Semantic interpolation *

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Abstract

We treat interpolation for various logics. In the full non-monotonic case, we connect the existence of interpolants to laws about abstract size, and to generalized Hamming relations. We also mention revision a la Parikh.

Contents

1 Introduction

The interpolation property is the ability to "squeeze" a new formula in the consequence relation where the new formula has only symbols common to both original formulas:

If $\alpha \mid\!\sim_1 \gamma$, then there is an interpolant β s.t. $\alpha \mid\!\sim_2 \beta \mid\!\sim_3 \gamma$ - and β has only symbols common to α and γ.

Here, the consequence relations $\mid\!\sim_i$ may be different. In the classical case, they are all the same, and equal to $\vdash$. We will see later that the more general case is important in other logics.

Interpolation for classical logic is well-known, see [Cra57], and there are also non-classical logics for which interpolation has been shown, e.g., for Circumscription, see [Ami02]. An extensive overview of interpolation is found in [GM05]. Chapter

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1 of this book gives a survey and a discussion and 11 points of view of interpolation are discussed. This paper is a new 12th point of view.

Omitting a variable in classical propositional logic means "liberating" it, i.e., semantically, any of the two values TRUE and FALSE may be taken. Thus, for the omitted variables, we take the full product. Thus, there is a straightforward translation from language to sequences of values TRUE, FALSE (for each variable), and "liberated" variables correspond to taking all possible combinations. We work immediately with such sequences. This has the advantage that we need not fix our approach to just 2 values, TRUE and FALSE, but can work with many truth values. We are also free from logically equivalent re-formulations, classical consequence becomes subset inclusion, etc., in short: life becomes simpler, and new perspectives open up naturally.

We will see that changing logic, e.g. from preferential logic to classical logic when "squeezing" β in will be important.

Most of these notes are purely semantical, or, better, even algebraic. In the background is (partly classical) propositional logic, but we tried to remain general in our approach.

Our approach has the advantage of short and elementary proofs. Perhaps more important, it separates algebraic from logical questions, and we see that there are logics with algebraic interpolation, but without logical interpolation, as the necessary sets of models are not definable in the language. This opens the way to making the language richer to obtain interpolation, when so desired.

2 Notation

Definition 2.1

Let $\Pi := \Pi\{X_i : i \in J\}$.

For $J' \subseteq J$ and $\sigma \in \Pi$, let $\sigma \upharpoonright J'$ be the restriction of σ to J', likewise for $\Sigma \subseteq \Pi$ and $\Sigma \upharpoonright J'$.

Definition 2.2

For $\Sigma \subseteq \Pi$ set

$I(\Sigma) := \{i \in J : \Sigma = \Sigma \upharpoonright (J - \{i\}) \times X_i\}$ (up to re-ordering), (the irrelevant or inessential i) and

$R(\Sigma) := J - I(\Sigma)$ (the relevant or essential i).

Fact 2.1

(1) $\Sigma = \Sigma \upharpoonright R(\Sigma) \times \Pi \upharpoonright I(\Sigma)$ (up to re-ordering)

(2) $\sigma \upharpoonright R(\Sigma) = \sigma' \upharpoonright R(\Sigma) \wedge \sigma \in \Sigma \Rightarrow \sigma' \in \Sigma$.

Proof

(1) Enumerate $I(\Sigma)$ somehow. Take the first $i \in I(\Sigma)$ where it fails. But i is irrelevant, contradiction.

$\square$

Notation 2.1

We consider products over X, let X', X'' be a disjoint cover of X. Let (by abuse of language) $\Pi := \Pi X$, $\Pi' := \Pi X'$, etc. σ will be some element of Π, σ' the restriction of σ to X', etc. Σ will be a set of such σ, etc.

3 The semantically monotonic case (upward or downward)

Note that the following interpolation results can be read upward (monotonic logic) or downward (non-monotonic logic, in the following sense: γ is the theory of the minimal models of α, and not just any formula which holds in the set of minimal models - which would be downward, and then upward again in the sense of set inclusion), in the latter case we have to be careful: we usually cannot go upward again, so we have the sharpest possible case in mind. The case of mixed movement - down and then up - as in full non-monotonic logic is treated in Section 4 (page 4).

Moreover, the logic can be many-valued. There is no restriction on the X_i.

Fact 3.1

Let $\Sigma' \subseteq \Sigma \subseteq \Pi$, where $\Pi = \Pi\{X_i : i \in J\}$.

