aleph_0$-categoricity of the associated theory of beautiful pairs. In particular, we show that this is the case for the theory of spaces of random variables.
(iii) Using these tools we give model theoretic proofs for results regarding sequences of random variables appearing in Berkes & Rosenthal [BR85].

Introduction

The motivation for the present paper comes from probability theory results of Berkes & Rosenthal [BR85]. These results have a strong model theoretic flavour to them: for example, the use of limit tail algebras (canonical bases of limit types), existence of almost exchangeable sequences (almost indiscernible sequences), distribution realisation (type realisation), compactness of the distribution space (type space compactness), and so on.

The appropriate model theoretic setting for this analysis is the continuous logic theory of atomless random variable spaces, ARV. Types in this theory correspond precisely to conditional distributions, and each of the notions of convergence of conditional distributions considered by Berkes & Rosenthal has a corresponding notion of convergence of types. It is easy to check that *weak convergence* of distributions corresponds to convergence in the logic topology (which is indeed the weakest natural topology on a type space). We also show that *strong convergence* of distributions corresponds to metric convergence of types, as well as to canonical base convergence. Indeed, showing that metric and canonical base convergence

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agree in ARV is an essential step in our proof that strong convergence corresponds to metric convergence. Modulo all these translations, the main theorem of [BR85] has a clear model theoretic counterpart, regarding existence of almost indiscernible sequences, which we prove in Section 2.

Generalising to an arbitrary stable theory, we see three topologies on the space of types which we compare:

(i) The logic topology is the weakest topology we consider (since it is compact, it is minimal among Hausdorff topologies).
(ii) The canonical base topology is defined in terms of convergence of the canonical bases of the types. It is stronger than the logic topology, and over a model it is strictly stronger.
(iii) The metric topology is defined in terms of convergence of realisations of types. It is the strongest of the three.

A theory for which the two last topologies agree is said to be SFB (strongly finitely based). Such theories are considered in Section 3. In particular, we prove a useful criterion for SFB under the assumption of $\aleph_0$-categoricity.

Theorem. *A stable theory T is $\aleph_0$-categorical and SFB if and only if the theory T_P of lovely pairs of models of T (as per Poizat* [Poi83]*) is $\aleph_0$-categorical.*

It follows easily that several familiar continuous theories, such as those of Hilbert spaces, probability algebras and random variable spaces, are SFB. On the other hand, it also follows from this theorem that the theory of atomless L_p Banach lattices is not SFB, contrary to the situation in classical logic where every $\aleph_0$-categorical, $\aleph_0$-stable theory is SFB (Zilber's Theorem). On the other hand, we do prove that the theory of beautiful pairs of L_p lattices is $\aleph_0$-categorical up to perturbation of the predicate defining the smaller structure. By analogy, one may say that the theory of L_p lattices is SFB up to perturbation. It is natural to ask whether every $\aleph_0$-stable, $\aleph_0$-categorical theory is SFB up to perturbation, and even to conjecture that this is always the case.

In Section 4 we go back to the article of Berkes & Rosenthal. This is where we establish a correspondence between probability theoretic notions and their model theoretic counterparts in the theory ARV, as alluded to above, using the fact that ARV is SFB. Under this translation, we prove several of Berkes and Rosenthal's results, including their main theorem, as special cases of model theoretic facts.

Throughout this paper we assume that T is a stable continuous theory. We assume that the reader is familiar with basic facts regarding stability and continuous logic, as presented in [BU]. For material regarding the theory ARV we refer the reader to [Benc]. Other background material includes Poizat [Poi83] for beautiful pairs and Pillay [Pil96] for Zilber's Theorem and its consequences for $\aleph_0$-categorical strongly minimal (and more generally, $\aleph_0$-stable) theories.

1. Convergence of types and canonical bases

Fix $m \in \mathbb{N}$, an m-tuple $\bar{x}$, and a set of parameters A. Then the space of types $\mathrm{S}_n(A)$ is equipped with the standard logic topology (i.e., the minimal topology in which all definable predicates are continuous), as well as with a metric

$$d(p, q) = \min\{d(\bar{a}, \bar{b}) \colon \bar{a} \vDash p \text{ and } \bar{b} \vDash q\}.$$

The distance between two finite tuples is defined as the maximum of the distances between coordinates. The metric on $\mathrm{S}_m(A)$ is stronger than the logic topology.

We obtain two notions of convergence in $\mathrm{S}_m(A)$: $p_n \to p$ and $p = \lim p_n$ will mean that the sequence $(p_n)_{n\in\mathbb{N}}$ converges to p in the logic topology, while $p_n \to^d p$ and $p = \lim^d p_n$ will mean convergence in the metric. Since the metric defines a stronger topology we have $p_n \to^d p \Longrightarrow p_n \to p$.

We can extend these notions of convergence to types of infinite tuples. The logic topology is defined as usual for spaces of types of infinite tuples, so there is nothing to worry about. On the other hand, there is no canonical metric on infinite tuples. Indeed, on uncountable tuples there is no (definable) metric at all. Instead we observe that in the finite case, $p_n \to^d p$ if and only they admit realisations $\bar{a}_n$ and $\bar{a}$ such that $\bar{a}_n \to \bar{a}$ in the product topology, i.e., such that $a_{n,k} \to a_k$ for all $k < m$, and we can define d-convergence of types of infinite tuples accordingly. In the case of countable tuples this does indeed correspond to a definable metric. Among the many equivalent possibilities, we (arbitrarily) choose

$$d(\bar{a}, \bar{b}) = \bigvee_{n\in\mathbb{N}} \frac{d(a_n, b_n)}{2^n}.$$

(Since the distance between any two singletons is at most 1, this converges to a value which is at most 1.)

Since we assume the theory to be stable, we can come up with yet another notion of convergence of stationary types (to be more precise, this is a notion of convergence of parallelism classes). Recall from [BU] that for every formula $\varphi(\bar{x}, \bar{y})$ there exists a definable predicate $d_{\bar{x}}\varphi(\bar{y}, Z)$, where $Z = (\bar{z}_n)_{n\in\mathbb{N}}$ consists of countably many copies of $\bar{x}$, such that for every stationary type $p(\bar{x})$ over a set A, its φ-definition is an instance $d_{\bar{x}}\varphi(\bar{y}, C)$ which is (equivalent to) an A-definable predicate. Moreover if p is over a model M then we can choose $C \subseteq M$, and (by [Bend]), if $(\bar{a}_n)_{n\in\mathbb{N}}$ is a Morley sequence in p then $C = (a_n)$ will do as well.

We define the *φ-canonical base* of p, denoted $\mathrm{Cb}_\varphi(p)$, as the canonical parameter of the definition $d_{\bar{x}}\varphi(\bar{y}, C)$. Let S_{Cb_φ} be the sort of canonical parameters of instances $d_{\bar{x}}\varphi(\bar{y}, Z)$. This sort is equipped with a natural metric: if c and c' are the canonical parameters of $d_{\bar{x}}\varphi(\bar{y}, C)$ and $d_{\bar{x}}\varphi(\bar{y}, C')$, respectively, then

$$d(c, c') = \sup_{\bar{y}} \bigl|d_{\bar{x}}\varphi(\bar{y}, C) - d_{\bar{x}}\varphi(\bar{y}, C')\bigr|.$$

Thus, if $M \vDash T$ and $p(\bar{x}), q(\bar{x}) \in \mathrm{S}_n(M)$, then:

$$d\bigl(\mathrm{Cb}_\varphi(p), \mathrm{Cb}_\varphi(q)\bigr) = \sup_{\bar{b}\in M} \bigl|\varphi(\bar{x}, \bar{b})^p - \varphi(\bar{x}, \bar{b})^q\bigr|.$$

Now let $\Phi(\bar{x})$ the set of all formulae of the form $\varphi(\bar{x}, \bar{y})$ (so $\bar{x}$ is fixed while $\bar{y}$ may vary with φ). The *canonical base* of $p(\bar{x})$ is defined as:

$$\mathrm{Cb}(p) = \bigl(\mathrm{Cb}_\varphi(p)\bigr)_{\varphi(\bar{x},\bar{y})\in\Phi(\bar{x})}.$$

This is usually viewed as a mere set (i.e., the minimal set to which p has a non-dividing stationary restriction), but we will rather view it as an infinite tuple indexed by $\Phi(\bar{x})$, i.e., living in the following infinite sort which only depends on the tuple $\bar{x}$:

$$S_{\mathrm{Cb}(\bar{x})} = \prod_{\varphi\in\Phi(\bar{x})} S_{\mathrm{Cb}_\varphi}.$$

Converge of infinite tuple is, as with finite tuples, defined through pointwise convergence:

$$\mathrm{Cb}\bigl(p_n(\bar{x})\bigr) \to \mathrm{Cb}\bigl(p(\bar{x})\bigr) \iff \mathrm{Cb}_\varphi(p_n) \to \mathrm{Cb}_\varphi(p) \text{ for all } \varphi \in \Phi(\bar{x})$$

Since the language is assumed to be countable, $\mathrm{Cb}(p)$ is a countable tuple. Let us enumerate $\Phi(\bar{x}) = \{\varphi_n\}_{n\in\mathbb{N}}$. As we pointed out above, the pointwise convergence topology (i.e., the product topology) is given by the metric:

$$d\bigl(\mathrm{Cb}(p), \mathrm{Cb}(q)\bigr) = \bigvee_n \frac{d\bigl(\mathrm{Cb}_{\varphi_n}(p), \mathrm{Cb}_{\varphi_n}(q)\bigr)}{2^n}. \tag{1}$$

In case the tuple of variables $\bar{x}$ is infinite there is no great difference. We define $\Phi(\bar{x})$ as the set of all formulae of the form $\varphi(\bar{x}', \bar{y})$, where $\bar{x}' \subseteq \bar{x}$ is a finite sub-tuple. For $\varphi \in \Phi(\bar{x})$ we may still define $\mathrm{Cb}_\varphi(p)$ as before (note only that the information contained in φ includes the precise sub-tuple $\bar{x}'$: otherwise we would need a more explicit notation such as $\mathrm{Cb}_{\varphi,\bar{x}'}\big(p(\bar{x})\big)$). We then define $\mathrm{Cb}(p) = \big(\mathrm{Cb}_\varphi(p)\big)_{\varphi\in\Phi(\bar{x})}$ and everything works essentially as in the finite case.

We can now sum up everything in the following definition.

Definition 1.1. Let $\big(p_n(\bar{x})\big)_{n\in\mathbb{N}}$ be a sequence of types in $\mathrm{S}_\alpha(A)$, and $p(\bar{x}) \in \mathrm{S}_\alpha(A)$ (α may be finite or infinite).

(i) If $\alpha = m < \omega$, we say that the sequence $(p_n)_n$ *d-converges* to p, in symbols $p_n \to^d p$ or $p = \lim^d p_n$, if p_n converges to p in the metric space $\big(\mathrm{S}_m(A), d\big)$. In case α is infinite, we say that $p_n \to^d p$ if for every finite sub-tuple $\bar{x}' \subseteq \bar{x}$, the restrictions to $\bar{x}'$ converge: $p_n{\restriction}_{\bar{x}'} \to^d p{\restriction}_{\bar{x}'}$.
It is not difficult to see that for $\alpha \leq \omega$ this is equivalent to the existence of realisations $\bar{a}_n \vDash p_n$ and $\bar{a} \vDash p$ such that $\bar{a}_n \to \bar{a}$.

(ii) If all the types are stationary then we say that the sequence $(p_n)_n$ Cb-*converges* to p, in symbols $p_n \to^{\mathrm{Cb}} p$ or $p = \lim^{\mathrm{Cb}} p_n$, if $\mathrm{Cb}(p_n) \to \mathrm{Cb}(p)$. If not all types are stationary, then $p_n \to^{\mathrm{Cb}} p$ if they have extensions to strong types over A, say q_n and q, respectively, such that $q_n \to^{\mathrm{Cb}} q$.

(iii) We say that the sequence $(p_n)_n$ *converges* to p, in symbols $p_n \to p$ or $p = \lim p_n$, if it converges in the standard logic topology, i.e., if $\varphi^{p_n} \to \varphi^p$ in $[0,1]$ for every formula $\varphi(\bar{x}) \in \mathcal{L}(A)$.

Remark 1.2. We have $p_n \to^d p$ if and only if they have extensions to strong types over A, say q_n and q, respectively, such that $q_n \to^d q$.

