On Some Procedures of Forming a Multi-partner Alliance*

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ABSTRACT

We study two different ways of forming multi-partner alliances between firms with the central idea that procedure is an important factor in multi-partner alliance formation. In the first procedure, an alliance is formed simultaneously, while in the second, step-by-step procedure, members are added one by one. In the model we present, each firm is assumed to have a multidimensional maneuvering space, which consists of all alliance positions acceptable to the firm, and an ideal position in this space. Alliances will form between the firms whose maneuvering spaces overlap. The results of the analysis confirm that procedure is an important factor in multi-partner alliance formation. Nevertheless, if ideal positions of firms are acceptable to all alliance partners, then the result of alliance formation does not depend on procedure. In addition, it is shown that it can be disadvantageous to be a first mover. Finally, we are able to provide sufficient conditions under which one procedure is preferred in a three-partner case. More specifically, a firm with its ideal position acceptable to the two other firms may prefer the simultaneous procedure to being a late mover if (1) there is a certain balance in the firms' degree of flexibility and their power and (2) if the agreed alliance position of the two other firms is acceptable to the firm in question.

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1 INTRODUCTION

Formation of strategic alliances between firms has been a widely studied phenomenon (e.g. Barringer and Harrison, 2000; Inkpen, 2001). Often, more than two firms form an alliance and hence we speak of a multi-partner alliance or a consortium instead of dyadic alliances. As Das and Teng (2002) point out, the formation and structure of such multi-party alliances are more complex than just a summation of dyadic relations. In real life, we find different kinds of multi-partner alliances. Within this variety of multi-partner alliances, different ways of alliance formation are observable. Take for example the One World Airline Alliance: after American Airlines, British Airways, Canadian Airlines, Cathay Pacific, and Qantas formed the initial group in February 1999, Finair and Iberia joined in September of that year. And only in June 2000, Aer Lingus and LAN Chile were added (www.oneworldalliance.com). However, when the semiconductor industry in the US bonded in SEMATECH, fourteen firms all together decided to form this alliance and negotiated all together on the alliance (Browning, Beyer, and Shelter, 1995). A final example is found in the mobile phone industry: the Open Mobile Alliance (OMA) was originated to achieve a certain standard (The WAP) to compete with American and Asian standards. From four original founding members, Nokia, Ericsson, Motorola, and Phone.com, this alliance evolved to an extensive group of firms supporting the standard (www.openmobilealliance.org).

Roughly speaking, two different ways of multi-party alliance formation can be discerned: a simultaneous and a step-by-step procedure. The simultaneous approach to alliance formation implies negotiating immediately together with all the members of the alliance. An alternative approach, the step-by-step procedure, means alliance formation as a process in which the alliance group incrementally forms: new members are added gradually. The distinction between these two procedures of alliance formation and its consequences are the central subject of this article. Does it matter which procedure is used if one compares the results of alliance formations and when is it advantageous for a firm to join an alliance? To answer these questions we will work out the two procedures in a formal spatial model. A multi-partner alliance is the term we will use to denote any voluntary inter-organizational relationship between more than two firms in which the firms remain independent. During the article, we will make the context of the model more clear and show in what kind of business situations the model holds.

In short, the model deals with firms (the players), their alliance preferences (their position in Euclidean space), and compromising with partners (changing position in space). A group of potential alliance members is placed in a multi-dimensional Euclidean space in which the alliance preferences of the firms on different negotiation issues (dimensions) form their positions. In the model, each firm has an ideal alliance position in a multidimensional Euclidean space, and around this position the firms have a certain flexibility in negotiating which we will call a *maneuvering space*. All the positions in this maneuvering space are acceptable to a firm. Firms' preferences over all positions are defined according to the rule: 'The closer to my ideal position the better'. Multi-party alliances are formed given the ideal alliance positions and the limits to the maneuvering spaces. To be able to form an alliance, all the members have to agree on the negotiation issues at stake, and this will result in one position in the space for that alliance. In this article, we formalize this alliance formation process by determining the position of an alliance by its *gravity center*, taking the weights of the firms in an alliance into account. We prove that the position of an alliance belongs to the intersection of the maneuvering spaces of its members, and, moreover, that the alliance position is *Pareto efficient*. To study the sequence of alliance formation, we consider different ways of reaching this alliance position: a simultaneous versus a step-by-step approach. We can prove that these two procedures lead to different results, and we show that in three-partner cases it sometimes pays for a firm to enter an alliance immediately, but sometimes a firm is better off entering the process later.

This article aims at contributing on two different fields. Our first contribution lies in the field of strategic alliance research. Although the formation of multi-partner alliances has been studied in alliance research (e.g. Gulati, 1998; Doz, Olk, and Smith Ring, 2000), the behavioral negotiations of alliance formation has - with some exceptions - remained understudied (Rao and Schmidt, 1998), let alone the lack of attention for the sequence of alliance formation. As seen above however, in real life, different ways of alliance formation are found. Here, we want to lay bare these differences in forming a multi-partner alliance and study the effects. We have chosen for a formal approach to shed light on these differences. Next to the virtues of such an approach, a more formal approach to alliance formation has so far lacked the field (some exceptions can be found in Parkhe, 1993; Axelrod, Mitchell, Thomas, Bennett, and Bruderer, 1995; and Park and Zhou, 2005). With our model, we aim at presenting a new and more formal vision on alliance formation. To explicitly show the use of such a formal model in the field of alliances we will extensively discuss its interpretations in section 2.1.

Second, we want to make a contribution in the field of coalition formation models, and more specific spatial coalition modelling (e.g. Grofman, 1982; Schofield, 1993; De Vries, 1999). Spatial models (Downs, 1957; Black, 1958; Enelow and Hinich, 1984; Hinich and Munger, 1994) use spatial analysis to forecast coalitions, with the central idea of 'cooperating with your closest neighbor'. Such models have mainly been applied to coalition formation in politics (Martin and Stevenson, 2001). These spatial coalition formation theories have so far for the most part lacked two dynamic aspects: changes of positions of players during the game and a lack of attention for different procedures. Players are usually assumed to have a fixed position which does not change during the coalition formation game. Also, spatial coalition theories often neglect the procedure of forming a coalition (c.f. Brams, Jones, and Kilgour, 2005; Bloch, 1996). In this paper, we will work on both assumptions: we will assume players can change their position during negotiations and we will take different procedures of forming an alliance into account.

The paper is organized as follows. In section 2.1, we introduce the main ideas of the model and its interpretation in the field of alliances. In section 2.2, we present definitions of the notions of gravity center and of Pareto efficiency. Section 3 concerns the simultaneous procedure and in section 4 we describe the step-by-step procedure. Those two different procedures are illustrated and compared in a numerical example presented in section 5. In section 6, we compare the procedures. We deliver sufficient conditions under which the procedures are equally attractive to a firm, and sufficient conditions under which one procedure is more attractive to a firm than another one. After the conclusions in section 7, the mathematical definitions and theorems needed for proving our results are stated in Appendix A, while the proofs themselves are presented in Appendix B.

2 SPATIAL MODEL OF ALLIANCE FORMATION

2.1 The model and its interpretations

In this section, we introduce the main concepts of our theory of alliance formation and its interpretation. Some ideas presented here are a formalized version of the spatial model of coalition formation introduced in De Ridder and Van Deemen (2004).

Let $N = \{1, ..., n\}$ be the set of players, here firms, which all want to form a coalition, or in this interpretation an alliance. Each firm i has a *weight* $w_i > 0$, where $i \in N$. One may think of the weight as a representation of the size of a firm by (a combination of) its returns, profit, or number of employees. An alternative interpretation of the weight could be the amount of bargaining power a firm has in the alliance (Yan and Gray, 1994). An x_i is a position of a player in a multidimensional Euclidean space $\mathbb{R}^m$, $m \geq 1$. It is assumed that each firm $i \in N$ chooses an *ideal position* x_i^* in this space.

In general, the ideal position of a firm refers to the most favorite position of a firm concerning the issues (i.e. dimensions) which have to be agreed on when forming an alliance. Relevant issues are not issues related to the raison-d'être of an alliance, but issues concerning the terms and conditions of the alliance which have to be settled by agreement (Douma, Bilderbeek, Idenburg, and Looise, 2000; Parkhe, 1991). Irrespective of the branch or the aim of an alliance, *every* alliance has to agree on these terms and specifications. Hence, the model in principle applies to every alliance. However, what the relevant issues are is of course different for each (kind of) alliance. Moreover, it is also context specific how many issues have to be settled, and hence how many dimensions play a role. Three general motives can be found in the literature to set up an alliance: to gain economies of scale, to gain access to particular resources, and collective lobbying (e.g. standard setting alliances) (Barringer and Harrison, 2000; Inkpen, 2001). Let us discuss per kind of alliance what an alliance position could imply.

Alliances that form due to economies of scale anticipate on efficiency advantages. Examples of such alliances are airline alliances (see the Introduction) and alliances in the health care sector. In the latter, the location of a new hospital might be an important issue. In airline alliances, think of the amount of planes each company provides, the amount of flight routes maintained in the alliance, and the exact combination of the air miles programme.

Access to resources is a very general motive for alliance formation. Resources can refer to tangible resources as money, buildings, machinery, but resources also include technology, market access, and capabilities. In this category, we find co-development alliances; firms complement each others resource profile so that new products or services can be developed. The alliance between IBM, Intel, and Microsoft set up to produce a computer in order to compete with Apple is an example of a co-development alliance. In such alliance producing goods or services together, the following issues can play a role: cultural and management issues (individualism versus collectivism, confrontation versus harmony style (Douma et al., 2000; Sarkar, Echambadi, Cavusgil, and Aulakh, 2001; Parkhe 1991)), marketing issues (amount of countries selling the product, amount of products developed together), and the set up of an alliance (kind of contract, relation with other brands, intended duration of alliance).

Finally, lobbying together includes for example setting a standard with alliance partners. The Open Mobile Alliance (OMA) from the introduction is such a standard setting alliance, but also think of a standard for recording a DVD. Especially in the early phase of such an alliance, partners have to negotiate on technical specifications in which each member has its own wishes given in by the technical expertise the firm has. In setting a standard for recording a DVD, technical issues concern the thickness of the laser, the angle of the laser, or the amount of gig. Non-technical disputes can be the amount of openness of the alliance (access for new members), the distribution of digital rights, or the communication style.

If all firms in the game have an ideal position, we can calculate a distance between each two positions $x_i = (x_{i1}, ..., x_{im})$ and $x_j = (x_{j1}, ..., x_{jm})$, which is given by

$$d(x_i, x_j) = \sqrt{\sum_{k=1}^{m} (x_{ik} - x_{jk})^2}. \quad (1)$$

Firms have their preferences defined in $\mathbb{R}^m$. Each firm $i \in N$ is assumed to have a complete, transitive, and reflexive *preference relation* $\succeq_i$ over $\mathbb{R}^m$; *the strict preference relation* $\succ_i$, which is irreflexive and transitive, and *the indifference preference relation* $\sim_i$, which is an equivalence relation, are defined in the standard way. The closer a position is to the ideal position of a firm, the better.

We assume that a firm is not willing to deviate too much from its ideal position. Each firm has a certain amount of flexibility on the negotiation issues and we formalize this with a maneuvering space $M_i \subset \mathbb{R}^m$ which consists of all alliance positions acceptable to firm i. For each $i \in N$, we assume

$$\forall i \in N \ \forall y, z \in \mathbb{R}^m \ [y \succeq_i z \ \Leftrightarrow \ d(x_i^*, y) \leq d(x_i^*, z)], \tag{2}$$

and, of course

$$\forall i \in N \ \forall y, z \in \mathbb{R}^m \ [y \succ_i z \ \Leftrightarrow \ d(x_i^*, y) < d(x_i^*, z)], \tag{3}$$

$$\forall i \in N \ \forall y, z \in \mathbb{R}^m \ [y \sim_i z \ \Leftrightarrow \ d(x_i^*, y) = d(x_i^*, z)]. \tag{4}$$

We assume the maneuvering space to be a ball in $\mathbb{R}^m$, that is,

$$M_i = \{y \in \mathbb{R}^m \mid d(x_i^*, y) \leq r_i\}, \tag{5}$$

where r_i is called the *radius* of the maneuvering space of firm i. The radius represents the size of the maneuvering space of a firm; in other words it indicates how flexible a firm is in negotiations. In the remainder of the paper we will use the notation M_i for the maneuvering space of firm i with middle point x_i^* and radius r_i.

Given the ideal positions x_i^* with $i \in S$, all the firms of the potential alliance S choose a position for that alliance S, called the alliance position and denoted as x_S. The alliance position is a formalization of the compromise which is reached between all the members of the alliance concerning the issues that have to be settled. In line with the idea of the maneuvering space, firm i will never agree on a position which does not belong to M_i. The positions that are outside the maneuvering space of the firm are unacceptable to the firm and hence a firm will never compromise outside its acceptable positions. Consequently, an alliance S has a chance to be formed if and only if

$$\bigcap_{i \in S} M_i \neq \emptyset. \tag{6}$$

Moreover, of course,

$$x_S \in \bigcap_{i \in S} M_i. \tag{7}$$

To sum up, we have the following ingredients in our model. A set of players, firms, which will form coalitions, alliances. Each firm has an ideal position in a multidimensional Euclidean space representing a firm's standpoint on the issues that have to be settled in the alliance. Furthermore, each firm has a spatial flexibility to negotiate (its maneuvering space). Alliances will only form between those firms whose maneuvering spaces overlap. If so, an alliance position will be decided on.

2.2 Definitions

Before we add different procedures of forming an alliance to the model introduced in section 2.1, we present here the definitions of a Gravity center and of Pareto efficiency, which we will use in the remainder of the paper.

To determine the positions of an alliance which is formed by one of the procedures we describe in this paper, we need the notion of a gravity center. In general, a gravity center is a weighted average of several positions. If firms form an alliance and have different ideal positions, the alliance position they will agree on will be dependent on their ideal positions and weights. The more weight a firm has relative to the others, the more this firm can 'pull' the alliance position to its own ideal position. The final alliance position will be somewhere in between all ideal positions, closest to the firm with the most weight and farthest to the smallest firm. The formal definition of the notion of gravity center is the following:

Definition 1 (Gravity center)
The coordinates of the gravity center $x^G = (x_1^G, ..., x_m^G)$ of a set of n alliance positions $x_i = (x_{i1}, ..., x_{im})$ in $\mathbb{R}^m$, $i = 1, ..., n$, with weights $w_1, ..., w_n$, where x_i and w_i denote the position and the weight of firm i, respectively, are given by the following formula

$$x_p^G = \frac{\sum_{i=1}^n w_i \cdot x_{ip}}{\sum_{i=1}^n w_i} \quad for \quad p = 1, ..., m. \tag{8}$$

Furthermore, we are interested in the alliance positions (belonging to the intersection of the maneuvering spaces) which are Pareto efficient. A position is Pareto efficient with respect to an alliance if there is no other position in the overlap of the alliance members' maneuvering spaces that is more preferred by all members of the alliance. The notion of Pareto efficiency is defined as follows:

Definition 2 (Pareto Efficiency)
Let $S \subseteq N$ and $M_i = M_i(x_i^*, r_i)$ for $i \in S$ be such that $\bigcap_{i \in S} M_i \neq \emptyset$. An alliance position

$y \in \bigcap_{i \in S} M_i$ is Pareto efficient with respect to alliance S if

$$\neg\ \exists z \in \bigcap_{i \in S} M_i\ \forall k \in S\ [z \succ_k y], \tag{9}$$

or equivalently,

$$\forall z \in \bigcap_{i \in S} M_i\ \exists k \in S\ [d(x_k^*, y) \leq d(x_k^*, z)]. \tag{10}$$

3 FORMING AN ALLIANCE SIMULTANEOUSLY

To find a solution to our basic alliance formation model, we will consider and compare two procedures: a simultaneous procedure and a step-by-step procedure. In both procedures we assume that the maneuvering spaces of all members of the envisaged alliance overlap. With this comparison we hope to shed some light on the problem underlying this article: does it matter for firms how they organize alliance negotiations, and if so which strategy is most advantageous to a firm? In this section, we study a non-hierarchial approach which considers alliance formation as a one-step procedure. In this simultaneous procedure there are no pre-alliances which form intermediate steps before a definitive coalition is reached: all the firms of an alliance sit round the table to negotiate simultaneously. The formal description of the simultaneous procedure looks as follows.

