

# ON THE WELL-POSEDNESS OF THE INCOMPRESSIBLE DENSITY-DEPENDENT EULER EQUATIONS IN THE $L^p$ FRAMEWORK

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ABSTRACT. The present paper is devoted to the study of the well-posedness issue for the density-dependent Euler equations in the whole space. We establish local-in-time results for the Cauchy problem pertaining to data in the Besov spaces embedded in the set of Lipschitz functions, including the borderline case  $B_{p,1}^{\frac{N}{p}+1}(\mathbb{R}^N)$ . A continuation criterion in the spirit of the celebrated one by Beale-Kato-Majda in [3] for the classical Euler equations, is also proved.

In contrast with the previous work dedicated to this system in the whole space, our approach is not restricted to the  $L^2$  framework or to small perturbations of a constant density state: we just need the density to be bounded away from zero. The key to that improvement is a new a priori estimate in Besov spaces for an elliptic equation with nonconstant coefficients.

The evolution of the density  $\rho = \rho(t, x) \in \mathbb{R}^+$  and of the velocity field  $u = u(t, x) \in \mathbb{R}^N$  of a nonhomogeneous incompressible fluid satisfies the following *density-dependent Euler equations*:

$$(1) \quad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0. \end{cases}$$

Above,  $f$  stands for a given body force and the gradient of the pressure  $\nabla \Pi$  is the Lagrangian multiplier associated to the divergence free constraint over the velocity. We assume the space variable  $x$  to belong to the whole  $\mathbb{R}^N$  with  $N \geq 2$ .

A plethora of recent mathematical works have been devoted to the study of the classical incompressible Euler equations

$$(2) \quad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla \Pi = f, \\ \operatorname{div} u = 0. \end{cases}$$

which may be seen as a special case of (1) (just take  $\rho \equiv 1$ ).

In contrast, not so many works have been devoted to the study of (1) in the nonconstant density case. In the situation where the equations are considered in a suitably smooth bounded domain of  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , the local well-posedness issue has been investigated by H. Beirão da Veiga and A. Valli in [4, 5, 6] for data with high enough Hölder regularity. The case of data with  $W^{2,p}$  regularity has been studied by A. Valli and W. Zajączkowski in [19] and by S. Itoh and A. Tani in [14]. The whole space case  $\mathbb{R}^3$  has been addressed by S. Itoh in [13]. There, the local existence for initial data  $(\rho_0, u_0)$  such that  $\rho_0$  is bounded, bounded away from 0 and such that  $\nabla \rho_0 \in H^2$ , in  $u_0$  is in  $H^3$  has been obtained. In the recent paper [11], we have generalized [13]'s result to any dimension  $N \geq 2$  and any Sobolev space  $H^s$  with  $s > 1 + N/2$ . Data in the limit Besov space  $B_{2,1}^{\frac{N}{2}+1}$  are also considered.

Let us also mention that, according to the work by J. Marsden in [16], the finite energy solutions to (1) may be interpreted in terms of the action of geodesics. This latter study is motivated by the fact that, as in the homogeneous situation, System (1) has a conserved energy, namely

$$(3) \quad \|(\sqrt{\rho} u)(t)\|_{L^2}^2 = \|\sqrt{\rho_0} u_0\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^N} (\rho f \cdot u)(\tau, x) d\tau dx.$$

Motivated by the fact that, in real life, a fluid is hardly homogeneous, we here want to study whether the classical results for homogeneous fluids remain true in the nonhomogeneous framework. More precisely, we aim at investigating the existence and uniqueness issue in the whole space and in the  $L^p$  framework for densities which may be large perturbations of a constant function: we only require the density to be bounded and bounded away from zero and to have enough regularity. We shall also establish blow-up criteria in the spirit of the celebrated one by Beale-Kato-Majda criterion for (2) (see [3]).

The functional framework that we shall adopt – Besov spaces embedded in the set  $C^{0,1}$  of bounded globally Lipschitz functions – is motivated by the fact that the density and velocity equations of (1) are transport equations by the velocity field. Hence no gain of smoothness may be expected during the evolution and conserving the initial regularity requires the velocity field to be at least locally Lipschitz with respect to the space variable. In fact, as regards the velocity field, the spaces that we shall use are exactly those that are suitable for (2). We thus believe our results to be optimal in terms of regularity.

Now, compared to the classical Euler equations, handling the gradient of the pressure is much more involved. To eliminate the pressure, the natural strategy consists in solving the elliptic equation

$$\operatorname{div}(a\nabla\Pi) = \operatorname{div} F \quad \text{with } F := \operatorname{div}(f - u \cdot \nabla u) \text{ and } a := 1/\rho.$$

If  $a$  is a small perturbation of a constant function  $\bar{a}$  then the above equation may be rewritten

$$\bar{a}\Delta\Pi = \operatorname{div}((\bar{a} - a)\nabla\Pi) + \operatorname{div} F.$$

Now, if  $1 < p < \infty$  then the standard  $L^p$  elliptic estimate may be used for absorbing the first term in the right-hand side. Hence we expect to get the same well-posedness results as for (2) in this situation. As a matter of fact, this strategy has been successfully implemented by Y. Zhou in [22].

In the general case of *large* perturbations of a constant density state, solving the above equation in the  $\mathbb{R}^N$  framework for  $F \in L^p$  may be a problem (unless  $p = 2$  of course). In fact, to our knowledge, even if  $a$  is smooth, bounded and bounded away from zero, there is no solution operator  $\mathcal{H} : F \rightarrow \nabla\Pi$  such that

$$\|\nabla\Pi\|_{L^p} \leq C\|F\|_{L^p}$$

unless  $p$  is “close” to 2 (see the work by N. Meyers in [17]). However that closeness is strongly related to whether  $a$  itself is close to a constant hence no result for all  $p$  may be obtained by taking advantage of Meyers’ result.

In the present work, we shall overcome this difficulty by requiring the data to satisfy a finite energy condition so as to ensure that  $F$  is in  $L^2$ . Indeed, this will enable us to use the classical  $L^2$  estimate and from that, it turns out to be possible to get estimates in high order Besov spaces  $B_{p,r}^s$ .

This observation is the conducting thread leading to the first two well-posedness results stated in the next section. The rest of the paper unfolds as follows. In section 2, we introduce the Littlewood-Paley decomposition and recall the definition of the nonhomogeneous Besov spaces  $B_{p,r}^s$ . Then, we define the paraproduct and remainder operators and state a few classical results in Fourier analysis. Section 3 is devoted to the proof of existence results and a priori estimates for an elliptic equation with nonconstant coefficients in the Besov space framework. To our knowledge, most of the results that are presented therein are new. Sections 4, 5 and 6 are dedicated to the proof of our main existence and continuation results. Some technical lemmas have been postponed in the appendix.

*Notation.* Throughout the paper,  $C$  stands for a harmless “constant” whose exact meaning depends on the context.

For all Banach space  $X$  and interval  $I$  of  $\mathbb{R}$ , we denote by  $\mathcal{C}(I; X)$  (resp.  $\mathcal{C}_b(I; X)$ ) the set of continuous (resp. continuous bounded) functions on  $I$  with values in  $X$ . If  $X$  has predual  $X^*$  then we denote by  $\mathcal{C}_w(I; X)$  the set of bounded measurable functions  $f : I \rightarrow X$  such that

for any  $\phi \in X^*$ , the function  $t \mapsto \langle f(t), \phi \rangle_{X \times X^*}$  is continuous over  $I$ . For  $p \in [1, \infty]$ , the notation  $L^p(I; X)$  stands for the set of measurable functions on  $I$  with values in  $X$  such that  $t \mapsto \|f(t)\|_X$  belongs to  $L^p(I)$ . We denote by  $L^p_{loc}(I)$  the set of those functions defined on  $I$  and valued in  $X$  which, restricted to any compact subset  $J$  of  $I$ , are in  $L^p(J)$ .

Finally, for any real valued function  $a$  over  $\mathbb{R}^N$ , we denote

$$a_* := \inf_{x \in \mathbb{R}^N} a(x) \quad \text{and} \quad a^* := \sup_{x \in \mathbb{R}^N} a(x).$$

## 1. MAIN RESULTS

As explained in the introduction, we hardly expect to get any well-posedness result if the initial velocity is not in  $C^{0,1}$ . It is well-known (see e.g. [2], Chap. 2) that the nonhomogeneous Besov space  $B^s_{p,r}$  is continuously embedded in  $C^{0,1}$  if and only if the triplet  $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$  satisfies the following condition:

$$(C) \quad s > 1 + N/p \quad \text{or} \quad s \geq 1 + N/p \quad \text{and} \quad r = 1.$$

This motivates the following statement concerning the existence of smooth solutions with finite energy:

**Theorem 1.** *Let  $(s, p, r)$  satisfy Condition (C) with  $1 < p < \infty$ . Let  $u_0$  be a divergence-free vector-field with coefficients in  $L^2 \cap B^s_{p,r}$ . Suppose that the body force  $f$  has coefficients in  $L^1([-T_0, T_0]; B^s_{p,r}) \cap \mathcal{C}([-T_0, T_0]; L^2)$  for some  $T_0 > 0$ . Assume that  $\rho_0$  is positive, bounded and bounded away from zero and that  $\nabla \rho_0 \in B^{s-1}_{p,r}$ . If  $p < 2$ , suppose in addition that  $(\rho_0 - \bar{\rho}) \in L^{p^*}$  with  $p^* := 2p/(2-p)$  for some positive real number  $\bar{\rho}$ .*

*There exists a time  $T \in (0, T_0]$  such that System (1) supplemented with initial data  $(\rho_0, u_0)$  has a unique local solution  $(\rho, u, \nabla \Pi)$  on  $[-T, T] \times \mathbb{R}^N$  with:*

- $\rho^{\pm 1} \in \mathcal{C}_b([-T, T] \times \mathbb{R}^N)$ ,  $D\rho \in \mathcal{C}_w([-T, T]; B^{s-1}_{p,r})$  (and  $(\rho - \bar{\rho}) \in \mathcal{C}([-T, T]; L^{p^*})$  if  $p < 2$ ),
- $u \in \mathcal{C}^1([-T, T]; L^2) \cap \mathcal{C}_w([-T, T]; B^s_{p,r})$ ,
- $\nabla \Pi \in \mathcal{C}([-T, T]; L^2) \cap L^1([-T, T]; B^s_{p,r})$ .

*Besides, the energy equality (3) is satisfied for all  $t \in [-T, T]$ , and time continuity holds with respect to the strong topology, if  $r < \infty$ .*

A few comments are in order:

- For the classical incompressible Euler equations (2), the above result statement (without the  $L^2$  assumption) belongs to the mathematical folklore. It has been established in e.g. [21] in the case  $1 < p < \infty$  and in e.g. [2], Chap. 7 in the case  $1 \leq p \leq \infty$ .
- Note that the above statement covers the borderline case  $B^{\frac{N}{p}+1}_{p,1}$  for Condition (C) without any smallness assumption. Thus, up to the lower order  $L^2$  assumption which is needed to control the low frequencies of the pressure, it extends the result [22] by Y. Zhou mentioned in the introduction.
- If one makes the stronger assumption that  $(\rho_0 - \bar{\rho}) \in B^s_{p,r}$  for some positive constant  $\bar{\rho}$  then we get in addition  $(\rho - \bar{\rho}) \in \mathcal{C}([-T, T]; B^s_{p,r})$  (or  $\mathcal{C}_w([-T, T]; B^s_{p,r})$  if  $r = \infty$ ).
- If  $1 < p \leq 2$  then  $u_0 \in B^s_{p,r}$  implies that  $u_0 \in L^2$ . Furthermore, in dimension  $N \geq 3$ , the assumption that  $(\rho_0 - \bar{\rho}) \in L^{p^*}$  may be omitted if  $p > N/(N-1)$ . Therefore, except if  $N = 2$  and  $p < 2$  or if  $N \geq 3$  and  $p \leq N/(N-1)$ , the density need not to tend to some constant at infinity.
- In contrast with the homogeneous case, in dimension  $N = 2$ , the global well-posedness issue for (1) with nonconstant density is an open (and challenging) problem. Indeed, the vorticity  $\omega := \partial_1 u^2 - \partial_2 u^1$  satisfies

$$\partial_t \omega + u \cdot \nabla \omega + \partial_1 \left( \frac{1}{\rho} \right) \partial_2 \Pi - \partial_2 \left( \frac{1}{\rho} \right) \partial_1 \Pi = 0,$$

hence is not transported by the flow of  $u$  if the density is not a constant.

Under the assumptions of Theorem 1, the solutions to (1) satisfy the following Beale-Kato-Majda type continuation criterion. For simplicity, we state the result for positive times only.

**Theorem 2.** *Under the hypotheses of Theorem 1, consider a solution  $(\rho, u, \nabla \Pi)$  to (1) on  $[0, T) \times \mathbb{R}^N$  with the properties described in Theorem 1. If in addition*

$$(4) \quad \int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla \Pi\|_{B_{p,r}^{s-1}}) dt < \infty$$

then  $(\rho, u, \nabla \Pi)$  may be continued beyond  $T$  into a solution of (1) with the same regularity.

Moreover, in the case  $s > 1 + N/p$ , the term  $\nabla u$  may be replaced by  $\operatorname{curl} u$  in (4).

**Remark 1.** *The above statement has two important consequences:*

- *First, as Condition (C) implies that  $B_{p,r}^{s-1}$  is embedded in  $L^\infty$ , one can show by means of an easy bootstrap argument that for data in  $B_{p,r}^s$ , the lifespan of a solution in  $B_{p,r}^s$  is the same as the lifespan in  $B_{p,1}^{\frac{N}{p}+1}$  (which is the larger space in this scale satisfying Condition (C)).*
- *Second, by combining the previous remark with an induction argument, we see that if we start with smooth data  $(a_0, u_0)$  such that the derivatives at any order of  $\nabla a_0$  and  $u_0$  are in  $L^p$  then we get a local-in-time smooth solution with the same properties. In addition, as above, the lifespan for that smooth solution is only determined by the  $B_{p,1}^{\frac{N}{p}+1}$  regularity. This generalizes prior results in the Hölder spaces framework in the case of a bounded domain obtained in [6].*

In the two-dimensional case, the assumption that  $u_0 \in L^2$  is somewhat restrictive since if, say, the initial vorticity is in the Schwartz class then  $u_0 \in L^2$  implies that the vorticity has average 0 over  $\mathbb{R}^N$ . This motivates the following statement which allows for *any* suitably smooth initial vector-field with compactly supported vorticity.

**Theorem 3.** *Let  $T_0$  be in  $]0, \infty[$  and let  $(s, p, r)$  satisfy Condition (C) with  $2 \leq p \leq 4$ . Let  $u_0$  be a divergence-free vector-field with coefficients in  $B_{p,r}^s$ . Assume that  $\rho_0$  is positive, bounded and bounded away from zero, and that  $\nabla \rho_0 \in B_{p,r}^{s-1}$ . Finally, suppose that the body force  $f$  has coefficients in  $L^1([T_0, T_0]; B_{p,r}^s)$  and that the potential part  $\mathcal{Q}f$  of  $f$  is in  $\mathcal{C}([-T_0, T_0]; L^2)$ .*

*There exists a time  $T \in (0, T_0]$  such that System (1) supplemented with initial data  $(\rho_0, u_0)$  has a unique local solution  $(\rho, u, \nabla \Pi)$  on  $[-T, T] \times \mathbb{R}^N$  with:*

- $\rho^{\pm 1} \in \mathcal{C}_b([-T, T] \times \mathbb{R}^N)$ ,  $D\rho \in \mathcal{C}_w([-T, T]; B_{p,r}^{s-1})$ ,
- $u \in \mathcal{C}_w([-T, T]; B_{p,r}^s)$ ,
- $\nabla \Pi \in \mathcal{C}([-T, T]; L^2) \cap L^1([-T, T]; B_{p,r}^s)$ .

*Besides, time continuity holds with respect to the strong topology, if  $r < \infty$ , and the continuation criteria stated in Theorem 2 also hold under the above assumptions.*

As regards the well-posedness theory, the study of the limit case  $p = \infty$  is of interest for different reasons. First, the Besov space  $B_{\infty,1}^1$  is the largest one for which Condition (C) holds. Second, the usual Hölder spaces belong to the family  $B_{\infty,r}^s$  (take  $r = \infty$ ) and are suitable for the study of the propagation of tangential regularity in (1), and of vortex patches than we plan to do in future works.

**Theorem 4.** *Assume that  $u_0 \in B_{\infty,r}^s \cap L^p$ ,  $f \in L^1([-T_0, T_0]; B_{\infty,r}^s \cap L^p)$  and  $\rho_0 \in B_{\infty,r}^s$  for some  $p \in (1, \infty)$  and some  $s > 1$  (or  $s \geq 1$  if  $r = 1$ ). There exists a constant  $\alpha > 0$  depending only on  $s$  and  $N$  such that if, for some positive real number  $\bar{p}$  we have*

$$(5) \quad \|\rho_0 - \bar{p}\|_{B_{\infty,r}^s} \leq \alpha \bar{p},$$

then there exists some  $T > 0$  such that System (1) has a unique solution  $(\rho, u, \nabla \Pi)$  with

- $\rho \in \mathcal{C}([-T, T]; B_{\infty,r}^s)$  (or  $\mathcal{C}_w([-T, T]; B_{\infty,r}^s)$  if  $r = \infty$ ),
- $u \in \mathcal{C}([-T, T]; L^p \cap B_{\infty,r}^s)$ , (or  $u \in \mathcal{C}([-T, T]; L^p) \cap \mathcal{C}_w([-T, T]; B_{\infty,r}^s)$  if  $r = \infty$ ),

- $\nabla \Pi \in L^1([-T, T]; B_{\infty, r}^s)$ .

Note that our result holds for small perturbations of a constant density state only. The reason why is that, in contrast with the previous statements, here bounding the pressure relies on estimates for the ordinary Laplace operator  $\Delta$ . In other words, the heterogeneity  $(a - \bar{a})\nabla \Pi$  is treated as a small perturbation term. We expect this smallness assumption to be just a technical artifact. However, removing it goes beyond the scope of this paper.

## 2. TOOLS

Our results mostly rely on the use of a nonhomogeneous dyadic partition of unity with respect to the Fourier variable, the so-called Littlewood-Paley decomposition. More precisely, fix a smooth radial function  $\chi$  supported in (say) the ball  $B(0, \frac{4}{3})$ , equals to 1 in a neighborhood of  $B(0, \frac{3}{4})$  and such that  $r \mapsto \chi(r e_r)$  is nondecreasing over  $\mathbb{R}^+$ , and set  $\varphi(\xi) = \chi(\frac{\xi}{2}) - \chi(\xi)$ .