(1) Let $\Sigma'' := \Sigma' \upharpoonright (R(\Sigma) \cap R(\Sigma')) \times \Pi \upharpoonright (I(\Sigma) \cup I(\Sigma'))$.

Then $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$.

The following two results concern "parallel interpolation", terminology introduced by D.Makinson in [KM07]. Thus in the first case, Σ' is a product, in the second case, Σ is a product.

(2) Let $\mathcal{J}$ be a disjoint cover of J.

Let $\Sigma' = \Pi\{\Sigma'_K : K \in \mathcal{J}\}$ with $\Sigma'_K \subseteq \Pi\{X_i : i \in K\}$.

Let $\Sigma''_K := \Sigma'_K \upharpoonright (R(\Sigma) \cap R(\Sigma'_K)) \times \Pi\{X_i : i \in K, i \in I(\Sigma) \cup I(\Sigma'_K)\}$.

Let $\Sigma'' := \Pi\{\Sigma''_K : K \in \mathcal{J}\}$ (re-ordered).

Then $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$.

(3) Let $\mathcal{J}$ be a disjoint cover of J.

Let $\Sigma = \Pi\{\Sigma_K : K \in \mathcal{J}\}$ with $\Sigma_K \subseteq \Pi\{X_i : i \in K\}$.

Let $\Sigma''_K := \Sigma' \upharpoonright (R(\Sigma') \cap R(\Sigma_K)) \times \Pi\{X_i : i \in K, i \in I(\Sigma_K) \cup I(\Sigma')\}$, so $\Sigma''_K \subseteq \Pi\{X_i : i \in K\}$.

Let $\Sigma'' := \Pi\{\Sigma''_K : K \in \mathcal{J}\}$ (re-ordered).

Then $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$.

Proof

(1)

(1.1) $\Sigma' \subseteq \Sigma''$ is trivial.

(1.2) $\Sigma'' \subseteq \Sigma$:

Let $\sigma \in \Sigma''$, so there is $\sigma' \in \Sigma'$ s.t. $\sigma \upharpoonright (R(\Sigma) \cap R(\Sigma')) = \sigma' \upharpoonright (R(\Sigma) \cap R(\Sigma'))$. By definition of $R(\Sigma')$ (or Fact 2.1 (page 2) (2) for Σ'), we may choose $\sigma' \in \Sigma'$ s.t. also $\sigma \upharpoonright I(\Sigma') = \sigma' \upharpoonright I(\Sigma')$, so there is $\sigma' \in \Sigma' \subseteq \Sigma$ s.t. $\sigma \upharpoonright R(\Sigma) = \sigma' \upharpoonright R(\Sigma)$, so by definition of $R(\Sigma)$ (or Fact 2.1 (page 2) (2) for Σ), $\sigma \in \Sigma$.

(2)

(2.1) $\Sigma' \subseteq \Sigma''$.

$\Sigma'_K \subseteq \Sigma''_K$, so by $\Sigma'' = \Pi\Sigma''_K$ the result follows.

(2.2) $\Sigma'' \subseteq \Sigma$.

Let $\sigma'' \in \Sigma''$, $\sigma'' = \circ\{\sigma''_K : K \in \mathcal{J}\}$ (concatenation) for suitable $\sigma''_K \in \Sigma''_K$. Consider σ''_K. By definition of Σ''_K, there is $\sigma'_K \in \Sigma'_K$ s.t. $\sigma''_K \upharpoonright (R(\Sigma'_K) \cap R(\Sigma)) = \sigma'_K \upharpoonright (R(\Sigma'_K) \cap R(\Sigma))$, so there is $\tau'_K \in \Sigma'_K$ s.t. $\sigma''_K \upharpoonright R(\Sigma) = \tau'_K \upharpoonright R(\Sigma)$. Let $\tau' := \circ\{\tau'_K : K \in \mathcal{J}\}$, so by $\Sigma' = \Pi\{\Sigma'_K : K \in \mathcal{J}\}$ $\tau' \in \Sigma' \subseteq \Sigma$. But $\sigma'' \upharpoonright R(\Sigma) = \tau' \upharpoonright R(\Sigma)$, so $\sigma'' \in \Sigma$.

(3)

We first show $R(\Sigma_K) = R(\Sigma) \cap K$.

Let $i \in I(\Sigma_K)$, then $\Sigma_K = \Sigma_K \upharpoonright (K - \{i\}) \times X_i$, but $\Sigma = \Pi\{\Sigma_K : K \in \mathcal{J}\}$, so $\Sigma = \Sigma \upharpoonright (J - \{i\}) \times X_i$, and $i \in I(\Sigma)$. Conversely, let $i \in I(\Sigma) \cap K$, then $\Sigma = \Sigma \upharpoonright (J - \{i\}) \times X_i$, so $\Sigma \upharpoonright K = \Sigma \upharpoonright (K - \{i\}) \times X_i$, so $i \in I(\Sigma_K)$.