Procedure S

Let $S \subseteq N$ and $M_i = M_i(x_i^*, r_i)$, for $i \in S$, be maneuvering spaces in $\mathbb{R}^m$ such that $\bigcap_{i \in S} M_i \neq \emptyset$.

- *First, each firm $i \in S$ chooses the position $\widetilde{x}_i^S$ such that*

$$\widetilde{x}_i^S = (\widetilde{x}_{i1}^S, ..., \widetilde{x}_{im}^S) = arg \min_{y \in \bigcap_{j \in S} M_j} d(x_i^*, y) \quad for \quad i \in S. \tag{11}$$

- *Next, the position $x_S = (x_{S1}, ..., x_{Sm})$ for the alliance S is chosen as the gravity center of the set of positions $\widetilde{x}_i^S$ defined in (11) with weights w_i, for $i \in S$, that is, $x_S = (x_{S1}, ..., x_{Sm})$ is defined by:*

$$x_{Sp} = \frac{\sum_{i \in S} w_i \cdot \widetilde{x}_{ip}^S}{\sum_{i \in S} w_i} \quad for \quad p = 1, ..., m. \tag{12}$$

In the simultaneous procedure, denoted above as Procedure S, each member i of the alliance S first chooses a position $\widetilde{x}_i^S$ in the intersection of the maneuvering spaces of all firms of S such that the distance of this position to the ideal point of i is minimal. Such a position $\widetilde{x}_i^S$ is called the *negotiation position* of party i and is defined by equation (11). The negotiation position is an intermediate step between a firm's ideal position and the alliance position. The choice of the negotiation positions means that on the one hand the firms express their readiness to form the alliance by taking a position in the intersection, but on the other hand, they like to stay as close as possible to their ideal position. Next, the firms decide together about the position of the alliance. Here the weights of the firms play a role, since the alliance position x_S is the gravity center of the set of the negotiation positions; see equation (12). So, a weighted average of the negotiation positions determines which position the alliance will take.

As a first finding, Proposition 1 describes some properties of x_S:

Proposition 1 *Let x_S be the position of the alliance S defined in (12). Then:*

(a) $x_S \in \bigcap_{i \in S} M_i$,
(b) *x_S is Pareto efficient with respect to alliance S.*

The first part of Proposition 1 confirms that the alliance position is in the overlap of all firms' maneuvering spaces, that is, the alliance position chosen by the simultaneous procedure is always acceptable by each member of the alliance. The second part says that there is no other position that is more preferred by all members of the alliance.
Secondly, we can prove that the simultaneous procedure is uniquely defined as it always leads to exactly one alliance position, i.e.,

Proposition 2 *Let $S \subseteq N$ and $M_i = M_i(x_i^*, r_i)$, for $i \in S$, be maneuvering spaces in $\mathbb{R}^m$ such that $\bigcap_{i \in S} M_i \neq \emptyset$. For each $i \in S$, there is always exactly one $\widetilde{x}_i^S \in \mathbb{R}^m$ satisfying (11). Consequently, Procedure S is uniquely defined, that is, there is always exactly one position x_S for the alliance S defined in (12).*

Hence, the defined procedure leads to a unique alliance position, which is acceptable for all alliance members and is Pareto efficient.

4 FORMING AN ALLIANCE STEP-BY-STEP

Instead of negotiating with all members of an alliance simultaneously, firms can also form an alliance step-by-step. The step-by-step procedure is an incremental and hierarchical approach which sees ‘ ... coalition building as a process in which actors with similar

... preferences first get together in some sort of provisional alliance and, only after this has been done ..., do they cast around for other coalition partners, adding these until the formation criterion is satisfied' (Laver and Schofield, 1990: 140). The proto-coalition model of Grofman (1982) is such a hierarchial model, and Seidman and Winter (1998) also present a gradual coalition formation model. In this section, we present the step-by-step procedure for forming an alliance with an arbitrary number of firms whose maneuvering spaces overlap. The provisional alliance which is used during negotiations will be denoted here with pre-alliance.

First of all, note that for each alliance $S \subseteq N$, there are $\frac{|S|!}{2}$ possible step-by-step procedures for forming an alliance. Each step-by-step procedure will be denoted by the order in which firms form an alliance. For instance, Procedure $\{\{i,j\},k\}$ means that first firms i and j form a pre-alliance $\{i,j\}$, and next firm k joins this alliance. In general, by $\overline{S}$ we will denote an order in which firms form an alliance S, that is

$$\overline{S} = \{\{...\{\{i_1, i_2\}, i_3\}...\}, i_{|S|}\}, \tag{13}$$

where $S = \{i_1, ..., i_{|S|}\}$ and $|S| \geq 3$. Moreover, let

$$S_p = \{i_1, ..., i_p\} \tag{14}$$

for $3 \leq p \leq |S|$, and let $\overline{S}_p$ denote an order in which the first p firms form an alliance S_p, that is,

$$\overline{S}_p = \{\{...\{\{i_1, i_2\}, i_3\}...\}, i_p\}. \tag{15}$$

Let $PROC(S)$ denote the set of all possible step-by-step procedures of forming alliance S. For instance, in case of a three-partner alliance, three step-by-step procedures are possible: k,j, and i joining as last members. Formally,

$$PROC(\{i,j,k\}) = \{\{\{i,j\},k\}, \{\{i,k\},j\}, \{\{j,k\},i\}\}.$$

Hence, as already mentioned, for each alliance S,

$$|PROC(S)| = \frac{|S|!}{2}.$$

Finally, $x_{\overline{S}}$ denotes the position of alliance S resulting from using Procedure $\overline{S} \in PROC(S)$, and $x_{\overline{S}_p}$ denotes the position of alliance $S_p = \{i_1, ..., i_p\}$ resulting from using Procedure $\overline{S}_p$. We propose the following step-by-step procedure for forming an alliance S:

Procedure $\overline{S}$

Let $S \subseteq N$ and $M_i = M_i(x_i^*, r_i)$, for $i \in S$, be maneuvering spaces in $\mathbb{R}^m$ such that $\bigcap_{i\in S} M_i \neq \emptyset$. Let $S = \{i_1, ..., i_{|S|}\}$, $|S| \geq 3$, and let $\overline{S}$ be an order in which firms form an alliance S, as defined in (13). Let $\overline{S}_p$ be an order in which the first p firms form an alliance S_p, $3 \leq p \leq |S|$, as defined in (15).
The firms $i_1, ..., i_{|S|}$ form the alliance S as follows:

- *First, firms i_1 and i_2 choose $\widetilde{x}_{i_1}^{\{i_1,i_2\}}$ and $\widetilde{x}_{i_2}^{\{i_1,i_2\}}$ defined as follows:*

$$\widetilde{x}_{i_1}^{\{i_1,i_2\}} = (\widetilde{x}_{i_1 1}^{\{i_1,i_2\}}, ..., \widetilde{x}_{i_1 m}^{\{i_1,i_2\}}) = arg \min_{z\in M_{i_1}\cap M_{i_2}} d(x_{i_1}^*, z), \tag{16}$$

$$\widetilde{x}_{i_2}^{\{i_1,i_2\}} = (\widetilde{x}_{i_2 1}^{\{i_1,i_2\}}, ..., \widetilde{x}_{i_2 m}^{\{i_1,i_2\}}) = arg \min_{z\in M_{i_1}\cap M_{i_2}} d(x_{i_2}^*, z). \tag{17}$$

- *Firms i_1 and i_2 form a pre-alliance $\{i_1, i_2\}$ and choose the position $x_{\{i_1,i_2\}}$ for the pre-alliance $\{i_1, i_2\}$. The position $x_{\{i_1,i_2\}}$ is the gravity center of $\widetilde{x}_{i_1}^{\{i_1,i_2\}}$ and $\widetilde{x}_{i_2}^{\{i_1,i_2\}}$, with weights w_{i_1} and w_{i_2}. That makes:*

$$x_{\{i_1,i_2\}l} = \frac{w_{i_1} \cdot \widetilde{x}_{i_1 l}^{\{i_1,i_2\}} + w_{i_2} \cdot \widetilde{x}_{i_2 l}^{\{i_1,i_2\}}}{w_{i_1} + w_{i_2}} \quad for \quad l = 1, ..., m. \tag{18}$$

- *Next, firm i_3 joins the pre-alliance $\{i_1, i_2\}$, and it chooses the position $\widetilde{x}_{i_3}^{\{i_1,i_2,i_3\}} \in \mathbb{R}^m$ such that*

$$\widetilde{x}_{i_3}^{\{i_1,i_2,i_3\}} = (\widetilde{x}_{i_3 1}^{\{i_1,i_2,i_3\}}, ..., \widetilde{x}_{i_3 m}^{\{i_1,i_2,i_3\}}) = arg \min_{z\in M_{i_1}\cap M_{i_2}\cap M_{i_3}} d(x_{i_3}^*, z). \tag{19}$$

- *The pre-alliance $\{i_1, i_2\}$ chooses a new position $\widetilde{x}_{\{i_1,i_2\}}^{\{i_1,i_2,i_3\}} = (\widetilde{x}_{\{i_1,i_2\}1}^{\{i_1,i_2,i_3\}}, ..., \widetilde{x}_{\{i_1,i_2\}m}^{\{i_1,i_2,i_3\}}) \in \mathbb{R}^m$ such that*

$$\widetilde{x}_{\{i_1,i_2\}}^{\{i_1,i_2,i_3\}} = (\widetilde{x}_{\{i_1,i_2\}1}^{\{i_1,i_2,i_3\}}, ..., \widetilde{x}_{\{i_1,i_2\}m}^{\{i_1,i_2,i_3\}}) = arg \min_{z\in M_{i_1}\cap M_{i_2}\cap M_{i_3}} d(x_{\{i_1,i_2\}}, z). \tag{20}$$

- Firms i_1, i_2, and i_3 choose the position $x_{\{\{i_1,i_2\},i_3\}} = (x_{\{\{i_1,i_2\},i_3\}1}, ..., x_{\{\{i_1,i_2\},i_3\}m}) \in \mathbb{R}^m$ for the pre-alliance $\{i_1, i_2, i_3\}$ such that

$$x_{\{\{i_1,i_2\},i_3\}l} = \frac{(w_{i_1} + w_{i_2}) \cdot \widetilde{x}_{\{i_1,i_2\}l}^{\{i_1,i_2,i_3\}} + w_{i_3} \cdot \widetilde{x}_{i_3 l}^{\{i_1,i_2,i_3\}}}{w_{i_1} + w_{i_2} + w_{i_3}} \quad for \quad l = 1, ..., m. \tag{21}$$

– If $|S| > 3$, then starting from $p = 3$ to $p = |S| - 1$ we repeat the following:

- *Firm* i_{p+1} *chooses the position* $\widetilde{x}_{i_{p+1}}^{S_p \cup \{i_{p+1}\}} \in \mathbb{R}^m$ *such that*

$$\widetilde{x}_{i_{p+1}}^{S_p \cup \{i_{p+1}\}} = arg \min_{z \in \bigcap_{i \in \{i_1, \ldots, i_{p+1}\}} M_i} d(x^*_{i_{p+1}}, z). \tag{22}$$

- *Pre-alliance* S_p *chooses a new position* $\widetilde{x}_{\overline{S}_p}^{S_p \cup \{i_{p+1}\}} \in \mathbb{R}^m$ *such that*

$$\widetilde{x}_{\overline{S}_p}^{S_p \cup \{i_{p+1}\}} = arg \min_{z \in \bigcap_{i \in \{i_1, \ldots, i_{p+1}\}} M_i} d(x_{\overline{S}_p}, z). \tag{23}$$

- *Firm* i_{p+1} *joins the pre-alliance* $\{i_1, ..., i_p\}$, *and firms* $i_1, ..., i_{p+1}$ *choose the position* $x_{\{\overline{S}_p, i_{p+1}\}} = (x_{\{\overline{S}_p, i_{p+1}\}1}, ..., x_{\{\overline{S}_p, i_{p+1}\}m})$ *for the alliance* $\{i_1, ..., i_{p+1}\}$ *such that, for each* $l = 1, ..., m$

$$x_{\{\overline{S}_p, i_{p+1}\}l} = \frac{\widetilde{x}_{\overline{S}_p l}^{S_p \cup \{i_{p+1}\}} \cdot \sum_{r=1}^{p} w_{i_r} + \widetilde{x}_{i_{p+1} l}^{S_p \cup \{i_{p+1}\}} \cdot w_{i_{p+1}}}{\sum_{r=1}^{p+1} w_{i_r}}. \tag{24}$$

Let us analyze the step-by-step procedure. First, a two-firm pre-alliance $\{i_1, i_2\}$ is formed. Firms i_1 and i_2 choose their negotiation positions $\widetilde{x}_{i_1}^{\{i_1,i_2\}}$ and $\widetilde{x}_{i_2}^{\{i_1,i_2\}}$ from the intersection of their maneuvering spaces; see equations (16) and (17). These negotiation positions minimize the distances between the ideal positions of the firms in question and the overlap of their maneuvering spaces. Next, the pre-alliance position $x_{\{i_1,i_2\}}$ is chosen which is the gravity center of the negotiation positions; see equation (18). So far, this coincides with the simultaneous procedure, but note the difference when a third party joins. In that case, this third firm i_3 chooses its negotiation position $\widetilde{x}_{i_3}^{\{i_1,i_2,i_3\}}$ which minimizes the distance between its ideal position and the overlap of the three firms' maneuvering spaces; see (19). Moreover, the pre-alliance $\{i_1, i_2\}$ chooses a new position $\widetilde{x}_{\{i_1,i_2\}}^{\{i_1,i_2,i_3\}}$ which minimizes a distance between the pre-alliance position $x_{\{i_1,i_2\}}$ and the overlap of the maneuvering spaces of all three firms; see (20). In contrast to the simultaneous procedure, the pre-alliance takes the agreed pre-alliance position as ideal position. Finally, firms i_1, i_2, i_3 choose the alliance position $x_{\{\{i_1,i_2\},i_3\}}$ as the gravity center of the negotiation positions of the old alliance $\{i_1, i_2\}$ ($\widetilde{x}_{\{i_1,i_2\}}^{\{i_1,i_2,i_3\}}$) and the firm i_3 ($\widetilde{x}_{i_3}^{\{i_1,i_2,i_3\}}$), with the weights $w_{i_1} + w_{i_2}$ and w_3, respectively; see (21). If there are more firms interested in joining, the procedure continues in an analogous way. The underlying idea is always the same: a new firm and the pre-alliance choose their negotiation positions. The negotiation position of the new firm minimizes the distance between its ideal position and the overlap of all firms in question (see (22)), while the negotiation position of the pre-alliance minimizes the distance between the position of this alliance and the overlap of all firms involved; see (23). As defined by (24), the alliance position is always the gravity center of the negotiation positions of the new firm and the pre-alliance. The weight of the pre-alliance is equal to the sum of the weights of the pre-alliance members.