The *dyadic blocks*  $(\Delta_q)_{q \in \mathbb{Z}}$  are defined by<sup>1</sup>

$$\Delta_q := 0 \text{ if } q \leq -2, \quad \Delta_{-1} := \chi(D) \quad \text{and} \quad \Delta_q := \varphi(2^{-q}D) \text{ if } q \geq 0.$$

We also introduce the following low frequency cut-off:

$$S_{qu} := \chi(2^{-q}D) = \sum_{p \geq q-1} \Delta_p \quad \text{for } q \geq 0.$$

The following classical properties will be used freely throughout in the paper:

- for any  $u \in \mathcal{S}'$ , the equality  $u = \sum_q \Delta_q u$  makes sense in  $\mathcal{S}'$ ;
- for all  $u$  and  $v$  in  $\mathcal{S}'$ , the sequence  $(S_{q-1}u \Delta_q v)_{q \in \mathbb{N}}$  is spectrally supported in dyadic annuli. Indeed, as  $\text{Supp } \chi \subset B(0, \frac{4}{3})$  and  $\text{Supp } \varphi \subset \{\xi \in \mathbb{R}^n / \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ , we have

$$\text{Supp } (\mathcal{F}(S_{q-1}u \Delta_q v)) \subset \{\xi \in \mathbb{R}^N / \frac{1}{12} \cdot 2^q \leq |\xi| \leq \frac{10}{3} \cdot 2^q\}.$$

One can now define what a Besov space  $B_{p,r}^s$  is:

**Definition 1.** Let  $u$  be a tempered distribution,  $s$  a real number, and  $1 \leq p, r \leq \infty$ . We set

$$\|u\|_{B_{p,r}^s} := \left( \sum_q 2^{rqs} \|\Delta_q u\|_{L^p}^r \right)^{\frac{1}{r}} \quad \text{if } r < \infty \quad \text{and} \quad \|u\|_{B_{p,\infty}^s} := \sup_q 2^{qs} \|\Delta_q u\|_{L^p}.$$

We then define the space  $B_{p,r}^s$  as the subset of distributions  $u \in \mathcal{S}'$  such that  $\|u\|_{B_{p,r}^s}$  is finite.

The Besov spaces have many interesting properties which will be recalled throughout the paper whenever they are needed. For the time being, let us just recall that if Condition (C) holds true then  $B_{p,r}^s$  is an algebra continuously embedded in the set  $C^{0,1}$  of bounded Lipschitz functions (see e.g. [2], Chap. 2), and that the gradient operator maps  $B_{p,r}^s$  in  $B_{p,r}^{s-1}$ . The following result will be also needed:

**Proposition 1.** Let  $F$  be a smooth homogeneous function of degree 0 on  $\mathbb{R}^N \setminus \{0\}$ . Then for all  $p \in (1, \infty)$ , Operator  $F(D)$  is a self-map on  $L^p$ . In addition, if  $r \in [1, \infty]$  and  $s \in \mathbb{R}$  then  $F(D)$  is a self-map on  $B_{p,r}^s$ .

*Proof.* The continuity on  $L^p$  stems from the Hörmander-Mihlin theorem (see e.g. [12]). The rest of the proposition follows from the fact that if  $u \in B_{p,r}^s$  then one may write, owing to  $F(2^{-q}\xi) = F(\xi)$  for all  $q \geq 0$  and  $\xi \neq 0$ ,

$$F(D)u = F(D)\Delta_{-1}u + \sum_{q \geq 0} (F\tilde{\varphi})(2^{-q}D)\Delta_q u$$

where  $\tilde{\varphi}$  is a smooth function with compact support away from the origin and value 1 on the support of  $\varphi$ . Note that  $\mathcal{F}^{-1}(F\tilde{\varphi})$  is in  $L^1$ . Therefore, the standard convolution inequality implies that

$$\|(F\tilde{\varphi})(2^{-q}D)\Delta_q u\|_{L^p} \leq C \|\Delta_q u\|_{L^p}$$

<sup>1</sup>Throughout we agree that  $f(D)$  stands for the pseudo-differential operator  $u \mapsto \mathcal{F}^{-1}(f\mathcal{F}u)$ .

while the  $L^p$  continuity result implies that

$$\|F(D)\Delta_{-1}u\|_{L^p} \leq \|\Delta_{-1}u\|_{L^p}.$$

Putting these two results together entails that  $F(D)$  maps  $B_{p,r}^s$  in itself.  $\square$

**Remark 2.** Both the Leray projector  $\mathcal{P}$  over divergence free vector-fields and  $\mathcal{Q} := \text{Id} - \mathcal{P}$  satisfy the assumptions of the above proposition. Indeed, in Fourier variables, we have for all vector-field  $u$  with coefficients in  $\mathcal{S}'(\mathbb{R}^N)$ ,

$$\widehat{\mathcal{Q}u}(\xi) = -\frac{\xi}{|\xi|^2} \xi \cdot \widehat{u}(\xi).$$

The following lemma (referred in what follows as *Bernstein's inequalities*) describe the way derivatives act on spectrally localized functions.

**Lemma 1.** Let  $0 < r < R$ . A constant  $C$  exists so that, for any nonnegative integer  $k$ , any couple  $(p, q)$  in  $[1, \infty]^2$  with  $q \geq p \geq 1$  and any function  $u$  of  $L^p$ , we have for all  $\lambda > 0$ ,

$$\begin{aligned} \text{Supp } \widehat{u} \subset B(0, \lambda R) &\implies \|D^k u\|_{L^q} \leq C^{k+1} \lambda^{k+N(\frac{1}{p}-\frac{1}{q})} \|u\|_{L^p}; \\ \text{Supp } \widehat{u} \subset \{\xi \in \mathbb{R}^N / r\lambda \leq |\xi| \leq R\lambda\} &\implies C^{-k-1} \lambda^k \|u\|_{L^p} \leq \|D^k u\|_{L^p} \leq C^{k+1} \lambda^k \|u\|_{L^p}. \end{aligned}$$

The first Bernstein inequality entails the following embedding result:

**Proposition 2.** The space  $B_{p_1,r}^{s_1}$  is embedded in the space  $B_{p_2,r}^{s_2}$  whenever

$$1 \leq p_1 \leq p_2 \leq \infty \quad \text{and} \quad s_2 \leq s_1 - N/p_1 + N/p_2.$$

**Remark 3.** Recall that for all  $s \in \mathbb{R}$ , the Besov space  $B_{2,2}^s$  coincides with the nonhomogeneous Sobolev space  $H^s$ . Furthermore if, for  $k \in \mathbb{N}$ , we denote by  $W^{k,p}$  the set of  $L^p$  functions with derivatives up to order  $k$  in  $L^p$  then we have the following chain of continuous embedding:

$$B_{p,1}^k \hookrightarrow W^{k,p} \hookrightarrow B_{p,\infty}^k.$$

Let us now recall a few nonlinear estimates in Besov spaces. Formally, any product of two tempered distributions  $u$  and  $v$ , may be decomposed into

$$(6) \quad uv = T_u v + T_v u + R(u, v)$$

with

$$T_u v := \sum_q S_{q-1} u \Delta_q v, \quad T_v u := \sum_q S_{q-1} v \Delta_q u \quad \text{and} \quad R(u, v) := \sum_q \sum_{|q'-q| \leq 1} \Delta_q u \Delta_{q'} v.$$

The above operator  $T$  is called ‘‘paraproduct’’ whereas  $R$  is called ‘‘remainder’’. The decomposition (6) has been introduced by J.-M. Bony in [7]. We shall sometimes use the notation

$$T'_u v := T_u v + R(u, v).$$

The paraproduct and remainder operators have many nice continuity properties. The following ones will be of constant use in this paper (see the proof in e.g. [2], Chap. 2):

**Proposition 3.** For any  $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$  and  $t < 0$ , the paraproduct operator  $T$  maps  $L^\infty \times B_{p,r}^s$  in  $B_{p,r}^s$ , and  $B_{\infty,\infty}^t \times B_{p,r}^s$  in  $B_{p,r}^{s+t}$ . Moreover, the following estimates hold:

$$\|T_u v\|_{B_{p,r}^s} \leq C \|u\|_{L^\infty} \|\nabla v\|_{B_{p,r}^{s-1}} \quad \text{and} \quad \|T_u v\|_{B_{p,r}^{s+t}} \leq C \|u\|_{B_{\infty,\infty}^t} \|\nabla v\|_{B_{p,r}^{s-1}}.$$

For any  $(s_1, p_1, r_1)$  and  $(s_2, p_2, r_2)$  in  $\mathbb{R} \times [1, \infty]^2$  such that  $s_1 + s_2 > 0$ ,  $1/p := 1/p_1 + 1/p_2 \leq 1$  and  $1/r := 1/r_1 + 1/r_2 \leq 1$  the remainder operator  $R$  maps  $B_{p_1,r_1}^{s_1} \times B_{p_2,r_2}^{s_2}$  in  $B_{p,r}^{s_1+s_2}$ .

Combining the above proposition with Bony’s decomposition (6), we easily get the following ‘‘tame estimate’’:

**Corollary 1.** *Let  $a$  be a bounded function such that  $\nabla a \in B_{p,r}^{s-1}$  for some  $s > 0$  and  $(p, r) \in [1, \infty]^2$ . Then for any  $b \in B_{p,r}^s \cap L^\infty$  we have  $ab \in B_{p,r}^s \cap L^\infty$  and there exists a constant  $C$  depending only on  $N, p$  and  $s$  such that*

$$\|ab\|_{B_{p,r}^s} \leq \|a\|_{L^\infty} \|b\|_{B_{p,r}^s} + \|b\|_{L^\infty} \|Da\|_{B_{p,r}^{s-1}}.$$

The following result pertaining to the composition of functions in Besov spaces will be needed for estimating the reciprocal of the density.

**Proposition 4.** *Let  $I$  be a bounded interval of  $\mathbb{R}$  and  $F : I \rightarrow \mathbb{R}$  a smooth function. Then for all compact subset  $J \subset I$ ,  $s > 0$  and  $(p, r) \in [1, \infty]^2$  there exists a constant  $C$  such that for all  $a \in B_{p,r}^s$  with values in  $J$ , we have  $F(a) \in B_{p,r}^s$  and*

$$\|F(a)\|_{B_{p,r}^s} \leq C \|a\|_{B_{p,r}^s}.$$

Our results concerning Equations (1) rely strongly on a priori estimates in Besov spaces for the transport equation

$$(T) \quad \begin{cases} \partial_t f + v \cdot \nabla f = g, \\ f|_{t=0} = f_0. \end{cases}$$

We shall often use the following result, the proof of which may be found in e.g. [2], Chap. 3, or in the appendix of [8].

**Proposition 5.** *Let  $1 \leq p, r \leq \infty$  and  $\sigma > 0$ . Let  $f_0 \in B_{p,r}^\sigma$ ,  $g \in L^1([0, T]; B_{p,r}^\sigma)$  and  $v$  be a time dependent vector-field in  $\mathcal{C}_b([0, T] \times \mathbb{R}^N)$  such that for some  $p_1 \geq p$ , we have*

$$\begin{aligned} \nabla v &\in L^1([0, T]; B_{p_1, \infty}^{\frac{N}{p_1}} \cap L^\infty) && \text{if } \sigma < 1 + \frac{N}{p_1}, \\ \nabla v &\in L^1([0, T]; B_{p_1, r}^{\sigma-1}) && \text{if } \sigma > 1 + \frac{N}{p_1}, \text{ or } \sigma = 1 + \frac{N}{p_1} \text{ and } r = 1. \end{aligned}$$

Then Equation (T) has a unique solution  $f$  in

- the space  $\mathcal{C}([0, T]; B_{p,r}^\sigma)$  if  $r < \infty$ ,
- the space  $\left(\bigcap_{\sigma' < \sigma} \mathcal{C}([0, T]; B_{p, \infty}^{\sigma'})\right) \cap \mathcal{C}_w([0, T]; B_{p, \infty}^\sigma)$  if  $r = \infty$ .

Moreover, for all  $t \in [0, T]$ , we have

$$(7) \quad e^{-CV(t)} \|f(t)\|_{B_{p,r}^\sigma} \leq \|f_0\|_{B_{p,r}^\sigma} + \int_0^t e^{-CV(t')} \|g(t')\|_{B_{p,r}^\sigma} dt'$$

$$\text{with } V'(t) := \begin{cases} \|\nabla v(t)\|_{B_{p_1, \infty}^{\frac{N}{p_1}} \cap L^\infty} & \text{if } \sigma < 1 + \frac{N}{p_1}, \\ \|\nabla v(t)\|_{B_{p_1, r}^{\sigma-1}} & \text{if } \sigma > 1 + \frac{N}{p_1}, \text{ or } \sigma = 1 + \frac{N}{p_1} \text{ and } r = 1. \end{cases}$$

If  $f = v$  then, for all  $\sigma > 0$ , Estimate (7) holds with  $V'(t) := \|\nabla v(t)\|_{L^\infty}$ .

### 3. ELLIPTIC ESTIMATES

In this section, we want to prove high regularity estimates in Besov spaces for the following elliptic equation

$$(8) \quad -\operatorname{div}(a \nabla \Pi) = \operatorname{div} F \quad \text{in } \mathbb{R}^N$$

where  $a = a(x)$  is a given suitably smooth bounded function satisfying

$$(9) \quad a_* := \inf_{x \in \mathbb{R}^N} a(x) > 0.$$

Let us recall that in the case  $a \equiv 1$  the following result is available:

**Proposition 6.** *If  $a \equiv 1$  and  $p \in (1, \infty)$  then there exists a solution map  $F \mapsto \nabla \Pi$  continuous on  $L^p$ .*

*Proof.* We set  $\nabla\Pi = \nabla(-\Delta)^{-1}\operatorname{div} F$ . Obviously the pseudo-differential operator  $\nabla(-\Delta)^{-1}\operatorname{div}$  satisfies the conditions of Proposition 1. Hence  $F \mapsto \nabla\Pi$  is a continuous self-map on  $L^p$ .  $\square$

We now turn to the study of (8) for nonconstant coefficients. For the convenience of the reader let us first establish the following classical result pertaining to the  $L^2$  case.

**Lemma 2.** *For all vector-field  $F$  with coefficients in  $L^2$ , there exists a tempered distribution  $\Pi$ , unique up to constant functions, such that  $\nabla\Pi \in L^2$  and Equation (8) is satisfied. In addition, we have*

$$(10) \quad a_* \|\nabla\Pi\|_{L^2} \leq \|F\|_{L^2}.$$

*Proof.* The existence part of the statement is a consequence of the Lax-Milgram theorem. Indeed, for  $\lambda > 0$ , consider the following bilinear map:

$$b_\lambda(u, v) = (a\nabla u \mid \nabla v)_{L^2} + \lambda(u \mid v)_{L^2} \quad \text{for } u \text{ and } v \text{ in } H^1(\mathbb{R}^N).$$

Obviously  $b_\lambda$  is continuous and coercive, hence, given  $F \in (L^2(\mathbb{R}^N))^N$ , there exists a unique  $\Pi_\lambda \in H^1(\mathbb{R}^N)$  so that

$$b_\lambda(u, \Pi_\lambda) = (u \mid F)_{L^2} \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Taking  $u = \Pi_\lambda$  and using the Cauchy-Schwarz inequality, we see that (10) is satisfied by  $\Pi_\lambda$ . Hence  $(\nabla\Pi_\lambda)_{\lambda>0}$  is bounded in  $L^2$  and there exist some  $Q \in (L^2(\mathbb{R}^N))^N$  and a sequence  $(\lambda_n)_{n \in \mathbb{N}}$  converging to 0, such that  $\nabla\Pi_{\lambda_n} \rightharpoonup Q$  weakly in  $L^2$ . Note that this implies that  $Q$  satisfies  $\operatorname{div}(aQ) = \operatorname{div} F$  in the distributional sense, and also that  $Q$  is the gradient of some tempered distribution  $\Pi$ . Besides, we have

$$\|\nabla\Pi\|_{L^2} = \|Q\|_{L^2} \leq \liminf \|\nabla\Pi_{\lambda_n}\|_{L^2} \leq a_*^{-1} \|F\|_{L^2}.$$

As regards uniqueness, it suffices to check that the constant functions are the only tempered solutions with gradient in  $L^2$  which satisfy (8) with  $F \equiv 0$ . So let us consider  $\Pi \in \mathcal{S}'$  with  $\nabla\Pi \in L^2$  and  $\operatorname{div}(a\nabla\Pi) = 0$ . We thus have

$$(11) \quad \int a\nabla u \cdot \nabla\Pi \, dx = 0 \quad \text{for all } u \in H^1.$$

By taking advantage of the Fourier transform and of Parseval equality, it is easy to check that for  $n > 0$ , the tempered distribution  $\Pi_n := (\operatorname{Id} - \chi(nD))\Pi$  (where the cut-off function  $\chi$  has been defined in Section 2) belongs to  $H^1$ . Hence one may take  $u = \Pi_n$  in (11) and we get

$$\int a\nabla\Pi \cdot \nabla\Pi_n \, dx = 0 \quad \text{for all } n > 0.$$

As  $\nabla\Pi_n$  tends to  $\nabla\Pi$  in  $L^2$  and  $a \geq a^* > 0$ , this readily implies that  $\nabla\Pi = 0$ .  $\square$

Let us now establish higher order estimates.

**Proposition 7.** *Let  $1 < p < \infty$  and  $1 \leq r \leq \infty$ . Let  $a$  be a bounded function satisfying (9) and such that  $Da \in B_{p,r}^{s-1}$  for some  $s > 1 + N/p$  or  $s \geq 1 + N/p$  if  $r = 1$ .*

- *If  $1 < p < \infty$ ,  $\sigma \in (1, s]$  and  $\nabla\Pi \in B_{p,r}^\sigma$  satisfies (8) for some function  $F$  such that  $\operatorname{div} F \in B_{p,r}^{\sigma-1}$  then we have for some constant  $C$  depending only on  $s, \sigma, p, N$ ,*

$$a_* \|\nabla\Pi\|_{B_{p,r}^\sigma} \leq C \left( \|\operatorname{div} F\|_{B_{p,r}^{\sigma-1}} + a_* \left(1 + a_*^{-1} \|Da\|_{B_{p,r}^{s-1}}\right)^\sigma \|\nabla\Pi\|_{L^p} \right).$$

- *If  $2 \leq p < \infty$  and  $F$  is in  $L^2$  and satisfies  $\operatorname{div} F \in B_{p,r}^{\sigma-1}$  for some  $\sigma \in (1 + N/p - N/2, s]$  then Equation (8) has a unique solution  $\Pi$  (up to constant functions) such that  $\nabla\Pi \in L^2 \cap B_{p,r}^\sigma$ . Furthermore, Inequality (10) is satisfied and there exists a positive exponent  $\gamma$  depending only on  $\sigma, p, N$  and a positive constant  $C$  depending only on  $s, \sigma, p, N$  such that*

$$a_* \|\nabla\Pi\|_{B_{p,r}^\sigma} \leq C \left( \|\operatorname{div} F\|_{B_{p,r}^{\sigma-1}} + \left(1 + a_*^{-1} \|Da\|_{B_{p,r}^{s-1}}\right)^\gamma \|F\|_{L^2} \right).$$

- If  $\sigma > 1$  and  $1 < p < \infty$  then the following inequality holds:

$$a_* \|\nabla \Pi\|_{B_{p,r}^\sigma} \leq C \left( \|\operatorname{div} F\|_{B_{p,r}^{\sigma-1}} + \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{p,r}^{\sigma-1}} + \|\nabla \Pi\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{\sigma-1}} \right).$$

*Proof.* Throughout,  $(c_q)_{q \geq -1}$  denotes a sequence in the unit sphere of  $\ell^r$ .

The proof relies on two ingredients:

- (i) the following commutator estimates (see Lemmas 6 and 7 in the appendix)

$$(12) \quad \|\operatorname{div} [a, \Delta_q] \nabla \Pi\|_{L^p} \leq C c_q 2^{-q(\sigma-1)} \|\nabla a\|_{B_{p,r}^{s-1}} \|\nabla \Pi\|_{B_{p,r}^{\sigma-1}},$$

$$(13) \quad \|\operatorname{div} [a, \Delta_q] \nabla \Pi\|_{L^p} \leq C c_q 2^{-q(\sigma-1)} (\|\nabla \Pi\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{\sigma-1}} + \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{p,r}^{\sigma-1}})$$

which hold true whenever  $\sigma \in (0, s]$  and  $(s, p, r)$  satisfies Condition (C) (as regards (12)) and whenever  $\sigma > 1$  (as concerns (13));

- (ii) a Bernstein type inequality (see Lemma 8 in the appendix).