Proof. A converging sequence $a_n \to a$ which witnesses that $\mathrm{tp}(a_n/A) \to^d \mathrm{tp}(a/A)$ also witnesses that $\mathrm{stp}(a_n/A) \to^d \mathrm{stp}(a/A)$, and vice versa. $\blacksquare_{1.2}$

Lemma 1.3. *Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of types in $\mathrm{S}_\alpha(A)$ and $p \in \mathrm{S}_\alpha(A)$ for some $\alpha \leq \omega$. Then the following imply one another from top to bottom:*

(i) $p_n \to^d p$.
(ii) $p_n \to^{\mathrm{Cb}} p$.
(iii) $p_n \to p$.

(Of course, if $p_n \to^d p$ then $p_n \to p$ without having to assume that T is stable.)

Proof. By Remark 1.2 and the definition of $p_n \to^{\mathrm{Cb}} p$, we may assume that all types are strong.

(i) $\Longrightarrow$ (ii). Since $p_n \to^d p$, we can find realisations $\bar{a}_n \vDash p_n$ and $\bar{a} \vDash p$ such that $\bar{a}_n \to \bar{a}$. Let $C = (\bar{a}_n)_{n\in\mathbb{N}}$ and let $(C^j\bar{a}^j)_{j\in\mathbb{N}}$ be a Morley sequence in $\mathrm{tp}(C\bar{a}/A)$, $C^j = (\bar{a}^j_n)_n$. Then each sequence $(\bar{a}^j_n)_j$ is a Morley sequence in p_n, $(\bar{a}^j)_j$ is a Morley sequence in p. We also have $(\bar{a}^j_n)_j \to (\bar{a}^j)_j$, whereby $\mathrm{Cb}(p_n) \to \mathrm{Cb}(p)$.

(ii) $\Longrightarrow$ (iii). Let $\varphi(\bar{x}, \bar{b})$ be a formula with parameters in A, and let $d_{\bar{x}}\varphi(\bar{y}, Z)$ be the uniform $\varphi(x,y)$-definition. Then

$$\varphi(\bar{x}, \bar{b})^{p_n} = d_{\bar{x}}\varphi(\bar{b}, \mathrm{Cb}_\varphi(p_n)) \to d_{\bar{x}}\varphi(\bar{b}, \mathrm{Cb}_\varphi(p)) = \varphi(\bar{x}, \bar{b})^p.$$

As this hold for every formula over A, we have $p_n \to p$. $\blacksquare_{1.3}$

The converse implications do not hold in general.

2. Almost indiscernible sequences and sub-sequences

For simplicity of notation, we state all the results in this section to (sequences of) singletons, but the same holds for arbitrary tuples.

Definition 2.1. A sequence $(a_n)_{n\in\mathbb{N}}$ is *almost indiscernible* if there exists an indiscernible sequence $(b_n)_{n\in\mathbb{N}}$ in the same sort such that $d(a_n, b_n) \to 0$.

Lemma 2.2. *Let $(a_n)_{n\in\mathbb{N}}$ be an almost indiscernible sequence, say witnessed by an indiscernible sequence $(b_n)_{n\in\mathbb{N}}$, and let $B \supseteq (a_n)_n$. Then $p = \lim \operatorname{tp}(a_n/B)$ exists and is stationary, and $\operatorname{Cb}(p) \subseteq \operatorname{dcl}(a_n)_n$. Moreover, $(b_n)_{n\in\mathbb{N}}$ is a Morley sequence in $p\restriction_{\operatorname{Cb}(p)}$.*

Proof. Let $B' = B \cup \{b_n\}_n$. Since T is stable, $r = \lim \operatorname{tp}(b_n/B')$ exists and is stationary, and $(b_n)_n$ is a Morley sequence in $r\restriction_{\operatorname{Cb}(r)}$. Moreover, for every k and formula $\varphi(x, y)$, $\operatorname{Cb}_\varphi(r)$ is the canonical parameter for $d_x\varphi\big(y, (b_n)_{n\geq k}\big)$. Consider now an automorphism of an ambient monster model which fixes $(a_n)_n$. For k large enough, it will move the tail $(b_n)_{n\geq k}$ as little as we wish, and therefore fix $\operatorname{Cb}(r)$. Therefore $\operatorname{Cb}(r) \subseteq \operatorname{dcl}(a_n)_n \subseteq \operatorname{dcl}(B)$.

Clearly $\lim \operatorname{tp}(a_n/B) = \lim \operatorname{tp}(b_n/B) = r\restriction_B$, so in particular the first limit exists, call it p. Since $\operatorname{Cb}(r) \subseteq B$, the type p is stationary, $\operatorname{Cb}(p) = \operatorname{Cb}(r) \subseteq \operatorname{dcl}(a_n)_n$. Finally, $r\restriction_{\operatorname{Cb}(r)} = p\restriction_{\operatorname{Cb}(p)}$. $\blacksquare_{2.2}$

In a discrete sort, an almost indiscernible sequence is just one which is indiscernible from some point onward, so having an almost indiscernible sub-sequence is the same as having an indiscernible sub-sequence. In metric sorts, however, the two notions may differ and it is the weaker one (namely, having an almost indiscernible sub-sequence) which we shall study.

Definition 2.3. Let B be a set containing a sequence $(a_n)_{n\in\mathbb{N}}$. We say that $(a_n)_n$ satisfies $(*_B)$ if $p = \lim \operatorname{tp}(a_n/B)$ exists and is stationary, and for $C = \operatorname{Cb}(p)$ and $c \vDash p$ we have:

$$\operatorname{tp}(Ba_n/C) \to^d \operatorname{tp}(Bc/C)$$

If $B = \{a_n\}_n$ we omit it and say that $(a_n)_n$ satisfies $(*)$.

Remark 2.4. If $B \supseteq B'$ then $(*_B) \implies (*_{B'})$, by Lemma 2.2 (so in particular $(*_B) \implies (*)$). In addition, the condition $(*_B)$ is equivalent to:

> There exists a stationary type $p \in \mathrm{S}(B)$ such that if $C = \operatorname{Cb}(p)$ and $c \vDash p$ then $\operatorname{tp}(Ba_n/C) \to^d \operatorname{tp}(Bc/C)$.

Indeed, this already implies that $\operatorname{tp}(a_n/B) \to \operatorname{tp}(c/B)$.

Our main result concerning existence of almost indiscernible sub-sequences is the following:

Theorem 2.5. *If T is stable and the sequence $(a_n)_{n\in\mathbb{N}} \subseteq B$ has a sub-sequence satisfying $(*_B)$ then $(a_n)_{n\in\mathbb{N}}$ also has an almost indiscernible sub-sequence. If T is superstable then the converse holds as well.*

Moreover, if in addition $q = \lim \operatorname{tp}(a_n/B)$ exists then it is stationary and the sequence witnessing almost indiscernibility is Morley over $\operatorname{Cb}(q)$.

Proof. We may assume that the sequence $(a_n)_n$ satisfies $(*_B)$, and therefore $(*)$. Let $A = \{a_n\}_n \subseteq B$ and let $p = \lim \operatorname{tp}(a_n/A)$, $C = \operatorname{Cb}(p)$, $c \vDash p$, so $c \mathop{\smile\!\!\!\!\!\!\mid}_C A$.

We construct by induction on $i \in \mathbb{N}$ an increasing sequence $(n_i)_i$ and copies $A^i c^i$ of Ac, $A^i = \{a^i_n\}_n$, such that:

(i) $d(a^i_{n_j}, a^{i+1}_{n_j}) \leq \frac{1}{2^i}$ for $j < i$.

(ii) $d(c^i, a^{i+1}_{n_i}) \leq \frac{1}{2^i}$.

(iii) $A^i c^i \equiv_C Ac$.

(iv) $c^i \mathop{\smile\!\!\!\!\!\!\mid}_C c^{<i}$.

We start with $A^0 c^0 = Ac$. At the ith step we already have A^i, c^i, and $n_{<i}$. By $(*)$ there exists k such that:

$$d\big(\operatorname{tp}(a_{n_{<i}} a_k/C), \operatorname{tp}(a_{n_{<i}} c/C)\big) \leq \tfrac{1}{2^i}.$$

We let $n_i = k$, and we may assume that $n_i > n_j$ for $j < i$. Since $Ac \equiv_C A^i c^i$, there exists $A^{i+1} \vDash \mathrm{tp}(A/C)$ such that

$$d(a^{i+1}_{n\le i}, a^i_{n<i} c^i) \le \tfrac{1}{2^i}.$$

This takes care of the first two requirements. Choose $c^{i+1} \vDash p{\restriction}_C$ such that $c^{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_C A^{i+1} c^{\le i}$. Then the two last requirements are satisfied as well, and the construction may proceed.

For each i, the sequence $(a^j_{n_i})_j$ is Cauchy, converging to a limit b^i, and we have $d(c^i, b^i) \le 2^{-i+2}$. Also, the sequence $(c^i)_i$ is indiscernible (being a Morley sequence in $p{\restriction}_C$), and $(b^i)_i \equiv_C (a_{n_i})_i$. Thus (a_n) admits an almost indiscernible sub-sequence $(a_{n_i})_i$.

For the moreover part, we may again assume that the entire sequence $(a_n)_n$ satisfies $(*_B)$, since the limit type, if it exists, must be equal to the limit type of any sub-sequence. Then the statement follows from Lemma 2.2.

For the converse we assume that T is superstable, and we may further assume that $(a_n)_n$ is almost indiscernible as witnessed by an indiscernible sequence $(c_n)_n$. By Lemma 2.2, $p = \lim \mathrm{tp}(a_n/B)$ exists and is stationary, and $(c_n)_n$ is a Morley sequence over $C = \mathrm{Cb}(p)$. Let $c \vDash p$, so $c \mathop{\smile\!\!\!\!\!\!\mid}_C B$.

Fix a finite tuple $\bar b \subseteq B$, and $\varepsilon > 0$. By superstability there exists $n \in \mathbb{N}$ such that $\bar b^\varepsilon \mathop{\smile\!\!\!\!\!\!\mid}_{Cc_{<n}} c_n$. In other words, there exists $\bar b' \equiv_{\mathrm{acl}(Cc_{<n})} \bar b$ such that $d(\bar b, \bar b') \le \varepsilon$ and $\bar b' \mathop{\smile\!\!\!\!\!\!\mid}_{Cc_{<n}} c_n$. By transitivity, $\bar b' \mathop{\smile\!\!\!\!\!\!\mid}_C c_n$, in which case $\bar b' c_n \equiv_C \bar b c$. This proves that $\mathrm{tp}(Bc_n/C) \to^d \mathrm{tp}(Bc/C)$. Since $d(a_n, c_n) \to 0$, it follows that $\mathrm{tp}(Ba_n/C) \to^d \mathrm{tp}(Bc/C)$ as desired. $\blacksquare_{2.5}$

Below we shall relate Theorem 2.5 to the main theorem of Berkes and Rosenthal [BR85].

3. Strongly finitely based (SFB) theories and lovely pairs

3.1. **Comparing type space topologies.** Fix a model M of T, $n \in \mathbb{N}$, and an n-tuple $\bar x$. The type space $\mathrm{S}_n(M)$ is usually equipped with two topologies, the logic topology $\mathcal{T}_\mathcal{L}$ and the metric topology $\mathcal{T}_d$. In Definition 1.1 we introduced (for a stable theory T) a third topology $\mathcal{T}_{\mathrm{Cb}}$ of Cb-convergence, and we showed that $\mathcal{T}_\mathcal{L} \subseteq \mathcal{T}_{\mathrm{Cb}} \subseteq \mathcal{T}_d$. It is natural to ask whether these inclusions are proper. First of all it is fairly clear that in the uninteresting case where M is compact, all three topologies agree. We shall therefore assume that M is not compact.

Lemma 3.1. *The first inclusion $\mathcal{T}_\mathcal{L} \subsetneq \mathcal{T}_{\mathrm{Cb}}$ is proper.*

Proof. Let $a \in N \succeq M$, $a \notin M$, and let $\varepsilon = d(a, M)/2 > 0$. Let $p = \mathrm{tp}(a/M)$, and consider the following $\mathcal{T}_{\mathrm{Cb}}$-neighbourhood of p:

$$\begin{aligned} S &= \{q \colon d\big(\mathrm{Cb}_d(p), \mathrm{Cb}_d(q)\big) \le \varepsilon\} \\ &= \{q \colon |d(x,b)^p - d(x,b)^q| \le \varepsilon \quad \forall b \in M\} \\ &\subseteq \{\mathrm{tp}(a'/M) \colon d(a', M) \ge \varepsilon\}. \end{aligned}$$

In particular S contains no realised type, and therefore has empty interior in $\mathcal{T}_\mathcal{L}$. $\blacksquare_{3.1}$

We are left with the question of when $\mathcal{T}_{\mathrm{Cb}} = \mathcal{T}_d$.