As in the simultaneous case, we can prove that the step-by-step procedure leads to an alliance position in the overlap of all maneuvering spaces which is acceptable to all firms in the alliance, and this alliance position is Pareto efficient. Formally:

Proposition 3 *Let $x_{\overline{S}}$ be the position of alliance S resulting from Procedure $\overline{S}$ (see (22), (23) and (24)), where $\overline{S} \in PROC(S)$. Then:*

(a) $x_{\overline{S}} \in \bigcap_{i \in S} M_i$,
(b) $x_{\overline{S}}$ *is Pareto efficient with respect to alliance S.*

Secondly, one may show that the step-by-step procedure leads to a unique prediction; it will lead to one alliance position.

Proposition 4 *Let $S \subseteq N$ and $\bigcap_{i \in S} M_i \neq \emptyset$. Procedure $\overline{S}$, where $\overline{S} \in PROC(S)$, is uniquely defined, that is, there is always exactly one position $x_{\overline{S}}$ defined in (24).*

In order to make the procedure more specific and show how calculations may be made, we will consider more in detail a two- and three-partner alliance formed with the step-by-step procedure. By virtue of Propositions 3 and 4, unique and Pareto efficient predictions will be made. In the case of a two-partner alliance formation, it is easy to calculate the negotiation positions.

Proposition 5 *Let $M_{i_1} = M_{i_1}(x^*_{i_1}, r_{i_1})$ and $M_{i_2} = M_{i_2}(x^*_{i_2}, r_{i_2})$ be two maneuvering spaces in $\mathbb{R}^m$ such that $M_{i_1} \cap M_{i_2} \neq \emptyset$. There is always exactly one $\widetilde{x}^{\{i_1,i_2\}}_{i_1} \in \mathbb{R}^m$ satisfying (16) and it looks as follows:*

– *If $r_{i_2} \geq d(x^*_{i_1}, x^*_{i_2})$, then $\widetilde{x}^{\{i_1,i_2\}}_{i_1} = x^*_{i_1}$,*
– *if $r_{i_2} < d(x^*_{i_1}, x^*_{i_2})$, then $\widetilde{x}^{\{i_1,i_2\}}_{i_1}$ is equal to*

$$\widetilde{x}^{\{i_1,i_2\}}_{i_1 p} = x^*_{i_2 p} + \frac{r_{i_2} \cdot (x^*_{i_1 p} - x^*_{i_2 p})}{\sqrt{\sum_{r=1}^{m} (x^*_{i_1 r} - x^*_{i_2 r})^2}} = x^*_{i_2 p} + \frac{r_{i_2} \cdot (x^*_{i_1 p} - x^*_{i_2 p})}{d(x^*_{i_1}, x^*_{i_2})} \qquad for\ p = 1, ..., m. \tag{25}$$

Proposition 5 gives two scenarios. First, if the ideal point of firm i_1 belongs to the maneuvering space of i_2, then the negotiation position of i_1 (when forming the pre-alliance $\{i_1, i_2\}$) coincides with its ideal point. In the second scenario, the ideal point of firm i_1 lies outside the maneuvering space of i_2 and in that case equation (25) gives the negotiation position of i_1.

Proposition 6 gives necessary and sufficient conditions for a Pareto efficient position in a two-partner pre-alliance: the set of all Pareto efficient positions with respect to the alliance is the segment between the negotiation positions of the firms involved.

Proposition 6 *Let $M_{i_1} = M_{i_1}(x^*_{i_1}, r_{i_1})$ and $M_{i_2} = M_{i_2}(x^*_{i_2}, r_{i_2})$ be two maneuvering spaces in $\mathbb{R}^m$ such that $M_{i_1} \cap M_{i_2} \neq \emptyset$. A position $y \in M_{i_1} \cap M_{i_2}$ is Pareto efficient with respect to $\{i_1, i_2\}$ if and only if $y \in \overline{\widetilde{x}^{\{i_1,i_2\}}_{i_1} \widetilde{x}^{\{i_1,i_2\}}_{i_2}}$, where $\widetilde{x}^{\{i_1,i_2\}}_{i_1}$ and $\widetilde{x}^{\{i_1,i_2\}}_{i_2}$ are defined in (16) and (17), respectively, and $\overline{\widetilde{x}^{\{i_1,i_2\}}_{i_1} \widetilde{x}^{\{i_1,i_2\}}_{i_2}}$ denotes the segment between $\widetilde{x}^{\{i_1,i_2\}}_{i_1}$ and $\widetilde{x}^{\{i_1,i_2\}}_{i_2}$.*

One can also calculate the negotiation position of a three-partner alliance.

Proposition 7 *Let $M_{i_1} = M_{i_1}(x^*_{i_1}, r_{i_1})$, $M_{i_2} = M_{i_2}(x^*_{i_2}, r_{i_2})$, and $M_{i_3} = M_{i_3}(x^*_{i_3}, r_{i_3})$ be three maneuvering spaces in $\mathbb{R}^m$ such that $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$. There is always exactly one $\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}} \in \mathbb{R}^m$ satisfying (20) and it looks as follows:*

- *If $r_{i_3} \geq d(x^*_{i_3}, x_{\{i_1,i_2\}})$, then $\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}} = x_{\{i_1,i_2\}}$,*
- *if $r_{i_3} < d(x^*_{i_3}, x_{\{i_1,i_2\}})$, then $\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}}$ is equal to*

$$\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}p} = x^*_{i_3p} + \frac{r_{i_3} \cdot (x_{\{i_1,i_2\}p} - x^*_{i_3p})}{d(x^*_{i_3}, x_{\{i_1,i_2\}})} \quad for \quad p = 1, ..., m. \tag{26}$$

Again, two scenarios are discussed in Proposition 7. If the position of pre-alliance $\{i_1, i_2\}$ belongs to the maneuvering space of a new joining firm i_3, then the negotiation position of $\{i_1, i_2\}$ (when forming a three-partner alliance) coincides with the pre-alliance position $x_{\{i_1,i_2\}}$ resulted from the pre-alliance formation. However, if the alliance position $x_{\{i_1,i_2\}}$ lies outside the maneuvering space of firm i_3, equation (26) holds.

5 A NUMERICAL EXAMPLE

In this section, we introduce a very simple numerical three-partner, two-dimensional example. From now on, we will often make use of three-partner alliances in our analyses.

The reason for that is twofold. First, analysis with more than three partners are theoretically complex. Second, three-partner alliances are formed more frequently in reality and larger multi-partner alliances fail more often (Parkhe, 1993). Hence, the relevance of three-partner alliances is larger.

This example illustrates all the procedures mentioned before and shows how these procedures can lead to different outcomes. The valuation of these outcomes by the firms in the game gives us the opportunity to compare these outcomes, and hence the procedures.

Let us consider three firms, $N = \{i, j, k\}$. Let the ideal alliance positions perceived by the parties be equal to $x_i^* = (-3, 1)$, $x_j^* = (3, 1)$, $x_k^* = (0, -3)$. Here, it is seen we work with two dimensions. Recall from Section 2.1 what these dimensions, and hence positions mean; issues to be agreed on. Furthermore, let the radii of the maneuvering spaces of the firms be $r_i = r_j = r_k = 4$. This means that all three firms are equally flexible and what to deviate to the same extent from their ideal alliance position. The weights of the parties are equal to $w_i = 5$, $w_j = 3$, $w_k = 2$. Firm i is the most powerful player, while firm k has the least power in the negotiations. From (1), we can calculate the distances between firms' ideal positions: $d(x_i^*, x_j^*) = 6$, $d(x_i^*, x_k^*) = d(x_j^*, x_k^*) = 5$. Moreover, we have seen that we need to calculate negotiation positions $\widetilde{x}$ as the intermediate step in the negotiations between ideal positions and alliance positions. For the step-by-step pre-alliances these are $\widetilde{x}_i^{\{i,j\}} = (-1, 1)$, $\widetilde{x}_j^{\{i,j\}} = (1, 1)$. By virtue of (18), we get

$$x_{\{i,j\}} = \left(-\frac{1}{4}, 1\right), \quad x_{\{i,k\}} = \left(-\frac{66}{35}, -\frac{17}{35}\right), \quad x_{\{j,k\}} = \left(\frac{42}{25}, -\frac{19}{25}\right).$$

In order to find the positions of the multi-partner alliance

$$\{i, j, k\}$$

resulting from different procedures, we will have to solve several systems of equations and inequalities. For instance, in order to find $\widetilde{x}_i^{\{i,j,k\}}$ we solve the following system of equations and inequality

$$\begin{cases} (\widetilde{x}_{i1}^{\{i,j,k\}} - 3)^2 + (\widetilde{x}_{i2}^{\{i,j,k\}} - 1)^2 = 16 \\ (\widetilde{x}_{i1}^{\{i,j,k\}})^2 + (\widetilde{x}_{i2}^{\{i,j,k\}} + 3)^2 = 16 \\ (\widetilde{x}_{i1}^{\{i,j,k\}} + 3)^2 + (\widetilde{x}_{i2}^{\{i,j,k\}} - 1)^2 \leq 16 \end{cases}$$

and get

$$\widetilde{x}_i^{\{i,j,k\}} = \left(\frac{3}{2} - \frac{2\sqrt{39}}{5}, -1 + \frac{3\sqrt{39}}{10}\right).$$

By analogy, we calculate

$$\widetilde{x}_j^{\{i,j,k\}} = \left(\frac{2\sqrt{39}}{5} - \frac{3}{2}, -1 + \frac{3\sqrt{39}}{10}\right), \quad \widetilde{x}_k^{\{i,j,k\}} = \left(0, 1 - \sqrt{7}\right).$$

In order to find $\widetilde{x}_{\{i,j\}}^{\{i,j,k\}}$ we solve the following system of equations

$$\begin{cases} (\widetilde{x}_{\{i,j\}1}^{\{i,j,k\}})^2 + (\widetilde{x}_{\{i,j\}2}^{\{i,j,k\}} + 3)^2 = 16 \\ \widetilde{x}_{\{i,j\}1}^{\{i,j,k\}} = t \cdot x_{\{i,j\}1} + (1-t) \cdot x_{k1}^* \\ \widetilde{x}_{\{i,j\}2}^{\{i,j,k\}} = t \cdot x_{\{i,j\}2} + (1-t) \cdot x_{k2}^* \quad for \quad t \in [0,1] \end{cases} \tag{27}$$

and get

$$\widetilde{x}_{\{i,j\}}^{\{i,j,k\}} = \left(-\frac{4}{\sqrt{257}}, \frac{64}{\sqrt{257}} - 3\right).$$

In an analogous way, we get

$$\widetilde{x}_{\{i,k\}}^{\{i,j,k\}} = \left(3 - \frac{684}{\sqrt{31945}}, 1 - \frac{208}{\sqrt{31945}}\right), \quad \widetilde{x}_{\{j,k\}}^{\{i,j,k\}} = \left(\frac{93}{125}, -\frac{51}{125}\right).$$

By virtue of (21) and the data above, we have

$$x_{\{\{i,j\},k\}} = \left(-\frac{16}{5\sqrt{257}}, \frac{256}{5\sqrt{257}} - \frac{11}{5} - \frac{\sqrt{7}}{5}\right) \tag{28}$$

$$x_{\{\{i,k\},j\}} = \left(\frac{33}{20} - \frac{2394}{5\sqrt{31945}} + \frac{3\sqrt{39}}{25}, \frac{2}{5} - \frac{728}{5\sqrt{31945}} + \frac{9\sqrt{39}}{100}\right) \tag{29}$$

$$x_{\{\{j,k\},i\}} = \left(\frac{561}{500} - \frac{\sqrt{39}}{5}, -\frac{88}{125} + \frac{3\sqrt{39}}{20}\right). \tag{30}$$

Finally, by virtue of (12) and the data above, we get

$$x_{\{i,j,k\}} = \left(\frac{3}{10} - \frac{2\sqrt{39}}{25}, -\frac{3}{5} + \frac{6\sqrt{39}}{25} - \frac{\sqrt{7}}{5}\right). \tag{31}$$

Next, to determine each firm's preferences we calculate the distance between its ideal alliance position and the different positions of the multi-partner alliance resulting from

the different procedures. To be complete, we also take the pre-alliances which form during the step-by-step procedure into account. For firm i, this means comparing x_i^* with the position of the simultaneously formed $\{i,j,k\}$ (see (31)) the three step-by-step alliances (see (28), (29), and (30)), and the two-partner pre-alliances i is part of. The same kind of comparisons are made for the other two firms. Based on these calculations, we create a ranking of (pre-)alliance positions for each firm. Figure 1 shows the results. Here, we can for example see that firm i prefers the two-partner pre-alliance with k ($\{i,k\}$) the most, and that joining $\{j,k\}$ later in the step-by-step procedure is its least favorite option.
Figure 1: Ranking of (pre-)alliances' positions

firm	preferences
i	$x_{\{i,k\}} \succ_i x_{\{i,j\}} \succ_i x_{\{\{i,k\},j\}} \succ_i x_{\{\{i,j\},k\}} \succ_i x_{\{i,j,k\}} \succ_i x_{\{\{j,k\},i\}}$
j	$x_{\{j,k\}} \succ_j x_{\{\{j,k\},i\}} \succ_j x_{\{\{i,j\},k\}} \succ_j x_{\{i,j\}} \succ_j x_{\{i,j,k\}} \succ_j x_{\{\{i,k\},j\}}$
k	$x_{\{j,k\}} \succ_k x_{\{i,k\}} \succ_k x_{\{\{i,k\},j\}} \succ_k x_{\{\{j,k\},i\}} \succ_k x_{\{i,j,k\}} \succ_k x_{\{\{i,j\},k\}}$

Based on this example, several implications can be stated. First, the example shows that different procedures yield different outcomes - alliance positions. Hence, when negotiating on an alliance position, the kind of procedure adopted matters. It is also seen that firms have different preferences over the different procedures. Recall that preference is determined by distance: the closer an alliance position to a firm's ideal position, the better. Since the distances between the different alliance positions and a firm's ideal position differ, the preferences over the outcomes and the procedures differ.

Second, in this example, we can draw some general conclusions for all three firms concerning their preferences over the procedures. There are three paths to reach the three-partner alliance: in a simultaneous way, step-by-step joining in the first round (first-mover), and step-by-step joining in the last round (late-mover). In this example, it happens that the preferences of all three firms over these three paths is the same. They prefer being a first mover the most. Second best is the simultaneous procedure, while the least preferred situation is the step-by-step alliance in which the specific firm joins as the last member. Disliking being a last-mover (for i this is $\{\{j,k\},i\}$) can be explained by the fact that not only the ideal position of a firm (x_i^*) is not acceptable by the other firms (j and k), but also the two-partner pre-alliance position ($x_{\{j,k\}}$) is not acceptable to the remaining third firm (i). This is why a firm is better off if it joins already in the beginning, either as one of the first movers or in the simultaneous procedure.

Third, in addition to distance to ideal positions, negotiation power plays a role in the model. In the example, the firms have different negotiating power: firm i is the most powerful one, while firm k is the least powerful. The role of negotiation power is for example seen in the attractiveness of firm k for both firm i and j: both firms i and j prefer to join first with the least powerful and 'closer' firm k. This is not only because firm k's ideal position is closer to i (and j) than the ideal position of j (and of i), but also because k has the smallest negotiating power. In general, in the example, we can see that for each firm, forming a two-partner pre-alliance with the less powerful and 'closer' of the remaining two firms is the most attractive option.