For proving the first part of the lemma, apply the spectral cut-off operator  $\Delta_q$  to (8). We get

$$-\operatorname{div} (a \Delta_q \nabla \Pi) = \operatorname{div} \Delta_q F + \operatorname{div} ([\Delta_q, a] \nabla \Pi) \quad \text{for all } q \geq 0.$$

Hence, multiplying both sides by  $|\Delta_q \Pi|^{p-2} \Delta_q \Pi$  and integrating over  $\mathbb{R}^N$ , we get

$$\begin{aligned} - \int |\Delta_q \Pi|^{p-2} \Delta_q \Pi \operatorname{div} (a \Delta_q \nabla \Pi) dx &= \int |\Delta_q \Pi|^{p-2} \Delta_q \Pi \operatorname{div} \Delta_q F dx \\ &\quad + \int |\Delta_q \Pi|^{p-2} \Delta_q \Pi \operatorname{div} ([\Delta_q, a] \nabla \Pi) dx. \end{aligned}$$

Apply Lemma 8 to bound by below the left-hand side of the above inequality. Using Hölder's inequality to handle the right-hand side, we get for all  $q \geq 0$ ,

$$(14) \quad a_* 2^{2q} \|\Delta_q \Pi\|_{L^p}^p \leq C \|\Delta_q \Pi\|_{L^p}^{p-1} \left( \|\operatorname{div} \Delta_q F\|_{L^p} + \|\operatorname{div} [\Delta_q, a] \nabla \Pi\|_{L^p} \right).$$

To deal with the last term, one may now take advantage of Inequality (12). Since, for  $q \geq 0$ , we have  $\|\Delta_q \nabla \Pi\|_{L^p} \approx 2^q \|\Delta_q \Pi\|_{L^p}$  according to Lemma 1, we get after our multiplying Inequality (14) by  $2^{q(\sigma-1)}$ :

$$a_* 2^{q\sigma} \|\Delta_q \nabla \Pi\|_{L^p} \leq C \left( 2^{q(\sigma-1)} \|\Delta_q \operatorname{div} F\|_{L^p} + c_q \|\nabla a\|_{B_{p,r}^{s-1}} \|\nabla \Pi\|_{B_{p,r}^{\sigma-1}} \right) \quad \text{for all } q \in \mathbb{N}.$$

Taking the  $\ell^r$  norm of both sides and adding up the low frequency block pertaining to  $\Delta_{-1} \nabla \Pi$ , we get

$$(15) \quad a_* \|\nabla \Pi\|_{B_{p,r}^\sigma} \leq C \left( \|\operatorname{div} F\|_{B_{p,r}^{\sigma-1}} + \|\nabla a\|_{B_{p,r}^{s-1}} \|\nabla \Pi\|_{B_{p,r}^{\sigma-1}} + a_* \|\Delta_{-1} \nabla \Pi\|_{L^p} \right).$$

Observe that  $\|\Delta_{-1} \nabla \Pi\|_{L^p} \leq C \|\nabla \Pi\|_{L^p}$ , and that the following interpolation inequality is available (recall that  $0 < \sigma - 1$ ):

$$\|\nabla \Pi\|_{B_{p,r}^{\sigma-1}} \leq C \|\nabla \Pi\|_{L^p}^{\frac{1}{\sigma}} \|\nabla \Pi\|_{B_{p,r}^\sigma}^{1-\frac{1}{\sigma}}.$$

Then, applying a suitable Young inequality completes the proof of the first part of the proposition.

Let us now tackle the proof of the second part of the Proposition. As  $F \in L^2$ , the existence of a solution  $\nabla \Pi$  in  $L^2$  is ensured by Lemma 2. Let us admit for a while that  $\nabla \Pi \in B_{p,r}^\sigma$  and let us prove the desired inequality. As  $p \geq 2$ , we have

$$L^2 \hookrightarrow B_{p,\infty}^{N(\frac{1}{p}-\frac{1}{2})}.$$

Hence, as  $B_{p,r}^{\sigma-1}$  is an interpolation space between  $B_{p,\infty}^{N(\frac{1}{p}-\frac{1}{2})}$  and  $B_{p,r}^\sigma$  (here comes the assumption that  $\sigma - 1 > N/p - N/2$ ), one may write for some convenient exponent  $\theta = \theta(p, \sigma, N) \in (0, 1)$ ,

$$\|\nabla \Pi\|_{B_{p,r}^{\sigma-1}} \leq C \|\nabla \Pi\|_{L^2}^\theta \|\nabla \Pi\|_{B_{p,r}^\sigma}^{1-\theta}.$$

In addition, as  $p \geq 2$ , Bernstein's inequality implies that

$$\|\Delta_{-1}\nabla\Pi\|_{L^p} \leq C\|\nabla\Pi\|_{L^2}.$$

Hence, plugging the last two inequalities in (15) and using (10) yields

$$a_*\|\nabla\Pi\|_{B_{p,r}^\sigma} \leq C\left(\|\operatorname{div} F\|_{B_{p,r}^{\sigma-1}} + \|F\|_{L^2} + a_*^{-1}\|F\|_{L^2}^\theta\|\nabla a\|_{B_{p,r}^{\sigma-1}}(a_*\|\nabla\Pi\|_{B_{p,r}^\sigma})^{1-\theta}\right).$$

Then applying Young's inequality completes the proof.

Remark that Inequality (15) remains valid whenever  $\nabla\Pi$  is in  $B_{p,r}^{\sigma-1}$ . Starting from the fact that the constructed solution  $\nabla\Pi$  is in  $B_{p,\infty}^{N(\frac{1}{p}-\frac{1}{2})}$ , a straightforward induction argument allows to state that  $\nabla\Pi$  is indeed in  $B_{p,r}^\sigma$ . This completes the second part of the proof.

For proving the last part of the proposition, the starting point is Inequality (14) which implies that

$$a_*2^{q\sigma}\|\nabla\Delta_q\Pi\|_{L^p} \leq C2^{q(\sigma-1)}(\|\Delta_q\operatorname{div} F\|_{L^p} + \|\operatorname{div}[\Delta_q, a]\nabla\Pi\|_{L^p}).$$

Now, taking advantage of Inequality (13) then summing up over  $q \geq -1$ , we readily obtain the desired result.  $\square$

#### 4. PROOF OF THE FIRST LOCAL WELL-POSEDNESS RESULT

As a preliminary step, let us observe that System (1) is *time reversible*. That is, changing  $(t, x)$  in  $(-t, -x)$  restricts the study of the Cauchy problem to the evolution for *positive* times. To simplify the presentation, we shall thus concentrate from now on to the unique solvability of the system for positive times only.

In the first part of this section, we establish the uniqueness part of Theorem 1. When proving existence, it is convenient to treat the two cases  $p \geq 2$  and  $p < 2$  separately. The reason why is that the proof strongly relies on Proposition 7 which enables to compute the pressure *only if*  $p \geq 2$ . Indeed, if  $p < 2$  then only an a priori estimate is stated.

So, in the second part of this section, we prove the existence in the case  $p \geq 2$ . The third subsection is devoted to the proof of Theorem 2 in the case  $p \geq 2$ . It will be needed for proving the existence part of Theorem 1 in the case  $p < 2$ . The following part of this section is devoted to the proof of Theorems 1 and 2 in the case  $p < 2$ . In the last paragraph, we justify the claim pertaining to the case  $p > N/(N-1)$  (see just after the statement of Theorem 1).

For expository purpose, we shall assume in this section and in the rest of the paper that  $r < \infty$ . For treating the case  $r = \infty$ , it is only a matter of replacing the strong topology by weak topology whenever regularity *up to index s* is involved.

**4.1. Uniqueness.** Uniqueness in Theorems 1 is a consequence of the following general stability result for solutions to (1).

**Proposition 8.** *Let  $(\rho_1, u_1, \nabla\Pi_1)$  and  $(\rho_2, u_2, \nabla\Pi_2)$  satisfy (1) with exterior forces  $f_1$  and  $f_2$ . Assume in addition that  $\rho_1$  and  $\rho_2$  are bounded and bounded away from zero, that  $\delta u := u_2 - u_1$  and  $\delta\rho := \rho_2 - \rho_1$  belong to  $C^1([0, T]; L^2)$ , that  $\delta f := f_2 - f_1$  is in  $C([0, T]; L^2)$  and that  $\nabla\Pi_1, \nabla\rho_1$  and  $\nabla u_1$  belong to  $L^1([0, T]; L^\infty)$ . Then for all  $t \in [0, T]$ , we have*

$$(16) \quad \|\delta\rho(t)\|_{L^2} + \|(\sqrt{\rho_2}\delta u)(t)\|_{L^2} \leq e^{A(t)}\left(\|\delta\rho(0)\|_{L^2} + \|(\sqrt{\rho_2}\delta u)(0)\|_{L^2} + \int_0^t e^{-A(\tau)}\|\sqrt{\rho_2}\delta f\|_{L^2} d\tau\right)$$

$$\text{with } A(t) := \int_0^t \left( \left\| \frac{\nabla\rho_1}{\sqrt{\rho_2}} \right\|_{L^\infty} + \left\| \frac{\nabla\Pi_1}{\rho_1\sqrt{\rho_2}} \right\|_{L^\infty} + \|\nabla u_1\|_{L^\infty} \right) d\tau.$$

*Proof.* On the one hand, as

$$\partial_t\delta\rho + u_2 \cdot \nabla\delta\rho = -\delta u \cdot \nabla\rho_1,$$

taking the  $L^2$  inner product with  $\delta\rho$  and integrating by parts in the second term of the left-hand side yields

$$(17) \quad \|\delta\rho(t)\|_{L^2} \leq \|\delta\rho(0)\|_{L^2} + \int_0^t \|(\sqrt{\rho_2}\delta u)\|_{L^2} \left\| \frac{\nabla\rho_1}{\sqrt{\rho_2}} \right\|_{L^\infty} d\tau.$$

On the other hand, denoting  $\nabla\delta\Pi := \nabla\Pi_2 - \nabla\Pi_1$ , we notice that

$$\rho_2(\partial_t\delta u + u_2 \cdot \nabla\delta u) + \nabla\delta\Pi = \rho_2 \left( \delta f + \frac{\delta\rho}{\rho_1\rho_2} \nabla\Pi_1 - \delta u \cdot \nabla u_1 \right).$$

So taking the  $L^2$  inner product of the second equation with  $\delta u$ , integrating by parts and using the fact that  $\operatorname{div}\delta u = 0$  and that

$$\partial_t\rho_2 + u_2 \cdot \nabla\rho_2 = 0,$$

we eventually get

$$\|(\sqrt{\rho_2}\delta u)(t)\|_{L^2} \leq \|(\sqrt{\rho_2}\delta u)(0)\|_{L^2} + \int_0^t \left( \|\sqrt{\rho_2}\delta f\|_{L^2} + \|\delta\rho\|_{L^2} \left\| \frac{\nabla\Pi_1}{\rho_1\sqrt{\rho_2}} \right\|_{L^\infty} + \|\nabla u_1\|_{L^\infty} \|\sqrt{\rho_2}\delta u\|_{L^2} \right) d\tau.$$

Adding up Inequality (17) to the above inequality and applying Gronwall lemma completes the proof of the proposition.  $\square$

*Proof of uniqueness in Theorem 1.* Consider two solutions  $(\rho_1, u_1, \nabla\Pi_1)$  and  $(\rho_2, u_2, \nabla\Pi_2)$  of (1) with the same data. Under the assumptions of Theorem 1, it is clear that the velocity and pressure fields satisfy the assumptions of the above proposition. As concerns the density, we notice that  $u_i \in \mathcal{C}([0, T]; L^2)$  and  $\nabla\rho_i \in \mathcal{C}([0, T]; L^\infty)$  for  $i = 1, 2$  implies that  $\partial_t\rho_i \in \mathcal{C}([0, T]; L^2)$ . Hence we have  $\delta\rho \in \mathcal{C}^1([0, T]; L^2)$ . Therefore Inequality (16) implies that  $(\rho_1, u_1, \nabla\Pi_1) \equiv (\rho_2, u_2, \nabla\Pi_2)$  on  $[0, T] \times \mathbb{R}^N$ . This completes the proof of the uniqueness in Theorem 1.

**4.2. The proof of existence in Theorem 1: the case  $2 \leq p < \infty$ .** We notice that, formally, the density-dependent incompressible Euler equations are equivalent to<sup>2</sup>

$$(18) \quad \begin{cases} \partial_t a + u \cdot \nabla a = 0 & \text{with } a := 1/\rho, \\ \partial_t u + u \cdot \nabla u + a \nabla\Pi = f, \\ -\operatorname{div}(a \nabla\Pi) = \operatorname{div}(u \cdot \nabla \mathcal{P}u) - \operatorname{div} f. \end{cases}$$

Let us give conditions under which this equivalence is rigorous.

**Lemma 3.** *Let  $u$  be a time-dependent vector-field with coefficients in  $\mathcal{C}^1([0, T] \times \mathbb{R}^N)$  and such that  $\mathcal{Q}u \in \mathcal{C}^1([0, T]; L^2)$ . Assume that  $\nabla\Pi \in \mathcal{C}([0, T]; L^2)$ . Let  $\rho$  be a continuous bounded function on  $[0, T] \times \mathbb{R}^N$  which is positive and bounded away from 0.*

*If in addition  $\operatorname{div}u(0, \cdot) \equiv 0$  in  $\mathbb{R}^N$  then  $(\rho, u, \nabla\Pi)$  is a solution to (1) if and only if  $(a, u, \nabla\Pi)$  satisfies (18).*

*Proof.* If  $(\rho, u, \nabla\Pi)$  satisfies (1) then, owing to  $\rho > 0$ , we see that  $a := 1/\rho$  satisfies the first equation of (18). Next, applying Operator  $\operatorname{div}$  to the velocity equation of (1) divided by  $\rho$ , and using that  $\mathcal{P}u = u$  yields the third equation of (18).

Conversely, if  $(a, u, \nabla\Pi)$  satisfies (18), it is obvious, owing to positivity, that  $\rho := 1/a$  satisfies the density equation of (1). In order to justify that the other two equations are satisfied, it is only a matter of proving that  $\operatorname{div}u \equiv 0$ . For that, one may apply  $\mathcal{Q}$  to the second equation. Then, using the third equation, we discover that

$$\partial_t \mathcal{Q}u + \mathcal{Q}(u \cdot \nabla \mathcal{Q}u) = 0.$$

Recall that  $\mathcal{Q}u \in \mathcal{C}^1([0, T]; L^2)$ . Therefore, taking the  $L^2$  inner product with  $\mathcal{Q}u$ , we get

$$\frac{1}{2} \frac{d}{dt} \|\mathcal{Q}u\|_{L^2}^2 + (\mathcal{Q}(u \cdot \nabla \mathcal{Q}u) | \mathcal{Q}u)_{L^2} = 0.$$

<sup>2</sup>Recall that  $\mathcal{P}$  stands for the Leray projector over divergence free vector-fields.

As  $\mathcal{Q}^T = \mathcal{Q}$  and  $\mathcal{Q}^2 = \mathcal{Q}$ , we thus get after integrating by parts in the second term:

$$\frac{d}{dt} \|\mathcal{Q}u\|_{L^2}^2 = \int |\mathcal{Q}u|^2 \operatorname{div} u \, dx,$$

and, as  $\mathcal{Q}u(0, \cdot) = 0$ , Gronwall lemma ensures that  $\mathcal{Q}u \equiv 0$ . Hence  $\operatorname{div} u = 0$ .  $\square$

As explained in Lemma 3, it suffices to solve System (18). So, for  $T > 0$ , let us introduce the set  $E_T$  of functions  $(a, u, \nabla \Pi)$  such that

$$\begin{aligned} a &\in \mathcal{C}_b([0, T] \times \mathbb{R}^N), & \nabla a &\in \mathcal{C}([0, T]; B_{p,r}^{s-1}), \\ u &\in \mathcal{C}^1([0, T]; L^2) \cap \mathcal{C}([0, T]; B_{p,r}^s), & \nabla \Pi &\in \mathcal{C}([0, T]; L^2) \cap L^1([0, T]; B_{p,r}^s). \end{aligned}$$

We denote

$$a_* := \inf_{x \in \mathbb{R}^N} a_0(x), \quad a^* := \sup_{x \in \mathbb{R}^N} a_0(x), \quad \rho_* := \inf_{x \in \mathbb{R}^N} \rho_0(x) \quad \text{and} \quad \rho^* := \sup_{x \in \mathbb{R}^N} \rho_0(x).$$

Note that if  $\rho$  is bounded and bounded away from zero, and satisfies  $\nabla \rho \in B_{p,r}^{s-1}$  then the same properties hold for  $a$  (and conversely). This may be easily shown by combining Propositions 2 and 4. Moreover, there exists some constant  $C$  depending only on  $a_*$ ,  $a^*$ ,  $N$  and on the regularity parameters such that

$$C^{-1} \|\nabla \rho\|_{B_{p,r}^{s-1}} \leq \|\nabla a\|_{B_{p,r}^{s-1}} \leq C \|\nabla \rho\|_{B_{p,r}^{s-1}}.$$

This fact will be used repeatedly in the rest of the paper.

*Step 1. Construction of a sequence of approximate solutions.* As a first step for solving (18), we construct a sequence  $(a^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  of global approximate solutions which belong to  $E_T$  for all  $T > 0$ .

For doing so, one may argue by induction. We first set  $(a^0, u^0, \nabla \Pi^0) := (a_0, u_0, 0)$ . Next, we assume that  $(a^n, u^n, \nabla \Pi^n)$  has been constructed over  $\mathbb{R}^+$ , belongs to the space  $E_T$  for all  $T > 0$  and that there exists a positive time  $T^*$  such that for all  $t \in [0, T^*]$ ,

$$(19) \quad a_* \leq a^n(t, x) \leq a^*,$$

$$(20) \quad \|\nabla a^n(t)\|_{B_{p,r}^{s-1}} \leq 2 \|\nabla a_0\|_{B_{p,r}^{s-1}} \quad \text{for all } t \in [0, T^*],$$

$$(21) \quad \|\sqrt{\rho^n(t)} u^n(t)\|_{L^2} \leq \sqrt{\rho^* a^*} (4 \|\sqrt{\rho_0} u_0\|_{L^2} + 8 \sqrt{\rho^*} \|f\|_{L_t^1(L^2)}) \quad \text{with } \rho^n := 1/a^n,$$

$$(22) \quad U^n(t) \leq 4U_0(t) + C_0 \rho^* A_0 \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} + C_0 (\rho^* A_0)^{\gamma+1} (\|u_0\|_{L^2} + \|f\|_{L_t^1(L^2)}),$$

$$(23) \quad a_* \|\nabla \Pi^n\|_{L_t^1(B_{p,r}^s)} \leq C \left( \int_0^t (U^n(\tau))^2 d\tau + \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} + (\rho^* A_0)^\gamma (\|u_0\|_{L^2} + \|f\|_{L_t^1(L^2)}) \right),$$

$$(24) \quad \|\nabla \Pi^n\|_{L_t^1(L^2)} d\tau \leq \sqrt{\rho^*} \|\sqrt{\rho_0} u_0\|_{L^2} + 3\rho^* \|f\|_{L_t^1(L^2)}$$

with  $A_0 := a^* + \|Da_0\|_{B_{p,r}^{s-1}}$ ,  $U_0(t) := \|u_0\|_{B_{p,r}^s} + \|f\|_{L_t^1(B_{p,r}^s)}$  and  $U^n(t) := \|u^n(t)\|_{B_{p,r}^s}$ . The positive exponent  $\gamma$  is given by Proposition 7. The constants  $C_0$  and  $C$  depend only on  $(s, p, r)$  and  $N$ , and may be made explicit from the following computations (in fact one can take  $C_0 = 2C^2$  with  $C$  large enough).