Definition 3.2. We say that a theory T is *strongly finitely based (SFB)* if for every model $M \vDash T$ and every n, the topologies $\mathcal{T}_{\mathrm{Cb}}$ and $\mathcal{T}_d$ agree on $\mathrm{S}_n(M)$.

For every $p \in \mathrm{S}_n(M)$, and every formula $\varphi(\bar x, \bar y)$, the canonical parameter $\mathrm{Cb}_\varphi(p)$ belongs to an imaginary sort M_φ of M. Our definition of canonical bases can now be viewed as a mapping $\mathrm{Cb}_{M,n} \colon \mathrm{S}_n(M) \to \prod_{\varphi \in \Phi(\bar x)} M_\varphi$. Let us denote the range of $\mathrm{Cb}_{M,n}$ by $\mathcal{C}_n(M)$. This is the set of all canonical bases of n-types over M; equivalently, this is the set of all tuples (in the appropriate sorts) in M which can be obtained as the canonical base of a stationary n-type (over any set of parameters). In fact it can be seen that $\mathcal{C}_n(M)$ is type-definable in M, where the defining partial type does not depend on M. Since the type can be recovered from its canonical base, the mapping $\mathrm{Cb}_{M,n}$ is injective, and by definition $\mathrm{Cb}_{M,n}\colon (\mathrm{S}_n(M), \mathcal{T}_{\mathrm{Cb}}) \to \mathcal{C}_n(M)$ is a homeomorphism.

Proposition 3.3. *Assume that T is SFB. Then T is $\aleph_0$-stable.*

Proof. Let M be a separable model. Then $\prod_{\varphi\in\Phi(\bar{x})} M_\varphi$ is a separable metrisable space (as the countable product of such spaces). As any subset of a separable metrisable space is separable, $\mathcal{C}_n(M) \subseteq \prod_{\varphi\in\Phi(\bar{x})} M_\varphi$ is separable, and therefore $(\mathrm{S}_n(M), \mathcal{T}_{\mathrm{Cb}})$ is separable. Since T is SFB, $(\mathrm{S}_n(M), d)$ is separable. $\blacksquare_{3.3}$

3.2. **Beautiful pairs and SFB.** In this section we give a characterisation of $\aleph_0$-categorical SFB theories in terms of the corresponding theory of beautiful pairs. Let us recall a few facts regarding definable sets in continuous logic.

Definition 3.4. Let M be any structure, $X \subseteq M$ a possibly large subset, $A \subseteq M$ a set of parameters. We say that X is *(A-)definable* in M if it is closed and the predicate $d(x, X)$ is definable (over A).

Definable subsets of M^n are defined similarly.

Fact 3.5 ([Bena] or [BBHU08]). *Let M be a structure, $X \subseteq M$ a closed, possibly large subset, $A \subseteq M$ a set of parameters. Then the following are equivalent:*

(i) *The set X is A-definable.*
(ii) *For every A-definable predicate $\varphi(x, \bar{y})$, the predicate $\psi(\bar{y}) = \inf_{x\in X} \varphi(x, \bar{y})$ is A-definable as well.*

In addition, every compact set is definable, and the union of any two definable sets is definable (by $d(x, X \cup Y) = d(x, X) \wedge d(x, Y)$).

(On the other hand, the intersection of two definable sets need not in general be definable.)

Let us also recall the following result, due to the third author. For a proof see [BU07].

Fact 3.6 (Ryll-Nardzewski Theorem for metric structures)**.** *Let T be a theory in a countable language. Then the following are equivalent:*

(i) *T is $\aleph_0$-categorical, i.e., admits a unique separable model up to isomorphism.*
(ii) *T is complete and for each $m \in \mathbb{N}$, the metric topology and the logic topology on $\mathrm{S}_m(T)$ agree.*
(iii) *T is complete and the metric topology and the logic topology on $\mathrm{S}_\omega(T)$ agree.*

In particular, if X is any type-definable set (without parameters) in an $\aleph_0$-categorical theory then the predicate $d(x, X)$ is metrically continuous. Thus the function $\mathrm{tp}(x) \mapsto d(x, X)$ is logically continuous, namely, a definable predicate, so X is definable.

(Notice that the separable models include any possible compact model of T, so $\aleph_0$-categoricity implies completeness by Vaught's Test.)

Concerning beautiful pairs we shall be brief, following [Poi83] fairly closely. We define an *elementary pair* of models of T to be a pair (M, N), where $N \prec M \vDash T$. We view such a pair as a structure (M, P) in $\mathcal{L}_P = \mathcal{L} \cup \{P\}$, where P is a new 1-Lipschitz unary predicate symbol measuring the distance to N, and we may also write $N = P(M)$. A *beautiful pair* of models of T is an elementary pair (M, P) such that $P(M)$ is $|\mathcal{L}|^+$-saturated, and M is $\aleph_0$-saturated over $P(M)$. We define T_P as the $\mathcal{L}_P$-theory of all beautiful pairs of models of T. If saturated models of T_P are not beautiful pairs (which may happen, for example, if T is a classical stable theory with the finite cover property) then continuous first order logic is not adequate for the consideration of the class of beautiful pairs. (On the other hand, *positive logic* always provides an adequate framework, see [Ben04].) If saturated models of T_P *are* beautiful pairs then continuous first order logic is adequate and we shall say that the class of beautiful pairs of models of T is *almost elementary.*

Lemma 3.7. *The class of elementary pairs of models of T is elementary. We shall denote its theory by $T_{P,0}$.*

Proof. Let $P(M)$ denote the zero set of P. The property $P(x) = d\big(x, P(M)\big)$ is elementary by [BBHU08, Theorem 9.12]. Modulo this property we may quantify over P (uniformly – see also [Bena, Proposition 1.8]), and we may express that $P(M) \preceq M$ by

$$\sup_{\bar{x}\in P} \left| \sup_{y} \varphi(y,\bar{x}) - \sup_{y\in P} \varphi(y,\bar{x}) \right| = 0.$$

$\blacksquare_{3.7}$

We need an analogue of [Poi83, Théorème 3]:

Lemma 3.8. *Let $A \subseteq M \vDash T$, $p \in \mathrm{S}_n(A)$, and let $\varphi(\bar{x},\bar{y})$ be a formula where $|\bar{x}| = n$. Let $p{\restriction}^M \subseteq \mathrm{S}_n(M)$ be the family of all non-dividing extensions of p. Then there exists an A-definable predicate $\psi(\bar{y})$ such that for all $\bar{b} \in M$:*

$$\psi(\bar{b}) = \sup_{q\in p\restriction^M} \varphi(\bar{x},\bar{b})^q.$$

Proof. Let $q_0 \in \mathrm{S}_n(M)$ be any non-dividing extension of p to M, and let $\chi(\bar{y}, c_0)$ be its φ-definition, where $c_0 = \mathrm{Cb}_\varphi(q_0)$ is the canonical parameter. Let C be the orbit of c_0 over A. Then C is compact, and by Fact 3.5 the predicate $\psi(\bar{y}) = \sup_{c\in C} \chi(\bar{y}, C)$ is definable. It is moreover A-invariant, and therefore definable over A.

On the other hand, the φ-definitions of the other non-dividing extensions of p to M are precisely those of the form $\chi(\bar{y}, c)$ where $c \in C$, whence the desired identity. $\blacksquare_{3.8}$

Let $\varphi(\bar{x},\bar{y})$ and $\psi(\bar{y},\bar{z})$ be formulae, and let

$$\pi_{\varphi,\psi}(\bar{z}) = \left\{ \sup_{\bar{y}_0,\ldots,\bar{y}_{n-1}} \inf_{\bar{x}} \bigvee_{i<n} \big(\varphi(\bar{x},\bar{y}_i) \dot{-} \psi(\bar{y}_i,\bar{z})\big) = 0 \right\}_{n\in\mathbb{N}}.$$

In other words, $M \vDash \pi_{\varphi,\psi}(\bar{c})$ if and only if the following partial type is consistent (and therefore realised in an elementary extension of M):

$$\big\{\varphi(\bar{x},\bar{b}) \leq \psi(\bar{b},\bar{c})\big\}_{\bar{b}\in M}.$$

Lemma 3.9. *Assume that the set defined by $\pi_{\varphi,\psi}$ is definable in models of T. Then the following is true in every beautiful pair of models of T.*

$$\sup_{\bar{z}\vDash\pi_{\varphi,\psi}} \inf_{\bar{x}} \sup_{\bar{y}\in P\cup\{\bar{z}\}} \big(\varphi(\bar{x},\bar{y}) \dot{-} \psi(\bar{y},\bar{z})\big) = 0. \tag{2}$$

(The set $P \cup \bar{z}$ is definable as the union of a definable set with a compact set.)

Proof. Let $\bar{c} \in M$ realise $\pi_{\varphi,\psi}$. Then the partial type $\big\{\varphi(\bar{x},\bar{b}) \leq \psi(\bar{b},\bar{c})\big\}_{\bar{b}\in P\cup\bar{c}}$ is consistent. Since M is $\aleph_0$-saturated over P, this partial type is realised by some $\bar{a} \in M$. $\blacksquare_{3.9}$

Lemma 3.10. *Assume again that the set defined by $\pi_{\varphi,\psi}$ is definable in models of T. Let T'_P consist of $T_{P,0}$ (namely the axioms saying that P is the distance to $P(M) \preceq M \vDash T$), as well as of all instances of (2). Then every $\aleph_1$-saturated model of T'_P is a beautiful pair.*

Proof. Let $(M, P) \vDash T'_P$ be $\aleph_1$-saturated. We only need to prove that M is $\aleph_0$-saturated over P. Let $\bar{c} \in M$ and let $p(\bar{x}) \in \mathrm{S}_n(P\bar{c})$. Fix a formula $\varphi(\bar{x},\bar{y})$. By Lemma 3.8 there exists a definable predicate $\psi_\varphi(\bar{y})$ with parameters in $P \cup \bar{c}$ such that for all $\bar{b} \in M$,

$$\psi_\varphi(\bar{b}) = \sup_{q\in p\restriction^M} \varphi(\bar{x},\bar{b})^q.$$

It follows that $\{\varphi(\bar{x},\bar{b}) \leq \psi_\varphi(\bar{b})\}_{\bar{b}\in M}$ is consistent. For $k \in \mathbb{N}$ choose a formula $\psi_{\varphi,k}(\bar{y},\bar{c}_k)$ which approximates $\psi_\varphi(\bar{y})$ up to 2^{-k}, where $\bar{c} \subseteq \bar{c}_k \subseteq P \cup \bar{c}$ (and $\psi_{\varphi,k}(\bar{y},\bar{z}_k)$ is parameter-free). We may further do this in such a manner that $\psi_\varphi(\bar{y}) \leq \psi_{\varphi,k}(\bar{y},\bar{c}_k)$. Thus $\{\varphi(\bar{x},\bar{b}) \leq \psi_{\varphi,k}(\bar{b},\bar{c}_k)\}_{\bar{b}\in M}$ holds as well. By our axioms, the conditions

$$\sup_{\bar{y}\in P\cup\bar{c}} \big(\varphi(\bar{x},\bar{y}) \dot{-} \psi_{\varphi,k}(\bar{y},\bar{c}_k)\big) = 0$$

are approximately realised in (M,P), and therefore so is the condition

$$(*_\varphi) \qquad \sup_{\bar{y}\in P\cup\bar{c}} \big(\varphi(\bar{x},\bar{y}) \dot{-} \psi_\varphi(\bar{y})\big) = 0.$$

We claim that every two conditions of the form $(*_\varphi)$ are a consequence of a third one of the same kind. We do this by the same coding trick as the one used in [BU]. So consider two formulae $\varphi_i(\bar{x},\bar{y})$, $i = 0,1$. For the sake of simplicity we shall assume that in every model of T there exist two elements of distance one, and define

$$\varphi(\bar{x},\bar{y},w,u) = \big[\neg d(w,u) \wedge \varphi_0(\bar{x},\bar{y})\big] \vee \big[d(w,u) \wedge \varphi_1(\bar{x},\bar{y})\big].$$