The question is whether the implications hold in all cases or are characteristic for this example. In the next section, we will check if these results hold in general or are just accidental.

6 COMPARING THE PROCEDURES

So far, we have shown the formalization of two procedures of alliance formation. In the previous section, we saw that these procedures indeed lead to different solutions, and that firms have clear preferences over these solutions. We will now investigate issues related to the role of procedure and try to find out whether the preference sequence from the example always hold. Three question are tackled: when does procedure not matter, is being a first mover always advantageous, and when is one procedure more advantageous? In order to answer those questions, we will provide some general results which hold for n players and m dimensions and be more specific and intuitive with three-partner m-dimensional specifications and examples.

6.1 When does procedure not matter?

In the presented example, the different procedures have all led to different results. A follow-up question would be to ask whether the procedures always lead to different alliance positions. In which case will an alliance formed by the simultaneous procedure lead to the same position as this alliance formed step-by-step? In Proposition 8, we deliver sufficient conditions under which the procedures of forming an n-partner alliance ($n \geq 3$) are equally attractive to a firm.

Proposition 8 *Let M_i for $i \in N$ be maneuvering spaces in $\mathbb{R}^m$,*
$S \subseteq N$, $|S| \geq 2$, $\widetilde{i} \in N \setminus S$.

(1) If $\bigcap_{j\in S} M_j \neq \emptyset$ *and for each* $i \in S$, $x_i^* \in \bigcap_{j\in S} M_j$, *then*

$$x_{\overline{S}} = x_S \text{ for each } \overline{S} \in PROC(S). \tag{32}$$

(2) If $\bigcap_{j\in S\cup\{\widetilde{i}\}} M_j \neq \emptyset$ *and for each* $i \in S$, $x_i^* \in \bigcap_{j\in S} M_j$, *then*

$$x_{\{\overline{S},\widetilde{i}\}} = x_{\{\overline{\overline{S}},\widetilde{i}\}} \text{ for any } \overline{S}, \overline{\overline{S}} \in PROC(S). \tag{33}$$

(3) If $\bigcap_{j\in S\cup\{\widetilde{i}\}} M_j \neq \emptyset$ *and for each* $i \in S$, $x_i^* \in \bigcap_{j\in S\cup\{\widetilde{i}\}} M_j$, *then*

$$x_{\{\overline{S},\widetilde{i}\}} = x_{S\cup\{\widetilde{i}\}} \text{ for each } \overline{S} \in PROC(S). \tag{34}$$

What does Proposition 8 say? According to part (1), if the ideal positions of all companies forming an alliance S belong to the (non-empty) intersection of the maneuvering spaces of the companies in question, then the simultaneous procedure and all step-by-step procedures of forming the alliance S do lead to the same alliance position. Parts (2) and (3) concern a situation in which a new firm $\widetilde{i}$ joins the alliance S, and therefore we assume that the intersection of the maneuvering spaces of the companies forming $S\cup\{\widetilde{i}\}$ is non-empty. Part (2) says that if the ideal positions of all members of S belong to the (non-empty) intersection of the maneuvering spaces of the companies of S, then all step-by-step procedures of forming $S\cup\{\widetilde{i}\}$ in which the firm $\widetilde{i}$ is the last mover also lead to the same alliance position. The final part (3) states that if the ideal positions of all members of S belong to the (non-empty) intersection of the maneuvering spaces of the companies of $S\cup\{\widetilde{i}\}$, then all step-by-step procedures of forming $S\cup\{\widetilde{i}\}$ in which the firm $\widetilde{i}$ is the last mover and the simultaneous procedure of forming $S\cup\{\widetilde{i}\}$ give the same alliance position.

In other words, to put it very simply and general: if ideal alliance positions of firms (either all firms forming an alliance or all first-movers) are acceptable to all firms, then procedure plays no role (either all procedures or certain step-by-step procedures). This very general finding can be fine-tuned according to the three sufficient conditions we have found in Proposition 8 (part (1), (2), and (3)). While the first part studies acceptability of all members of an alliance, the second and third part consider acceptability between first movers.

First, if the ideal alliance positions of the members of an alliance are already acceptable by all members of the alliance, then procedure plays no role (part (1)). That means that either the ideal positions themselves are really close and acceptable by the alliance members or the firms are very flexible such that they also find a distant ideal position of another firm still acceptable. The firms already agree to such an extent, that neither negotiations nor negotiation procedures are that important anymore.

Second, we can relax this rather strict condition and consider a situation in which firms with their ideal positions acceptable by all (pre-)alliance members (as in (1)) have already formed a pre- alliance with the step-by-step procedure and are now joined by an 'outlier' (part (2)). The joining firm is an outlier as its ideal position may be outside the overlap of the other firms' maneuvering spaces. This means that the ideal alliance position of the joining firm may be not acceptable by (some of) the members of the pre-alliance. However, this does not mean that all of the positions acceptable for the firm joining last are unacceptable to the members of the pre-alliance. As in all (pre-)alliances we study, all firms involved have a non-empty intersection of their maneuvering space, that is, an agreement is possible: there is an alliance position which is acceptable to all firms involved. In the situation in which the ideal alliance positions of all firms, except of the last mover, are acceptable by all members of the alliance, then all the step-by-step procedures with the outlier as last mover lead to the same outcome.

Third, if, additionally, the ideal alliance positions of all alliance members are acceptable to the last mover, then all the step-by-step procedures with this last mover and the simultaneous procedure lead to the same result (part (3)).

As one can see, the found conditions under which procedure plays no role do not depend on the negotiating power of the firms involved. The only important element here is whether the alliance members can already accept the ideal alliance positions of either all the members or only of the first-movers. This acceptability depends on the relation between the ideal positions of the firms and the flexibility of the alliance members.

In conclusion, different procedures do not always lead to different alliance positions. However, in most cases they do. If, in particular, the ideal positions of the firms are acceptable to all alliance members, the different procedures lead to the same result. Also, if first movers in a procedure have ideal positions which are mutually acceptable, the procedure in which a late-mover outlier joins them does not play a role. Both these situations will not be very common in business reality: usually alliance members have their differences and need to negotiate firmly to overcome those differences. Therefore, we can realistically say that, except for some special and rare conditions, the procedures lead to different alliance positions.

6.2 Is being a first mover always advantageous?

In the example of the previous section, we saw that each firm preferred the step-by-step procedure in which it joins first (first mover) over a simultaneous alliance over a step-by-step alliance in which the firm joins last (late mover). This seems an intuitive result, but does it always hold? Or, is it also possible that the following holds $x_{\{\{i_2,i_3\},i_1\}} \succ_{i_1} x_{\{i_1,i_2,i_3\}} \succ_{i_1} x_{\{\{i_1,i_2\},i_3\}}$? In other words, (1) can a situation occur in which a firm favors alliance formation such that it joins two partners which have already agreed together on an alliance (i.e. being a last mover)? At first sight this seems a disadvantageous situation: joining later in the process means coping with a strong block which already has an agreement. And, (2) is it possible that a firm prefers to negotiate simultaneously over being a first mover in a step-by-step approach? If firms want to keep control over negotiations, this also seems counter-intuitive: bargaining in a large group is more difficult as more compromises have to be made in order to reach consensus with all partners. In order to study those two questions, we present a counter-example.

Example 1 Let us again consider forming a three-partner alliance $N = \{1, 2, 3\}$. This is a one-dimensional example, that is, there is only one issue the three firms will have to agree on. The ideal alliance positions perceived by the firms and the radii are as follows: $x_1^* = 0$, $x_2^* = -3$, $x_3^* = 3$, $r_1 = 2$, $r_2 = r_3 = 4$. Hence, the maneuvering spaces here are equal to the following intervals: $M_1 = [-2, 2]$, $M_2 = [-7, 1]$, $M_3 = [-1, 7]$, and consequently, $M_1 \cap M_2 = [-2, 1]$, $M_2 \cap M_3 = M_1 \cap M_2 \cap M_3 = [-1, 1]$. We assume the firms to be 'equally strong', that is $w_1 = w_2 = w_3 = 1$. When applying the step-by-step procedures and the simultaneous procedure we get the following results:

$$x_1^{\{1,2\}} = x_1^* = 0, \quad x_2^{\{1,2\}} = -2, \quad x_{\{1,2\}} = -1$$

$$x_{\{1,2\}}^{\{1,2,3\}} = x_{\{1,2\}} = -1, \quad x_3^{\{1,2,3\}} = 1, \quad x_{\{\{1,2\},3\}} = -\frac{1}{3}$$

$$x_2^{\{2,3\}} = -1, \quad x_3^{\{2,3\}} = 1, \quad x_{\{2,3\}} = 0 = x_1^*$$

$$x_{\{2,3\}}^{\{1,2,3\}} = x_1^* = x_1^{\{1,2,3\}} = 0, \quad x_{\{\{2,3\},1\}} = 0 = x_1^*$$

$$x_1^{\{1,2,3\}} = x_1^* = 0, \quad x_2^{\{1,2,3\}} = -1, \quad x_3^{\{1,2,3\}} = 1, \quad x_{\{1,2,3\}} = 0 = x_1^*$$

Hence, we get:

$$d(x_1^*, x_{\{\{1,2\},3\}}) = \frac{1}{3} > 0 = d(x_1^*, x_{\{\{2,3\},1\}}) = d(x_1^*, x_{\{1,2,3\}})$$

which means that $x_{\{1,2,3\}} \succ_1 x_{\{\{1,2\},3\}}$ and $x_{\{\{2,3\},1\}} \succ_1 x_{\{\{1,2\},3\}}$.

Let us now adjust the example by assuming that the first firm is much more powerful than the others, let $w_1 = 8$, $w_2 = w_3 = 1$. The remaining assumptions are the same as in Example 1. Hence, we get:

$$d(x_3^*, x_{\{\{1,2\},3\}}) = \frac{31}{10} < \frac{37}{10} = d(x_3^*, x_{\{1,2,3\}})$$

which means that $x_{\{\{1,2\},3\}} \succ_3 x_{\{1,2,3\}}$.

To interpret the findings of this example, we see that in the original version of the example, firm 1 prefers the simultaneous procedure over the step-by-step procedure $\{\{1,2\},3\}$. Here, we answer the second question posed at the beginning of this section. According to this example, it is possible that a simultaneous procedure is preferred by a firm over any of the step-by-step procedures. Moreover, we can also answer the first question phrased at the beginning of this section; can a firm prefer being a late-mover? Some counter-intuitive findings in the example shows that it is possible. Player 1 chooses step-by-step procedure $\{\{2,3\},1\}$ in which it joins as the last over step-by-step procedure $\{\{1,2\},3\}$ in which 1 is one of the first movers. Also in the adjusted example, firm 3 rather joins in later: the step-by-step procedure in which 3 joins as last partner is preferred by 3 over the simultaneous procedure.

With this example, we have shown that preferences of firms over procedures might not always be as straightforward as one expects. It is possible that a firm prefers being a late mover over simultaneous formation over being a first-mover. It appears that in some cases it might be worth to wait to enter alliance negotiations. As an explanation, one may think of delaying entrance until the other partners have settled their disputes. For instance, if the other alliance partners can reach a compromise close to the ideal position of the firm in question, it might pay for this firm to enter later. In other instances, forming the alliance with the whole group simultaneously is the wisest thing to do. This could be the case, for example, if a firm is a first mover in a multi-partner alliance, first trying to settle with a firm which is very 'far' from its ideal point. In that case, negotiating at once with all members of the alliance could appear to be more advantageous. Together with its close neighbors it could be able to put more weight in the game to pull the alliance position away from the outlier.

6.3 When is one procedure more advantageous?

Next, we consider under which conditions one procedure is more attractive to a firm than another. We will study the attractiveness of the procedures in two steps. First, we present a general fact and second, we zoom in on three-partner alliances. For the three-partner alliance case, we present two specific situations (Propositions 9 and 10) under which players have preferences over procedures.

To start with, using equations (2), (3) and (4), we immediately get necessary and sufficient conditions for the attractiveness of one procedure over another. Taking any two procedures, labelled $\overline{S}$ and $\overline{\overline{S}}$, a firm will prefer procedure $\overline{S}$ over procedure $\overline{\overline{S}}$ if the alliance position predicted by procedure $\overline{S}$ is closer to the ideal position of the firm than the alliance position predicted by procedure $\overline{\overline{S}}$. In a similar way, we can define equal attractiveness and strict preference. Formally, the following fact holds:

Fact 1 *For any $S \subseteq N$, $\overline{S}, \overline{\overline{S}} \in PROC(S) \cup \{S\}$, $i \in S$*

$$x_{\overline{S}} \succeq_i x_{\overline{\overline{S}}} \quad iff \quad \sum_{l=1}^{m} (x_{\overline{S}l} - x_{il}^*)^2 \leq \sum_{l=1}^{m} \left(x_{\overline{\overline{S}}l} - x_{il}^*\right)^2 \tag{35}$$

$$x_{\overline{S}} \succ_i x_{\overline{\overline{S}}} \quad iff \quad \sum_{l=1}^{m} (x_{\overline{S}l} - x_{il}^*)^2 < \sum_{l=1}^{m} \left(x_{\overline{\overline{S}}l} - x_{il}^*\right)^2 \tag{36}$$

$$x_{\overline{S}} \sim_i x_{\overline{\overline{S}}} \quad iff \quad \sum_{l=1}^{m} (x_{\overline{S}l} - x_{il}^*)^2 = \sum_{l=1}^{m} \left(x_{\overline{\overline{S}}l} - x_{il}^*\right)^2 \tag{37}$$

The presented model contains three important elements which determine the outcome of a procedure: the weights of the involved firms, the radii of their maneuvering spaces, and their ideal positions. To put it differently, it matters how much power the firms have, how flexible the firms are, and what their ideal alliance ideas are. The last two elements determine a related element which is important in the model: the configuration of positions of all firms in the game. What is essential then is how the alliance ideas of the firms are related to one another. In order to determine which procedure is better in which case, all these elements play a role. Predicting in which kind of situation e.g. a simultaneous procedure is always the best option implies finding specific combinations of these elements. For a general case, finding such predictions for different procedures including all elements involving n players and m dimensions is very technical and complicated.

However, as a second step to finding out when one procedure is better than the other, we can provide some sufficient conditions on three-partner alliances under which a firm prefers one procedure over another. In this way, we show how the conditions of Fact 1 look like in a specific situation. Consider the following two propositions (Proposition 9 and 10). Proposition 9 focusses on the choice between the simultaneous procedure and being a late mover. The choice between the simultaneous procedure and being a first mover is studied in Proposition 10.

Proposition 9 *Let $M_{i_1} = M_{i_1}(x_{i_1}^*, r_{i_1})$, $M_{i_2} = M_{i_2}(x_{i_2}^*, r_{i_2})$ and $M_{i_3} = M_{i_3}(x_{i_3}^*, r_{i_3})$ be three maneuvering spaces in $\mathbb{R}^m$ such that $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$, $x_{i_1}^* \notin M_{i_2} \cup M_{i_3}$, $x_{i_2}^* \notin M_{i_1} \cup M_{i_3}$, $x_{i_3}^* \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$, and*

$$\frac{w_{i_1}}{w_{i_2}} = \frac{r_{i_1}}{r_{i_2}}. \tag{38}$$

(1) If $d(x_{i_3}^*, x_{\{i_1,i_2\}}) \leq r_{i_3}$ and

$$\sum_{l=1}^{m} (x_{i_3 l}^* - x_{i_1 l}^*)(x_{i_3 l}^* - x_{i_2 l}^*) \geq \sum_{l=1}^{m} \left(x_{i_3 l}^* - \widetilde{x}_{i_1 l}^{\{i_1,i_2,i_3\}}\right) \left(x_{i_3 l}^* - \widetilde{x}_{i_2 l}^{\{i_1,i_2,i_3\}}\right) \tag{39}$$

then

$$x_{\{i_1,i_2,i_3\}} \succ_{i_3} x_{\{\{i_1,i_2\},i_3\}}. \tag{40}$$

(2) If $d(x^*_{i_3}, x_{\{i_1,i_2\}}) > r_{i_3}$ *then* $x_{\{i_1,i_2,i_3\}} \succeq_{i_3} x_{\{\{i_1,i_2\},i_3\}}$.