Denoting by  $\psi^n$  the flow of  $u^n$ , (which belongs to  $\mathcal{C}^1(\mathbb{R}^+ \times \mathbb{R}^N)$  owing to  $u^n \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$  and to  $B_{p,r}^{s-1} \hookrightarrow \mathcal{C}_b$ ), we set

$$a^{n+1}(t, x) := a_0((\psi_t^n)^{-1}(x)) \quad \text{and} \quad \rho^{n+1}(t, x) := \rho_0((\psi_t^n)^{-1}(x)).$$

As  $\psi_t^n$  is a diffeomorphism over  $\mathbb{R}^N$  for all  $t \geq 0$ , we have

$$\|a^{n+1}(t)\|_{L^\infty} = \|a_0\|_{L^\infty} = a^* \quad \text{and} \quad \|\rho^{n+1}(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty} = \rho^*.$$

Hence (19) is satisfied by  $a^{n+1}$ . In addition, we have

$$\partial_t a^{n+1} + u^n \cdot \nabla a^{n+1} = 0$$

so that for all  $i \in \{1, \dots, N\}$ ,

$$\partial_t \partial_i a^{n+1} + u^n \cdot \nabla \partial_i a^{n+1} = -\partial_i u^n \cdot \nabla a^{n+1}.$$

As  $\partial_i a|_{t=0}^{n+1} = \partial_i a_0 \in B_{p,r}^{s-1}$  by assumption, (a slight generalization of) Proposition 5 combined with Gronwall lemma guarantees that  $\nabla a^{n+1} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^{s-1})$  and that

$$(25) \quad \|\nabla a^{n+1}(t)\|_{B_{p,r}^{s-1}} \leq e^{C \int_0^t U^n(\tau) d\tau} \|\nabla a_0\|_{B_{p,r}^{s-1}}.$$

So if we assume that  $T^*$  has been chosen so that

$$(26) \quad C \int_0^{T^*} U^n(t) dt \leq \log 2$$

then  $a^{n+1}$  satisfies (20).

Next, we want to define  $u^{n+1}$  as the unique solution in  $\mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$  of the transport equation:

$$(27) \quad \partial_t u^{n+1} + u^n \cdot \nabla u^{n+1} = -a^{n+1} \nabla \Pi^n + f, \quad u|_{t=0}^{n+1} = u_0.$$

That the right-hand side belongs to  $L_{loc}^1(\mathbb{R}^+; B_{p,r}^s)$  is a consequence of Corollary 1 and of the embedding  $B_{p,r}^{s-1} \hookrightarrow L^\infty$ . In addition, we have for a.e. positive time

$$(28) \quad \|a^{n+1} \nabla \Pi^n\|_{B_{p,r}^s} \leq C(\|a^{n+1}\|_{L^\infty} + \|\nabla a^{n+1}\|_{B_{p,r}^{s-1}}) \|\nabla \Pi^n\|_{B_{p,r}^s}.$$

So finally, the existence of  $u^{n+1} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$  is ensured by Proposition 5, and we have

$$(29) \quad \begin{aligned} \|u^{n+1}(t)\|_{B_{p,r}^s} &\leq e^{C \int_0^t U^n(\tau) d\tau} \left( \|u_0\|_{B_{p,r}^s} \right. \\ &\left. + \int_0^t e^{-C \int_0^\tau U^n(\tau') d\tau'} \left( (\|a^{n+1}\|_{L^\infty} + \|\nabla a^{n+1}\|_{B_{p,r}^{s-1}}) \|\nabla \Pi^n\|_{B_{p,r}^s} + \|f\|_{B_{p,r}^s} \right) d\tau \right). \end{aligned}$$

Therefore, if we restrict our attention to those  $t$  that are in  $[0, T^*]$  with  $T^*$  satisfying (26), we see that for all  $t \in [0, T^*]$ ,

$$U^{n+1}(t) \leq 2U_0(t) + CA_0 \int_0^t \|\nabla \Pi^n\|_{B_{p,r}^s} d\tau \quad \text{with } A_0 := a^* + \|\nabla a_0\|_{B_{p,r}^{s-1}}.$$

So if we assume that  $T^*$  and  $C_0$  have been chosen so that

$$(30) \quad 2C^2 \rho^* A_0 \int_0^{T^*} U^n(t) dt \leq 1 \quad \text{and } C_0 = 2C^2$$

then taking advantage of Inequalities (23) and (22), we see that  $u^{n+1}$  satisfies (22) on  $[0, T^*]$ .

Let us now prove (21) for  $u^{n+1}$ . First, we notice that the right-hand side of (27) belongs to  $\mathcal{C}(\mathbb{R}^+; L^2)$  so that  $u^{n+1}$  is in  $\mathcal{C}^1(\mathbb{R}^+; L^2)$ . As  $\rho^{n+1}$  is bounded and  $\mathcal{C}^1$  with respect to the time and space variables, this allows us to take the  $L^2$  inner product of the equation for  $u^{n+1}$  with  $\rho^{n+1} u^{n+1}$ . We readily get

$$(31) \quad \frac{1}{2} \frac{d}{dt} \|\sqrt{\rho^{n+1}} u^{n+1}\|_{L^2}^2 - \int \rho^{n+1} u^{n+1} \cdot f dx = \frac{1}{2} \int \rho^{n+1} |u^{n+1}|^2 \operatorname{div} u^n dx - (\nabla \Pi^n | u^{n+1})_{L^2}.$$

Let us point out that  $u^n$  and  $u^{n+1}$  need not be divergence-free, so that the right-hand side may be nonzero. However, from the above inequality, it is easy to get

$$\begin{aligned} \|(\sqrt{\rho^{n+1}} u^{n+1})(t)\|_{L^2} &\leq \|\sqrt{\rho_0} u_0\|_{L^2} \\ &\quad + \int_0^t \left( \sqrt{a^*} \|\nabla \Pi^n\|_{L^2} + \sqrt{\rho^*} \|f\|_{L^2} + \frac{1}{2} \|\sqrt{\rho^{n+1}} u^{n+1}\|_{L^2} \|\operatorname{div} u^n\|_{L^\infty} \right) dx. \end{aligned}$$

So, if we assume that  $C$  has been taken large enough in (26) then Gronwall's lemma implies that

$$(32) \quad \|(\sqrt{\rho^{n+1}} u^{n+1})(t)\|_{L^2} \leq 2(\|\sqrt{\rho_0} u_0\|_{L^2} + \sqrt{\rho^*} \|f\|_{L_t^1(L^2)} + \sqrt{a^*} \|\nabla \Pi^n\|_{L_t^1(L^2)}).$$

Now, putting the above inequality together with Inequality (24) ensures that Inequality (21) is also satisfied by  $u^{n+1}$  on  $[0, T^*]$ .

To finish with, we have to construct the approximate pressure  $\Pi^{n+1}$ . For that, we aim at solving the following elliptic equation

$$(33) \quad \operatorname{div}(a^{n+1} \nabla \Pi^{n+1}) = \operatorname{div}(f - u^{n+1} \cdot \nabla \mathcal{P}u^{n+1})$$

for every positive time.

We have already proved that  $a^{n+1}$  satisfies the required ellipticity condition through (19). Moreover, as  $u^{n+1} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$ , Remark 2 ensures that  $\nabla \mathcal{P}u^{n+1}$  is in  $\mathcal{C}(\mathbb{R}^+; B_{p,r}^{s-1})$ . As  $B_{p,r}^{s-1} \hookrightarrow L^\infty$  and  $u^{n+1} \in \mathcal{C}(\mathbb{R}^+; L^2)$ , we thus have  $u^{n+1} \cdot \nabla \mathcal{P}u^{n+1} \in \mathcal{C}(\mathbb{R}^+; L^2)$  and

$$\begin{aligned} \|u^{n+1} \cdot \nabla \mathcal{P}u^{n+1}\|_{L^2} &\leq \sqrt{a^*} \|\sqrt{\rho^{n+1}} u^{n+1}\|_{L^2} \|\nabla \mathcal{P}u^{n+1}\|_{L^\infty}, \\ &\leq C \sqrt{a^*} \|\sqrt{\rho^{n+1}} u^{n+1}\|_{L^2} \|\nabla u^{n+1}\|_{B_{p,r}^{s-1}}. \end{aligned}$$

Therefore Lemma 2 guarantees that (33) has a solution  $\nabla \Pi^{n+1}$  in  $\mathcal{C}(\mathbb{R}^+; L^2)$  which satisfies

$$(34) \quad a_* \|\nabla \Pi^{n+1}\|_{L_t^1(L^2)} \leq \|f\|_{L_t^1(L^2)} + C \sqrt{a^*} \int_0^t U^{n+1} \|\sqrt{\rho^{n+1}} u^{n+1}\|_{L^2} d\tau.$$

Let us insert Inequality (32) in the above inequality. We see that if  $T^*$  has been chosen so that

$$(35) \quad 4Ca^* \rho^* \int_0^{T^*} U^{n+1} d\tau \leq 1$$

then Inequality (34) implies that

$$\|\nabla \Pi^{n+1}\|_{L_t^1(L^2)} \leq \frac{3}{2} \rho^* \|f\|_{L_t^1(L^2)} + \frac{1}{2} \sqrt{\rho^*} \|\sqrt{\rho_0} u_0\|_{L^2} + \frac{1}{2} \|\nabla \Pi^n\|_{L_t^1(L^2)},$$

hence Inequality (24) is satisfied by  $\nabla \Pi^{n+1}$  on  $[0, T^*]$ .

In order to prove that  $\nabla \Pi^{n+1}$  belongs to  $L_{loc}^1(\mathbb{R}^+; B_{p,r}^s)$ , one may apply the second part of Proposition 7. Indeed, because, owing to  $\operatorname{div} \mathcal{P}u^{n+1} = 0$ , we have

$$\operatorname{div}(u^{n+1} \cdot \nabla \mathcal{P}u^{n+1}) = \nabla u^{n+1} : \nabla \mathcal{P}u^{n+1}$$

and as  $B_{p,r}^{s-1}$  is an algebra, the term  $\operatorname{div}(u^{n+1} \cdot \nabla \mathcal{P}u^{n+1})$  is in  $B_{p,r}^{s-1}$  and

$$\|\operatorname{div}(u^{n+1} \cdot \nabla \mathcal{P}u^{n+1})\|_{B_{p,r}^{s-1}} \leq C(U^{n+1})^2.$$

Hence Proposition 7 implies that for all  $t \in \mathbb{R}^+$ ,

$$\begin{aligned} a_* \|\nabla \Pi^{n+1}\|_{L_t^1(B_{p,r}^s)} &\leq C \left( \int_0^t (U^{n+1})^2 d\tau + \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} \right. \\ &\quad \left. + (1 + \rho^* \|Da^{n+1}\|_{L_t^\infty(B_{p,r}^{s-1})})^\gamma (\|f\|_{L_t^1(L^2)} + \|u^{n+1} \cdot \nabla \mathcal{P}u^{n+1}\|_{L_t^1(L^2)}) \right), \end{aligned}$$

whence, using (20) at rank  $n+1$  and Hölder inequality, we get

$$\begin{aligned} a_* \|\nabla \Pi^{n+1}\|_{L_t^1(B_{p,r}^s)} &\leq C \left( \int_0^t (U^{n+1})^2 d\tau + \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} \right. \\ &\quad \left. + (\rho^* A_0)^\gamma (\|f\|_{L_t^1(L^2)} + \sqrt{a^*} \|\sqrt{\rho^{n+1}} u^{n+1}\|_{L_t^\infty(L^2)} \int_0^t U^{n+1} d\tau) \right). \end{aligned}$$

Taking advantage of Inequality (21) at rank  $n+1$  one can now conclude that if (35) holds then  $\nabla \Pi^{n+1}$  satisfies (23).

At this stage we have proved that if Inequalities (19) to (24) hold for  $(a^n, u^n, \nabla \Pi^n)$  then they also hold for  $(a^{n+1}, u^{n+1}, \nabla \Pi^{n+1})$  provided  $T^*$  satisfies Inequalities (26), (30) and (35). Note that (30) is the strongest condition. Obviously it is satisfied if we set

(36)

$$T^* := \sup \left\{ t > 0 / \rho^* t A_0 \left( U_0(t) + \rho^* A_0 \|\operatorname{div} f\|_{L_t^1(B_{\rho, r}^{s-1})} + (\rho^* A_0)^{\gamma+1} (\|u_0\|_{L^2} + \|f\|_{L_t^1(L^2)}) \right) \leq c \right\}$$

for a small enough constant  $c$  depending only on  $s$ ,  $p$  and  $N$ .

*Step 2 . Convergence of the sequence.* Let  $\tilde{a}^n := a^n - a_0$ . In this step, we shall establish that  $(\tilde{a}^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{C}([0, T^*]; L^2)$ .

Let  $\delta a^n := \tilde{a}^{n+1} - \tilde{a}^n$ ,  $\delta u^n := u^{n+1} - u^n$  and  $\delta \Pi^n := \Pi^{n+1} - \Pi^n$ . We have for  $n \geq 2$ ,

$$(37) \quad \begin{cases} \partial_t \delta a^n + u^n \cdot \nabla \delta a^n = -\delta u^{n-1} \cdot \nabla a^n, \\ \partial_t \delta u^n + u^n \cdot \nabla \delta u^n = -\delta u^{n-1} \cdot \nabla u^n - a^n \nabla \delta \Pi^{n-1} - \delta a^n \nabla \Pi^n, \\ \operatorname{div} (a^{n-1} \nabla \delta \Pi^{n-1}) = -\operatorname{div} (\delta u^{n-1} \cdot \nabla \mathcal{P} u^n + u^{n-1} \cdot \nabla \mathcal{P} \delta u^{n-1} + \delta a^{n-1} \nabla \Pi^n). \end{cases}$$

For all  $n \in \mathbb{N}$ , we have  $\partial_t \tilde{a}^{n+1} = -u^n \cdot \nabla a^{n+1}$ . So, given that, according to the previous step,  $u^n \in \mathcal{C}([0, T^*]; L^2)$  and  $\nabla a^{n+1} \in \mathcal{C}_b([0, T^*] \times \mathbb{R}^N)$ , and that  $\tilde{a}_{|t=0}^{n+1} = 0$ , we discover that  $\tilde{a}^{n+1}$ , and thus also  $\delta a^n$ , are in  $\mathcal{C}^1([0, T^*]; L^2)$ . Taking the  $L^2$  inner product of the equation for  $\delta a^n$  with  $\delta a^n$ , we thus get

$$\frac{1}{2} \frac{d}{dt} \|\delta a^n\|_{L^2}^2 = \frac{1}{2} \int (\delta a^n)^2 \operatorname{div} u^n dx - \int \delta u^{n-1} \cdot \nabla a^n \delta a^n dx,$$

whence for all  $t \in [0, T^*]$ ,

$$(38) \quad \|\delta a^n(t)\|_{L^2} \leq \frac{1}{2} \int_0^t \|\operatorname{div} u^n\|_{L^\infty} \|\delta a^n\|_{L^2} d\tau + \int_0^t \|\nabla a^n\|_{L^\infty} \|\delta u^{n-1}\|_{L^2} d\tau.$$

Next, taking the  $L^2$  inner product of the equation for  $\delta u^n$  with  $\rho^{n+1} \delta u^n$ , performing integration by parts and using the equation for  $\rho^{n+1}$ , we get

$$\frac{1}{2} \frac{d}{dt} \int \rho^{n+1} |\delta u^n|^2 dx = \frac{1}{2} \int \rho^{n+1} |\delta u^n|^2 \operatorname{div} u^n dx - \int \rho^{n+1} \delta u^n \cdot (\delta u^{n-1} \cdot \nabla u^n + a^n \nabla \delta \Pi^{n-1} + \delta a^n \nabla \Pi^n) dx.$$

Hence

$$(39) \quad \begin{aligned} \|\sqrt{\rho^{n+1}(t)} \delta u^n(t)\|_{L^2} &\leq \frac{1}{2} \int_0^t \|\operatorname{div} u^n\|_{L^\infty} \|\sqrt{\rho^{n+1}} \delta u^n\|_{L^2} d\tau \\ &+ \int_0^t (\|\nabla u^n\|_{L^\infty} \|\delta u^{n-1}\|_{L^2} + \|a^n\|_{L^\infty} \|\nabla \delta \Pi^{n-1}\|_{L^2} + \|\nabla \Pi^n\|_{L^\infty} \|\delta a^n\|_{L^2}) d\tau. \end{aligned}$$

Adding up Inequalities (38) and (39), applying Gronwall lemma and using the fact that  $\rho^{n+1} \geq \rho_*$  and the bounds stated in the first step, we thus get for all  $t \in [0, T^*]$ ,

$$(40) \quad \|(\delta a^n, \delta u^n)(t)\|_{L^2} \leq C_{T^*} \left( \int_0^t \|(\delta a^{n-1}, \delta u^{n-1})(\tau)\|_{L^2} d\tau + \int_0^t \|\nabla \delta \Pi^{n-1}(\tau)\|_{L^2} d\tau \right),$$

where the constant  $C_{T^*}$  depends only on  $T^*$  and on the initial data.

In order to bound  $\nabla \delta \Pi^{n-1}$ , we shall use that for any  $\mathcal{C}^1$  vector-fields  $a$  and  $b$ , we have

$$\operatorname{div} (a \cdot \nabla b) = \operatorname{div} (b \cdot \nabla a) + \operatorname{div} (a \operatorname{div} b) - \operatorname{div} (b \operatorname{div} a).$$

Applying this to  $a = u^{n-1}$  and  $b = \mathcal{P} \delta u^{n-1}$  and bearing in mind that  $\operatorname{div} \mathcal{P} \delta u^{n-1} = 0$ , we deduce from the third equation of (37) that

$$\operatorname{div} (a^{n-1} \nabla \delta \Pi^{n-1}) = \operatorname{div} (\mathcal{P} \delta u^{n-1} \operatorname{div} u^{n-1} - \mathcal{P} \delta u^{n-1} \cdot \nabla u^{n-1} - \delta u^{n-1} \cdot \nabla \mathcal{P} u^n - \delta a^{n-1} \nabla \Pi^n).$$

Therefore, Lemma 2 and the fact that  $\|\mathcal{P}\|_{\mathcal{L}(L^2; L^2)} = 1$  guarantee that

$$a_* \|\nabla \delta \Pi^{n-1}\|_{L^2} \leq \|\delta u^{n-1}\|_{L^2} (\|\operatorname{div} u^{n-1}\|_{L^\infty} + \|\nabla u^{n-1}\|_{L^\infty} + \|\nabla \mathcal{P} u^n\|_{L^\infty}) + \|\delta a^{n-1}\|_{L^2} \|\nabla \Pi^n\|_{L^\infty}.$$

Using the uniform bounds of the previous step, we thus get for all  $t \in [0, T^*]$ ,

$$(41) \quad \|\nabla \delta \Pi^{n-1}\|_{L^2} \leq C_{T^*} (\|\delta u^{n-1}\|_{L^2} + \|\delta a^{n-1}\|_{L^2}).$$

Plugging Inequality (41) in Inequality (40), we end up with (up to a change of  $C_{T^*}$ ),

$$\|(\delta a^n, \delta u^n)(t)\|_{L^2} \leq C_{T^*} \int_0^t \|(\delta a^{n-1}, \delta u^{n-1})(\tau)\|_{L^2} d\tau.$$

Arguing by induction, one may conclude that

$$\sup_{t \in [0, T^*]} \|(\delta a^n, \delta u^n)(t)\|_{L^2} \leq \frac{(C_{T^*} T^*)^n}{n!} \sup_{t \in [0, T^*]} \|(\delta a^0, \delta u^0)(t)\|_{L^2}.$$

It is now obvious that both  $(\tilde{a}^n)_{n \in \mathbb{N}}$  and  $(u^n)_{n \in \mathbb{N}}$  are Cauchy sequences in  $\mathcal{C}([0, T^*]; L^2)$ , hence converge to some functions  $\tilde{a}$  and  $u$  in  $\mathcal{C}([0, T^*]; L^2)$ . Taking advantage of (41), it is also clear that  $(\nabla \Pi^n)_{n \in \mathbb{N}}$  converges to some function  $\nabla \Pi$  in  $\mathcal{C}([0, T^*]; L^2)$ .