Now let $\bar{a} \in M$ realise $(*_\varphi)$ (such $\bar{a}$ exists by saturation) and let $t,s \in P$, $d(t,s) = 1$. Thus $\varphi_0(\bar{x},\bar{y}) = \varphi(\bar{x},\bar{y},t,t)$ and $\varphi_1(\bar{x},\bar{y}) = \varphi(\bar{x},\bar{y},t,s)$, whereby $\psi_\varphi(\bar{y},t,t) = \psi_{\varphi_0}(\bar{y})$ and $\psi_\varphi(\bar{y},t,s) = \psi_{\varphi_1}(\bar{y})$. Therefore, for every $\bar{b} \in P \cup \bar{c}$:

$$\varphi_0(\bar{a},\bar{b}) \dot{-} \psi_{\varphi_0}(\bar{b}) = \varphi(\bar{a},\bar{b},t,t) \dot{-} \psi_\varphi(\bar{b},t,t) = 0,$$
$$\varphi_1(\bar{a},\bar{b}) \dot{-} \psi_{\varphi_1}(\bar{b}) = \varphi(\bar{a},\bar{b},t,s) \dot{-} \psi_\varphi(\bar{b},t,s) = 0.$$

Proceeding in this manner, we see that the collection of all the conditions of the form $(*_\varphi)$ is consistent, and by saturation realised in (M,P), say by $\bar{a}$. Then for every formula $\varphi(\bar{x},\bar{y})$ and every $\bar{b} \in P \cup \bar{c}$ we have

$$\varphi(\bar{a},\bar{b}) \leq \psi_\varphi(\bar{b}) = \sup_{q\in p\restriction^M} \varphi(\bar{x},\bar{b})^p = \varphi(\bar{x},\bar{b})^p.$$

Replacing φ with $\neg\varphi$ we obtain $\varphi(\bar{a},\bar{b}) = \varphi(\bar{x},\bar{b})^p$, i.e., $\bar{a} \vDash p$, as desired. $\blacksquare_{3.10}$

Proposition 3.11. *Assume T is $\aleph_0$-categorical (and stable, as is assumed throughout this paper). Then the class of beautiful pairs is almost elementary.*

Proof. Since T is $\aleph_0$-categorical, the set defined by $\pi_{\varphi,\psi}$ above is definable. The assertion now follows from the two preceding Lemmas, and we obtain in addition that $T'_P \equiv T_P$. $\blacksquare_{3.11}$

Lemma 3.12. *Let (M,P) be a pair of models, $\bar{x}$ an n-tuple, and $\varphi(\bar{x},\bar{y}) \in \Phi(\bar{x})$ a formula. Then the mapping $\bar{a} \mapsto \mathrm{Cb}_\varphi(\bar{a}/P)$ is uniformly definable in (M,P), i.e., its graph is definable by a partial type which does not depend on (M,P).*

Proof. The graph of $z = \mathrm{Cb}_\varphi(\bar{x}/P)$ is defined by:

$$P(z) = 0 \quad \& \quad \sup_{\bar{y}\in P}\big|\varphi(\bar{x},\bar{y}) - d_{\bar{x}}\varphi(\bar{y},z)\big| = 0. \qquad \blacksquare_{3.12}$$

It follows that for every m we have a uniformly definable mapping $\theta\colon (M,P)^m \to \mathcal{C}_m(P)$ inducing a continuous function $\hat{\theta}\colon \mathrm{S}_m(T_P) \to \mathrm{S}_{\mathrm{Cb}(m)}(T)$ given as follows (here $\mathrm{S}_{\mathrm{Cb}(m)}(T)$ is the space of types in the sort $S_{\mathrm{Cb}(\bar{x})}$, where $\bar{x}$ is an m-tuple).

$$(3) \qquad \begin{aligned} \theta\colon &\quad \bar{a} &\mapsto&\quad \mathrm{Cb}(\bar{a}/P), \\ \hat{\theta}\colon &\quad \mathrm{tp}^{\mathcal{L}_P}(\bar{a}) &\mapsto&\quad \mathrm{tp}^{\mathcal{L}}\big(\mathrm{Cb}(\bar{a}/P)\big) = \mathrm{tp}\big(\theta(\bar{a})\big). \end{aligned}$$

Fact 3.13. *Let (M, P) and (N, P) be two beautiful pairs of models of T and let $\bar{a} \in M$ and $\bar{b} \in N$ be two tuples of the same length. Let $C = \theta(\bar{a}) = \mathrm{Cb}(\bar{a}/P(M))$ and $D = \theta(\bar{b})$. Then $\mathrm{tp}^{\mathcal{L}_P}(\bar{a}) = \mathrm{tp}^{\mathcal{L}_P}(\bar{b})$ if and only if $\mathrm{tp}(C) \equiv \mathrm{tp}(D)$.*

Proof. As in [Poi83, Théorème 4] (notice that $\mathrm{tp}(C) \equiv \mathrm{tp}(D)$ if and only if $\mathrm{tp}^M(\bar{a}/P(M))$ and $\mathrm{tp}^N(\bar{b}/P(N))$ are equivalent in the fundamental order of T). $\blacksquare_{3.13}$

Proposition 3.14. *Assume the class of beautiful pairs of models of T is almost elementary. Then the mapping $\hat{\theta}$ defined above is a homeomorphic embedding.*

Proof. Since $\hat{\theta}$ is a continuous mapping from a compact space into a Hausdorff space, all we need to show is that it is injective. So let $(M, P), (N, P) \vDash T_P$, $\bar{a} \in M$, $\bar{b} \in N$, and $C = \theta(\bar{a})$, $D = \theta(\bar{b})$. Then we need to show that $C \equiv^{\mathcal{L}} D \Longrightarrow \bar{a} \equiv^{\mathcal{L}_P} \bar{b}$. We may replace both (M, P) and (N, P) by $|\mathcal{L}|^+$-saturated elementary extensions. By assumption (M, P) and (N, P) are beautiful pairs and we may apply Fact 3.13. $\blacksquare_{3.14}$

Theorem 3.15. *Let T be any stable continuous first order theory. Then T_P is $\aleph_0$-categorical if and only if T is $\aleph_0$-categorical and SFB.*

Proof. Assume first that T_P is $\aleph_0$-categorical. Then clearly T is $\aleph_0$-categorical (indeed, if $\mathrm{S}_m(T_P)$ is metrically compact then so is $\mathrm{S}_m(T)$).

So fix $m \in \mathbb{N}$ and let $\bar{x}$ be an m-tuple. We shall in fact prove a uniform version of SFB, namely that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $N \vDash T$ and $p, q \in \mathrm{S}_m(N)$ are such that $d(\mathrm{Cb}(p), \mathrm{Cb}(q)) < \delta$ (where the distance between canonical bases is as defined in (1)), then $d(p, q) \leq \varepsilon$. For simplicity of notation we shall assume that $m = 1$ and drop the bars.

Recall from Lemma 3.12 that the mapping $\theta \colon a \mapsto \mathrm{Cb}(a/P)$ is uniformly definable in T_P. Let $r(x, y)$ be the partial $\mathcal{L}_P$-type saying that $x \equiv_P y$. Since T_P is $\aleph_0$-categorical, the distance $d(xy, r)$ is a definable predicate. Consider now the partial $\mathcal{L}_P$-type consisting of $\{d(xy, r) \geq \varepsilon/2\} \cup \{d\big(\theta(x), \theta(y)\big) < \delta\}_{\delta > 0}$. This partial type is contradictory, whence we obtain a $\delta > 0$ such that

$$d\big(\theta(x), \theta(y)\big) < \delta \vdash d(xy, r) < \varepsilon/2.$$

We claim that this δ is as required, i.e., if $N \vDash T$, $p, q \in \mathrm{S}_1(N)$, and $d\big(\mathrm{Cb}(p), \mathrm{Cb}(q)\big) < \delta$, then $d(p, q) \leq \varepsilon$. Indeed, passing to an elementary extension and taking non-dividing extensions of the types we may assume that N is $|\mathcal{L}|^+$-saturated, and then find $M \succ N$ which is $|N|^+$-saturated, so $(M, N) = (M, P) \vDash T_P$ is a beautiful pair. Let $C = \mathrm{Cb}(p)$, $D = \mathrm{Cb}(q)$, so $d(C, D) < \delta$.

By our saturation assumption there exist $a, b \in M$ such that $a \vDash p$ and $b \vDash q$, so $\theta(a) = C$, $\theta(b) = D$, and therefore $d(ab, r) < \varepsilon/2$. In other words, there exist $a'b' \in M$ such that $d(ab, a'b') < \varepsilon/2$ and $a'b' \vDash r$, i.e., $\mathrm{tp}(a'/N) = \mathrm{tp}(b'/N) = p'$, say, and

$$d(p, q) \leq d(p, p') + d(p', q) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Conversely, assume that T is $\aleph_0$-categorical and is SFB. By the metric Ryll-Nardzewsky Theorem, we need to show that for each m, the logic topology and the metric topology on $\mathrm{S}_m(T_P)$ coincide. In other words, we need to show that if $p_n \to p$ in $\mathrm{S}_m(T_P)$, then $p_n \to^d p$ there.

Assume then that $p_n \to p$. Since T is $\aleph_0$-categorical, the class of beautiful pairs of models of T is almost elementary (Proposition 3.11). By Proposition 3.14, the mapping $\hat{\theta} \colon \mathrm{tp}^{\mathcal{L}_P}(a) \mapsto \mathrm{tp}^{\mathcal{L}}\big(\mathrm{Cb}(a/P)\big)$ is a topological embedding, so $\hat{\theta}(p_n) \to \hat{\theta}(p)$, and since T is $\aleph_0$-categorical we have $\hat{\theta}(p_n) \to^d \hat{\theta}(p)$. In other words, in a sufficiently saturated model $N \vDash T$ we can find infinite tuples $C_n \vDash \hat{\theta}(p_n)$ and $C \vDash \hat{\theta}(p)$ such that $C_n \to C$.

Write $C = \{c_\varphi\}_{\varphi \in \Phi(\bar{x})}$, and let $q \in \mathrm{S}_m(N)$ be the unique type over N such that $\mathrm{Cb}(q) = C$, i.e.,

$$\varphi(x, b)^q = d_x\varphi(b, c_\varphi), \qquad b \in N, \varphi \in \Phi(x).$$

Define $q_n \in \mathrm{S}_m(N)$ such that $\mathrm{Cb}(q_n) = C_n$ similarly. Then $q_n \to^{\mathrm{Cb}} q$ by definition, and since T is SFB, $q_n \to^d q$. Let $a_n \vDash q_n$ and $a \vDash q$ witness this, so $a_n \to a$ in some $M \succeq N$, which we may assume to be $|N|^+$-saturated, so (M, N) is a beautiful pair. Write it rather as $(M, P) \vDash T_P$ (where $P(M) = N$). Then $\theta(a) = C \vDash \hat{\theta}(p)$ implies $p = \mathrm{tp}^{\mathcal{L}_P}(a)$, and similarly $\mathrm{tp}^{\mathcal{L}_P}(a_n) = p_n$. Thus $a_n \to a$ witnesses that $p_n \to p$, and the proof is complete. $\blacksquare_{3.15}$

3.3. **Examples.** First, we consider a class of examples which relates our SFB property with classical first order logic. Recall an important corollary of Zilber's Theorem regarding $\aleph_0$-categorical strongly minimal sets.

Fact 3.16 ([Pil96, Theorem 5.12])**.** *An $\aleph_0$-categorical, $\aleph_0$-stable classical theory is one-based.*

Proposition 3.17. *Let T be a classical $\aleph_0$-categorical (and stable) theory. Then the following are equivalent:*

(i) *T is SFB.*
(ii) *T is $\aleph_0$-stable.*
(iii) *T is one based.*
(iv) *T is finitely based (meaning that for every m there exists k such that every indiscernible sequence of m-tuples, is a Morley sequence over its first k elements).*

Proof. (i) $\Longrightarrow$ (ii). We have already seen that SFB implies $\aleph_0$-stability.

(ii) $\Longrightarrow$ (iii). By Fact 3.16.

(iii) $\Longrightarrow$ (iv). Immediate ($k = 1$).

(iv) $\Longrightarrow$ (i). Let us fix $m = 1$ and the corresponding k. Then the type of an indiscernible sequence (of singletons) is determined by the type of the first $k+1$ members of that sequence, so only finitely many types of indiscernible sequences exist. On the other hand, if $(M, P) \vDash T_P$ and $a \in M$, then $\mathrm{tp}^{\mathcal{L}_P}(a)$ is determined by the $\mathcal{L}$-type of $\mathrm{Cb}(a/P)$, which in turn is determined by the type of a Morley sequence in $\mathrm{tp}(a/P)$. We conclude that $\mathrm{S}_1(T_P)$ is finite, and by similar reasoning so is $\mathrm{S}_m(T_P)$ for all m. Therefore T_P is $\aleph_0$-categorical, so T is SFB by Theorem 3.15. $\blacksquare_{3.17}$

This result does not generalise to the case of a stable $\aleph_0$-categorical continuous theory. For the direction "one-based $\Longrightarrow$ SFB" we merely observe that the proof given above does not carry over to the metric setting. For the direction "SFB $\Longrightarrow$ one-based" we present below several counter-examples.