What does Proposition 9 literally say? It assumes the ideal position of one of the three firms (labelled here as firm i_3) belongs to the maneuvering spaces of all firms in question (i.e. i_1, i_2 and i_3), while the ideal positions of the remaining firms (firms i_1 and i_2) lie outside the maneuvering spaces of the other firms. Condition (38) says that the ratio of the weights of firms i_1 and i_2 is equal to the ratio of the radii of these firms.

According to part (1), if the position ($x_{\{i_1,i_2\}}$) of alliance $\{i_1, i_2\}$ belongs to the maneuvering space of firm i_3 and condition (39) is satisfied, then firm i_3 strongly prefers the simultaneous procedure to the step-by-step procedure in which firm i_3 joins in the last step. Condition (39) is a technical condition which says that a certain expression of the ideal positions of all the three firms creates a value, which is not smaller than the value of the analogous expression, in which we replace the ideal positions of firms i_1 and i_2 by the negotiation positions of these firms.

Part (2) says that if the position of pre-alliance $\{i_1, i_2\}$ lies outside the maneuvering space of firm i_3, then firm i_3 weakly prefers the simultaneous procedure to the step-by-step procedure in which i_3 joins in the end.

To be more intuitive, let us explain this proposition more in detail. The proposition holds under a specific situation. The first characteristic of this situation is that the ideal alliance position of firm i_3 is acceptable by firms i_1 and i_2, but the ideal alliance ideas of i_1 and i_2 are not acceptable by the remaining two firms. Firm i_3 will usually have to move less than its fellow alliance members, as its ideal position is already acceptable to them. Secondly, firms i_1 and i_2 have a certain balance in their degree of flexibility and their power. If firm i_1 is twice as powerful (powerless) as i_2, this characteristic says that firm i_1 is also twice as flexible (non-flexible) as firm i_2.

Now, firms i_1 and i_2 form a pre-alliance as a intermediate step in the step-by-step procedure. Of course, firms i_1 and i_2 together will form a pre-alliance position they will use in the remaining step of the procedure (negotiation with i_3). Two scenarios are possible. First, the position which i_1 and i_2 agree on ($x_{\{i_1,i_2\}}$) is acceptable by firm i_3. In this first scenario, a certain technical condition should hold, which demands a specific relation between the ideal positions and negotiation positions of the firms. If this technical condition holds in the first scenario, then firm i_3 rather forms the three-player alliance with the simultaneous procedure than joins as last member in the step-by-step procedure. The second scenario describes the situation in which the position the pre-alliance $\{i_1, i_2\}$ agrees on is not acceptable by i_3. In that case, firm i_3 is either indifferent between the step-by-step procedure with i_3 as last member and the simultaneous procedure or firm i_3 still prefers the simultaneous procedure.

What makes the difference between the two scenarios and the preferences of firm i_3 in it? In the second scenario, firm i_3 does not strictly prefer the simultaneous procedure over being the last mover, while firm i_3 is rather strict in its preferences in the first scenario; there the simultaneous procedure is always its most preferred procedure over the late mover procedure. In the second scenario something is different which makes that firm i_3 'starts to like' the last mover procedure more. In addition to the technical condition needed in the first scenario, the two scenarios differ in the acceptability of the pre-alliance position for i_3. Whereas scenario one leads to a pre-alliance position acceptable for i_3, the pre-alliance position reached in the second scenario is not acceptable for i_3. Hence, in the second scenario, firm i_1 and i_2 need to move their pre-alliance position more to reach i_3. In the late mover procedure of scenario two, firm i_3 is relatively better capable of pulling the other two players towards its ideal alliance position than in scenario one. It is exactly the relative distant pre-alliance position which makes this difference: pulling is needed more.

The second proposition (Proposition 10) which is presented now, analyzes when being a first mover is preferred over simultaneously forming an alliance.

Proposition 10 *Let* $M_{i_1} = M_{i_1}(x^*_{i_1}, r_{i_1})$*,* $M_{i_2} = M_{i_2}(x^*_{i_2}, r_{i_2})$ *and* $M_{i_3} = M_{i_3}(x^*_{i_3}, r_{i_3})$ *be three maneuvering spaces in* $\mathbb{R}^m$ *such that* $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$*,* $x^*_{i_1} \notin M_{i_2} \cup M_{i_3}$*,* $x^*_{i_2} \notin M_{i_1} \cup M_{i_3}$*,* $x^*_{i_3} \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$*. Moreover, let* $d(x^*_{i_2}, x_{\{i_1,i_3\}}) \leq r_{i_2}$ *and*

$$\sum_{l=1}^{m} \left(x^*_{i_1 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l}\right)\left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l}\right) \geq 0. \tag{41}$$

Then

$$x_{\{\{i_1,i_3\},i_2\}} \succ_{i_3} x_{\{i_1,i_2,i_3\}}. \tag{42}$$

In words, Proposition 10 also concerns the situation in which the ideal position of firm i_3 belongs to the maneuvering spaces of all three firms in question, while the ideal positions of firms i_1 and i_2 lie outside the maneuvering spaces of the other firms. Moreover, we assume that the position of alliance $\{i_1, i_3\}$ belongs to the maneuvering space of firm i_2. Finally, we impose condition (41), which says that a certain expression involving the ideal positions of i_1, i_3, and the negotiation positions of i_1 and i_2 has a non-negative value. If all of this holds, than firm i_3 strictly prefers the step-by-step procedure with i_2 as last mover over the simultaneous procedure.

In order to explain this further, let us study more in detail the situation under which this proposition holds. Partly this situation is the same as the one we have seen in

Proposition 9. The ideal alliance position of i_3 is acceptable for i_1 and i_2, but the ideal alliance positions of i_1 and i_2 are not acceptable by the other two firms. Note that the second characteristic mentioned in Proposition 9 is not imposed in this proposition.

As said earlier, in this proposition, we focus on the choice between the simultaneous procedure and the step-by-step procedure with being a first mover. We will again use the perspective of firm i_3.

Furthermore, two conditions are met in the proposition. First, the position of the pre-alliance between firms i_1 and i_3 is acceptable for i_2. The pre-alliance position is close enough for i_2 to be acceptable. So, although i_2 does not accept the ideal alliance position of i_1, it does accept the position of the pre-alliance formed by i_1 and i_3. The second condition demands a certain relation between the ideal positions and negotiation positions of the firms, again a rather technical condition.

If the sketched situation and the two conditions hold, then firm i_3 prefers the step-by-step procedure in which it is involved in the first step (and i_2 as last mover) over the simultaneous procedure. The situation and conditions make that firm i_3 now prefers being a first mover over forming the alliance with all members simultaneously. Apparently, the position i_3 can reach with i_1 makes i_3's status against i_2 better than negotiating with i_2 in the simultaneous procedure.

7 CONCLUSION

In this article, we have proposed a dynamic spatial alliance model with two different procedures to reach a multi-partner alliance. We have shown several results: the procedure of forming an alliance matters, when procedure does not matter and that late moving in the alliance process might pay off. On a less general level, but for a three-partner case, we have analyzed when which procedure is more advantageous for a firm: when being a first mover is more attractive than forming the alliance simultaneously, and when the simultaneous formation is better than being a late mover. Innovations of the paper are to model flexibility of players in achieving a certain (alliance) position, as opposed to the static fixation of positions of most spatial coalition models. Also, to model two different procedures (simultaneous and step-by-step) enriches the literature, as most studies neglect procedure or adopt one procedure. Moreover, through formalizing alliance positions into a multi-dimensional space, we can achieve *endogenous* preference formation for alliances, while preferences over alliances are often assumed exogenously.

With this formal model, some three-partner examples, and general results, we can draw several conclusions. First, the procedure adopted for reaching a multi-partner alliance does matter: different procedures usually lead to different alliance positions. We can also answer the question when it does not matter. If alliance partners' ideal positions are acceptable by all members of the alliance, the simultaneous procedure and all step-by-step procedures yield the same alliance position. Also, if the ideal positions of all firms except the last mover are acceptable by all alliance members, the simultaneous procedure and the step-by-step procedures with the given last mover lead to the same outcome.

Second, we conclude that it can be disadvantageous for a firm to be a first mover in negotiations. Although one would expect that a firm can better pull the negotiations towards its own ideas when involved in an early stage, it can happen that joining later is a better strategy. Also, a simultaneous procedure can be more advantageous than a step-by-step procedure.

As a third conclusion, several elements play a role in determining when which procedure is better: the power, the flexibility, the ideal alliance position of each player, and an element related to these ones - the configuration of the positions. The *combination* of those elements in a specific empirical situation determine what is the best procedure. The best procedure for a firm is the procedure that yields an alliance position closest to the firm's ideal position. In a three-partner case, firm k, with its ideal position acceptable to i and j, prefers the simultaneous procedure over being a late mover (1) if there is a certain balance in the firms' degree of flexibility and their power and (2) if the pre-alliance position of i and j is acceptable to k. In the same situation, firm k prefers being a first mover with i over the simultaneous procedure (1) if the pre-alliance position of i and k is not acceptable to j.

The implications of these findings are relevant for the two fields of contributions we have mentioned in the introduction: multi-partner alliance formation and formal coalition research. Although the literature on multi-partner alliances (e.g. Das and Teng, 2002; Doz et al., 2000) has acknowledged the complexity of multi-partner alliances as compared to dyadic alliances, no study has paid attention to the different formation paths of multi-partner alliance formation. As we saw with some examples in the introduction, formations of such multi-partner constellations differ. By simplifying those differences to two procedures, we have shown that procedure influences the result of alliance formation. Therefore, it really pays for an alliance manager to carefully consider his or her strategic options when entering an alliance. Next to this insight, we have contributed with our model by filling the gap of a lack of a more formal approach. The model has shown that it is capable of laying bare different procedures and the implications those have. More use of formal models might lead to more of such 'hidden' insights.

Concerning the second field of application, it also holds that almost no coalition model has so far taken the aspect of procedure into account. Either the models neglected the procedure (Schofield, 1993), or only one specific procedure was considered (as e.g. in

Grofman, 1982). Since it appears that the choice for a procedure is crucial for the result of a (spatial) coalition model, we strongly encourage future research to take this into account.

Although no empirical testing has been done in this paper, we do want to discuss how one could perform empirical tests with the model. In a second and related paper, De Ridder et al. (2006), more attention is paid to testing and making calculations with the model. These calculations have been done with an algorithm developed and reported in Sáiz and Hendrix (2006). The algorithm calculates which alliances are feasible, what the different procedures are for each feasible alliance, and to which alliance position these procedure lead. As input for the model, three kinds of data are needed: ideal positions of players, a weight and a radius for each player.

In the De Ridder et al. article, the field of application is political science and data from Dutch elections have been used. The ideal positions are ideal policy positions and have been derived from a data set with policy positions of Dutch political parties on 56 dimensions from 1998 and 2003 (Klingemann et al., 2006). The weights of the parties have been determined by the number of seats each party had in Dutch parliament. Due to the lack of empirical data on the radii of the maneuvering spaces, two different ways to determine the radii have been used: a radius similar for each party and a radius different for each party, randomly generated.

In a more business setting, the following kind of data could be used. The ideal positions of firms could be determined by empirically gathering these positions, for example via qualitative research. A more easy approach is to used secondary data, as annual reports, websites, and patents. The weights of firms can be determined in several ways, depending on the specific use of weight (see Section 2.1). Information on amount of employees and profit is usually public, while using bargaining power as interpretation would need additional empirical research. Finally, for the maneuvering space the same holds as in political science. So, either a fixed or randomly generated radius can be used.

We want to end with some suggestions for further research. The main drawbacks of the model can be explained by the focus we have aimed at: revealing the procedures of alliance formation and its consequences. We have therefore not been able to deal with other aspects of the model. The condition for an alliance to be able to form is for example quite strong: overlapping maneuvering spaces. No solution has been offered in the case that firms' maneuvering spaces do not overlap. Also, more research should be done on which procedure is the best option in a specific situation. Instead of looking for mathematical proofs, we suggest computer simulations due to the complexity of the problem under study. And finally, future research should test the model in a business setting, as the analogous model of coalition formation has been tested using data from Dutch politics.

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APPENDIX A: DEFINITIONS AND THEOREMS

First of all, we recall the *parametric equation of a segment*:

Definition 3 (Parametric equation of a segment)
Let $x_A = (x_{A1}, ..., x_{Am})$ *and* $x_B = (x_{B1}, ..., x_{Bm})$ *be two points in* $\mathbb{R}^m$*. Then,*

$$\overline{x_A x_B} = \{(x_1, ..., x_m) \in \mathbb{R}^m \mid x_k = t \cdot x_{Ak} + (1-t) \cdot x_{Bk} \; for \; k = 1, ..., m; \; t \in [0,1]\}, \quad (43)$$

where $\overline{x_A x_B}$ *denotes the segment between* x_A *and* x_B*.*

In some calculations below, we need some inequalities well-known in the literature in mathematics, that is, *Buniakowski-Cauchy inequality*:

Theorem 1 (Buniakowski-Cauchy inequality)
Let $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ be two sequences of n (arbitrary) numbers. Then, the following inequality holds:

$$\sum_{i=1}^{n} a_i \cdot b_i \leq \sqrt{\sum_{i=1}^{n} a_i^2} \cdot \sqrt{\sum_{i=1}^{n} b_i^2} \tag{44}$$

Definition 4 *A convex set A is a bounded and closed set such that if a and b are any two points belonging to A, then all points of the segment $\overline{ab}$ also belong to A.*

Theorem 2 *If A is a closed and bounded set and f is continuous in A, then there exist points in A where f has its minimum and there exist points in A where f has its maximum.*

APPENDIX B: PROOFS

Proof of Proposition 1

(a) We have to show that for each $i \in S$

$$\sum_{p=1}^{m} (x_{ip}^* - x_{Sp})^2 \leq r_i^2. \tag{45}$$

Since $\widetilde{x}_j^S \in \bigcap_{i \in S} M_i$ for each $j \in S$, we have for each $i, j \in S$

$$\sum_{p=1}^{m} (x_{ip}^* - \widetilde{x}_{jp}^S)^2 \leq r_i^2. \tag{46}$$

For each $i, j, k \in S$, by virtue of Buniakowski-Cauchy inequality for the sequences $(x_{i1}^* - \widetilde{x}_{j1}^S, ..., x_{im}^* - \widetilde{x}_{jm}^S)$ and $(x_{i1}^* - \widetilde{x}_{k1}^S, ..., x_{im}^* - \widetilde{x}_{km}^S)$, we have

$$\sum_{p=1}^{m} (x_{ip}^* - \widetilde{x}_{jp}^S) \cdot (x_{ip}^* - \widetilde{x}_{kp}^S) \leq \sqrt{\sum_{p=1}^{m} (x_{ip}^* - \widetilde{x}_{jp}^S)^2} \cdot \sqrt{\sum_{p=1}^{m} (x_{ip}^* - \widetilde{x}_{kp}^S)^2}. \tag{47}$$