*Step 3. Final checking.* Let  $a := a_0 + \tilde{a}$ . We now have to check that  $(a, u, \nabla \Pi)$  is indeed a solution to (1) and that it has the properties stated in Theorem 1. From the previous step, we already know that  $(a - a_0)$ ,  $u$  and  $\nabla \Pi$  are in  $\mathcal{C}([0, T^*]; L^2)$ . Moreover:

- As  $(\nabla a^n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty([0, T^*]; B_{p,r}^{s-1})$  and as Besov spaces have the Fatou property, we deduce that  $\nabla a$  belongs to  $L^\infty([0, T^*]; B_{p,r}^{s-1})$ . Since  $(a^n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty([0, T^*] \times \mathbb{R}^N)$ , we also have  $a \in L^\infty([0, T^*] \times \mathbb{R}^N)$ .
- As  $(u^n)_{n \in \mathbb{N}}$  is bounded in  $L^\infty([0, T^*]; B_{p,r}^s)$ , we deduce that  $u \in L^\infty([0, T^*]; B_{p,r}^s)$ .
- Finally, as  $(\nabla \Pi^n)_{n \in \mathbb{N}}$  is bounded in  $L^1([0, T^*]; B_{p,r}^s)$  we deduce that  $\nabla \Pi$  belongs to  $L^1([0, T^*]; B_{p,r}^s)$ .

Arguing by interpolation, we see that the above sequences converge strongly in every intermediate space between  $\mathcal{C}([0, T^*]; L^2)$  and  $\mathcal{C}([0, T^*]; B_{p,r}^s)$  which is more than enough to pass to the limit in the equations satisfied by  $(a^n, u^n, \nabla \Pi^n)$ . Hence  $(a, u, \nabla \Pi)$  satisfies (18).

Passing to the limit in (31), we see that, in addition,  $(\rho, u)$  satisfies the energy equality (3).

Finally, the continuity properties of the solution with respect to the time may be recovered by using the equations satisfied by  $(a, u, \nabla \Pi)$ , and Proposition 5.

**4.3. A continuation criterion.** The key to the proof of Theorem 2 is the following lemma:

**Lemma 4.** *Let  $(s, p, r)$  satisfy Condition (C) with  $1 < p < \infty$ . Consider a solution  $(\rho, u, \nabla \Pi)$  to (1) on  $[0, T] \times \mathbb{R}^N$  such that<sup>3</sup>  $u \in \mathcal{C}([0, T]; B_{p,r}^s)$  and*

$$\rho_* \leq \rho \leq \rho^*, \quad \rho \in \mathcal{C}([0, T] \times \mathbb{R}^N) \quad \text{and} \quad \nabla \rho \in \mathcal{C}([0, T]; B_{p,r}^{s-1}).$$

*If in addition*

$$(42) \quad \int_0^T (\|\nabla u\|_{L^\infty} + \|\nabla \Pi\|_{B_{p,r}^{s-1}}) dt < \infty$$

*then*

$$\int_0^t \|\nabla \Pi\|_{B_{p,r}^s} d\tau + \sup_{0 \leq t < T} (\|u(t)\|_{B_{p,r}^s} + \|\nabla \rho\|_{B_{p,r}^{s-1}}) < \infty.$$

*Proof.* Note that  $a := 1/\rho$  satisfies the same assumptions as  $\rho$ . Therefore we shall rather work with  $a$ , for convenience. Recall that

$$(43) \quad \partial_t \partial_k a + u \cdot \nabla \partial_k a = -\partial_k u \cdot \nabla a \quad \text{for } k = 1, \dots, N.$$

So, applying Operator  $\Delta_q$  to the above equality and using that  $\operatorname{div} u = 0$ , one may write (with the summation convention)

$$\partial_t \Delta_q \partial_k a + u \cdot \nabla \Delta_q \partial_k a = -\Delta_q (\partial_k u \cdot \nabla a) + \partial_j [u^j, \Delta_q] \partial_k a.$$

Therefore for all  $t \in [0, T)$ ,

$$(44) \quad \|\Delta_q \partial_k a(t)\|_{L^p} \leq \|\Delta_q \partial_k a_0\|_{L^p} + \int_0^t \|\Delta_q (\partial_k u \cdot \nabla a)\|_{L^p} d\tau + \int_0^t \|\partial_j [u^j, \Delta_q] \partial_k a\|_{L^p} d\tau.$$

<sup>3</sup>With the usual convention if  $r = \infty$

According to Proposition 3, the term  $\partial_k u \cdot \nabla a$  belongs to  $B_{p,r}^{s-1}$  and satisfies

$$\|\partial_k u \cdot \nabla a\|_{B_{p,r}^{s-1}} \leq C(\|\partial_k u\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{s-1}} + \|\nabla a\|_{L^\infty} \|\partial_k u\|_{B_{p,r}^{s-1}})$$

while Lemma 7 ensures that for all  $q \geq -1$ ,

$$\|\partial_j [u^j, \Delta_q] \partial_k a\|_{L^p} \leq C c_q 2^{q(s-1)} (\|\partial_k a\|_{L^\infty} \|\nabla u\|_{B_{p,r}^{s-1}} + \|\nabla u\|_{L^\infty} \|\partial_k a\|_{B_{p,r}^{s-1}}).$$

Using the definition of the norm in  $B_{p,r}^{s-1}$ , we thus get after summation in (44) that

$$(45) \quad \|\nabla a(t)\|_{B_{p,r}^{s-1}} \leq \|\nabla a_0\|_{B_{p,r}^{s-1}} + C \int_0^t (\|\nabla u\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{s-1}} + \|\nabla a\|_{L^\infty} \|\nabla u\|_{B_{p,r}^{s-1}}) d\tau.$$

In order to bound the velocity, let us apply the last part of Proposition 5 to the velocity equation, and the following inequality (which stems from Corollary 1):

$$\|a \nabla \Pi\|_{B_{p,r}^s} \leq C(a^* \|\nabla \Pi\|_{B_{p,r}^s} + \|\nabla \Pi\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{s-1}}).$$

We get for all  $t \in [0, T)$ ,

$$(46) \quad \begin{aligned} \|u(t)\|_{B_{p,r}^s} &\leq e^{C \int_0^t \|\nabla u\|_{L^\infty} d\tau} \left( \|u_0\|_{B_{p,r}^s} \right. \\ &\quad \left. + \int_0^t e^{-C \int_0^\tau \|\nabla u\|_{L^\infty} d\tau'} \left( \|f\|_{B_{p,r}^s} + C a^* \|\nabla \Pi\|_{B_{p,r}^s} + C \|\nabla \Pi\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{s-1}} \right) d\tau \right). \end{aligned}$$

In order to bound the pressure term, one may use the fact that

$$\operatorname{div}(a \nabla \Pi) = \operatorname{div} f - \operatorname{div}(u \cdot \nabla u)$$

and apply the last part of Proposition 7. Performing a time integration and using the fact that

$$\|\operatorname{div}(u \cdot \nabla u)\|_{B_{p,r}^{s-1}} = \|\nabla u : \nabla u\|_{B_{p,r}^{s-1}} \leq C \|\nabla u\|_{L^\infty} \|\nabla u\|_{B_{p,r}^{s-1}},$$

we get

$$\begin{aligned} a_* \|\nabla \Pi\|_{L_t^1(B_{p,r}^s)} &\leq C \left( \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} \right. \\ &\quad \left. + \int_0^t (\|\nabla u\|_{L^\infty} \|\nabla u\|_{B_{p,r}^{s-1}} + \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{p,r}^{s-1}} + \|\nabla \Pi\|_{L^\infty} \|\nabla a\|_{B_{p,r}^{s-1}}) d\tau. \right) \end{aligned}$$

Let us insert this latter inequality in (46). Then adding up Inequality (45) and applying Gronwall lemma we end up with

$$(47) \quad \begin{aligned} \|\nabla a(t)\|_{B_{p,r}^{s-1}} + \|u(t)\|_{B_{p,r}^s} &\leq C \exp \left( \int_0^t \|(\nabla a, \nabla u, \nabla \Pi)\|_{L^\infty} d\tau \right) \\ &\quad \left( \|\nabla a_0\|_{B_{p,r}^{s-1}} + \|u_0\|_{B_{p,r}^s} + \|f\|_{L_t^1(B_{p,r}^s)} + \int_0^t \|\nabla a\|_{L^\infty} \|\nabla \Pi\|_{B_{p,r}^{s-1}} d\tau \right) \end{aligned}$$

for some constant  $C$  depending only on the regularity parameters and on  $N$ ,  $a_*$  and  $a^*$ .

Now, let us notice that  $\nabla a$  is bounded on  $[0, T) \times \mathbb{R}^N$ . Indeed, from Equation (43) and Gronwall lemma, we see that

$$\|\nabla a(t)\|_{L^\infty} \leq e^{\int_0^t \|\nabla u\|_{L^\infty} d\tau} \|\nabla a_0\|_{L^\infty}.$$

As  $\nabla \Pi$  is in  $L^1([0, T); B_{p,r}^{s-1})$  and  $\nabla u$  is in  $L^1([0, T); L^\infty)$  by assumption and as  $B_{p,r}^{s-1} \hookrightarrow L^\infty$ , we discover that both the last term in (47) and the exponential term are bounded on  $[0, T)$ . This completes the proof of the lemma.  $\square$

The following lemma implies the first part of Theorem 2 in the case  $p \geq 2$ .

**Lemma 5.** *Let  $(s, p, r)$  satisfy Condition (C) with  $2 \leq p < \infty$ . Consider a solution  $(\rho, u, \nabla \Pi)$  to (1) on  $[0, T] \times \mathbb{R}^N$  such that<sup>4</sup>*

- $\rho_* \leq \rho \leq \rho^*$ ,  $\rho \in \mathcal{C}([0, T) \times \mathbb{R}^N)$  and  $\nabla \rho \in \mathcal{C}([0, T); B_{p,r}^{s-1})$ ,

<sup>4</sup>With the usual convention if  $r = \infty$

- $u \in \mathcal{C}([0, T]; B_{p,r}^s) \cap \mathcal{C}^1([0, T]; L^2)$ ,
- $\nabla \Pi \in \mathcal{C}([0, T]; L^2) \cap L^1([0, T]; B_{p,r}^s)$ .

If in addition Condition (42) is satisfied then  $(\rho, u, \nabla \Pi)$  may be continued beyond  $T$  into a solution of (1) with the above regularity.

*Proof.* Lemma 4 ensures that  $\|u\|_{L_T^\infty(B_{p,r}^s)}$  and  $\|Da\|_{L_T^\infty(B_{p,r}^{s-1})}$  are finite. So one may set

$$\varepsilon := c(\rho^* A_0)^{-1} \left( U_0(T) + \rho^* A_0 \|\operatorname{div} f\|_{L_T^1(B_{p,r}^{s-1})} + (\rho^* A_0)^{\gamma+1} (\|u_0\|_{L^2} + \|f\|_{L_T^1(L^2)}) \right)^{-1}$$

where  $c$  is the small constant (depending only on  $N$  and  $(s, p, r)$ ) defined in (36).

Then we know from the proof of Theorem 1 in the case  $p \geq 2$  that for any  $T' < T$ , System (1) with data  $(\rho(T'), u(T'), f(T' + \cdot))$  has a unique solution up to time  $\varepsilon$ . Taking  $T' = T - \varepsilon/2$  we thus get a continuation of  $(\rho, u, \nabla \Pi)$  up to time  $T + \varepsilon/2$ .  $\square$

Let us now justify the last part of Theorem 2. It stems from the following logarithmic interpolation inequality (see e.g. [15]):

$$\|\nabla u\|_{L^\infty} \leq C \left( 1 + \|\nabla u\|_{\dot{B}_{\infty,\infty}^0} \log(e + \|\nabla u\|_{B_{p,r}^{s-1}}) \right) \quad \text{with} \quad \|\nabla u\|_{\dot{B}_{\infty,\infty}^0} := \sup_{q \in \mathbb{Z}} \|\varphi(2^{-q} D) \nabla u\|_{L^\infty}$$

which holds true whenever the embedding of  $B_{p,r}^{s-1}$  is not critical (that is  $s > 1 + N/p$ ).

Then, arguing exactly as in Proposition 5.3 of [11], we discover that Condition (42) may be replaced by the following *weaker* condition:

$$(48) \quad \int_0^T (\|\nabla u\|_{\dot{B}_{\infty,\infty}^0} + \|\nabla \Pi\|_{B_{p,r}^{s-1}}) dt < \infty.$$

Now, it is classical (see e.g. [2], Chap. 7) that there exists some constant  $C$  such that

$$\|\nabla u\|_{\dot{B}_{\infty,\infty}^0} \leq C \|\operatorname{curl} u\|_{L^\infty}.$$

This completes the proof of Theorem 2 in the case  $p \geq 2$ .

**4.4. The case  $1 < p < 2$ .** Note that by virtue of Proposition 2, the data satisfy the assumptions of the theorem for the triplet  $(s - N/p + N/2, 2, r)$ . Hence, applying the theorem in the case  $p = 2$  supplies a local solution with the  $B_{2,r}^{s-N/p+N/2}$  regularity. However, proving that the  $B_{p,r}^s$  regularity is also preserved, is not utterly obvious. For proving that, we shall proceed as follows:

- i) first, we smooth out the data so as to get a solution in  $H^\infty := \cap_\sigma H^\sigma$  for which the  $B_{p,r}^s$  regularity is also preserved;
- ii) second, we establish uniform bounds in  $B_{p,r}^s$  on a fixed suitably small time interval;
- iii) third, we show the convergence of the sequence of smooth solutions and that the limit has the required properties.

*Step 1: smooth solutions.* Set

$$a_0^n := S_n a_0, \quad u_0^n := S_n u_0 \quad \text{and} \quad f^n := S_n f$$

where  $S_n$  is the low frequency cut-off introduced in Section 2.

Note that for all large enough  $n \in \mathbb{N}$  and  $t \in [0, T_0]$ , we have

$$(49) \quad a_*/2 \leq a_0^n \leq 2a^*, \quad \|Da_0^n\|_{B_{p,r}^{s-1}} \leq C \|Da_0\|_{B_{p,r}^{s-1}},$$

$$(50) \quad \|u_0^n\|_{B_{p,r}^s} \leq C \|u_0\|_{B_{p,r}^s},$$

$$(51) \quad \|f^n\|_{L_t^1(B_{p,r}^s)} \leq C \|f\|_{L_t^1(B_{p,r}^s)} \quad \text{and} \quad \|f^n(t)\|_{L^2} \leq \|f(t)\|_{L^2}.$$

It is also clear that (with obvious notation)  $\nabla a_0^n$  and  $u_0^n$  are in  $B_{p,r}^\infty$  (hence also in  $H^\infty$ ) and that  $f^n \in \mathcal{C}([0, T_0]; H^\infty) \cap L^1([0, T_0]; B_{p,r}^\infty)$ .

Finally, taking advantage of Lebesgue's dominated convergence theorem one may prove, if  $r < \infty$ , that<sup>5</sup>

$$\begin{aligned} Da_0^n &\longrightarrow Da_0 \text{ in } B_{p,r}^{s-1} \quad \text{and} \quad (a_0^n - \bar{a}) \rightarrow (a_0 - \bar{a}) \text{ in } L^{p^*} \text{ with } \bar{a} := 1/\bar{\rho}, \\ u_0^n &\longrightarrow u_0 \text{ in } B_{p,r}^{s-1}, \\ f^n &\longrightarrow f \text{ in } L^1([0, T_0]; B_{p,r}^s) \cap \mathcal{C}([0, T_0]; L^2). \end{aligned}$$

As usual, the strong convergence has to be replaced by the weak convergence if  $r = \infty$ .

Applying Theorem 1 in the case  $p = 2$  and using the fact that the lifespan does not depend on the index of regularity (see Remark 1), we get a local maximal solution  $(a^n, u^n, \nabla \Pi^n)$  with  $Da^n$ ,  $u^n$  and  $\nabla \Pi^n$  in  $\mathcal{C}([0, T_n^*]; H^\infty)$ , and

$$(52) \quad a_*/2 \leq a^n \leq 2a^*.$$

Note that as  $a^n$  and  $\rho^n$  are just transported by the (smooth) flow of  $u^n$ , we also have

$$(53) \quad \|(\rho^n(t) - \bar{\rho})\|_{L^{p^*}} = \|\rho_0^n - \bar{\rho}\|_{L^{p^*}} \leq C\|\rho_0 - \bar{\rho}\|_{L^{p^*}} \quad \text{for all } t \in [0, T_n^*]$$

(and similarly for  $a^n$ ) and  $\nabla a^n$  and  $\nabla \rho^n$  belong to  $\mathcal{C}([0, T_n^*]; B_{p,r}^{s-1})$ .

Let us now establish that  $\nabla \Pi^n$  is in  $L^1([0, T]; B_{p,r}^s)$  for all  $T \in [0, T_n^*)$ . Fix some  $T \in [0, T_n^*)$ . Applying Operator  $\operatorname{div}$  to the momentum equation of (1) and using that  $\operatorname{div} u^n = 0$  yields

$$(54) \quad \Delta \Pi^n = \operatorname{div}(\rho^n f^n) - \operatorname{div}(\rho^n u^n \cdot \nabla u^n) - \nabla \rho^n \cdot \partial_t u^n.$$

According to Proposition 1,  $F \mapsto D^2 \Pi$  is a self-map on  $B_{p,r}^s$ . Hence, in order to show that  $\nabla \Pi^n \in L^1([0, T]; B_{p,r}^s)$ , it suffices to establish that  $\nabla \Pi^n \in L^1([0, T]; L^p)$  and that  $\Delta \Pi^n \in L^1([0, T]; B_{p,r}^{s-1})$ .