(3.1) $\Sigma' \subseteq \Sigma''$.

$\Sigma' \upharpoonright K \subseteq \Sigma''_K$, so by $\Sigma'' = \Pi\{\Sigma''_K : K \in \mathcal{J}\}$ $\Sigma' \subseteq \Sigma''$.

(3.2) $\Sigma'' \subseteq \Sigma$.

By $\Sigma = \Pi\{\Sigma_K : K \in \mathcal{J}\}$, it suffices to show $\Sigma''_K \subseteq \Sigma_K$.

Let $\sigma''_K \in \Sigma''_K$. So there is $\sigma' \in \Sigma'$ s.t. $\sigma' \upharpoonright (R(\Sigma_K) \cap R(\Sigma')) = \sigma''_K \upharpoonright (R(\Sigma_K) \cap R(\Sigma'))$, so there is $\tau' \in \Sigma' \subseteq \Sigma$ s.t. $\tau' \upharpoonright R(\Sigma_K) = \sigma''_K \upharpoonright R(\Sigma_K)$, so there is $\sigma \in \Sigma$ s.t. $\sigma \upharpoonright K = \sigma''_K$, so $\sigma''_K \in \Sigma_K$.

$\square$

We have shown *semantic* interpolation, this is not yet *syntactic* interpolation. We still need that the set of sequences is definable. (The importance of definability in the context of non-monotonic logics was examined by one of the authors in [Sch92].) Note that we "simplified" the set of sequences, but perhaps the logic at hand does not share this perspective. Consider, e.g., intuitionistic logic with three worlds, linearly ordered. This is a monotonic logic, so by our result, it has semantic interpolation. But it has no syntactic interpolation, so the created set of models must not be definable. In classical propositional logic, the created set *is* definable, as we will see in Fact 3.2 (page 4).

We describe now above example in detail.

Example 3.1

The semantics is a set of 3 worlds, w, w', w'', linearly ordered. The logic is intuitionistic, so knowledge "grows". Thus, ϕ can be true from w on, from w' on, from w'' on, or never, resulting in 4 truth values.

The formulas to consider are

$\alpha(p,q,r) := \Big(p \rightarrow (((q \rightarrow r) \rightarrow q) \rightarrow q)\Big) \rightarrow p,$

$\beta(p,s) := ((s \rightarrow p) \rightarrow s) \rightarrow s$

We have provable $\alpha \rightarrow \beta$, but no syntactic interpolant (which could use only p).

Introducing a new operator Jp meaning "from next moment onwards p holds and if now is the last moment then p holds now" gives enough definability to have also syntactic interpolation.

We may also see this definability property as natural, and as a criterion for well-behaviour of a logic. In some cases, introducing new constants analogous to TRUE, FALSE - in the cited case e.g. ONE, TWO when truth starts at world one or two - might help, but we did not investigate this question. This question is also examined in [ABM03].

We can go further with a logic in language L0 for which there is no interpolation. For every pair of formulas which give a counterexample to interpolation we introduce a new connective which corresponds to the semantic interpolant. Now we have a language L1 which allows for interpolation for formulas in the original language L0. L1 itself may or may not have interpolation. So we might have to continue to L2, L3, etc.

This leads to the following definition for the cases where we have semantic, but not necessarily syntactic interpolation:

Definition 3.1

A logic has level 0 semantic interpolation, iff it has interpolation.

A logic has level $n+1$ semantic interpolation iff it has not level n semantic interpolation, but introducing new elements into the language (of level n) results in interpolation also for the new language.

We have not examined this notion.

The case of full non-monotonic logic is, of course, different, as the logics might not even have semantic interpolation, so above repairing is not possible.

Fact 3.2

Simplification preserves definability in classical propositional logic:

Let $\Gamma = \Sigma \upharpoonright X' \times \Pi X''$. Then, if Σ is formula definable, so is Γ.