Example 3.18. The theory of infinite dimensional Hilbert spaces IHS is SFB.

Proof. Let IHS'_P consist $IHS_{P,0}$ together with the axiom scheme expressing, for each k, that there exist k orthonormal vectors which are orthogonal to P (we leave the details to the reader, pointing out that since P is definable modulo IHS_P, one may quantify over it). It is then not difficult to check that every beautiful pair of models of IHS is a model of IHS'_P, so $IHS'_P \subseteq IHS_P$. On the other hand, IHS'_P admits a unique separable model $(H \oplus H_1, H)$ where $H \cong H_1 \vDash IHS$ are separable. Thus IHS'_P is complete, so $IHS'_P = IHS_P$, and IHS is SFB by Theorem 3.15. $\blacksquare_{3.18}$

Example 3.19. The theory of atomless probability algebras APr is SFB.

Proof. The argument is essentially the same as above. We define APr'_P to consist of $APr_{P,0}$ along with the axiom saying that M is atomless over P, expressible as

$$\sup_x \inf_y \sup_{z \in P} \left| \tfrac{1}{2}\mu[x \cap z] - \mu[x \cap y \cap z] \right| = 0.$$

Then again every beautiful pair is a model of APr'_P and APr'_P admits a unique separable model, namely $(\mathfrak{B}(X \times Y), \mathfrak{B}(X))$ where $X = Y = [0, 1]$ equipped with the Lebesgue measure, and the embedding $\mathfrak{B}(X) \hookrightarrow \mathfrak{B}(X \times Y)$ is induced by the projection $X \times Y \twoheadrightarrow X$. $\blacksquare_{3.19}$

Both of the examples above are $\aleph_0$-categorical and $\aleph_0$-stable, so it is natural to expect them to satisfy some continuous analogue of finite basedness. It is not difficult to verify that none of them is one-based. In fact, no known continuous stable theory is one based, except for those constructed trivially from classical ones. Given the examples above, and in analogy with Proposition 3.17, it stands to reason to contend that at least for $\aleph_0$-categorical theories, SFB is the correct continuous logic analogue of a classical one-based theory, and one may further formalise it as a conjecture:

Conjecture 3.20 (Zilber's Theorem for continuous logic, naïve version)**.** Every $\aleph_0$-categorical $\aleph_0$-stable theory is SFB.

Unfortunately, this conjecture has an easy counterexample:

Example 3.21. The theory $ALpL$ of atomless L_p Banach lattices for $p \in [1, \infty)$ (see [BBH]) is not SFB.

Indeed, a model of $ALpL_P$ is of the form $\big(L_p(X, \mathfrak{B}_X, \mu_X), L_p(Y, \mathfrak{B}_Y, \mu_Y)\big)$, where $\mathfrak{B}_Y \subseteq \mathfrak{B}_X$ (so in particular $Y \subseteq X$) and $\mu_Y = \mu_X\restriction_{\mathfrak{B}_Y}$, such that in addition μ_Y is atomless and μ_X is atomless over $\mathfrak{B}_Y$. The theory $ALpL_P$ has precisely two non isomorphic separable models, one where $Y = X$ and the other where $\mu(X \smallsetminus Y) > 0$.

We may construct them explicitly as $\big(L_p(X \times Y), L_p(X)\big)$ and $\big(L_p(Z \times Y), L_p(X)\big)$, where $X = Y = [0, 1] \subseteq Z = [0, 2]$ are equipped with the Lebesgue measure, the embedding $L_p(X) \subseteq L_p(X \times Y)$ is given by $f'(x, y) = f(x)$ and $L_p(X \times Y) \subseteq L_p(Z \times Y)$ is given by $f'(w) = f(w)$ for $w \in X \times Y$, $f'(w) = 0$ otherwise.

It is worthwhile to point out that this last example is disturbing on several other "counts":

- It is a counter-example for Vaught's no-two-models theorem in continuous logic.
- Since $ALpL$ is $\aleph_0$-stable, $ALpL_P$ is superstable by [Ben06], and we get a counter-example to Lachlan's theorem on the number of countable models of a first order superstable theory.

Nonetheless, one may still hope to recover a version of Zilber's Theorem for continuous logic using the notion of perturbations of metric structures (as introduced in [Benb, Ben08]). Natural considerations suggest that whenever adding symbols to a language (especially to the language of an $\aleph_0$-categorical theory) one should also study the expanded structures up to arbitrarily small perturbations of the new symbol. Thus, the question should not be whether $ALpL_P$ is $\aleph_0$-categorical, but rather, whether it is $\aleph_0$-categorical up to small perturbations of the predicate P (the positive non-perturbed results for IHS_P and APr_P should be viewed witnessing the exceptional structural simplicity of these theories).

Proposition 3.22. *The theory $ALpL_P$ is $\aleph_0$-categorical up to arbitrarily small perturbations of P.*

Proof. We need to show that if $(M, P), (N, P) \vDash ALpL_P$ are separable then there exists an isomorphism $\rho\colon M \to N$ such that $|d(f, P) - d(\rho(f), P)| < \varepsilon$ for all $f \in M$. Since $ALpL_P$ has precisely two non-isomorphic separable models, it will suffice to show this for those two models.

Let N, M_1 and M_2 be the closed unit balls of $L_p([0, 1])$, $L_p([0, 1] \times [0, 1])$ and $L_p([0, 2] \times [0, 1])$, respectively (with the Lebesgue measure). As in the example above, we consider that $N \subseteq M_1 \subseteq M_2$. In particular, N is the set of all $g \in M_1$ such that the value of $g(x, y)$ depends only on x. Then the two non-isomorphic models are (M_1, N) and (M_2, N).

Define $\rho_1\colon L_p([0, 1] \times [0, 1]) \to L_p([0, 1] \times [\varepsilon, 1])$ and $\rho_2\colon L_p([1, 2] \times [0, 1]) \to L_p([0, 1] \times [0, \varepsilon])$ by:

$$(\rho_1 f)(x, y) = (1 - \varepsilon)^{-1/p} f\big(x, (y - \varepsilon)/(1 - \varepsilon)\big)$$
$$(\rho_2 f)(x, y) = \varepsilon^{-1/p} f(x + 1, y/\varepsilon).$$

Then ρ_1 and ρ_2 are isomorphisms of Banach lattices, which can be combined into an isomorphism $\rho = \rho_1 \oplus \rho_2\colon L_p([0, 2] \times [0, 1]) \to L_p([0, 1] \times [0, 1])$. This restricts to an isomorphism of the unit balls which will also be denoted by $\rho\colon M_2 \to M_1$. Let also $D = [0, 1] \times [0, \varepsilon]$ and $E = [0, 1] \times [\varepsilon, 1]$, namely the supports of the images of ρ_2 and ρ_1, respectively.

We claim that $\rho{\restriction}_N \colon N \to M_1$ is not too far from the identity. Indeed, let $g \in N$. Then $\|g\| \leq 1$, and we can write it as a function of the first coordinate $g(x)$. Then $\rho(g) = \rho_1(g)$ can be written as $(1-\varepsilon)^{-1/p} g(x)\chi_E(x,y)$. For $r \in [0,1]$ let:

$$\zeta(r) = 1 - (1-r)^{1/p} + r^{1/p}.$$

Then:

$$\begin{aligned}\|g - \rho(g)\| &\leq \|g\chi_E - (1-\varepsilon)^{-1/p} g\chi_E\| + \|g\chi_D\| \\ &= \Big((1-\varepsilon)^{-1/p} - 1\Big)\|g\chi_E\| + \|g\|\varepsilon^{1/p} \\ &= \|g\|\Big((1-\varepsilon)^{-1/p} - 1\Big)(1-\varepsilon)^{1/p} + \|g\|\varepsilon^{1/p} \\ &= \|g\|\zeta(\varepsilon) \leq \zeta(\varepsilon).\end{aligned}$$

Now let $f \in M_2$. Then:

$$\begin{aligned}\big|\|f-g\| - \|\rho(f) - g\|\big| &= \big|\|\rho(f) - \rho(g)\| - \|\rho(f) - g\|\big| \\ &\leq \|g - \rho(g)\| \leq \zeta(\varepsilon).\end{aligned}$$

Fixing $f \in M_2$ while letting $g \in N$ vary, we conclude that:

$$\begin{aligned}\big|d(\rho f, P)^{(M_1,N)} - d(f,P)^{(M_2,N)}\big| &= \big|d(f,N) - d(\rho(f),N)\big| \\ &\leq \sup_{g\in N}\big|\|f-g\| - \|\rho(f)-g\|\big| \\ &\leq \zeta(\varepsilon).\end{aligned}$$

Since ζ is continuous and $\zeta(0) = 0$, by taking $\varepsilon > 0$ small enough we can get $\rho\colon (M_2,N) \to (M_1,N)$ to be as small a perturbation of the predicate $P(x) = d(x,N)$ as we wish. $\blacksquare_{3.22}$

We therefore propose the following:

Conjecture 3.23 (Zilber's Theorem for continuous logic)**.** Whenever T is an $\aleph_0$-categorical $\aleph_0$-stable theory (in a countable language) T_P is $\aleph_0$-categorical up to arbitrarily small perturbations of the predicate P.

4. The model theoretic contents of Berkes & Rosenthal [BR85]

The motivation for this paper has its origins in a theorem from probability theory characterising sequences of random variables possessing almost exchangeable sub-sequences. This is the main result (Theorem 2.4) of Berkes & Rosenthal [BR85], and has a strong model theoretic flavour to it. In this section we translate some of [BR85] to model-theoretic language, and show how their main result is essentially a special case of Theorem 2.5 and the fact that the theory of atomless probability algebras (or rather, of spaces of random variables over such algebras, see below) is SFB.

4.1. **The theory of** $[0,1]$**-valued random variables.** Let us recall a few facts from [Benc, Section 2]. Let Ω be a probability space, and $M = L_1(\Omega,[0,1])$ the space of all $[0,1]$-valued random variables, equipped with the L_1 distance. We view M as a metric structure $(M, 0, \neg, \frac{1}{2}, \dot{-})$ where the function symbols $\neg$, $\frac{1}{2}$ and $\dot{-}$ are interpreted naturally by composition. We shall also use $E(X)$ as an abbreviation for $d(X,0)$, namely the expectation of X. The class of all such structures is elementary, axiomatised by a universal theory RV.

The probability algebra associated with Ω can be identified with the set of all characteristic functions in $L_1(\Omega,[0,1])$, and this set is uniformly quantifier-free definable in models of RV, and will be denoted by $\mathscr{F}$. For $A \subseteq M$, let $\sigma(A) \subseteq \mathscr{F}^M$ denote the minimal complete sub-algebra with respect to which every $X \in A$ is measurable (so in particular $\sigma(M) = \mathscr{F}^M$).

The theory RV admits a model companion ARV, whose models are the spaces of the form $L_1(\Omega, [0,1])$ where Ω is atomless. The theory ARV is $\aleph_0$-categorical (whereby complete), $\aleph_0$-stable and it eliminates quantifiers. Furthermore, non forking in models of ARV coincides with probabilistic independence. In other words, $A \mathop{\smile\!\!\!\!\!\!\;|}_B C$ if and only if $\mathbb{P}[X|\sigma(BC)] = \mathbb{P}[X|\sigma(B)]$ for every $X \in \sigma(A)$ (or, equivalently, for every $X \in \sigma(AB)$).

The theories RV and Pr (the theory of probability algebras) are biïnterpretable. Indeed we have already mentioned that the probability algebra is definable in the corresponding random variable space. Conversely, using a somewhat more involved argument, one can interpret, in a probability algebra $\mathscr{F}$, the space of random variables $L_1(\mathscr{F}, [0,1])$, such that for $M \vDash Pr$ and $N \vDash RV$:

$$M = \mathscr{F}^{L_1(M,[0,1])}, \qquad N = L_1\big(\mathscr{F}^N, [0,1]\big).$$

This biïnterpretability specialises to the theories ARV and APr, and transfers to the corresponding classes of elementary pairs and of lovely pairs. It follows that ARV_P is $\aleph_0$-categorical as well, its unique separable model being $\big(L_1(\mathscr{F}, [0,1]), L_1(P(\mathscr{F}), [0,1])\big)$ where $(\mathscr{F}, P)$ is the unique separable model of APr_P. Thus ARV is SFB. (One can also show directly that SFB is preserved under biïnterpretability, without using Theorem 3.15, and thus without any $\aleph_0$-categoricity assumption.)