Let us first show that all the terms of the right-hand side of (54) are in  $L^1([0, T]; B_{p,r}^{s-1})$ . Since, by assumption,  $f^n \in L^1([0, T]; B_{p,r}^s)$  and as it has been established that  $\rho^n \in L^\infty$  and  $\nabla \rho^n \in B_{p,r}^{s-1}$ , Corollary 1 implies that  $\operatorname{div}(\rho^n f^n) \in L^1([0, T]; B_{p,r}^{s-1})$ . For the next term, we use that for all  $i \in \{1, \dots, N\}$ ,

$$(\rho^n u^n \cdot \nabla u^n)^i = \sum_j T'_{\rho^n(u^n)_j} \partial_j u^n + T_{\partial_j(u^n)_i} \rho^n (u^n)^j.$$

By embedding,  $\rho^n u^n$  and  $\nabla u^n$  are in  $L^{p^*}$  (recall that  $p^* > 2 > p$ ) and, arguing as for  $\rho^n f^n$ , one can check that  $\rho^n u^n$  is in  $H^\infty$ . Of course,  $\nabla u^n$  is also in  $H^\infty$ . Given that  $1/p = 1/p^* + 1/2$ , continuity results for the paraproduct and remainder in the spirit of Proposition 3 (see [18]) ensure that  $\rho^n u^n \cdot \nabla u^n$  is in  $B_{p,r}^{s-1}$ .

For the last term in (54), one may write that

$$\nabla \rho^n \cdot \partial_t u^n = T'_{\partial_t u^n} \cdot \nabla \rho^n + T_{\nabla \rho^n} \cdot \partial_t u^n.$$

As, by embedding,  $\partial_t u^n \in L^\infty$ , and as  $\nabla \rho^n \in B_{p,r}^{s-1}$ , continuity results for the paraproduct ensure that the first term in the right-hand side is in  $B_{p,r}^{s-1}$ . Concerning the second term, one may use that  $\nabla \rho^n \in L^{p^*}$  (by embedding) and that  $\partial_t u^n \in H^\infty$  (from the equation). Hence  $\Delta \Pi^n$  is indeed in  $L^1([0, T]; B_{p,r}^{s-1})$ , as claimed above.

In order to establish that  $\nabla \Pi^n \in L^1([0, T]; L^p)$ , we use the fact that, owing to  $\operatorname{div} \partial_t u^n = 0$ , one may write

$$(55) \quad \Delta \Pi^n = \operatorname{div}(\rho^n f^n - \rho^n u^n \cdot \nabla u^n - (\rho^n - \bar{\rho}) \partial_t u^n).$$

Hence, it suffices to check that  $\rho^n f^n$ ,  $\rho^n u^n \cdot \nabla u^n$  and  $(\rho^n - \bar{\rho}) \partial_t u^n$  are in  $L^1([0, T]; L^p)$ . For  $\rho^n u^n$  this is obvious as, by embedding,  $f^n \in L^1([0, T]; L^p)$  and  $u^n \in \mathcal{C}_b([0, T] \times \mathbb{R}^N)$ . By embedding, we also have  $\nabla u^n \in \mathcal{C}([0, T]; L^\infty)$  and  $u^n \in \mathcal{C}([0, T]; L^p)$ , hence  $\rho^n u^n \cdot \nabla u^n$  is in  $L^1([0, T]; L^p)$ .

To deal with the last term in (55) the property that  $(\rho^n - \bar{\rho}) \in \mathcal{C}([0, T]; L^{p^*})$  comes into play. Indeed, from the velocity equation, as the solution is in  $H^\infty$ , one easily gathers that  $\partial_t u^n$

<sup>5</sup>Recall that  $(\rho_0 - \bar{\rho}) \in L^{p^*}$  by assumption

belongs to  $\mathcal{C}([0, T]; L^2)$ . Hence Hölder's inequality (note that  $1/2 + 1/p^* = 1/p$ ) ensures that  $(\rho^n - \bar{\rho})\partial_t u^n \in \mathcal{C}([0, T]; L^p)$ .

To finish this step, one has to prove that  $u^n$  is in  $\mathcal{C}([0, T_n^*]; B_{p,r}^s)$ . In fact, from the product laws in Besov spaces and the properties of regularity that have been just established for the pressure and the density, we get

$$\partial_t u^n + u^n \cdot \nabla u^n = f^n - a^n \nabla \Pi^n \in L_{loc}^1([0, T_n^*]; B_{p,r}^s).$$

As  $u_0^n \in B_{p,r}^s$ , Proposition 5 ensures that  $u^n \in \mathcal{C}([0, T_n^*]; B_{p,r}^s)$ .

*Step 2: Uniform estimates.* Let us remark that, by Sobolev embedding and owing to (49), (50), (51), one may find some index  $\sigma > d/2 + 1$  such that  $(Da_0^n)_{n \in \mathbb{N}}$ ,  $(u_0^n)_{n \in \mathbb{N}}$  and  $(f^n)_{n \in \mathbb{N}}$  are bounded in  $H^{\sigma-1}$ ,  $H^\sigma$  and  $\mathcal{C}([0, T_0]; L^2) \cap L^1([0, T_0]; H^\sigma)$ , respectively. Taking advantage of Theorem 1 in the case  $p = 2$  and of the lower bound provided by (36) we thus deduce that there exists some time  $T > 0$  and some  $M > 0$  such that for all  $n \in \mathbb{N}$ , we have  $T_n^* > T$  and

$$(56) \quad \|\nabla a^n\|_{L_T^\infty(H^{\sigma-1})} + \|u^n\|_{L_T^\infty(H^\sigma)} + \|\nabla \Pi^n\|_{L_T^1(H^\sigma)} \leq M.$$

Of course the energy equality (3) is satisfied on  $[0, T]$  by any solution  $(a^n, u^n, \nabla \Pi^n)$ . Recall that in addition, according to the previous step of the proof, (53) is satisfied and

$$\nabla a^n \in \mathcal{C}([0, T]; B_{p,r}^{s-1}), \quad u^n \in \mathcal{C}([0, T]; B_{p,r}^s) \quad \text{and} \quad \nabla \Pi^n \in L^1([0, T]; B_{p,r}^s).$$

We claim that, up to a change of  $T$ , the norm of the solution may be bounded *independently* of  $n$  in the space  $E_T$  defined in Subsection 4.2. In all that follows, we denote by  $C_M$  a “constant” depending only on  $(s, p, r, N, a_*, a^*)$  and on  $M$ .

From Proposition 5, we have

$$(57) \quad \|\nabla a^n(t)\|_{B_{p,r}^s} \leq \|\nabla a_0^n\|_{B_{p,r}^s} e^{C \int_0^t \|\nabla u^n\|_{B_{p,r}^{s-1}} d\tau}$$

and, arguing as for proving Inequality (28),

$$(58) \quad \begin{aligned} \|u^n(t)\|_{B_{p,r}^s} &\leq e^{C \int_0^t \|\nabla u^n\|_{B_{p,r}^{s-1}} d\tau} \left( \|u_0^n\|_{B_{p,r}^s} \right. \\ &\quad \left. + \int_0^t e^{-C \int_0^\tau \|\nabla u^n\|_{B_{p,r}^{s-1}} d\tau'} \left( \|f^n\|_{B_{p,r}^s} + (a^* + \|\nabla a^n\|_{B_{p,r}^{s-1}}) \|\nabla \Pi^n\|_{B_{p,r}^s} \right) d\tau \right). \end{aligned}$$

In order to bound  $\nabla \Pi^n$ , we apply the first part of Proposition 7 to the following equation:

$$\operatorname{div}(a^n \nabla \Pi^n) = \operatorname{div}(f^n - u^n \cdot \nabla u^n).$$

Using the fact that  $B_{p,r}^{s-1}$  is an algebra and the relation  $\operatorname{div}(u^n \cdot \nabla u^n) = \nabla u^n : \nabla u^n$ , we end up with

$$(59) \quad a_* \|\nabla \Pi^n\|_{L_t^1(B_{p,r}^s)} \leq C \left( \|f^n\|_{L_t^1(B_{p,r}^s)} + \int_0^t \|u^n\|_{B_{p,r}^s}^2 d\tau + a_* \left( 1 + \frac{\|Da^n\|_{L_t^\infty(B_{p,r}^{s-1})}}{a_*} \right)^s \|\nabla \Pi^n\|_{L_t^1(L^p)} \right).$$

In order to “close the estimate”, we now have to bound  $\nabla \Pi^n$  in  $L^p$ . For that, we apply the standard  $L^p$  elliptic estimates stated in Proposition 6 to (55), and Hölder inequality so as to get

$$\|\nabla \Pi^n\|_{L_t^1(L^p)} \leq C \left( \rho^* \left( \|f^n\|_{L_t^1(L^p)} + \int_0^t \|u^n\|_{L^{p^*}} \|\nabla u^n\|_{L^2} d\tau \right) + \|\rho^n - \bar{\rho}\|_{L_t^\infty(L^{p^*})} \|\partial_t u^n\|_{L_t^1(L^2)} \right).$$

Note that, by Sobolev embedding, we have

$$\|u^n\|_{L^{p^*}} \leq C \|u^n\|_{H^\sigma}.$$

So finally, there exists some constant  $C_M$  such that

$$\|\nabla \Pi^n\|_{L_T^1(L^p)} \leq C_M.$$

Plugging this latter inequality in (59), we thus get

$$a_* \|\nabla \Pi^n\|_{L_t^1(B_{p,r}^s)} \leq C \left( \|f^n\|_{L_t^1(B_{p,r}^s)} + \int_0^t \|\nabla u^n\|_{B_{p,r}^{s-1}}^2 d\tau + a_* C_M \left( 1 + \frac{\|Da^n\|_{L_t^\infty(B_{p,r}^{s-1})}}{a_*} \right)^s \right).$$

It is now easy to conclude this step: denoting

$$U^n(t) := \|u^n(t)\|_{B_{p,r}^s} \quad \text{and} \quad A^n(t) := a^* + \|Da^n(t)\|_{B_{p,r}^{s-1}},$$

and assuming that  $\bar{T} \leq T$  has been chosen so that

$$(60) \quad C \int_0^{\bar{T}} \|\nabla u^n\|_{B_{p,r}^{s-1}} d\tau \leq \log 2,$$

the above inequalities and (49), (50), (51) imply that for all  $t \in [0, \bar{T}]$  we have

$$A^n(t) \leq 2A_0 \quad \text{with} \quad A_0 := a^* + \|Da_0\|_{B_{p,r}^{s-1}}$$

and

$$U^n(t) \leq 2(U_0(t) + C\rho^* A_0 \int_0^t (\|f\|_{B_{p,r}^s} + (U^n)^2 + C_M a_* A_0^s) d\tau).$$

So finally, there exists a nondecreasing function  $F$  depending only on the norm of the data and such that for all  $t \in [0, \bar{T}]$ , we have

$$U^n(t) \leq 2F(t) + C\rho^* A_0 \int_0^t (U^n(\tau))^2 d\tau.$$

Therefore, if in addition

$$(61) \quad 2CA_0 \int_0^{\bar{T}} U^n(\tau) d\tau \leq a_*$$

then we have  $U^n \leq 4F$  on  $[0, \bar{T}]$ .

By arguing exactly as in the case  $p \geq 2$ , it is easy to see that Condition (61) is satisfied if  $\bar{T}$  is small enough (an explicit lower bound may be obtained in terms of the data). So finally, we have found a positive time  $T$  so that  $(a^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  is bounded in the space  $E_T$ .

*Step 3: Convergence of the sequence.* Let  $\delta u^n := u^{n+1} - u^n$ ,  $\delta \rho^n := \rho^{n+1} - \rho^n$  and  $\delta \Pi^n = \Pi^{n+1} - \Pi^n$ . Applying Inequality (16) to the solutions  $(\rho^n, u^n, \nabla \Pi^n)$  and  $(\rho^{n+1}, u^{n+1}, \nabla \Pi^{n+1})$  and using the uniform bounds that have been established in the previous step, and (52) ensures that there exists some  $M > 0$  such that for all  $t \in [0, T]$  and  $n \in \mathbb{N}$ , we have

$$(62) \quad \|\delta \rho^n(t)\|_{L^2} + \|\delta u^n(t)\|_{L^2} \leq M \left( \|\delta \rho^n(0)\|_{L^2} + \|\delta u^n(0)\|_{L^2} + \int_0^t \|\delta f^n\|_{L^2} d\tau \right).$$

Now, from the definition of  $a_0^n$  and the mean value theorem, we get for large enough  $n$ ,

$$\|\delta \rho^n(0)\|_{L^2} \leq C2^{-n} \|Da_0\|_{L^2}.$$

Similarly, we have

$$\|\delta u^n(0)\|_{L^2} + \|\delta f^n\|_{L_T^1(L^2)} \leq C2^{-n} (\|\delta u_0\|_{L^2} + \|\delta f\|_{L_T^1(L^2)}).$$

So Inequality (62) entails that  $(\rho^n - \rho^0)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{C}([0, T]; L^2)$  and that  $(u^n)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{C}([0, T]; L^2)$ . Then, using for instance (55), we see that  $(\nabla \Pi^n)_{n \in \mathbb{N}}$  is also a Cauchy sequence in  $\mathcal{C}([0, T]; L^2)$ .

Finally, from the bounds in large norm that have been stated in the previous step, and the Fatou property for the Besov space, one may conclude that the limit  $(a, u, \nabla \Pi)$  to  $(a^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  converges to some solution (1) and has the desired properties of regularity. As similar arguments have been used for handling the case  $p \geq 2$ , the details are left to the reader.

Let us now establish Theorem 2 in the case  $p < 2$ . Let  $(\rho, u, \nabla \Pi)$  be a solution with the properties described in Theorem 1. Note that Lemma 4 is also true if  $p < 2$ . So the only change lies in the proof of Lemma 5 which now uses the (new) lower bound for the lifespan that may be obtained from the computations of step 2, instead of (36). This gives the first part of Theorem 2. As in the case  $p \geq 2$ , the last part of the proof of the theorem is a mere consequence of the logarithmic interpolation inequality stated in [15].

**4.5. Removing the assumptions on the low frequency of the data.** As pointed out in Section 1, in dimension  $N \geq 3$ , the supplementary assumption that  $(\rho_0 - \bar{\rho}) \in L^{p^*}$  is not needed if  $p > N/(N-1)$ .

In order to see that, one may repeat the proof of the theorem in the case  $1 < p \leq 2$ . As before, bounding  $\nabla \Pi^n$  in  $L_T^1(L^p)$  is the main difficulty. For that, one may decompose  $\nabla \Pi^n$  into two terms  $\nabla \Pi_1^n$  and  $\nabla \Pi_2^n$  such that

$$\Delta \Pi_1^n = \operatorname{div}(\rho^n f^n - \rho^n u^n \cdot \nabla u^n) \quad \text{and} \quad \Delta \Pi_2^n = \nabla \rho^n \cdot \partial_t u^n.$$

On the one hand, as before, one may write that

$$\|\nabla \Pi_1^n\|_{L^p} \leq C \rho^* (\|f^n\|_{L^p} + \|u^n\|_{L^{p^*}} \|\nabla u^n\|_{L^2}).$$

On the other hand, we have

$$\nabla \Pi_2^n = (-\Delta)^{-1} \nabla (\nabla \rho^n \cdot \partial_t u^n).$$

Recall that in dimension  $N \geq 2$ , the kernel of Operator  $(-\Delta)^{-1} \nabla$  behaves as  $|x|^{1-N}$ . Hence, according to the Hardy-Littlewood-Sobolev inequality, if  $1/p + 1/N < 1$  then we have

$$\|\nabla \Pi_2^n\|_{L^p} \leq C \|\nabla \rho^n \cdot \partial_t u^n\|_{L^q} \quad \text{with} \quad \frac{1}{q} = \frac{1}{p} + \frac{1}{N}.$$

As  $(\nabla \rho^n)_{n \in \mathbb{N}}$  may be bounded in  $\mathcal{C}([0, T]; L^p)$  and, by embedding,  $(\partial_t u^n)_{n \in \mathbb{N}}$  may be bounded in  $L^1([0, T]; L^N)$ , *in terms of Sobolev norms only*, it is thus possible to get a bound of  $\nabla \Pi_2^n$  in  $L^1([0, T]; L^p)$  in terms of the initial data. The rest of the proof goes by the steps that we used before.

## 5. THE PROOF OF THEOREM 3

For  $T > 0$ , let us introduce the set  $F_T$  of functions  $(a, u, \nabla \Pi)$  such that

$$\begin{aligned} a &\in \mathcal{C}_b([0, T] \times \mathbb{R}^N), & Da &\in \mathcal{C}([0, T]; B_{p,r}^{s-1}), \\ u &\in \mathcal{C}([0, T]; B_{p,r}^s), & \nabla \Pi &\in \mathcal{C}([0, T]; L^2) \cap L^1([0, T]; B_{p,r}^s). \end{aligned}$$

Uniqueness in Theorem 3 stems from Proposition 8. Indeed, we see that, as  $p \leq 4$ , any solution  $(\rho, u, \nabla \Pi)$  in  $F_T$  satisfies  $u \in \mathcal{C}([0, T]; W^{1,4})$  (according to Proposition 2 and to the remark that follows) and  $\nabla \Pi \in \mathcal{C}([0, T]; L^2)$ . Therefore, using the velocity equation and Hölder's inequality, we get

$$(63) \quad (\partial_t u - f) = -(u \cdot \nabla u + a \nabla \Pi) \in \mathcal{C}([0, T]; L^2).$$

Note that, as  $u$  and  $\nabla \rho$  are in  $\mathcal{C}([0, T]; L^4)$ , we have

$$(64) \quad \partial_t \rho = -u \cdot \nabla \rho \in \mathcal{C}([0, T]; L^2).$$

Now, consider two solutions  $(\rho_1, u_1, \nabla \Pi_1)$  and  $(\rho_2, u_2, \nabla \Pi_2)$  in  $F_T$ , corresponding to the same data. Then (63) implies that  $\delta u := u_2 - u_1$  belongs to  $\mathcal{C}^1([0, T]; L^2)$  while (64) guarantees that  $\delta \rho := \rho_2 - \rho_1$  is in  $\mathcal{C}^1([0, T]; L^2)$ . So Proposition 8 applies and yields uniqueness.

Let us now tackle the proof of the existence part of the theorem. We claim that if we restrict our attention to solutions which are  $F_T$  then the assumptions of Lemma 3 are fulfilled so that it suffices to solve System (18). Indeed, it is only a matter of checking whether  $\mathcal{Q}u$  is in  $\mathcal{C}([0, T]; L^2)$ . Applying  $\mathcal{Q}$  to the velocity equation of (18), we get

$$\partial_t \mathcal{Q}u = \mathcal{Q}f - \mathcal{Q}(a \nabla \Pi) - \mathcal{Q}(u \cdot \nabla u).$$

From the assumptions on  $f$ , the definition of  $F_T$  and the fact that  $\mathcal{Q}$  maps  $L^2$  in  $L^2$ , we see that the first two terms in the right-hand side are in  $\mathcal{C}([0, T]; L^2)$ . Concerning the last term, we just use the fact that, as pointed out above,  $u \cdot \nabla u$  belongs to  $\mathcal{C}([0, T]; L^2)$  so and  $\mathcal{Q}(u \cdot \nabla u)$ , too.

Let us now go to the proof of the existence of a local-in-time solution for (18) under the assumptions of Theorem 3. Compared to Theorem 1, the main change is that we do not expect to have  $u \in \mathcal{C}([0, T]; L^2)$  any longer (i.e. the energy may be infinite). However, as the pressure satisfies

$$-\operatorname{div}(a \nabla \Pi) = \operatorname{div}(u \cdot \nabla \mathcal{P}u) - \operatorname{div} f,$$

Lemma 2 will ensure that  $\nabla \Pi \in \mathcal{C}([0, T]; L^2)$  anyway if  $u \cdot \nabla \mathcal{P}u$  belongs to  $\mathcal{C}([0, T]; L^2)$ . In view of Proposition 2, Remark 2 and Hölder's inequality, this latter property is guaranteed by the fact that  $u \in \mathcal{C}([0, T]; B_{p,r}^s)$  for some  $p \leq 4$ .