Hence, from (12), (46) and (47) we get for an arbitrary $i \in S$

$$\sum_{p=1}^{m} (x_{ip}^* - x_{Sp})^2 = \sum_{p=1}^{m} \left(x_{ip}^* - \frac{\sum_{j \in S} w_j \cdot \widetilde{x}_{jp}^S}{\sum_{j \in S} w_j} \right)^2 =$$

$$= \frac{\sum_{p=1}^{m}\left[\sum_{j\in S} w_j \cdot (x_{ip}^* - \widetilde{x}_{jp}^S)\right]^2}{\left(\sum_{j\in S} w_j\right)^2} = \frac{\sum_{p=1}^{m}\left[\sum_{j\in S} w_j^2 \cdot (x_{ip}^* - \widetilde{x}_{jp}^S)^2\right]}{\left(\sum_{j\in S} w_j\right)^2} +$$

$$+ \frac{2\sum_{p=1}^{m}\left[\sum_{j,k\in S, j\neq k} w_j \cdot w_k \cdot (x_{ip}^* - \widetilde{x}_{jp}^S)\cdot(x_{ip}^* - \widetilde{x}_{kp}^S)\right]}{\left(\sum_{j\in S} w_j\right)^2} =$$

$$= \frac{\sum_{j\in S}\left[w_j^2 \cdot \sum_{p=1}^{m}(x_{ip}^* - \widetilde{x}_{jp}^S)^2\right]}{\left(\sum_{j\in S} w_j\right)^2} +$$

$$+ \frac{2\sum_{j,k\in S, j\neq k}\left[w_j \cdot w_k \cdot \sum_{p=1}^{m}(x_{ip}^* - \widetilde{x}_{jp}^S)\cdot(x_{ip}^* - \widetilde{x}_{kp}^S)\right]}{\left(\sum_{j\in S} w_j\right)^2} \leq \frac{\sum_{j\in S} w_j^2 \cdot r_i^2}{\left(\sum_{j\in S} w_j\right)^2} +$$

$$+ \frac{2}{\left(\sum_{j\in S} w_j\right)^2} \cdot \sum_{j,k\in S, j\neq k} w_j \cdot w_k \cdot \sqrt{\sum_{p=1}^{m}(x_{ip}^* - \widetilde{x}_{jp}^S)^2} \cdot \sqrt{\sum_{p=1}^{m}(x_{ip}^* - \widetilde{x}_{kp}^S)^2}$$

$$\leq \frac{\sum_{j\in S} w_j^2 \cdot r_i^2 + 2\sum_{j,k\in S, j\neq k} w_j \cdot w_k \cdot r_i^2}{\left(\sum_{j\in S} w_j\right)^2} =$$

$$= \frac{r_i^2 \cdot \left[\sum_{j\in S} w_j^2 + 2\sum_{j,k\in S, j\neq k} w_j \cdot w_k\right]}{\left(\sum_{j\in S} w_j\right)^2} = r_i^2$$

Hence, we have proven (45). □

(b) Let $S \subseteq N$ and $M_p(x_p^*, r_p)$, for $p \in S$, be maneuvering spaces in $\mathbb{R}^m$ such that $\bigcap_{p\in S} M_p \neq \emptyset$. Let us consider an arbitrary S such that $|S| = 2$, say $S = \{i, j\}$. Let us consider an arbitrary $y \in M_i \cap M_j$ such that $y \in \overline{\widetilde{x}_i^{\{i,j\}}\widetilde{x}_j^{\{i,j\}}}$. Hence, $d(x_i^*, y) + d(x_j^*, y) = d(x_i^*, x_j^*)$. First, we will show that y is Pareto efficient with respect to $\{i, j\}$. Suppose that y is not Pareto efficient with respect to $\{i, j\}$. From definition (9) this means that there exists $z \in M_i \cap M_j$ such that $z \succ_i y$ and $z \succ_j y$. Hence, by virtue of (3), $d(x_i^*, z) < d(x_i^*, y)$ and $d(x_j^*, z) < d(x_j^*, y)$, and therefore $d(x_i^*, z) + d(x_j^*, z) < d(x_i^*, x_j^*)$. But this gives a contradiction, since $d(x_i^*, z) + d(x_j^*, z) = d(x_i^*, x_j^*)$ if $z \in \overline{x_i^* x_j^*}$, and $d(x_i^*, z) + d(x_j^*, z) > d(x_i^*, x_j^*)$ (triangle inequality) if $z \notin \overline{x_i^* x_j^*}$.

$x_{\{i,j\}}$ lies on the segment $\overline{\widetilde{x}_i^{\{i,j\}}\widetilde{x}_j^{\{i,j\}}}$, and therefore $x_{\{i,j\}}$ is Pareto efficient with respect to $\{i, j\}$.

Let us consider an arbitrary S such that $|S| \geq 3$. If all $\widetilde{x}_p^*$ for $p \in S$ lie on one segment, the proof that x_S is Pareto efficient is analogous to the case mentioned above. If not all $\widetilde{x}_p^*$ for $p \in S$ lie on one segment, then the proof that x_S is Pareto efficient is the following. Since not all $\widetilde{x}_p^*$ for $p \in S$ lie on one segment, there are three parties $i, j, k \in S$ such that x_i^*, x_j^* and x_k^* do not lie on one segment. Consider a plane containing x_i^*, x_j^* and x_k^*. Create segments $\overline{x_S x_i^*}$, $\overline{x_S x_j^*}$, and $\overline{x_S x_k^*}$, and lines perpendicular to these segments. The lines form a triangle, let us call it, the triangle ABC, where A, B, and C are the intersection points of the lines containing x_i^* and x_j^* (point A), x_i^* and x_k^* (point B), and x_j^* and x_k^* (point C), respectively. Let $|AB|$, $|AC|$ and $|BC|$ denote the lengths of the sides of the triangle ABC. The area of the triangle ABC, denoted by $|ABC|$ is equal to

$$|ABC| = \frac{d(x_i^*, x_S) \cdot |AB|}{2} + \frac{d(x_j^*, x_S) \cdot |AC|}{2} + \frac{d(x_k^*, x_S) \cdot |BC|}{2}. \tag{48}$$

Suppose x_S is not Pareto efficient with respect to S. This means that there is $y \in \bigcap_{p \in S} M_p$ such that for each $r \in S$, $y \succ_r x_S$, that is, for each $r \in S$, $d(x_r^*, y) < d(x_r^*, x_S)$. Hence, we have,

$$d(x_i^*, y) < d(x_i^*, x_S), \quad d(x_j^*, y) < d(x_j^*, x_S), \quad d(x_k^*, y) < d(x_k^*, x_S). \tag{49}$$

Let h_{AB}, h_{AC} and h_{BC} be the lengths of the heights starting in y and perpendicular to AB, AC, and BC, respectively. Hence, we can write

$$|ABC| = \frac{h_{AB} \cdot |AB|}{2} + \frac{h_{AC} \cdot |AC|}{2} + \frac{h_{BC} \cdot |BC|}{2}. \tag{50}$$

Note that, of course,

$$h_{AB} \leq d(x_i^*, y), \quad h_{AC} \leq d(x_j^*, y), \quad h_{BC} \leq d(x_k^*, y), \tag{51}$$

and therefore, together with (49), we get

$$h_{AB} < d(x_i^*, x_S), \quad h_{AC} < d(x_j^*, x_S), \quad h_{BC} < d(x_k^*, x_S). \tag{52}$$

Applying (52) to (50), we have

$$|ABC| < \frac{d(x_i^*, x_S) \cdot |AB|}{2} + \frac{d(x_j^*, x_S) \cdot |AC|}{2} + \frac{d(x_k^*, x_S) \cdot |BC|}{2}, \tag{53}$$

which gives contradiction with (48). Hence, there is no $y \in \bigcap_{p \in S} M_p$ such that for each $r \in S$, $y \succ_r x_S$, which means that x_S is Pareto efficient with respect to S. □

Proof of Proposition 2

Let $S \subseteq N$ and $M_i = M_i(x_i^*, r_i)$, for $i \in S$, be maneuvering spaces in $\mathbb{R}^m$ such that $\bigcap_{i\in S} M_i \neq \emptyset$. First of all, note that $\bigcap_{i\in S} M_i$ is convex as an intersection of convex sets M_i, $i \in S$.

The proof of the well-known fact that the intersection of an arbitrary number of convex sets is convex is very straightforward. Let us take two arbitrary positions y and z such that $y \in \bigcap_{i\in S} M_i$ and $z \in \bigcap_{i\in S} M_i$. Hence, $y \in M_i$ and $z \in M_i$ for each $i \in S$. Since M_i is convex (as a ball) for each $i \in S$, we have $\overline{yz} \in M_i$ for each $i \in S$, where $\overline{yz}$ denotes the segment between y and z. But hence $\overline{yz} \in \bigcap_{i\in S} M_i$, this also means that $\bigcap_{i\in S} M_i$ is convex.

Note that $d(x_i^*, y)$ is continuous on the convex set $\bigcap_{k\in S} M_k$. Hence, by virtue of Theorem 2, there is at least one position in $\bigcap_{k\in S} M_k$ in which $d(x_i^*, y)$ has its minimum, that is there is at least one solution of (11). Suppose that for a certain $i \in S$ there are two positions $\widetilde{x}_i^S \in \bigcap_{k\in S} M_k$ and $\widetilde{y}_i^S \in \bigcap_{k\in S} M_k$ satisfying (11). Hence,

$$d(x_i^*, \widetilde{x}_i^S) = d(x_i^*, \widetilde{y}_i^S) = \min_{y\in\bigcap_{k\in S} M_k} d(x_i^*, y). \tag{54}$$

Take a plane containing the positions x_i^*, $\widetilde{x}_i^S$ and $\widetilde{y}_i^S$. In this plane, the positions x_i^*, $\widetilde{x}_i^S$ and $\widetilde{y}_i^S$ create a triangle with sides of lengths $d(x_i^*, \widetilde{x}_i^S) = d(x_i^*, \widetilde{y}_i^S)$ and $|\overline{\widetilde{x}_i^S\widetilde{y}_i^S}|$. Of course, since $\bigcap_{k\in S} M_k$ is convex, $\overline{\widetilde{x}_i^S\widetilde{y}_i^S} \in \bigcap_{k\in S} M_k$. Let us create the height of the triangle which is a segment between the apex x_i^* and a certain point $\widetilde{z} \in \overline{\widetilde{x}_i^S\widetilde{y}_i^S}$, perpendicular to $\overline{\widetilde{x}_i^S\widetilde{y}_i^S}$. Since $\bigcap_{k\in S} M_k$ is convex, in particular $\widetilde{z} \in \bigcap_{k\in S} M_k$. But this means that

$$d(x_i^*, \widetilde{z}) < d(x_i^*, \widetilde{x}_i^S) = d(x_i^*, \widetilde{y}_i^S),$$

which together with (54) gives a contradiction

$$d(x^*, \widetilde{z}) < \min_{y\in\bigcap_{k\in S} M_k} d(x_i^*, y).$$

□

Proof of Proposition 3 is analogous to the proof of Proposition 1.

Proof of Proposition 4 is analogous to the proof of Proposition 2.

Proof of Proposition 5

Let $M_{i_1} = M_{i_1}(x^*_{i_1}, r_{i_1})$ and $M_{i_2} = M_{i_2}(x^*_{i_2}, r_{i_2})$ be two maneuvering spaces in $\mathbb{R}^m$ such that $M_{i_1} \cap M_{i_2} \neq \emptyset$. If $r_{i_2} \geq d(x^*_{i_1}, x^*_{i_2})$, then by virtue of (5), $x^*_{i_1} \in M_{i_2}$, and therefore

$$\min_{z \in M_{i_1} \cap M_{i_2}} d(x^*_{i_1}, z) = d(x^*_{i_1}, x^*_{i_1}) = 0.$$

Hence, $\widetilde{x}^{\{i_1,i_2\}}_{i_1} = x^*_{i_1}$. Suppose now that $r_{i_2} < d(x^*_{i_1}, x^*_{i_2})$. In order to find $\widetilde{x}^{\{i_1,i_2\}}_{i_1}$, we have to solve the following set of $m+1$ equations:

$$\begin{cases} \sum_{p=1}^m (\widetilde{x}^{\{i_1,i_2\}}_{i_1p} - x^*_{i_2p})^2 = r^2_{i_2} \\ \widetilde{x}^{\{i_1,i_2\}}_{i_1p} = t \cdot x^*_{i_1p} + (1-t) \cdot x^*_{i_2p} \quad for \quad p = 1, ..., m \end{cases} \tag{55}$$

with $m+1$ unknown quantities $\widetilde{x}^{\{i_1,i_2\}}_{i_11}, ..., \widetilde{x}^{\{i_1,i_2\}}_{i_1m}, t$, where $t \in [0,1]$. Hence, we get

$$\sum_{p=1}^m (t \cdot x^*_{i_1p} - t \cdot x^*_{i_2p})^2 = r^2_{i_2}.$$

Since $0 \leq r_{i_2} < d(x^*_{i_1}, x^*_{i_2})$, we have $d(x^*_{i_1}, x^*_{i_2}) > 0$, and therefore, we can write equivalently

$$t = \frac{r_{i_2}}{\sqrt{\sum_{p=1}^m (x^*_{i_1p} - x^*_{i_2p})^2}} = \frac{r_{i_2}}{d(x^*_{i_1}, x^*_{i_2})}. \tag{56}$$

Applying (56) to (55), we get (25), because for each $p = 1, ..., m$

$$\widetilde{x}^{\{i_1,i_2\}}_{i_1p} = \frac{r_{i_2}}{d(x^*_{i_1}, x^*_{i_2})} \cdot x^*_{i_1p} + x^*_{i_2p} - \frac{r_{i_2}}{d(x^*_{i_1}, x^*_{i_2})} \cdot x^*_{i_2p} = x^*_{i_2p} + \frac{r_{i_2} \cdot (x^*_{i_1p} - x^*_{i_2p})}{d(x^*_{i_1}, x^*_{i_2})}.$$

□

Proof of Proposition 6

Let $M_{i_1} = M_{i_1}(x^*_{i_1}, r_{i_1})$ and $M_{i_2} = M_{i_2}(x^*_{i_2}, r_{i_2})$ be such that $M_{i_1} \cap M_{i_2} \neq \emptyset$. If $M_{i_1} \cap M_{i_2} = \{y\}$, then $\overline{\widetilde{x}^{\{i_1,i_2\}}_{i_1} \widetilde{x}^{\{i_1,i_2\}}_{i_2}} = \{y\}$ and (10) is satisfied, which means that y is Pareto efficient. Suppose that $M_{i_1} \cap M_{i_2}$ contains more than one element. Let us take an arbitrary $y \in M_{i_1} \cap M_{i_2}$ such that $y \in \overline{\widetilde{x}^{\{i_1,i_2\}}_{i_1} \widetilde{x}^{\{i_1,i_2\}}_{i_2}}$. Hence, from the proof of Proposition 1(b), we have that y is Pareto efficient.