Definition 4.1. Let $\mathscr{A}$ be a probability algebra. An *n-dimensional distribution* over $\mathscr{A}$ $\vec{\mu}$ is an $L_1(\mathscr{A}, [0,1])$-valued Borel probability measure on $\mathbb{R}^n$ (σ-additive in the L_1 topology, and $\vec{\mu}(\mathbb{R}^n)$ is the constant function $1 \in L_1(\mathscr{A}, [0,1])$). The space of all n-dimensional distributions over $\mathscr{A}$ will be denoted $\mathfrak{D}_{\mathbb{R}^n}(\mathscr{A})$. For a Borel set $B \subseteq \mathbb{R}^n$, we denote by $\mathfrak{D}_B(\mathscr{A})$ the space of n-dimensional conditional distributions which, as measures, are supported by B (we shall only use this notation for $B = [0,1]^n$).

Let $\bar{X}$ be an n-tuple of real-valued random variables. The joint conditional distribution of $\bar{X}$ over $\mathscr{A}$ denoted here by $\vec{\mu} = \operatorname{dist}(\bar{X}|\mathscr{A})$ (and by $c \cdot (\mathscr{A}) \operatorname{dist}(\bar{X})$ in [BR85]) is the n-dimensional distribution over $\mathscr{A}$ given by

$$\vec{\mu}(B) = \mathbb{P}[\bar{X} \in B|\mathscr{A}], \qquad B \subseteq \mathbb{R}^n \text{ Borel.}$$

Recall that a net $(X_i)_{i\in I} \subseteq L_1(\mathscr{A}, [0,1])$ converges in the *weak topology* to X if for every $Y \in L_1(\mathscr{A}, [0,1])$, $E[X_iY] \to E[XY]$. The net (X_i) converges to X in the *strong topology* if it converges in L_1.

Definition 4.2. Following [BR85, Proposition 1.8], say that a net $(\vec{\mu}_i)_{i\in I}$ of n-dimensional distributions over $\mathscr{A}$ *converges weakly (strongly)* to $\vec{\mu}$ if for every continuous function $\theta\colon \mathbb{R}^n \to [0,1]$ we have $\int \theta(\bar{x})\, d\vec{\mu}_i(\bar{x}) \to \int \theta(\bar{x})\, d\vec{\mu}(\bar{x})$ weakly (strongly).

Let us make two remarks regarding this last condition. First, if $\theta\colon [0,1]^n \to [0,1]$ is continuous then the mapping $\bar{X} \mapsto \theta(\bar{x})$ is uniformly definable in models of RV. This follows immediately from the fact that θ can be approximated arbitrarily well by $\mathcal{L}_{RV}$-terms (i.e., by the operations $\neg$, $\frac{1}{2}$ and $\dot{-}$). Second, by the Stone-Weierstrass Theorem, every continuous θ can be arbitrarily well approximated by polynomials. It follows that it is enough to consider only monomial test functions $\bar{x}^\alpha = \prod x_i^{\alpha_i}$, where $\alpha \in \mathbb{N}^n$.

Theorem 4.3. *Let $\bar{X}$ be an n-tuple in a model of ARV, A a set. Then the joint conditional distribution* $\operatorname{dist}(\bar{X}|\sigma(A))$ *depends only on* $\operatorname{tp}(\bar{X}/A)$. *Moreover, the mapping*

$$\zeta\colon \operatorname{tp}(\bar{X}/A) \mapsto \operatorname{dist}(\bar{X}|\sigma(A))$$

is a homeomorphism between $\mathrm{S}_n(A)$ *(equipped with the logic topology) and* $\mathfrak{D}_{[0,1]^n}(\sigma(A))$ *equipped with the topology of weak convergence.*

Proof. The first assertion, as well as the injectivity of ζ, are shown in [Benc].

For surjectivity of ζ let $\mathscr{A} = \sigma(A)$ and $\vec{\mu} \in \mathfrak{D}_{[0,1]^n}(\mathscr{A})$. Let Ω be the Stone space of the underlying Boolean algebra of $\mathscr{A}$. This is a compact, totally disconnected space, and $\mathscr{A}$ is canonically identified with the algebra of clopen sets there. Let also $K = 2^{\mathbb{N}}$ be the Cantor space (where $2 = \{0,1\}$), also compact and totally disconnected. We have a canonical continuous surjection $\pi \colon K \to [0,1]$ sending $\bar{a} \mapsto \sum a_n 2^{-n-1}$. Let $\theta \colon [0,1] \to K$ be the section which avoids all infinite sequences of ones on $[0,1)$. While it is not continuous, θ is a Borel mapping.

Let $\Omega' = \Omega \times K^n$. For a product $B \times C \subseteq \Omega'$, where $B \in \mathscr{A}$ and $C \subseteq K^n$ are clopen, the set $\theta^{-1}C \subseteq [0,1]^n$ is Borel and we may define $\nu(B \times C) = \mathbb{E}[\mathbb{1}_B \vec{\mu}(\theta^{-1}C)] \in [0,1]$. More generally, any clopen set $D \subseteq \Omega'$ can be written as a disjoint union $D = \bigcup_{i<m} B_i \times C_i$ of such products. We then define $\nu(D) = \sum_{i<n} \nu(B_i \times C_i)$, observing that this does not depend on the specific presentation as a finite disjoint union (by passing to a common refinement). Since all the clopen sets in Ω' are compact, the hypotheses of Carathéodory's theorem are verified, and ν extends to a probability measure on Ω'. For a measurable set $B \in \mathscr{A}$ we have $\nu(B \times K) = \mathbb{P}[B]$, so the projection $\Omega' \to \Omega$ is measure preserving. In particular, we may identify $\mathscr{A}$ with a sub-algebra of the probability algebra of (Ω', ν).

Let $Y_i \colon \Omega' \to K$ be the projection on the ith copy of K and let $X_i = \pi Y_i$. Let $B \in \mathscr{A}$ and $C \subseteq K^n$ be clopen. Then

$$\mathbb{E}\big[\mathbb{1}_B \mathbb{P}[\bar{Y} \in C|\mathscr{A}]\big] = \nu(B \times C) = \mathbb{E}\big[\mathbb{1}_B \vec{\mu}(\theta^{-1}C)\big].$$

The same follows for every closed C (since it can be expressed as a countable intersection of clopen sets) and therefore for every Borel C. If $C \subseteq [0,1]^n$ is any Borel set then $\pi^{-1}C \subseteq K$ is Borel as well, whereby

$$\mathbb{E}\big[\mathbb{1}_B \mathbb{P}[\bar{X} \in C|\mathscr{A}]\big] = \nu(B \times \pi^{-1}C) = \mathbb{E}\big[\mathbb{1}_B \vec{\mu}(\theta^{-1}\pi^{-1}C)\big] = \mathbb{E}\big[\mathbb{1}_B \vec{\mu}(C)\big].$$

Thus $\mathbb{P}[\bar{X} \in C|\mathscr{A}] = \vec{\mu}(C)$ and $\vec{\mu} = \operatorname{dist}(\bar{X}|\mathscr{A})$. Possibly replacing (Ω', ν) with $(\Omega', \nu) \times ([0,1], \lambda)$ (where λ is the Lebesgue measure) we obtain an atomless probability space. Thus $M = L_1(\Omega', [0,1])$ is a model of ARV containing a copy of $\mathscr{A} = \sigma(A)$, and therefore a copy of A (since $\operatorname{dcl}(A) = \operatorname{dcl}(\sigma(A))$), as well as a tuple $\bar{X} \in M^n$ such that $\zeta\big(\operatorname{tp}(\bar{X}/A)\big) = \vec{\mu}$. Therefore ζ is surjective.

Consider now a random variable $Y \in L_1(\mathscr{A}, [0,1]) = \operatorname{dcl}(A)$ and $\alpha \in \mathbb{N}^n$. Since the mapping $\bar{X} \mapsto \bar{X}^\alpha$ is uniformly definable, the mapping $\bar{X} \mapsto E[Y\bar{X}^\alpha]$ is an A-definable predicate. If $p = \operatorname{tp}(\bar{X}/A)$ and $\vec{\mu} = \operatorname{dist}(\bar{X}|\mathscr{A}) = \zeta(p)$ then

$$E[Y\bar{x}^\alpha]^{p(\bar{x})} = E[Y\bar{X}^\alpha] = \mathbb{E}\left[Y \int \bar{x}^\alpha \, d\vec{\mu}(\bar{x})\right].$$

Thus the mapping $p \mapsto \mathbb{E}\left[Y \int \bar{x}^\alpha \, d\zeta(p)(\bar{x})\right]$ is continuous in p. By definition of weak convergence, ζ is continuous. Since $\mathrm{S}_n(A)$ is compact and $\mathfrak{D}_{[0,1]^n}(\mathscr{A})$ Hausdorff, ζ is a homeomorphism. $\blacksquare_{4.3}$

From this point onwards we identify n-types over A with n-dimensional conditional distributions over $\sigma(A)$. In particular, from now on we shall omit ζ from the notation, writing $\int dp(\bar{x})$ where before we wrote $\int d\zeta(p)(\bar{x})$. Strong convergence of conditional distributions also has a model theoretic counterpart.

Corollary 4.4 (Quantifier Elimination to Moments)**.** *Modulo the theory ARV, every formula $\varphi(\bar{x})$ can be expressed as a (possibly infinite) continuous combination of the definable predicates $E[\bar{x}^\alpha]$.*

Proof. By the theorem, the mapping $p \mapsto \big(E[\bar{x}^\alpha]^p\big)_{\alpha \in \mathbb{N}^n}$ is a topological embedding $\iota \colon \mathrm{S}_n(\varnothing) \hookrightarrow [0,1]^{\mathbb{N}^n}$. If $\varphi(\bar{x})$ is any formula, then it can be identified with a continuous function $\varphi \colon \mathrm{S}_n(\varnothing) \to [0,1]$, which, by Tietze's Extension Theorem, can be written as $\hat{\varphi} \circ \iota$ for some continuous $\hat{\varphi} \colon [0,1]^{\mathbb{N}^n} \to [0,1]$. The statement follows. $\blacksquare_{4.4}$

In particular, every formula can be approximated arbitrarily well by finite continuous combinations of the definable predicates $E[\bar{x}^\alpha]$.

Corollary 4.5. *Every sequence $(\vec{\mu}_n)_n \subseteq \mathfrak{D}_{[0,1]^m}(\mathscr{A})$ admits a sub-sequence which converges weakly.*

Proof. First of all, we may assume that $\mathscr{A}$ is separable, since we may replace it with $\sigma\Big(\big\{\bar{\mu}_n\big(\prod_{i<m}[0,q_i]\big)\big\}_{n\in\mathbb{N},\bar{q}\in\mathbb{Q}^m}\Big)$. Then $\mathrm{S}_m(\mathscr{A})$ is compact *and* admits a countable basis, so every sequence there admits a converging sub-sequence. $\blacksquare_{4.5}$

In case we wish to consider distributions of $\mathbb{R}$-valued random variables we need to be a little more careful.

Definition 4.6. A family of distributions $\mathfrak{C} \subseteq \mathfrak{D}_{\mathbb{R}^m}(\mathscr{A})$ is *tight* if for every $\varepsilon > 0$ there is $R \in \mathbb{R}$ such that $\|\vec{\mu}([-R,R]^m)\|_1 > 1-\varepsilon$ for all $\vec{\mu} \in \mathfrak{C}$.

We say that a family of m-tuples of random variables is *bounded in measure* if their respective joint distributions form a tight family.

Let $\theta\colon [-\infty,\infty] \to [0,1]$ be any Borel mapping. For $\vec{\mu} \in \mathfrak{D}_{\mathbb{R}^m}(\mathscr{A})$, we may view $\vec{\mu}$ as a member of $\mathfrak{D}_{[-\infty,\infty]^m}$ and then let $\theta_*\vec{\mu} \in \mathfrak{D}_{[0,1]^m}(\mathscr{A})$ denote the image measure under θ, i.e., $\rho_*\vec{\mu}(B) = \vec{\mu}\big((\theta\times\cdots\times\theta)^{-1}[B]\big)$.