Once this has been noticed, one may use the same approximation scheme as in Theorem 1: we first set  $(a^0, u^0, \nabla \Pi^0) := (a_0, u_0, 0)$ . Next, we assume that  $(a^n, u^n, \nabla \Pi^n)$  has been constructed over  $\mathbb{R}^+$ , belongs to the space  $F_T$  for all  $T > 0$  and that there exists a positive time  $T^*$  such that (19) is satisfied for all  $t \in [0, T^*]$  and, for suitable constants  $C_0$  and  $C$  (one can take  $C_0 = 2C^2$ ),

$$(65) \quad \|\nabla a^{n+1}(t)\|_{B_{p,r}^{s-1}} \leq 2\|\nabla a_0\|_{B_{p,r}^{s-1}} \quad \text{for all } t \in [0, T^*],$$

$$(66) \quad U^n(t) \leq 4U_0(t) + C_0 \rho^* A_0 \left( \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} + (\rho^* A_0)^\gamma \|\mathcal{Q}f\|_{L_t^1(L^2)} \right),$$

$$(67) \quad a_* \|\nabla \Pi^n\|_{L_t^1(B_{p,r}^s)} \leq C \left( \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} + (\rho^* A_0)^\gamma \int_0^t ((U^n)^2 + \|\mathcal{Q}f\|_{L^2}) d\tau \right)$$

with  $A_0 := a^* + \|Da_0\|_{B_{p,r}^{s-1}}$ ,  $U_0(t) := \|u_0\|_{B_{p,r}^s} + \|f\|_{L_t^1(B_{p,r}^s)}$  and  $U^n(t) := \|u^n(t)\|_{B_{p,r}^s}$ .

Arguing exactly as in the proof of Theorem 1, we see that if we define  $a^{n+1}$  as the solution to

$$\partial_t a^{n+1} + u^n \cdot \nabla a^{n+1}, \quad a|_{t=0}^{n+1} = a_0$$

then (19) is satisfied for all time,  $\nabla a^{n+1} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^{s-1})$  and

$$(68) \quad \|\nabla a^{n+1}(t)\|_{B_{p,r}^{s-1}} \leq e^{C \int_0^t U^n(\tau) d\tau} \|\nabla a_0\|_{B_{p,r}^{s-1}}.$$

So if we assume that  $T^*$  has been chosen so that

$$(69) \quad C \int_0^{T^*} U^n(t) dt \leq \log 2$$

then  $a^{n+1}$  satisfies (65).

Next, we take  $u^{n+1}$  to be the unique solution in  $\mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$  of the transport equation (27). As before, the right-hand side of (27) belongs to  $\mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$  and one may use (28). So finally, the existence of  $u^{n+1} \in \mathcal{C}(\mathbb{R}^+; B_{p,r}^s)$  is ensured by Proposition 5, and we have

$$\begin{aligned} \|u^{n+1}(t)\|_{B_{p,r}^s} &\leq e^{C \int_0^t U^n(\tau) d\tau} \left( \|u_0\|_{B_{p,r}^s} \right. \\ &\quad \left. + \int_0^t e^{-C \int_0^\tau U^n(\tau') d\tau'} \left( \|a^{n+1}\|_{L^\infty} + \|\nabla a^{n+1}\|_{B_{p,r}^{s-1}} \right) \|\nabla \Pi^n\|_{B_{p,r}^s} + \|f\|_{B_{p,r}^s} d\tau \right). \end{aligned}$$

Therefore, if we restrict our attention to those  $t$  that are in  $[0, T^*]$  with  $T^*$  satisfying (69), and use Inequality (68), we see that for all  $t \in [0, T^*]$ ,

$$U^{n+1}(t) \leq 2U_0(t) + CA_0 \int_0^t \|\nabla \Pi^n\|_{B_{p,r}^s} d\tau \quad \text{with } A_0 := a^* + \|\nabla a_0\|_{B_{p,r}^{s-1}}.$$

So if we assume that  $C_0 = 2C^2$  and that  $T^*$  has been chosen so that

$$(70) \quad C^2 \rho^* A_0 \int_0^{T^*} U^n(t) dt \leq \frac{1}{2}$$

then taking advantage of Inequality (67), we see that  $u^{n+1}$  satisfies (66) on  $[0, T^*]$ .

To finish with, in order to construct the approximate pressure  $\Pi^{n+1}$ , we solve the elliptic equation (33) for every positive time. Recall that we have  $\operatorname{div}(u^{n+1} \cdot \nabla \mathcal{P}u^{n+1}) \in B_{p,r}^{s-1}$  and that

$$\|\operatorname{div}(u^{n+1} \cdot \nabla \mathcal{P}u^{n+1})\|_{B_{p,r}^{s-1}} \leq C(U^{n+1})^2.$$

Next, given our assumptions on  $(s, p, r)$  we have  $B_{p,r}^s \hookrightarrow W^{1,4}$ . Therefore, since  $\mathcal{P}$  maps  $L^4$  in  $L^4$ , one may write

$$\begin{aligned} \|u^{n+1} \cdot \nabla \mathcal{P}u^{n+1}\|_{L^2} &\leq \|u^{n+1}\|_{L^4} \|\mathcal{P}u^{n+1}\|_{L^4}, \\ &\leq C\|u^{n+1}\|_{L^4} \|u^{n+1}\|_{W^{1,4}}, \\ &\leq C(U^{n+1})^2. \end{aligned}$$

Therefore, the second part of Proposition 7 ensures that  $\nabla \Pi^{n+1}$  is well defined in  $\mathcal{C}(\mathbb{R}^+; L^2) \cap L_{loc}^1(\mathbb{R}^+; B_{p,r}^s)$  and that

$$a_* \|\nabla \Pi^{n+1}\|_{L_t^1(B_{p,r}^s)} \leq C \left( \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} + (1 + \rho^* \|Da^{n+1}\|_{L_t^\infty(B_{p,r}^{s-1})})^\gamma \int_0^t (U^{n+1})^2 + \|\mathcal{Q}f\|_{L^2} d\tau \right).$$

Taking advantage of Inequality (65) at rank  $n+1$ , one can now conclude that  $\nabla \Pi^{n+1}$  satisfies (67).

At this stage we have proved that if Inequalities (65), (66) and (67) hold for  $(a^n, u^n, \nabla \Pi^n)$  then they also hold for  $(a^{n+1}, u^{n+1}, \nabla \Pi^{n+1})$  provided  $T^*$  satisfies Inequality (70). One may easily check that this is indeed the case if we set

$$(71) \quad T^* := \sup \left\{ t > 0 / \rho^* A_0 t \left( U_0(t) + \rho^* A_0 \|\operatorname{div} f\|_{L_t^1(B_{p,r}^{s-1})} + (\rho^* A_0)^{\gamma+1} \|\mathcal{Q}f\|_{L_t^1(L^2)} \right) \leq c \right\}$$

for a small enough constant  $c$  depending only on  $s, p$  and  $N$ .

Once the bounds in  $F_{T^*}$  have been established, the last steps of the proof are almost identical to those of Theorem 1. Indeed, introducing

$$\tilde{a}^n(t, x) := a^n(t, x) - a_0(x) \quad \text{and} \quad \tilde{u}^n(t, x) := u^n(t, x) - u_0(x) - \int_0^t f(\tau, x) d\tau$$

and observing that  $\delta u^n := u^{n+1} - u^n = \tilde{u}^{n+1} - \tilde{u}^n$ , one can use exactly the same computations as before for bounding  $\delta a^n$ ,  $\delta u^n$  and  $\nabla \delta \Pi^n$ . As a consequence  $(\tilde{a}^n, \tilde{u}^n, \nabla \Pi^n)$  is a Cauchy sequence in  $\mathcal{C}([0, T^*]; L^2)$ . Next, the bounds (65), (66) and (67) enable us to show that the limit is indeed in  $F_T$  and satisfies (18). The details are left to the reader.

Let us finally establish the continuation criterion. Note that Lemma 4 still applies in the context of infinite energy solutions. Hence, repeating the proof of Lemma 5 and using the logarithmic interpolation inequality of [15] yields the result. This completes the proof of Theorem 3.

## 6. THE PROOF OF THEOREM 4

In this section, we aim at investigating the well-posedness issue of System (1) in Hölder spaces  $\mathcal{C}^s$  (which coincide with the Besov spaces  $B_{\infty, \infty}^s$  if  $s$  is not an integer), and, more generally, in Besov spaces of type  $B_{\infty, r}^s$ . Of particular interest is the case of the Besov space  $B_{\infty, 1}^1$  which is the largest one for which Condition (C) holds.

The main difficulty is that the previous proofs were based on the elliptic estimate stated in Proposition 7 which fails in the limit case  $p = \infty$ . In this section, we shall see that the case of a small perturbation of a constant density state may be handled by a different approach.

**6.1. The proof of uniqueness.** Note that in the case  $p > 4$  the solution provided by Theorem 4 need not satisfy  $\nabla\Pi \in L^1([0, T]; L^2)$ . Hence  $\partial_t u$  need not be in  $L^1([0, T]; L^2)$  and the assumptions of Proposition 8 are not satisfied.

So, in order to prove uniqueness, we shall prove stability estimates in  $L^p$  rather than in  $L^2$ . These estimates will be also needed in the last step of the proof of the existence part of Theorem 4.

Consider two solutions  $(\rho_1, u_1, \nabla\Pi_1)$  and  $(\rho_2, u_2, \nabla\Pi_2)$  of (1). Let  $a_1 := 1/\rho_1$  and  $a_2 := 1/\rho_2$ . As usual, denote  $\delta a := a_2 - a_1$ ,  $\delta u := u_2 - u_1$ ,  $\nabla\delta\Pi := \nabla\Pi_2 - \nabla\Pi_1$  and  $\delta f := f_2 - f_1$ . First, as  $\operatorname{div} u_2 = 0$  and

$$\partial_t \delta a + u_2 \cdot \nabla \delta a = -\delta u \cdot \nabla a_1,$$

one may write

$$(72) \quad \|\delta a(t)\|_{L^p} \leq \|\delta a(0)\|_{L^p} + \int_0^t \|\nabla a_1\|_{L^\infty} \|\delta u\|_{L^p} d\tau.$$

Next, as

$$\partial_t \delta u + u_2 \cdot \nabla \delta u = \delta f - \delta u \cdot \nabla u_1 - \delta a \nabla \Pi_1 - a_2 \nabla \delta \Pi,$$

we have

$$(73) \quad \|\delta u(t)\|_{L^p} \leq \|\delta u(0)\|_{L^p} + \int_0^t \left( \|\delta f\|_{L^p} + \|\nabla u_1\|_{L^\infty} \|\delta u\|_{L^p} + \|\nabla \Pi_1\|_{L^\infty} \|\delta a\|_{L^p} + a^* \|\nabla \delta \Pi\|_{L^p} \right) d\tau.$$

Finally, we notice that  $\nabla \delta \Pi$  satisfies the elliptic equation

$$(74) \quad \operatorname{div} (a_2 \nabla \delta \Pi) = \operatorname{div} \delta f - \operatorname{div} (\delta a \nabla \Pi_1) - \operatorname{div} (\delta u \cdot \nabla u_1) - \operatorname{div} (u_2 \cdot \nabla \delta u).$$

The key point here is that, owing to  $\operatorname{div} u_2 = \operatorname{div} \delta u = 0$ , we have

$$\operatorname{div} (u_2 \cdot \nabla \delta u) = \operatorname{div} (\delta u \cdot \nabla u_2).$$

Hence Equality (74) rewrites

$$a_* \Delta \delta \Pi = \operatorname{div} \delta f + \operatorname{div} ((a_* - a_2) \nabla \delta \Pi) - \operatorname{div} (\delta a \nabla \Pi_1) - \operatorname{div} (\delta u \cdot \nabla (u_1 + u_2))$$

so that the  $L^p$  elliptic estimate stated in Proposition 6 implies that

$$a_* \|\nabla \delta \Pi\|_{L^p} \leq C \left( \|\delta f\|_{L^p} + (a^* - a_*) \|\nabla \delta \Pi\|_{L^p} + \|\nabla \Pi_1\|_{L^\infty} \|\delta a\|_{L^p} + \|\nabla (u_1 + u_2)\|_{L^\infty} \|\delta u\|_{L^p} \right).$$

If the quantity  $a^*/a_* - 1$  is small enough then we thus have, up to a change of  $C$ ,

$$a_* \|\nabla \delta \Pi\|_{L^p} \leq C \left( \|\delta f\|_{L^p} + \|\nabla \Pi_1\|_{L^\infty} \|\delta a\|_{L^p} + \|\nabla (u_1 + u_2)\|_{L^\infty} \|\delta u\|_{L^p} \right).$$

Plugging this latter inequality in (73) then adding up Inequality (72), we get

$$\begin{aligned} \|(\delta a, \delta u)(t)\|_{L^p} &\leq \|(\delta a, \delta u)(0)\|_{L^p} \\ &+ C \int_0^t \left( \|\delta f\|_{L^p} + (\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty} + \|\nabla a_1\|_{L^\infty} + \|\nabla \Pi_1\|_{L^\infty}) \|(\delta a, \delta u)\|_{L^p} \right) d\tau. \end{aligned}$$

Applying Gronwall's lemma yields the following result which obviously implies the uniqueness part of Theorem 4:

**Proposition 9.** *Let  $(\rho_1, u_1, \nabla\Pi_1)$  and  $(\rho_2, u_2, \nabla\Pi_2)$  be two solutions of (1) on  $[0, T] \times \mathbb{R}^N$  such that for some  $p \in (1, \infty)$ ,*

- $\delta a := a_2 - a_1$  and  $\delta u := u_2 - u_1$  are in  $\mathcal{C}([0, T]; L^p)$ ,
- $\nabla \delta \Pi := \nabla \Pi_2 - \nabla \Pi_1$  is in  $L^1([0, T]; L^p)$ ,

and for some positive real numbers  $a_*$  and  $a^*$  such that  $a_* \leq a^*$ ,

$$a_* \leq a_1, a_2 \leq a^*.$$

There exists a constant  $c$  depending only on  $p$  and on  $N$  such that if

$$a^* - a_* \leq ca_*$$

and if for all  $t \in [0, T]$ ,

$$V(t) := \int_0^t (\|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty} + \|\nabla a_1\|_{L^\infty} + \|\nabla \Pi_1\|_{L^\infty}) d\tau < \infty$$

then the following inequality is satisfied:

$$\|(\delta a, \delta u)(t)\|_{L^p} \leq e^{CV(t)} \left( \|(\delta a, \delta u)(0)\|_{L^p} + C \int_0^t e^{-CV(\tau)} \|\delta f(\tau)\|_{L^p} d\tau \right).$$

**6.2. A priori estimates.** Here we assume that  $(\rho, u, \nabla \Pi)$  is a solution to (1) on the time interval  $[0, T]$  with the  $B_{\infty, r}^s$  regularity. We want to show that if  $T$  has been chosen small enough then the size of the solution at time  $t \leq T$  is of the same order as the size of the data.

First, it is clear that we have

$$a_* \leq a \leq a^*.$$

Moreover, one may write thanks to Proposition 5:

$$(75) \quad \|\nabla a(t)\|_{B_{\infty, r}^{s-1}} \leq e^{C \int_0^t \|u\|_{B_{\infty, r}^s} d\tau} \|\nabla a_0\|_{B_{\infty, r}^{s-1}},$$

and for the velocity, we have, as in the case  $p < \infty$ ,

$$(76) \quad \|u(t)\|_{B_{\infty, r}^s} \leq e^{C \int_0^t \|u\|_{B_{\infty, r}^s} d\tau} \left( \|u_0\|_{B_{\infty, r}^s} + \int_0^t e^{-C \int_0^\tau \|u\|_{B_{\infty, r}^s} d\tau'} \left( \|f\|_{B_{\infty, r}^s} + \|a\|_{B_{\infty, r}^s} \|\nabla \Pi\|_{B_{\infty, r}^s} \right) d\tau \right).$$

Note that applying standard  $L^p$  estimates for the transport equation yields

$$(77) \quad \|u(t)\|_{L^p} \leq \|u_0\|_{L^p} + \int_0^t \|f\|_{L^p} d\tau + a^* \int_0^t \|\nabla \Pi\|_{L^p} d\tau.$$

As Propositions 6 and 7 fail in the limit case  $p = \infty$ , in order to bound the pressure, we have to resort to other arguments. Now, dividing the velocity equation of (1) by  $\rho$  and applying  $\operatorname{div}$ , we get

$$(78) \quad \bar{a} \Delta \Pi = \operatorname{div} f - \operatorname{div} (u \cdot \nabla u) + \operatorname{div} ((\bar{a} - a) \nabla \Pi) \quad \text{with } \bar{a} := 1/\bar{\rho}$$

and, by virtue of the Bernstein inequality, we have

$$\begin{aligned} \|\nabla \Pi\|_{B_{\infty, r}^s} &\leq \|\Delta_{-1} \nabla \Pi\|_{B_{\infty, r}^s} + \|(\operatorname{Id} - \Delta_{-1}) \nabla \Pi\|_{B_{\infty, r}^s}, \\ &\leq C \|\nabla \Pi\|_{L^p} + \|(\operatorname{Id} - \Delta_{-1}) \nabla \Pi\|_{B_{\infty, r}^s}. \end{aligned}$$

On the one hand, in order to bound the  $L^p$  norm of  $\nabla \Pi$ , we simply apply the standard  $L^p$  elliptic estimate (see Proposition 6) to (78). We get

$$\bar{a} \|\nabla \Pi\|_{L^p} \leq C \left( \|\mathcal{Q}f\|_{L^p} + \|u\|_{L^p} \|\nabla u\|_{L^\infty} + \|\bar{a} - a\|_{L^\infty} \|\nabla \Pi\|_{L^p} \right).$$

Hence, if  $a^*/\bar{a} - 1$  is small enough then

$$(79) \quad \bar{a} \|\nabla \Pi\|_{L^p} \leq C \left( \|\mathcal{Q}f\|_{L^p} + \|u\|_{L^p} \|\nabla u\|_{L^\infty} \right).$$

On the other hand, for bounding the high frequency part of the pressure, one can use the fact that Operator  $\nabla(-\Delta)^{-1}(\operatorname{Id} - \Delta_{-1})$  is homogeneous of degree  $-1$  away from a ball centered at the origin, hence maps  $B_{p, r}^{s-1}$  in  $B_{p, r}^s$  (see e.g. [2], Chap. 2). Therefore we have

$$\begin{aligned} \bar{a} \|(\operatorname{Id} - \Delta_{-1}) \nabla \Pi\|_{B_{\infty, r}^s} &\leq C \bar{a} \|\Delta \Pi\|_{B_{\infty, r}^{s-1}}, \\ &\leq C \left( \|\operatorname{div} f\|_{B_{\infty, r}^{s-1}} + \|\operatorname{div} (u \cdot \nabla u)\|_{B_{\infty, r}^{s-1}} + \|\operatorname{div} ((\bar{a} - a) \nabla \Pi)\|_{B_{\infty, r}^{s-1}} \right). \end{aligned}$$

In order to bound the second term, one may combine the Bony decomposition and the fact that  $\operatorname{div} u = 0$ . This gives

$$\operatorname{div} (u \cdot \nabla u) = \sum_{i, j} (2T_{\partial_i u^j} \partial_j u^i + \partial_i R(u^j, \partial_j u^i)).$$

Thus applying Proposition 3, we may write

$$\|\operatorname{div}(u \cdot \nabla u)\|_{B_{\infty,r}^{s-1}} \leq C \|\nabla u\|_{L^\infty} \|u\|_{B_{\infty,r}^s}.$$

Finally, as  $B_{\infty,r}^s$  is a Banach algebra, we have

$$\|\operatorname{div}((\bar{a} - a)\nabla \Pi)\|_{B_{\infty,r}^{s-1}} \leq C \|a - \bar{a}\|_{B_{\infty,r}^s} \|\nabla \Pi\|_{B_{\infty,r}^s}.$$

Putting this together with (79), one may conclude that there exists a constant  $c$  such that if

$$(80) \quad \|a - \bar{a}\|_{L_T^\infty(B_{\infty,r}^s)} \leq c\bar{a}$$

then

$$(81) \quad \bar{a}(\|\nabla \Pi\|_{L^p} + \|\nabla \Pi\|_{B_{\infty,r}^s}) \leq C(\|\mathcal{Q}f\|_{L^p} + \|\operatorname{div} f\|_{B_{p,r}^{s-1}} + \|u\|_{L^p \cap B_{\infty,r}^s} \|\nabla u\|_{L^\infty}).$$

Let us assume that  $T$  has been chosen so that

$$(82) \quad C \int_0^T \|\nabla u\|_{B_{\infty,r}^{s-1}} \leq \log 2$$

and that the initial density is such that

$$\|a_0 - \bar{a}\|_{B_{\infty,r}^{s-1}} \leq \frac{c}{2} \bar{a}.$$

Then (80) is fulfilled and, combining Inequalities (76), (77) and (81), we get

$$U(t) \leq 2U_0(t) + C\bar{p}\|a_0\|_{B_{\infty,r}^s} \int_0^t (\|\mathcal{Q}f\|_{L^p} + \|\operatorname{div} f\|_{B_{p,r}^{s-1}} + U^2) d\tau$$

with

$$U(t) := \|u(t)\|_{L^p \cap B_{\infty,r}^s} \quad \text{and} \quad U_0(t) := \|u_0\|_{L^p \cap B_{\infty,r}^s} + \int_0^t \|f\|_{L^p \cap B_{\infty,r}^s} d\tau.$$

It is now easy to find a time  $T > 0$  depending only on the data and such that both Condition (82) and

$$U(t) \leq 4U_0(t) \quad \text{for all } t \in [0, T]$$

are satisfied.