Let us consider now an arbitrary $y \in M_{i_1} \cap M_{i_2}$ such that $y \notin \overline{\widetilde{x}_{i_1}^{\{i_1,i_2\}} \widetilde{x}_{i_2}^{\{i_1,i_2\}}}$. Hence, $y \notin \overline{x_{i_1}^* x_{i_2}^*}$. It holds $d(x_{i_1}^*, y) + d(x_{i_2}^*, y) > d(x_{i_1}^*, x_{i_2}^*)$ (triangle inequality). We create two balls $H_{i_1}(x_{i_1}^*, d(x_{i_1}^*, y)) \subseteq M_{i_1}$ (the ball with the middle point $x_{i_1}^*$ and the radius $d(x_{i_1}^*, y)$) and $H_{i_2}(x_{i_2}^*, d(x_{i_2}^*, y)) \subseteq M_{i_2}$ (the ball with the middle point $x_{i_2}^*$ and the radius $d(x_{i_2}^*, y)$). Note that $|H_{i_1} \cap H_{i_2}| > 1$. Moreover, there is always $z \in H_{i_1} \cap H_{i_2} \subseteq M_{i_1} \cap M_{i_2}$ such that $d(x_{i_1}^*, z) + d(x_{i_2}^*, z) = d(x_{i_1}^*, x_{i_2}^*)$, $d(x_{i_1}^*, z) < d(x_{i_1}^*, y)$ and $d(x_{i_2}^*, z) < d(x_{i_2}^*, y)$. But this means, by virtue of (3), that $z \succ_{i_1} y$ and $z \succ_{i_2} y$. Hence, from (9), y is not Pareto efficient with respect to $\{i_1, i_2\}$.

□

Proof of Proposition 7 is analogous to the proof of Proposition 5.

Proof of Proposition 8

(1) Let $S \subseteq N$, $|S| \geq 2$, $\bigcap_{j \in S} M_j \neq \emptyset$, and for each $i \in S$, $x_i^* \in \bigcap_{j \in S} M_j$. Hence, by virtue of (11), $\widetilde{x}_i^S = x_i^*$ for each $i \in S$, and therefore by virtue of (12), we have for each $l = 1, ..., m$

$$x_{Sl} = \frac{\sum_{i \in S} w_i \cdot \widetilde{x}_{il}^S}{\sum_{i \in S} w_i} = \frac{\sum_{i \in S} w_i \cdot x_{il}^*}{\sum_{i \in S} w_i}. \tag{57}$$

Take an arbitrary $\overline{S} = \{\{...\{\{i_1, i_2\}, i_3\}...\}, i_{|S|}\} \in PROC(S)$. First, we consider an alliance $\{i_1, i_2\}$. Since each $i \in S$, $x_i^* \in \bigcap_{j \in S} M_j$, we have from (16),

$$\widetilde{x}_{i_1}^{\{i_1,i_2\}} = x_{i_1}^*, \quad \widetilde{x}_{i_2}^{\{i_1,i_2\}} = x_{i_2}^*$$

and from (18), for each $l = 1, ..., m$

$$x_{\{i_1,i_2\}l} = \frac{w_{i_1} \cdot \widetilde{x}_{i_1 l}^{\{i_1,i_2\}} + w_{i_2} \cdot \widetilde{x}_{i_2 l}^{\{i_1,i_2\}}}{w_{i_1} + w_{i_2}} = \frac{w_{i_1} \cdot x_{i_1 l}^* + w_{i_2} \cdot x_{i_2 l}^*}{w_{i_1} + w_{i_2}}.$$

Since $\bigcap_{j \in S} M_j$ is a convex set, and $x_{i_1}^*, x_{i_2}^* \in \bigcap_{j \in S} M_j$, also $x_{\{i_1,i_2\}} \in \bigcap_{j \in S} M_j$. Hence, by virtue of (20), $\widetilde{x}_{\{i_1,i_2\}}^{\{i_1,i_2,i_3\}} = x_{\{i_1,i_2\}}$. Moreover, $\widetilde{x}_{i_3}^{\{i_1,i_2,i_3\}} = x_{i_3}^*$. We have then from (21), for each $l = 1, ..., m$

$$x_{\{\{i_1,i_2\},i_3\}l} = \frac{(w_{i_1} + w_{i_2}) \cdot \widetilde{x}_{\{i_1,i_2\}l}^{\{i_1,i_2,i_3\}} + w_{i_3} \cdot \widetilde{x}_{i_3 l}^{\{i_1,i_2,i_3\}}}{w_{i_1} + w_{i_2} + w_{i_3}}$$

$$= \frac{(w_{i_1}+w_{i_2})\cdot x_{\{i_1,i_2\}l} + w_{i_3}\cdot x^*_{i_3 l}}{w_{i_1}+w_{i_2}+w_{i_3}} = \frac{w_{i_1}\cdot x^*_{i_1 l} + w_{i_2}\cdot x^*_{i_2 l} + w_{i_3}\cdot x^*_{i_3 l}}{w_{i_1}+w_{i_2}+w_{i_3}}$$

Next we will show that for each alliance S_p, where $3 \le p < |S|$, if for $l = 1, ..., m$

$$x_{\overline{S}_p l} = \frac{\sum_{r=1}^{p} w_{i_r}\cdot x^*_{i_r l}}{\sum_{r=1}^{p} w_{i_r}} \tag{58}$$

then

$$x_{\{\overline{S}_p, i_{p+1}\}l} = \frac{\sum_{r=1}^{p+1} w_{i_r}\cdot x^*_{i_r l}}{\sum_{r=1}^{p+1} w_{i_r}}.$$

First of all, note that $x_{\overline{S}_p} \in \bigcap_{j\in S} M_j$, and therefore from (23), $\widetilde{x}^{S_p\cup\{i_{p+1}\}}_{\overline{S}_p} = x_{\overline{S}_p}$. Moreover, since $x^*_{i_{p+1}} \in \bigcap_{j\in S} M_j$, we have from (22), $\widetilde{x}^{S_p\cup\{i_{p+1}\}}_{i_{p+1}} = x^*_{i_{p+1}}$. Hence, from (24) and (58), we have for each $l = 1, ..., m$

$$x_{\{\overline{S}_p, i_{p+1}\}l} = \frac{\widetilde{x}^{S_p\cup\{i_{p+1}\}}_{\overline{S}_p l}\cdot \sum_{r=1}^{p} w_{i_r} + \widetilde{x}^{S_p\cup\{i_{p+1}\}}_{i_{p+1} l}\cdot w_{i_{p+1}}}{\sum_{r=1}^{p+1} w_{i_r}} = \frac{\sum_{r=1}^{p+1} w_{i_r}\cdot x^*_{i_r l}}{\sum_{r=1}^{p+1} w_{i_r}}.$$

If we take $p = |S|-1$, that is, an alliance $S_p = S\setminus\{i_{|S|}\}$, then we get for each $l = 1, ..., m$,

$$x_{\overline{S}l} = \frac{\sum_{i\in S} w_i\cdot x^*_{il}}{\sum_{i\in S} w_i} = x_{Sl}.$$

□

(2) Let $S \subseteq N$, $|S| \ge 2$, $\bigcap_{j\in S\cup\{\widetilde{i}\}} M_j \ne \emptyset$, and for each $i \in S$, $x^*_i \in \bigcap_{j\in S} M_j$. Let $\widetilde{i} \in N\setminus S$. Take arbitrary $\overline{S}, \overline{\overline{S}} \in PROC(S)$. By virtue of (32), $x_{\overline{S}} = x_S$ and $x_{\overline{\overline{S}}} = x_S$. Moreover, from (57), for each $l = 1, ..., m$

$$x_{Sl} = \frac{\sum_{i\in S} w_i\cdot x^*_{il}}{\sum_{i\in S} w_i}.$$

Since $x_{\overline{S}}, x_{\overline{\overline{S}}} \in \bigcap_{j\in S} M_j$, from (23), we have

$$\widetilde{x}^{S\cup\{\widetilde{i}\}}_{\overline{S}} = x_{\overline{S}} = x_S, \quad \widetilde{x}^{S\cup\{\widetilde{i}\}}_{\overline{\overline{S}}} = x_{\overline{\overline{S}}} = x_S. \tag{59}$$

Hence, by virtue of (24) and (59), for each $l = 1, ..., m$, we have

$$x_{\{\overline{S},\widetilde{i}\}l} = \frac{\widetilde{x}^{S\cup\{\widetilde{i}\}}_{\overline{S}l}\cdot \sum_{i\in S} w_i + \widetilde{x}^{S\cup\{\widetilde{i}\}}_{\widetilde{i}l}\cdot w_{\widetilde{i}}}{\sum_{i\in S\cup\{\widetilde{i}\}} w_i} = \frac{\sum_{i\in S} w_i\cdot x^*_{il} + w_{\widetilde{i}}\cdot \widetilde{x}^{S\cup\{\widetilde{i}\}}_{\widetilde{i}l}}{\sum_{i\in S\cup\{\widetilde{i}\}} w_i} \tag{60}$$

$$x_{\{\overline{\overline{S}},\widetilde{i}\}l} = \frac{\widetilde{x}^{S\cup\{\widetilde{i}\}}_{\overline{\overline{S}}l} \cdot \sum_{i\in S} w_i + \widetilde{x}^{S\cup\{\widetilde{i}\}}_{\widetilde{i}l} \cdot w_{\widetilde{i}}}{\sum_{i\in S\cup\{\widetilde{i}\}} w_i} = \frac{\sum_{i\in S} w_i \cdot x^*_{il} + w_{\widetilde{i}} \cdot \widetilde{x}^{S\cup\{\widetilde{i}\}}_{\widetilde{i}l}}{\sum_{i\in S\cup\{\widetilde{i}\}} w_i},$$

and therefore $x_{\{\overline{S},\widetilde{i}\}} = x_{\{\overline{\overline{S}},\widetilde{i}\}}$. □

(3) Let $S \subseteq N$, $|S| \geq 2$, $\bigcap_{j\in S\cup\{\widetilde{i}\}} M_j \neq \emptyset$, $\widetilde{i} \in N \setminus S$, and for each $i \in S$, $x^*_i \in \bigcap_{j\in S\cup\{\widetilde{i}\}} M_j$. Take an arbitrary $\overline{S} \in PROC(S)$. From (11), $\widetilde{x}^{S\cup\{\widetilde{i}\}}_i = x^*_i$ for each $i \in S$, and therefore by virtue of (12) and (60), we have for each $l = 1, ..., m$

$$x_{S\cup\{\widetilde{i}\}l} = \frac{\sum_{i\in S\cup\{\widetilde{i}\}} w_i \cdot \widetilde{x}^{S\cup\{\widetilde{i}\}}_{il}}{\sum_{i\in S\cup\{\widetilde{i}\}} w_i} = \frac{\sum_{i\in S} w_i \cdot x^*_{il} + w_{\widetilde{i}} \cdot \widetilde{x}^{S\cup\{\widetilde{i}\}}_{\widetilde{i}l}}{\sum_{i\in S\cup\{\widetilde{i}\}} w_i} = x_{\{\overline{S},\widetilde{i}\}l}.$$

□

Proof of Fact 1

Let $S \subseteq N$. Take arbitrary $\overline{S}, \overline{\overline{S}} \in PROC(S) \cup \{S\}$ and $i \in S$. By virtue of (2), (3) and (4), and taking into account that distances are non-negative, we have respectively

$$x_{\overline{S}} \succeq_i x_{\overline{\overline{S}}} \;\Leftrightarrow\; d(x_{\overline{S}}, x^*_i) \leq d(x_{\overline{\overline{S}}}, x^*_i) \;\Leftrightarrow\; d^2(x_{\overline{S}}, x^*_i) \leq d^2(x_{\overline{\overline{S}}}, x^*_i)$$

$$\Leftrightarrow\; \sum_{l=1}^{m} (x_{\overline{S}l} - x^*_{il})^2 \leq \sum_{l=1}^{m} \left(x_{\overline{\overline{S}}l} - x^*_{il}\right)^2$$

$$x_{\overline{S}} \succ_i x_{\overline{\overline{S}}} \;\Leftrightarrow\; d(x_{\overline{S}}, x^*_i) < d(x_{\overline{\overline{S}}}, x^*_i) \;\Leftrightarrow\; \sum_{l=1}^{m} (x_{\overline{S}l} - x^*_{il})^2 < \sum_{l=1}^{m} \left(x_{\overline{\overline{S}}l} - x^*_{il}\right)^2$$

$$x_{\overline{S}} \sim_i x_{\overline{\overline{S}}} \;\Leftrightarrow\; d(x_{\overline{S}}, x^*_i) = d(x_{\overline{\overline{S}}}, x^*_i) \;\Leftrightarrow\; \sum_{l=1}^{m} (x_{\overline{S}l} - x^*_{il})^2 = \sum_{l=1}^{m} \left(x_{\overline{\overline{S}}l} - x^*_{il}\right)^2$$

□

Proof of Proposition 9

(1) Let $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$, $x^*_{i_1} \notin M_{i_2} \cup M_{i_3}$, $x^*_{i_2} \notin M_{i_1} \cup M_{i_3}$, $x^*_{i_3} \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$,

$$\frac{w_{i_1}}{w_{i_2}} = \frac{r_{i_1}}{r_{i_2}}, \quad d(x^*_{i_3}, x_{\{i_1,i_2\}}) \leq r_{i_3}$$

$$\sum_{l=1}^{m}\left(x^*_{i_2l}-x^*_{i_3l}\right)\left(x^*_{i_1l}-x^*_{i_3l}\right)\geq\sum_{l=1}^{m}\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right).$$

Note that $\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_3}=x^*_{i_3}$, $d(x^*_{i_3},\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1})=r_{i_3}$, $d(x^*_{i_3},\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2})=r_{i_3}$. We have for each $l=1,...,m$

$$x_{\{i_1,i_2,i_3\}l}=\frac{w_{i_1}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}+w_{i_2}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}+w_{i_3}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_3l}}{w_{i_1}+w_{i_2}+w_{i_3}}$$

$$=\frac{w_{i_1}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}+w_{i_2}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}+w_{i_3}\cdot x^*_{i_3l}}{w_{i_1}+w_{i_2}+w_{i_3}}$$

and hence

$$d^2(x^*_{i_3},x_{\{i_1,i_2,i_3\}})=\sum_{l=1}^{m}\left(x^*_{i_3l}-\frac{w_{i_1}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}+w_{i_2}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}+w_{i_3}\cdot x^*_{i_3l}}{w_{i_1}+w_{i_2}+w_{i_3}}\right)^2$$

$$=\frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\sum_{l=1}^{m}\left[w_{i_1}\cdot\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)+w_{i_2}\cdot\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)\right]^2=$$

$$\frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}[w^2_{i_1}\cdot d^2(x^*_{i_3},\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1})+w^2_{i_2}\cdot d^2(x^*_{i_3},\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2})+$$

$$2w_{i_1}w_{i_2}\sum_{l=1}^{m}\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)]$$

$$=\frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r^2_{i_3}\cdot(w^2_{i_1}+w^2_{i_2})+2w_{i_1}w_{i_2}\sum_{l=1}^{m}\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)\right].$$