Lemma 4.7. *Let $(\vec{\mu}_n)_n \subseteq \mathfrak{D}_{\mathbb{R}^m}(\mathscr{A})$ be any sequence, and let $\rho\colon [-\infty,\infty] \to [0,1]$ be a homeomorphism. Then $(\vec{\mu}_n)_n$ converges weakly in $\mathfrak{D}_{\mathbb{R}^m}(\mathscr{A})$ if and only if it is tight and $(\rho_*\vec{\mu}_n)_n$ converges weakly in $\mathfrak{D}_{[0,1]^m}(\mathscr{A})$.*

Proof. For $R > 0$, let $\chi_R\colon \mathbb{R}^m \to [0,1]$ be continuous with $\mathbb{1}_{[-R,R]^m} \leq \chi_R \leq \mathbb{1}_{[-R-1,R+1]^m}$. Notice that the sequence is tight if and only if, for every $\varepsilon > 0$ there is an R such that $\|\int \chi_R\,d\vec{\mu}_n\|_1 > 1-\varepsilon$ for all n.

For left to right, assume that $\vec{\mu}_n \to \vec{\mu}$ weakly. Then $\rho_*\vec{\mu}_n \to \rho_*\vec{\mu}$ weakly. For tightness, for each $\varepsilon > 0$ there exists R_0 such that $\|\int \chi_{R_0}\,d\vec{\mu}\|_1 > 1-\varepsilon$. By assumption $\|\int \chi_{R_0}\,d\vec{\mu}_n\|_1 \to \|\int \chi_{R_0}\,d\vec{\mu}\|_1$, so for some n_0 we have $\|\int \chi_{R_0}\,d\vec{\mu}_n\|_1 > 1-\varepsilon$ for all $n \geq n_0$. Since n_0 is finite, we can also find R_1 such that $\|\int \chi_{R_1}\,d\vec{\mu}_n\|_1 > 1-\varepsilon$ for all $n < n_0$. Let $R = \max(R_0,R_1)$. Then $\|\int \chi_R\,d\vec{\mu}_n\|_1 > 1-\varepsilon$ for all n.

For right to left, we assume that the sequence is tight and that $\rho_*\vec{\mu}_n \to \vec{\nu}$ weakly in $\mathfrak{D}_{[0,1]^m}(\mathscr{A})$. Then there exists $\vec{\mu} \in \mathfrak{D}_{[-\infty,\infty]^m}(\mathscr{A})$ such that $\rho_*\vec{\mu} = \vec{\nu}$ and $\vec{\mu}_n \to \vec{\mu}$ weakly in $\mathfrak{D}_{[-\infty,\infty]^m}(\mathscr{A})$. By tightness, for each $\varepsilon > 0$ there is R such that $\|\int \chi_R\,d\vec{\mu}_n\|_1 > 1-\varepsilon$ for all n. By weak convergence we obtain $\|\int \chi_R\,d\vec{\mu}\|_1 \geq 1-\varepsilon$. We conclude that $\vec{\mu}(\mathbb{R}^m) = 1$, i.e., $\mu \in \mathfrak{D}_{\mathbb{R}^m}(\mathscr{A})$, as desired. $\blacksquare_{4.7}$

Corollary 4.8 ([BR85, Theorem 1.7])**.** *Every tight sequence in $\mathfrak{D}_{\mathbb{R}^m}(\mathscr{A})$ has a weakly converging sub-sequence.*

Theorem 4.9. *Let A be a set of parameters, and identify $\mathrm{S}_n(A)$ with $\mathfrak{D}_n(\sigma(A))$ as above. Then the topologies of d-convergence,* Cb*-convergence (of types) and strong convergence (of distributions) agree.*

Proof. Since ARV is SFB, we already know that d-convergence and Cb-convergence agree. Let us first show that d-convergence implies strong convergence. Let M be a large model containing A, and let $\alpha \in \mathbb{N}^n$. Then the mapping $\bar{X} \mapsto E[\bar{X}^\alpha|\sigma(A)]$ is a continuous mapping $M \to L_1(\sigma(A))$, where both are equipped with the usual L_1 metric. It follows that the mapping $p \mapsto \int \bar{x}^\alpha\,dp(\bar{x})$ is a continuous mapping $(\mathrm{S}_n(A),d) \to L_1(\sigma(A))$. Thus d-converges implies strong convergence.

We now prove that strong convergence implies Cb-convergence. For this purpose we need to show that for every formula $\varphi(\bar{x},\bar{y})$, the mapping that associates $p \mapsto \mathrm{Cb}_\varphi(p)$ is continuous when equipping $\mathrm{S}_n(A)$ with the topology of strong convergence. By our observation above, it is enough to show this where $\varphi(\bar{x},\bar{y}) = E[\bar{x}^\alpha\bar{y}^\beta]$ (strictly speaking, this is a definable predicate and not a formula, but this makes no difference). If $\bar{X} \in M$ and $\bar{Y} \in \mathrm{dcl}(A)$ then

$$\varphi(\bar{X},\bar{Y}) = E[\bar{X}^\alpha\bar{Y}^\beta] = E\big[\bar{Y}^\beta\mathbb{E}[\bar{X}^\alpha|\sigma(A)]\big]$$

Thus the mapping $\mathrm{tp}(\bar{X}/A) \mapsto \mathbb{E}[\bar{X}^\alpha|\sigma(A)]$ uniformly yields a parameter for the $E[\bar{x}^\alpha\bar{y}^\beta]$-definition of $\mathrm{tp}(\bar{X}/A)$, and it is continuous when equipping $\mathrm{S}_n(A)$ with the topology of strong convergence. $\blacksquare_{4.9}$

4.2. **Exchangeable and indiscernible sequences.** We are about to compare various model theoretic definitions and results with their probability theoretic counterparts which appear in Berkes & Rosenthal [BR85]. A word of caution is in place, regarding the fact that Berkes and Rosenthal consider $\mathbb{R}$-valued random variables, whereas we prefer to consider $[0,1]$-valued ones. This can be overcome, observing that (with one exception, which we shall treat explicitly) only topological (and not, say, algebraic) properties of $\mathbb{R}$ are actually used, and that $\mathbb{R}$ can be topologically identified with $(0,1) \subseteq [0,1]$.

The following definitions were given in [BR85] for sequences of single random variables. We give the obvious extensions to sequences of tuples of a fixed length.

Definition 4.10. Let $(\bar{X}_n)_{n\in\mathbb{N}}$ be a sequence of m-tuples of random variables, $\bar{X}_n = (X_{n,0}, \ldots, X_{n,m-1})$.

(i) Let $\mathscr{C} \supseteq \sigma\big(\{X_{n,i}\}_{n,i}\big)$ be any probability algebra with respect to which all the $X_{n,i}$ are measurable. Then the sequence is *determining in* $\mathscr{C}$ if the sequence $\operatorname{dist}(\bar{X}_n|\mathscr{C})$ converges weakly in $\mathfrak{D}_{\mathbb{R}^m}(\mathscr{C})$. (We use an alternative characterisation from [BR85, Proposition 2.1].)

(ii) Let $(\bar{X}_n)_n$ be a determining sequence of $\mathscr{C}$-measurable random variables, and let $\vec{\mu} \in \mathfrak{D}_{\mathbb{R}^m}(\mathscr{C})$ be the limit distribution. Then the *limit tail algebra* [BR85, p. 474] of $(\bar{X}_n)_n$ is $\sigma(\vec{\mu}) = \sigma\Big(\big\{\vec{\mu}\big(\prod(-\infty, q_i)\big)\big\}_{\bar{q}\in\mathbb{Q}^m}\Big) \subseteq \mathscr{C}$.

(iii) The sequence is *exchangeable* if the joint distribution (over the trivial algebra) of any k distinct tuples of the sequence depends only on k.

(iv) The sequence is *almost exchangeable* if there is an exchangeable sequence $(\bar{Y}_n)_{n\in\mathbb{N}}$ such that $\sum_{n,i} |X_{n,i} - Y_{n,i}| < \infty$ almost surely.

While Berkes & Rosenthal consider the ambient probability algebra as fixed (this is in particular apparent in their definition of a determining sequence), the model theoretic setting suggests that we allow it to vary. Conveniently, this has no effect on the definitions:

Fact 4.11. *A sequence $(\bar{X}_n)$ is determining in some $\mathscr{C} \geq \mathscr{C}_0 = \sigma\big(\{\bar{X}_n\}_n\big)$ if and only if it is determining in $\mathscr{C}_0$. Therefore, from now we just say that a sequence is* determining.

Proof. Follows from the fact that if $\vec{\mu}_n$ are conditional distributions over $\mathscr{C}_0$ and $\vec{\mu}$ a conditional distribution over $\mathscr{C} \supseteq \mathscr{C}_0$, then $\vec{\mu}_n \to \vec{\mu}$ weakly as conditional distributions over $\mathscr{C}$ if and only if $\vec{\mu}$ is in fact over $\mathscr{C}_0$ and $\vec{\mu}_n \to \vec{\mu}$ weakly as conditional distributions over $\mathscr{C}_0$. $\blacksquare_{4.11}$

By Theorem 4.3, $(\bar{X}_n)_n$ is determining (in $\mathscr{C}$, say) if and only if the sequence $\big(\operatorname{tp}(\rho\bar{X}_n/\mathscr{C})\big)_n$ converges in $\mathrm{S}_m(\mathscr{C})$ to some $\operatorname{tp}(\rho\bar{Y}/\mathscr{C})$, where $\bar{Y}$ is $\mathbb{R}^m$-valued random variables. On the other hand, if we only know that $\big(\operatorname{tp}(\rho\bar{X}_n/\mathscr{C})\big)_n$ converges in $\mathrm{S}_m(\mathscr{C})$, say with limit $\operatorname{tp}(\bar{Z}/\mathscr{C})$, then $\bar{Z}$ consists of $[0,1]^m$-valued random variables, so $\bar{Y} = \rho^{-1}\bar{Z}$ consists of $[-\infty,\infty]^m$-valued random variables, which need not necessarily be $\mathbb{R}^m$-valued.

Lemma 4.12. *Let $(\bar{X}_n)_n$ be an $\mathbb{R}^m$-valued sequence, and let $\mathscr{C} \supseteq \sigma(\{\bar{X}_n\}_n)$. Then the sequence is determining if and only if:*

(i) *The sequence $\big(\operatorname{tp}(\rho\bar{X}_n/\mathscr{C})\big)_n$ converges in $\mathrm{S}_m(\mathscr{C})$; and:*

(ii) *The sequence $(\bar{X}_n)_n$ is bounded in measure.*

Proof. Immediate from Lemma 4.7. $\blacksquare_{4.12}$

Proposition 4.13. [BR85, Theorem 2.2] *Every sequence of $\mathbb{R}^m$-valued random variables which is bounded in measure has a determining sub-sequence.*

Proof. Immediate from Corollary 4.8. $\blacksquare_{4.13}$

Clearly, a sequence $(\bar{X}_n)_n$ is exchangeable if and only if it is an indiscernible set (or more precisely, if and only if the $(0,1)^m$-valued sequence $(\rho\bar{X}_n)_n$ is). Of course, since ARV is a stable theory, every indiscernible sequence is indiscernible as a set, so exchangeable is synonymous with indiscernible. This observation is part of the statement of [BR85, Theorem 1.1]. The full statement is that an indiscernible sequence of random variables is conditionally i.i.d. over its tail field. This can also be obtained as an application to *ARV* of the following three facts: that in a stable theory, if $(\bar{a}_n)_n$ is any indiscernible sequence, then it is a Morley sequence over $C = \bigcap_n \operatorname{dcl}^{eq}\big(\{\bar{a}_k\}_{k\geq n}\big)$; the characterisation of the definable closure given above; and that in models of *ARV*, canonical bases exist in the real sort.

On the other hand, being almost exchangeable is *not* invariant under a homeomorphism of $\mathbb{R}$ with $(0,1)$, so something needs to be said. Recall first that if $(X_n)_n$ is a sequence of uniformly bounded random variables then $X_n \to 0$ in L_p, for any $1 \leq p < \infty$, if and only if $X_n \to 0$ in measure.

Lemma 4.14. *Let $\rho\colon \mathbb{R} \to (0,1)$ be any continuous injective function, and let (X_n) and (Y_n) be sequences of $\mathbb{R}$-valued random variables.*

(i) *If $\sum |X_n - Y_n| < \infty$ a.s. then $|\rho X_n - \rho Y_n| \to 0$ in L_1.*
(ii) *Assume conversely that $|\rho X_n - \rho Y_n| \to 0$ in L_1, and that the sequence $(Y_n)_n$ is bounded in measure. Then there exists a sub-sequence for which $\sum |X_{n_k} - Y_{n_k}| < \infty$ a.s.*

Proof. For the first item, notice that ρ is necessarily uniformly continuous. If $\sum |X_n - Y_n| < \infty$ a.s. then by standard arguments $|X_n - Y_n| \to 0$ in measure, in which case $|\rho X_n - \rho Y_n| \to 0$ in measure. Since $|\rho X_n - \rho Y_n|$ are bounded random variables, $\|\rho X_n - \rho Y_n\|_1 \to 0$.