**6.3. The proof of existence.** This is mainly a matter of making the above estimates rigorous. We have to be a bit careful though since the data which are considered here do not enter in the framework of Theorems 1 and 3.

As a first step, we construct a sequence of smooth solutions. In order to enter in the Sobolev spaces framework, one may proceed as follows.

For the density, one may consider  $\rho_0^n := \bar{\rho} + S_n(\phi(n^{-1}\cdot)(\rho_0 - \bar{\rho}))$  where  $\phi$  is a smooth compactly supported cut-off function with value 1 on the unit ball of  $\mathbb{R}^N$ . Obviously,  $\rho_0^n - \bar{\rho}$  is in  $H^\infty$  and converges weakly to  $\rho_0 - \bar{\rho}$  when  $n$  goes to infinity. In addition, by using the fact that  $\phi$  is smooth and that  $B_{\infty,r}^s$  is an algebra, one may establish that there exists some constant  $C$  such that for all  $n \in \mathbb{N}$ ,

$$\|\rho_0^n - \bar{\rho}\|_{B_{\infty,r}^s} \leq C \|\rho_0 - \bar{\rho}\|_{B_{\infty,r}^s}.$$

Similarly, for the velocity, one may set  $u_0^n := S_n(\phi(n^{-1}\cdot)u_0)$  and for the source term,  $f^n := \alpha_n \star_t (S_n(\phi(n^{-1}\cdot) \cdot f))$  where the convolution is taken with respect to the time variable only and  $(\alpha_n)_{n \in \mathbb{N}}$  is a sequence of mollifiers on  $\mathbb{R}$ .

Applying Theorem 1 thus provides a sequence of continuous-in-time solutions with values in  $H^\infty$ , defined on a fixed time interval. Then applying the above a priori estimates, it is easy to find a time  $T$  independent of  $n$  for which the sequence  $(\rho^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$  is bounded in the desired space.

For proving convergence, one may take advantage of the stability estimates in  $L^p$ . The proof is similar to that of Theorem 1 in the case  $1 < p \leq 2$  and is thus omitted.

**6.4. A continuation criterion.** This paragraph is dedicated to the proof of the following continuation criterion:

**Proposition 10.** *Assume that  $s > 1$  (or that  $s \geq 1$  if  $r = 1$ ). Consider a solution  $(\rho, u, \nabla \Pi)$  to (1) on  $[0, T] \times \mathbb{R}^N$  such that for some  $p \in (1, \infty)$  we have*

- $\rho \in \mathcal{C}([0, T]; B_{\infty, r}^s)$ ,
- $u \in \mathcal{C}([0, T]; B_{\infty, r}^\infty \cap L^p)$ ,
- $\nabla \Pi \in L^1([0, T]; B_{\infty, r}^s \cap L^p)$ .

*There exists a constant  $c$  depending only on  $N$  and  $s$  such that if for some  $\bar{\rho} > 0$  we have*

$$\sup_{0 \leq t < T} \|\rho(t) - \bar{\rho}\|_{B_{\infty, r}^s} \leq c\bar{\rho} \quad \text{and} \quad \int_0^T \|\nabla u\|_{L^\infty} dt < \infty$$

*then  $(\rho, u, \nabla \Pi)$  may be continued beyond  $T$  into a  $B_{\infty, r}^s$  solution of (1).*

*Proof.* Applying the last part of Proposition 5 and product estimates to the velocity equation of (1) yields for all  $t \in [0, T)$ ,

$$\begin{aligned} \|u(t)\|_{B_{\infty, r}^s} &\leq e^{C \int_0^t \|\nabla u\|_{L^\infty} d\tau} \left( \|u_0\|_{B_{\infty, r}^s} \right. \\ &\quad \left. + \int_0^t e^{-C \int_0^\tau \|\nabla u\|_{L^\infty} d\tau'} \left( \|f\|_{B_{\infty, r}^s} + \|a\|_{B_{\infty, r}^s} \|\nabla \Pi\|_{B_{\infty, r}^s} \right) d\tau \right). \end{aligned}$$

Let us bound the pressure term according to Inequality (81). Combining with (77) and (79), we eventually get

$$\begin{aligned} \|u(t)\|_{L^p \cap B_{\infty, r}^s} &\leq e^{C \int_0^t \|\nabla u\|_{L^\infty} d\tau} \left( \|u_0\|_{B_{\infty, r}^s} \right. \\ &\quad \left. + \bar{\rho} \int_0^t e^{-C \int_0^\tau \|\nabla u\|_{L^\infty} d\tau'} \|a\|_{B_{\infty, r}^s} \left( \|f\|_{L^p \cap B_{\infty, r}^s} + \|u\|_{L^p \cap B_{\infty, r}^s} \|\nabla u\|_{L^\infty} \right) d\tau \right). \end{aligned}$$

So applying Gronwall's lemma ensures that  $u$  belongs to  $L^\infty([0, T]; L^p \cap B_{\infty, r}^s)$ . From this point, completing the proof is similar as for the previous continuation criteria.  $\square$

**Remark 4.** *As in the  $B_{p, r}^s$  framework, an improved continuation criterion involving  $\|\nabla u\|_{\dot{B}_{\infty, \infty}^0}$  instead of  $\|\nabla u\|_{L^\infty}$  may be proved for the  $B_{\infty, r}^s$  regularity, if  $s > 1$ . The details are left to the reader.*

#### APPENDIX A. COMMUTATOR ESTIMATES

Here we prove two estimates that have been used for estimating the pressure. The first result reads:

**Lemma 6.** *Let  $(s, p, r)$  satisfy Condition (C). Let  $\varsigma$  be in  $(-1, s - 1]$ . There exists a constant  $C$  depending only on  $s, p, r, \varsigma$  and  $N$  such that for all  $k \in \{1, \dots, N\}$ , we have*

$$\|\partial_k [a, \Delta_q] w\|_{L^p} \leq C c_q 2^{-q\varsigma} \|\nabla a\|_{B_{p, r}^{s-1}} \|w\|_{B_{p, r}^s} \quad \text{for all } q \geq -1$$

*with  $\|(c_q)_{q \geq -1}\|_{\ell^r} = 1$ .*

*Proof.* We follow the proof of Lemma 8.8 in [11]. Let  $\tilde{a} := a - \Delta_{-1} a$ . Taking advantage of the Bony decomposition (6), we rewrite the commutator as<sup>6</sup>

$$(83) \quad \partial_k ([a, \Delta_q] w) = \underbrace{\partial_k ([T_a^{\sim}, \Delta_q] w)}_{R_q^1} + \underbrace{\partial_k T'_{\Delta_q} w \tilde{a}}_{R_q^2} - \underbrace{\partial_k \Delta_q T'_w \tilde{a}}_{R_q^3} + \underbrace{\partial_k [\Delta_{-1} a, \Delta_q] w}_{R_q^4}.$$

From the localization properties of the Littlewood-Paley decomposition, we gather that

$$R_q^1 = \sum_{|q' - q| \leq 4} \partial_k ([S_{q'-1} \tilde{a}, \Delta_q] \Delta_{q'} w).$$

<sup>6</sup>Recall the notation  $T'_u v := T_u v + R(u, v)$ .

Note that  $R_q^1$  is spectrally supported in an annulus of size  $2^q$ . Hence, combining Bernstein's inequality and Lemma 2.97 in [2], we get

$$\|R_q^1\|_{L^p} \leq C \sum_{|q'-q| \leq 4} \|\nabla S_{q'-1} \tilde{a}\|_{L^\infty} \|\Delta_{q'} w\|_{L^p},$$

whence for some sequence  $(c_q)_{q \geq -1}$  in the unit sphere of  $\ell^r$ ,

$$(84) \quad \|R_q^1\|_{L^p} \leq C c_q 2^{-q\varsigma} \|\nabla a\|_{L^\infty} \|w\|_{B_{p,r}^\varsigma}.$$

To deal with  $R_q^2$ , we use the fact that, owing to the localization properties of the Littlewood-Paley decomposition, we have

$$R_q^2 = \sum_{q' \geq q-2} \partial_k (S_{q'+2} \Delta_q w \Delta_{q'} \tilde{a}).$$

Hence, using the Bernstein and Hölder inequalities and the fact that  $\tilde{a}$  has no low frequencies,

$$\begin{aligned} \|R_q^2\|_{L^p} &\leq C \sum_{q' \geq q-2} \|S_{q'+2} \Delta_q w\|_{L^\infty} \|\Delta_{q'} \nabla \tilde{a}\|_{L^p}, \\ &\leq C 2^{-q\varsigma} 2^{q(\frac{N}{p}+1-s)} \sum_{q' \geq q-2} 2^{(q-q')(s-1)} (2^{q(\varsigma-\frac{N}{p})} \|\Delta_q w\|_{L^\infty}) (2^{q'(s-1)} \|\Delta_{q'} \nabla \tilde{a}\|_{L^p}). \end{aligned}$$

Therefore, by virtue of convolution inequalities for series and because  $N/p + 1 - s \leq 0$ ,

$$(85) \quad \|R_q^2\|_{L^p} \leq C c_q 2^{-q\varsigma} \|\nabla a\|_{B_{p,r}^{s-1}} \|w\|_{B_{\infty,r}^{\varsigma-\frac{N}{p}}}.$$

Next, Proposition 3 ensures that, under the assumptions of the lemma, the paraproduct and the remainder map  $B_{p,r}^\varsigma \times B_{p,r}^s$  in  $B_{p,r}^\varsigma$ . As moreover we have

$$(86) \quad \|\tilde{a}\|_{B_{p,r}^\varsigma} \leq C \|\nabla a\|_{B_{p,r}^{s-1}},$$

one may conclude that

$$(87) \quad \|R_q^3\|_{L^p} \leq C c_q 2^{-q\varsigma} \|\nabla a\|_{B_{p,r}^{s-1}} \|w\|_{B_{p,r}^\varsigma}.$$

Finally, as the last term  $R_q^4$  is spectrally localized in a ball of size  $2^q$ , Bernstein's inequality ensures that

$$\|R_q^4\|_{L^p} \leq C 2^q \|[\Delta_{-1} a, \Delta_q] w\|_{L^p}.$$

Then, resorting again to Lemma 2.97 in [2], we get

$$(88) \quad \|R_q^4\|_{L^p} \leq C c_q 2^{-q\varsigma} \|\nabla a\|_{L^\infty} \|w\|_{B_{p,r}^\varsigma}.$$

Putting Inequalities (84), (85), (87) and (88) together and using Proposition 2 completes the proof of the lemma.  $\square$

**Lemma 7.** *Let  $\varsigma > 0$  and  $1 \leq p, r \leq \infty$ . There exists a constant  $C$  such that*

$$\|\partial_k [a, \Delta_q] w\|_{L^p} \leq C c_q 2^{-q\varsigma} \left( \|\nabla a\|_{L^\infty} \|w\|_{B_{p,r}^\varsigma} + \|w\|_{L^\infty} \|\nabla a\|_{B_{p,r}^\varsigma} \right) \quad \text{for all } q \geq -1$$

with  $\|(c_q)_{q \geq -1}\|_{\ell^r} = 1$ .

*Proof.* We use again Decomposition (83). We have already proved in (84) and (88) that  $R_q^1$  and  $R_q^4$  satisfy the desired inequality. Concerning  $R_q^2$ , recall that

$$\|R_q^2\|_{L^p} \leq C \sum_{q' \geq q-2} \|S_{q'-1} \Delta_q w\|_{L^\infty} \|\Delta_{q'} \nabla a\|_{L^p},$$

whence

$$\|R_q^2\|_{L^p} \leq C 2^{-q\varsigma} \sum_{q' \geq q-2} 2^{(q-q')\varsigma} \|w\|_{L^\infty} 2^{q'\varsigma} \|\Delta_{q'} \nabla a\|_{L^p}.$$

As  $\varsigma > 0$ , convolution inequalities for series yield the desired inequality for  $R_q^2$ .

According to Proposition 3, we have

$$\|T'_w \tilde{a}\|_{B_{p,r}^{\zeta+1}} \leq C \|w\|_{L^\infty} \|\tilde{a}\|_{B_{p,r}^{\zeta+1}}.$$

Hence, as  $\|\tilde{a}\|_{B_{p,r}^{\zeta+1}} \leq C \|\nabla a\|_{B_{p,r}^{\zeta}}$ , the term  $R_q^3$  satisfies the required inequality.  $\square$

#### APPENDIX B. A BERNSTEIN-TYPE INEQUALITY

**Lemma 8.** *Let  $1 < p < \infty$  and  $u \in L^p$  such that  $\text{Supp } \hat{u} \subset \{\xi \in \mathbb{R}^N / R_1 \leq |\xi| \leq R_2\}$  for some real numbers  $R_1$  and  $R_2$  such that  $0 < R_1 < R_2$ . Let  $a$  be a bounded measurable function over  $\mathbb{R}^N$  such that  $a \geq a_* > 0$  a.e. There exists a constant  $c$  depending only on  $N$  and  $R_2/R_1$ , and such that*

$$(89) \quad ca_* \left( \frac{p-1}{p^2} \right) R_1^2 \int |u|^p dx \leq (p-1) \int a |\nabla u|^2 |u|^{p-2} dx = - \int \text{div} (a \nabla u) |u|^{p-2} u dx.$$

*Proof.* The case  $a \equiv 1$  has been treated in [9] and readily entails the left inequality in (89) for one may write, owing to the case  $a \equiv 1$ ,

$$c a_* R_1^2 \int |u|^p dx \leq p^2 \int a_* |\nabla u|^2 |u|^{p-2} dx \leq p^2 \int a |\nabla u|^2 |u|^{p-2} dx.$$

Let us now justify the right equality in (89). In the case  $p \geq 2$ , it stems from a straightforward integration by parts.

Let us focus on the case  $1 < p < 2$  which is more involved. Smoothing out  $a$  if needed, one may assume with no loss of generality that  $a$  is in  $W^{1,\infty}$ . Let  $T_\varepsilon(x) = \sqrt{x^2 + \varepsilon^2}$  for  $x \in \mathbb{R}$  and  $\varepsilon > 0$ . We have

$$\begin{aligned} - \int_{\mathbb{R}^N} \text{div} (a \nabla u) (T_\varepsilon(u))^{p-1} T'_\varepsilon(u) dx &= (p-1) \int_{\mathbb{R}^N} a |\nabla u|^2 |T'_\varepsilon(u)|^2 (T_\varepsilon(u))^{p-2} dx \\ &\quad + \int_{\mathbb{R}^N} a |\nabla u|^2 T''_\varepsilon(u) (T_\varepsilon(u))^{p-1} dx. \end{aligned}$$

In view of the monotonous convergence theorem,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} a |\nabla u|^2 |T'_\varepsilon(u)|^2 (T_\varepsilon(u))^{p-2} dx = \int_{\mathbb{R}^N} a |\nabla u|^2 |u|^{p-2} dx \in \overline{\mathbb{R}}^+.$$

Next, we notice that

$$(90) \quad |\text{div} (a \nabla u) (T_\varepsilon(u))^{p-1} T'_\varepsilon(u)| \leq |u|^{p-1} |\text{div} (a \nabla u)|.$$

Now, as  $a \in W^{1,\infty}$  and  $u$  is a smooth function with all derivatives in  $L^p$  (owing to the spectral localization), one may write

$$\text{div} (a \nabla u) = a \Delta u + \nabla a \cdot \nabla u,$$

hence  $\text{div} (a \nabla u)$  is in  $L^p$  and the right-hand side of (90) is an integrable function. So finally Lebesgue's dominated convergence theorem entails that

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} \text{div} (a \nabla u) (T_\varepsilon(u))^{p-1} T'_\varepsilon(u) dx = \int_{\mathbb{R}^N} u |u|^{p-2} \text{div} (a \nabla u) dx.$$

Therefore

$$(91) \quad (p-1) \int_{\mathbb{R}^N} a |\nabla u|^2 |u|^{p-2} dx \leq - \int_{\mathbb{R}^N} u |u|^{p-2} \text{div} (a \nabla u) dx < \infty.$$

In fact, equality does hold. Indeed, whenever  $x \neq 0$ , the term  $T''_\varepsilon(x) T_\varepsilon(x)^{p-1}$  tends to 0 when  $\varepsilon$  goes to 0 and

$$T''_\varepsilon(x) T_\varepsilon(x)^{p-1} = |x|^{p-2} \frac{(\varepsilon/x)^2}{(1 + (\varepsilon/x)^2)^{2-\frac{p}{2}}} \leq |x|^{p-2}.$$

Therefore, as, according to (91), the function  $|\nabla u|^2 |u|^{p-2}$  is integrable over  $\mathbb{R}^N$ , we get

$$\lim_{\varepsilon \rightarrow 0} \int_{u \neq 0} a |\nabla u|^2 T''_\varepsilon(u) (T_\varepsilon(u))^{p-1} dx = 0.$$

On the other hand, as  $u$  is real analytic,

$$\int_{u=0} a|\nabla u|^2 T_\varepsilon''(u)(T_\varepsilon(u))^{p-1} dx = \varepsilon^{p-2} \int_{u=0} a|\nabla u|^2 = 0.$$

□

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