Since $w_{i_1}\cdot r_{i_2}=w_{i_2}\cdot r_{i_1}$, for each $l=1,...,m$

$$x_{\{i_1,i_2\}l}=\frac{w_{i_1}\cdot\widetilde{x}^{\{i_1,i_2\}}_{i_1l}+w_{i_2}\cdot\widetilde{x}^{\{i_1,i_2\}}_{i_2l}}{w_{i_1}+w_{i_2}}$$

$$=\frac{w_{i_1}}{w_{i_1}+w_{i_2}}\left[x^*_{i_2l}+\frac{r_{i_2}\cdot(x^*_{i_1l}-x^*_{i_2l})}{d(x^*_{i_1},x^*_{i_2})}\right]+\frac{w_{i_2}}{w_{i_1}+w_{i_2}}\left[x^*_{i_1l}+\frac{r_{i_1}\cdot(x^*_{i_2l}-x^*_{i_1l})}{d(x^*_{i_1},x^*_{i_2})}\right]$$

$$=\frac{1}{w_{i_1}+w_{i_2}}\left[w_{i_1}\cdot x^*_{i_2l}+w_{i_2}\cdot x^*_{i_1l}+\frac{(w_{i_1}\cdot r_{i_2}-w_{i_2}\cdot r_{i_1})(x^*_{i_1l}-x^*_{i_2l})}{d(x^*_{i_1},x^*_{i_2})}\right]=\frac{w_{i_1}\cdot x^*_{i_2l}+w_{i_2}\cdot x^*_{i_1l}}{w_{i_1}+w_{i_2}}$$

If $d(x^*_{i_3}, x_{\{i_1,i_2\}}) \le r_{i_3}$, then $\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}} = x_{\{i_1,i_2\}}$. For each $l = 1, ..., m$ we have

$$x_{\{\{i_1,i_2\},i_3\}l} = \frac{(w_{i_1} + w_{i_2}) \cdot \widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}l} + w_{i_3} \cdot \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_3 l}}{w_{i_1} + w_{i_2} + w_{i_3}} = \frac{w_{i_1} \cdot x^*_{i_2 l} + w_{i_2} \cdot x^*_{i_1 l} + w_{i_3} \cdot x^*_{i_3 l}}{w_{i_1} + w_{i_2} + w_{i_3}}$$

and therefore, from (39) and $d(x^*_{i_1}, x^*_{i_3}) > r_{i_3}$, $d(x^*_{i_2}, x^*_{i_3}) > r_{i_3}$, we have

$$d^2(x^*_{i_3}, x_{\{\{i_1,i_2\},i_3\}}) = \frac{1}{(w_{i_1} + w_{i_2} + w_{i_3})^2} \sum_{l=1}^{m} \left[w_{i_1} \cdot (x^*_{i_2 l} - x^*_{i_3 l}) + w_{i_2} \cdot (x^*_{i_1 l} - x^*_{i_3 l})\right]^2$$

$$= \frac{1}{(w_{i_1} + w_{i_2} + w_{i_3})^2} \left[w^2_{i_1} \cdot d^2(x^*_{i_2}, x^*_{i_3}) + w^2_{i_2} \cdot d^2(x^*_{i_1}, x^*_{i_3}) + 2w_{i_1} \cdot w_{i_2} \sum_{l=1}^{m} (x^*_{i_2 l} - x^*_{i_3 l})(x^*_{i_1 l} - x^*_{i_3 l})\right]$$

$$> \frac{1}{(w_{i_1} + w_{i_2} + w_{i_3})^2} \left[r^2_{i_3} \cdot (w^2_{i_1} + w^2_{i_2}) + 2w_{i_1} \cdot w_{i_2} \sum_{l=1}^{m} \left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l}\right)\left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l}\right)\right]$$

$$= d^2(x^*_{i_3}, x_{\{i_1,i_2,i_3\}})$$

This means that $x_{\{i_1,i_2,i_3\}} \succ_{i_3} x_{\{\{i_1,i_2\},i_3\}}$. □

(2) Let $M_{i_1} \cap M_{i_2} \cap M_{i_3} \ne \emptyset$, $x^*_{i_1} \notin M_{i_2} \cup M_{i_3}$, $x^*_{i_2} \notin M_{i_1} \cup M_{i_3}$, $x^*_{i_3} \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$,

$$\frac{w_{i_1}}{w_{i_2}} = \frac{r_{i_1}}{r_{i_2}}, \quad d(x^*_{i_3}, x_{\{i_1,i_2\}}) > r_{i_3}.$$

As in part (1), we have

$$d^2(x^*_{i_3}, x_{\{i_1,i_2,i_3\}}) =$$

$$\frac{1}{(w_{i_1} + w_{i_2} + w_{i_3})^2} \left[r^2_{i_3} \cdot (w^2_{i_1} + w^2_{i_2}) + 2w_{i_1} w_{i_2} \sum_{l=1}^{m} \left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l}\right)\left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l}\right)\right],$$

and for each $l = 1, ..., m$

$$x_{\{i_1,i_2\}l} = \frac{w_{i_1} \cdot x^*_{i_2 l} + w_{i_2} \cdot x^*_{i_1 l}}{w_{i_1} + w_{i_2}}.$$

$$d^2(x^*_{i_3}, x_{\{i_1,i_2\}}) = \sum_{l=1}^{m} \left(\frac{w_{i_1} \cdot x^*_{i_2 l} + w_{i_2} \cdot x^*_{i_1 l}}{w_{i_1} + w_{i_2}} - x^*_{i_3 l}\right)^2$$

$$= \frac{1}{(w_{i_1} + w_{i_2})^2} \sum_{l=1}^{m} \left[w_{i_1} \cdot (x^*_{i_2 l} - x^*_{i_3 l}) + w_{i_2} \cdot (x^*_{i_1 l} - x^*_{i_3 l})\right]^2$$

$$= \frac{1}{(w_{i_1}+w_{i_2})^2}\left[w_{i_1}^2\cdot d^2(x^*_{i_2},x^*_{i_3})+w_{i_2}^2\cdot d^2(x^*_{i_1},x^*_{i_3})+2w_{i_1}\cdot w_{i_2}\cdot\sum_{l=1}^{m}(x^*_{i_2l}-x^*_{i_3l})(x^*_{i_1l}-x^*_{i_3l})\right]$$

If $d(x^*_{i_3}, x_{\{i_1,i_2\}}) > r_{i_3}$, then for each $l = 1, ..., m$

$$\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}l} = x^*_{i_3l} + \frac{r_{i_3}\cdot(x_{\{i_1,i_2\}l}-x^*_{i_3l})}{d(x^*_{i_3},x_{\{i_1,i_2\}})}$$

$$= x^*_{i_3l} + \frac{r_{i_3}\cdot[w_{i_1}\cdot(x^*_{i_2l}-x^*_{i_3l})+w_{i_2}\cdot(x^*_{i_1l}-x^*_{i_3l})]}{\sqrt{w_{i_1}^2\cdot d^2(x^*_{i_2},x^*_{i_3})+w_{i_2}^2\cdot d^2(x^*_{i_1},x^*_{i_3})+2w_{i_1}\cdot w_{i_2}\cdot\sum_{l=1}^{m}(x^*_{i_2l}-x^*_{i_3l})(x^*_{i_1l}-x^*_{i_3l})}}$$

Since $x^{\{i_1,i_2,i_3\}}_{i_3} = x^*_{i_3}$, for each $l = 1, ..., m$ we have

$$x_{\{\{i_1,i_2\},i_3\}l} = \frac{(w_{i_1}+w_{i_2})\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_2\}l}+w_{i_3}\cdot\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_3l}}{w_{i_1}+w_{i_2}+w_{i_3}} = \frac{1}{w_{i_1}+w_{i_2}+w_{i_3}}[(w_{i_1}+w_{i_2}+w_{i_3})\cdot x^*_{i_3l}+$$

$$\frac{(w_{i_1}+w_{i_2})\cdot r_{i_3}\cdot[w_{i_1}\cdot(x^*_{i_2l}-x^*_{i_3l})+w_{i_2}\cdot(x^*_{i_1l}-x^*_{i_3l})]}{\sqrt{w_{i_1}^2\cdot d^2(x^*_{i_2},x^*_{i_3})+w_{i_2}^2\cdot d^2(x^*_{i_1},x^*_{i_3})+2w_{i_1}\cdot w_{i_2}\cdot\sum_{l=1}^{m}(x^*_{i_2l}-x^*_{i_3l})(x^*_{i_1l}-x^*_{i_3l})}}]$$

and hence

$$d^2(x^*_{i_3}, x_{\{\{i_1,i_2\},i_3\}}) = \sum_{l=1}^{m}(x_{\{\{i_1,i_2\},i_3\}l}-x^*_{i_3l})^2 = \frac{(w_{i_1}+w_{i_2})^2\cdot r_{i_3}^2}{(w_{i_1}+w_{i_2}+w_{i_3})^2}$$

$$= \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r_{i_3}^2\cdot(w_{i_1}^2+w_{i_2}^2)+2w_{i_1}\cdot w_{i_2}\cdot r_{i_3}^2\right]$$

By virtue of the Buniakowski-Cauchy inequality we have

$$d^2(x^*_{i_3}, x_{\{i_1,i_2,i_3\}}) =$$

$$\frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r_{i_3}^2\cdot(w_{i_1}^2+w_{i_2}^2)+2w_{i_1}\cdot w_{i_2}\sum_{l=1}^{m}\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)\right]$$

$$\leq \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r_{i_3}^2\cdot(w_{i_1}^2+w_{i_2}^2)+2w_{i_1}\cdot w_{i_2}\cdot d(x^*_{i_3},\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1})\cdot d(x^*_{i_3},\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2})\right]$$

$$= \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r_{i_3}^2\cdot(w_{i_1}^2+w_{i_2}^2)+2w_{i_1}\cdot w_{i_2}\cdot r_{i_3}^2\right] = d^2(x^*_{i_3}, x_{\{\{i_1,i_2\},i_3\}})$$

which means that $x_{\{i_1,i_2,i_3\}} \succeq_{i_3} x_{\{\{i_1,i_2\},i_3\}}$. □

Proof of Proposition 10

Let $M_{i_1} \cap M_{i_2} \cap M_{i_3} \neq \emptyset$, $x^*_{i_1} \notin M_{i_2} \cup M_{i_3}$, $x^*_{i_2} \notin M_{i_1} \cup M_{i_3}$, $x^*_{i_3} \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$, $d(x^*_{i_2}, x_{\{i_1,i_3\}}) \leq r_{i_2}$ and

$$\sum_{l=1}^{m} \left(x^*_{i_1 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l} \right) \left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l} \right) \geq 0.$$

From the proof of Proposition 9, we know that

$$d^2(x^*_{i_3}, x_{\{i_1,i_2,i_3\}}) =$$

$$\frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2} \left[r^2_{i_3} \cdot (w^2_{i_1} + w^2_{i_2}) + 2 w_{i_1} w_{i_2} \sum_{l=1}^{m} \left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l} \right) \left(x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l} \right) \right].$$

Since $x^*_{i_3} \in M_{i_1} \cap M_{i_2} \cap M_{i_3}$ and $x^*_{i_1} \notin M_{i_2} \cup M_{i_3}$, we have $\widetilde{x}^{\{i_1,i_3\}}_{i_3} = x^*_{i_3}$, $d(x^*_{i_1}, x^*_{i_3}) > r_{i_3}$, for each $l = 1, ..., m$

$$\widetilde{x}^{\{i_1,i_3\}}_{i_1 l} = x^*_{i_3 l} + \frac{r_{i_3} \cdot (x^*_{i_1 l} - x^*_{i_3 l})}{d(x^*_{i_1}, x^*_{i_3})}$$

and for each $l = 1, ..., m$

$$x_{\{i_1,i_3\} l} = \frac{w_{i_1} \cdot \widetilde{x}^{\{i_1,i_3\}}_{i_1 l} + w_{i_3} \cdot \widetilde{x}^{\{i_1,i_3\}}_{i_3 l}}{w_{i_1} + w_{i_3}}$$

$$= \frac{1}{w_{i_1} + w_{i_3}} \left[(w_{i_1} + w_{i_3}) \cdot x^*_{i_3 l} + \frac{r_{i_3} \cdot w_{i_1} \cdot (x^*_{i_1 l} - x^*_{i_3 l})}{d(x^*_{i_1}, x^*_{i_3})} \right].$$

If $d(x^*_{i_2}, x_{\{i_1,i_3\}}) \leq r_{i_2}$, then $\widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_3\}} = x_{\{i_1,i_3\}}$. Hence, for each $l = 1, ..., m$ we have

$$x_{\{\{i_1,i_3\},i_2\} l} = \frac{(w_{i_1} + w_{i_3}) \cdot \widetilde{x}^{\{i_1,i_2,i_3\}}_{\{i_1,i_3\} l} + w_{i_2} \cdot \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l}}{w_{i_1} + w_{i_2} + w_{i_3}}$$

$$= \frac{1}{w_{i_1} + w_{i_2} + w_{i_3}} \left[(w_{i_1} + w_{i_3}) \cdot x^*_{i_3 l} + \frac{r_{i_3} \cdot w_{i_1} \cdot (x^*_{i_1 l} - x^*_{i_3 l})}{d(x^*_{i_1}, x^*_{i_3})} + w_{i_2} \cdot \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2 l} \right]$$

Note that $\frac{r_{i_3}}{d(x^*_{i_1}, x^*_{i_3})} < 1$, and $d(x^*_{i_3}, \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2}) = r_{i_3}$. Moreover, note that for each $l = 1, ..., m$,

$$x^*_{i_1 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l} = x^*_{i_1 l} - x^*_{i_3 l} + x^*_{i_3 l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1 l}$$

and therefore

$$(x^*_{i_1l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l})(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}) = (x^*_{i_1l}-x^*_{i_3l})(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l})+(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l})(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l})$$

Hence,

$$0 \leq \sum_{l=1}^{m} \left(x^*_{i_1l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)$$

$$= \sum_{l=1}^{m} \left(x^*_{i_1l} - x^*_{i_3l}\right)\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right) + \sum_{l=1}^{m}\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)$$

which means that

$$\sum_{l=1}^{m} \left(x^*_{i_3l} - x^*_{i_1l}\right)\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right) \leq \sum_{l=1}^{m}\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)$$

We have then

$$d^2(x^*_{i_3}, x_{\{\{i_1,i_3\},i_2\}}) = \sum_{l=1}^{m}\left(x_{\{\{i_1,i_3\},i_2\}l} - x^*_{i_3l}\right)^2$$

$$= \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\sum_{l=1}^{m}\left[\frac{r_{i_3}\cdot w_{i_1}\cdot(x^*_{i_1l}-x^*_{i_3l})}{d(x^*_{i_1},x^*_{i_3})} + w_{i_2}\cdot\left(\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l} - x^*_{i_3l}\right)\right]^2$$

$$= \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r^2_{i_3}\cdot(w^2_{i_1}+w^2_{i_2}) + \frac{2r_{i_3}\cdot w_{i_1}\cdot w_{i_2}}{d(x^*_{i_1},x^*_{i_3})}\sum_{l=1}^{m}(x^*_{i_1l}-x^*_{i_3l})\left(\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l} - x^*_{i_3l}\right)\right]$$

$$< \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r^2_{i_3}\cdot(w^2_{i_1}+w^2_{i_2}) + 2w_{i_1}\cdot w_{i_2}\sum_{l=1}^{m}(x^*_{i_3l}-x^*_{i_1l})\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)\right]$$

$$\leq \frac{1}{(w_{i_1}+w_{i_2}+w_{i_3})^2}\left[r^2_{i_3}\cdot(w^2_{i_1}+w^2_{i_2}) + 2w_{i_1}\cdot w_{i_2}\sum_{l=1}^{m}\left(x^*_{i_3l}-\widetilde{x}^{\{i_1,i_2,i_3\}}_{i_1l}\right)\left(x^*_{i_3l} - \widetilde{x}^{\{i_1,i_2,i_3\}}_{i_2l}\right)\right]$$

$$= d^2(x^*_{i_3}, x_{\{i_1,i_2,i_3\}})$$

which means that $x_{\{\{i_1,i_3\},i_2\}} \succ_{i_3} x_{\{i_1,i_2,i_3\}}$. □