For the second item we assume that $|\rho X_n - \rho Y_n| \to 0$ in L_1, or equivalently, in measure, and that $(Y_n)_n$ is bounded in measure. We first claim that $|X_n - Y_n| \to 0$ in measure. Indeed, let $\varepsilon > 0$. Then there exists $R \in \mathbb{R}$ such that $\mathbb{P}[|Y_n| > R] < \varepsilon$ for all n. Let $K_0 = \rho\big[[-R,R]\big]$, $K_1 = \rho\big[[-R-\varepsilon, R+\varepsilon]\big]$. Then $K_0 \subseteq K_1^\circ \subseteq K_1 \subseteq (0,1)$, and since K_1 is compact, ρ^{-1} is uniformly continuous on K_1. In particular, there exists $\delta > 0$ such that if $x, y \in K_1$ and $|x - y| < \delta$ then $|\rho^{-1}(x) - \rho^{-1}(y)| < \varepsilon$. Possibly taking a smaller δ, we may assume that K_1 contains a δ-neighbourhood of K_0. Since $|\rho X_n - \rho Y_n| \to 0$ in measure, for n big enough we have $|\rho X_n - \rho Y_n| < \delta$ outside a set of probability ε. Thus, outside a set of probability 2ε we have both $|\rho X_n - \rho Y_n| < \delta$ and $\operatorname{rng}(\rho Y_n) \subseteq K_0$, whereby $\operatorname{rng}(\rho X_n) \subseteq K_1$ and therefore $|X_n - Y_n| < \varepsilon$. This concludes the proof that $X_n - Y_n \to 0$ in measure. We may now pass to a sub-sequence such that $\mathbb{P}[|X_{n_k} - Y_{n_k}| > 2^{-k}] < 2^{-k}$, and it is easy to check that then $\sum |X_{n_k} - Y_{n_k}| < \infty$ a.s. $\blacksquare_{4.14}$

Lemma 4.15. *Let $(\bar{X}_n)_n$ be a sequence of $\mathbb{R}^m$-valued random variables.*

(i) *Assume that $(\bar{X}_n)$ is almost exchangeable. Then it is bounded in measure.*
(ii) *Assume that $(\bar{X}_n)$ is bounded in measure. Then it has an almost exchangeable sub-sequence if and only if the sequence $(\rho\bar{X}_n)_n$ has an almost indiscernible one. Moreover, in that case, the indiscernible sequence witnessing almost indiscernibility is $(0,1)^m$-valued.*

Proof. For the first item, it is clear that an exchangeable (and more generally, an identically distributed) sequence is bounded in measure. Assume now that $(\bar{Y}_n)_n$ witnesses that $(\bar{X}_n)_n$ is almost exchangeable. As in the proof of Lemma 4.14 we have $\bar{X}_n - \bar{Y}_n \to 0$ in measure, and the statement follows.

We now prove the second item. For left to right, we may assume that $(\bar{X}_n)_n$ is almost exchangeable, as witnessed by $(\bar{Y}_n)_n$. By Lemma 4.14 the sequence $(\rho\bar{Y}_n)_n$ witnesses that $(\rho\bar{X}_n)_n$ is almost indiscernible.

For right to left, we may assume that $(\rho\bar{X}_n)_n$ is almost indiscernible, as witnessed by an indiscernible sequence $(\rho\bar{Y}_n)_n$ (where the $Y_{n,i}$ are, *a priori*, $[-\infty,\infty]$-valued). Let $\mathscr{C} = \sigma\big(\{\bar{X}_n\}_n\big)$. Then the limit $p = \lim \operatorname{tp}(\rho\bar{Y}_n/\mathscr{C})$ exists, whereby the limit $\lim \operatorname{tp}(\rho\bar{X}_n/\mathscr{C}) = p$ exists as well. Let $\rho\bar{Y} \vDash p$. By Lemma 4.12 $(\bar{X}_n)_n$ is determining and $\bar{Y}$ is $\mathbb{R}^m$-valued. Since $\bar{Y}_n \equiv \bar{Y}$ (over $\varnothing$, even though not necessarily over $\mathscr{C}$), each $\bar{Y}_n$ is $\mathbb{R}$-valued as well. Now, again by Lemma 4.14, there exist sub-sequences $(\bar{X}_{n_k})_k$, $(\bar{Y}_{n_k})_k$ such that $\sum |X_{n_k,i} - Y_{n_k,i}| < \infty$ a.s. $\blacksquare_{4.15}$

Finally, a word regarding the limit tail algebra of a determining sequence. Let $M = L^1(\mathscr{F}, [0,1])$ be a big saturated model of ARV, $\mathscr{C} \subseteq \mathscr{F}$ a sub-algebra, and let $(\bar{X}_n)_n$ be a determining sequence of $\mathscr{C}$-measurable random variables. Let $\bar{Y}$ realise the limit distribution over $\mathscr{C}$, measurable in $\mathscr{F}$ (although not necessarily in $\mathscr{C}$). Then the tail measure algebra of $(\bar{X}_n)_n$ is precisely $\mathscr{A} = \sigma\big(\{\mathbb{E}[(\rho\bar{Y})^\alpha|\mathscr{C}]\}_{\alpha\in\mathbb{N}^m}\big) \subseteq \mathscr{C}$, which is interdefinable with $\mathrm{Cb}(\rho\bar{Y}/\mathscr{C}) = \mathrm{Cb}(\rho\bar{Y}/M)$.

After all this translation work it is relatively easy to show that the main theorem of [BR85] is indeed a special case of our Theorem 2.5.

Theorem 4.16 ([BR85, Main Theorem (2.4)])**.** *Let $(\bar{X}_n)_n$ be a sequence of random variables in a probability space $(\Omega, \mathscr{C}, \mu)$. Then $(\bar{X}_n)_n$ has an almost exchangeable sub-sequence if and only if it has a determining sub-sequence whose conditional distributions (with respect to the limit tail algebra of the sequence), relative to every set of positive measure, converge strongly.*

Moreover, if in addition $(\bar{X}_n)$ is determining then the sequence witnessing almost exchangeability is i.i.d. over the limit tail algebra.

Proof. We may view $\mathscr{C}$ as a sub-algebra of a rich atomless probability algebra $\mathscr{F}$ and work in $M = L^1(\mathscr{F}, [0,1]) \models ARV$. Since almost exchangeable and determining sequences are bounded in measure, we may assume that $(\bar{X}_n)_n$ is bounded in measure.

Under this assumption, the first condition is equivalent to saying that $(\rho\bar{X}_n)_n$ admits an almost indiscernible sub-sequence. Regarding the second condition, a sub-sequence $(\bar{X}_{n_k})_k$ is determining if and only if $\big(\mathrm{tp}(\bar{X}_{n_k}/\mathscr{C})\big)_k$ converge to some type $p \in \mathrm{S}_m(\mathscr{C})$. In this case the limit tail algebra $\mathscr{A}$ is interdefinable with $\mathrm{Cb}(p)$, and the conditional distributions $\mathrm{dist}(\bar{X}_{n_k}, S|\mathscr{A})$ converge strongly for every $S \in \mathscr{C}$ if and only if $\mathrm{tp}(\rho\bar{X}_{n_k}, \mathscr{C}/\mathscr{A})$ converge in $(\mathrm{S}(\mathscr{A}), \mathscr{T}_{\mathrm{Cb}})$, or equivalently, in $(\mathrm{S}(\mathscr{A}), \mathscr{T}_d)$.

Thus the statement of the theorem is equivalent to saying that the sequence $(\rho\bar{X}_n)_n$ has an almost indiscernible sequence if and only if it has a sub-sequence with the property $(*_\mathscr{C})$. This is just a special case of Theorem 2.5 (and the same for the moreover part). $\blacksquare_{4.16}$

Corollary 4.17 ([BR85, Theorem 3.1])**.** *A sequence of random variables has an almost i.i.d. sub-sequence if and only if it has a sub-sequence whose distributions relative to any set of positive measure converge to the same limit.*

Proof. Let $(\bar{X}_n)_n$ be the sequence, and let $\mathscr{C}$ denote the ambient probability algebra in the statement, and we may embed $\mathscr{C}$ in a model $M \models ARV$.

Following the same translation as above, if $(\bar{X}_n)_n$ is almost i.i.d., say as witnessed by $(\bar{Y}_n)_n$, then $\lim \mathrm{tp}\big(\rho\bar{X}_n/\mathscr{C}\big) = \lim \mathrm{tp}\big(\rho\bar{Y}_n/\mathscr{C}\big) = p$, say, and $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(\varnothing)$. In other words, if $\rho\bar{Z} \models p$ then $\mathscr{C} \perp \bar{Z}$, meaning precisely that the distribution of $\bar{Z}$ relative to any non zero member of $\mathscr{C}$ is the same.

Conversely, assume that $\lim \mathrm{dist}(\bar{X}_n|S) = \lim \mathrm{dist}(\bar{X}_n) = \mu$, say, for every $0 \neq S \in \mathscr{C}$. Let $\bar{Z}$ realise μ independently of $\mathscr{C}$. Then $\mathrm{dist}(\bar{X}_n|\mathscr{C}) \to \mathrm{dist}(\bar{Z}|\mathscr{C})$ weakly, so the sequence is determining, and since $\bar{Z} \perp \mathscr{C}$ the limit tail algebra is trivial. Also, since $\mathrm{dist}(\bar{X}_n|\mathscr{C}) \to \mathrm{dist}(\bar{Z}|\mathscr{C})$ weakly, we have $\mathrm{dist}(\bar{X}_n, \mathscr{C}) \to \mathrm{dist}(\bar{Z}, \mathscr{C})$ (weakly or strongly, over the trivial algebra it is the same thing), so passing to a sub-sequence we may assume that $(\bar{X}_n)_n$ is almost exchangeable, say witnessed by $(\bar{Y}_n)_n$. By the moreover part of the theorem, this sequence is i.i.d. $\blacksquare_{4.17}$

References

[BBH] Itaï Ben Yaacov, Alexander Berenstein, and C. Ward Henson, *Model-theoretic independence in the Banach lattices $L_p(\mu)$*, submitted.

[BBHU08] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov, *Model theory for metric structures*, Model theory with Applications to Algebra and Analysis, volume 2 (Zoé Chatzidakis, Dugald Macpherson, Anand Pillay, and Alex Wilkie, eds.), London Math Society Lecture Note Series, vol. 350, Cambridge University Press, 2008, pp. 315–427.

[Bena] Itaï Ben Yaacov, *Definability of groups in $\aleph_0$-stable metric structures*, submitted, arXiv:0802.4286.

[Benb] ______, *On perturbations of continuous structures*, submitted, arXiv:0802.4388.
[Benc] ______, *On theories of random variables*, submitted, arXiv:0901.1584.
[Bend] ______, *Stability and stable groups in continuous logic*, submitted, arXiv:0810.4087.
[Ben04] ______, *Lovely pairs of models: the non first order case*, Journal of Symbolic Logic **69** (2004), no. 3, 641–662, arXiv:0902.0119.
[Ben06] ______, *On supersimplicity and lovely pairs of cats*, Journal of Symbolic Logic **71** (2006), no. 3, 763–776, arXiv:0902.0118.
[Ben08] ______, *Topometric spaces and perturbations of metric structures*, Logic and Analysis **1** (2008), no. 3–4, 235–272, arXiv:0802.4458.
[BR85] István Berkes and Haskell P. Rosenthal, *Almost exchangeable sequences of random variables*, Z. Wahrsch. Verw. Gebiete **70** (1985), no. 4, 473–507.
[BU] Itaï Ben Yaacov and Alexander Usvyatsov, *Continuous first order logic and local stability*, Transactions of the American Mathematical Society, to appear, arXiv:0801.4303.
[BU07] ______, *On d-finiteness in continuous structures*, Fundamenta Mathematicæ **194** (2007), 67–88.
[Pil96] Anand Pillay, *Geometric stability theory*, Oxford Logic Guides, vol. 32, The Clarendon Press Oxford University Press, New York, 1996, Oxford Science Publications.
[Poi83] Bruno Poizat, *Paires de structures stables*, Journal of Symbolic Logic **48** (1983), no. 2, 239–249.

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