Proof

As Σ is formula definable, it is defined by $\phi_1 \vee \ldots \vee \phi_n$, where $\phi_i = \psi_{i,1} \wedge \ldots \wedge \psi_{i,n_i}$. Let $\Phi_i := \{\psi_{i,1}, \ldots, \psi_{i,n_i}\}$, $\Phi'_i := \{\psi \in \Phi_i : \psi \in X'\}$ (more precisely, ψ or $\neg\psi \in X'$), $\Phi''_i := \Phi_i - \Phi'_i$. Let $\phi'_i := \bigwedge \Phi'_i$. Thus $\phi_i \vdash \phi'_i$. We show that $\phi'_1 \vee \ldots \vee \phi'_n$ defines Γ. (Alternatively, we may replace all $\psi \in \Phi''_i$ by TRUE.)

(1) $\Gamma \models \phi'_1 \vee \ldots \vee \phi'_n$

Lt $\sigma \in \Gamma$, then there is $\tau \in \Sigma$ s.t. $\sigma \upharpoonright X' = \phi \upharpoonright X'$. By prerequisite, $\tau \models \phi_1 \vee \ldots \vee \phi_n$, so $\tau \models \phi'_1 \vee \ldots \vee \phi'_n$, so $\sigma \models \phi'_1 \vee \ldots \vee \phi'_n$.

(2) Suppose $\sigma \not\in \Gamma$, we have to show $\sigma \not\models \phi'_1 \vee \ldots \vee \phi'_n$.

Suppose then $\sigma \not\in \Gamma$, but $\sigma \models \phi'_1 \vee \ldots \vee \phi'_n$, wlog. $\sigma \models \phi'_1 = \bigwedge \Phi'_1$. As $\sigma \not\in \Gamma$, there is no $\tau \in \Sigma$ $\tau \upharpoonright X' = \sigma \upharpoonright X'$. Choose τ s.t. $\sigma \upharpoonright X' = \tau \upharpoonright X'$ and $\tau \models \psi$ for all $\psi \in \Phi''_i$. By $\sigma \models \psi$ for $\psi \in \Phi'_1$, and $\sigma \upharpoonright X' = \tau \upharpoonright X'$ $\tau \models \psi$ for $\psi \in \Phi'_1$. By prerequisite, $\tau \models \psi$ for $\psi \in \Phi''_1$, so $\tau \models \psi$ for all $\psi \in \Phi_1$, so $\tau \models \phi_1 \vee \ldots \vee \phi_n$, and $\tau \in \Sigma$, as $\phi \vee \ldots \vee \phi$ defines Σ, contradiction.

□

Corollary 3.3

The same result holds if Σ is theory definable.

Proof

(Outline). Define Σ by a (possibly infinite) conjunction of (finite) disjunctions. Transform this into a possibly infinite disjunction of possibly infinite conjunctions. Replace all $\phi \in \Phi''_i$ by TRUE. The same proof as above shows that this defines Γ (finiteness was nowhere needed). Transform backward into a conjunction of finite disjunctions, where the $\phi \in \Phi''_i$ are replaced by TRUE.

□

4 The full non-monotonic case, i.e., downward and upward

4.1 Discussion

We consider here a non-monotonic logic $\mid\sim$. $\mid\sim$ will be defined by a principal filter, generated by a set operator $\mu(X) \subseteq X$:

$\beta \mid\sim \gamma$ iff $\mu(M(\beta)) \subseteq M(\gamma)$, where $M(\alpha)$ is the set of models of α.

Often, μ will be generated by a binary relation $\prec$ on the model set,

$\mu(X) := \{x \in X : \neg\exists x' \prec x, x' \in X\}$.

The following example shows that interpolation does not always exist for the full non-monotonic case.

Example 4.1

Full non-monotonic logics, i.e. down and up, has not necessarily interpolation.

Consider the model order $pq \prec p\neg q \prec \neg p\neg q \prec \neg pq$. Then $\neg p \mid\sim \neg q$, there are no common variables, and $true \mid\sim q$ (and, of course, $\neg p \not\mid\sim false$). (Full consequence of $\neg p$ is $\neg p\neg q$, so this has trivial interpolation.)

We look at the interpolation problem in 3 ways.

Given $\phi \mid\sim \psi$, there is an interpolant α s.t.

(1) $\phi \mid\sim \alpha \vdash \psi$

(2) $\phi \vdash \alpha \mid\sim \psi$

(3) $\phi \mid\sim \alpha \mid\sim \psi$

The first variant will be fully characterized below.

The second variant has no full characterization at the time of writing (to the authors' knowledge), but is connected to very interesting properties about multiplication of size and componentwise independent relations.

The third variant stays unexplored for the moment.

We turn to variant (1) and (2).

4.2 $\phi \mid\!\sim \alpha \vdash \psi$

Fact 4.1

Let $\Sigma \subseteq \Pi X$, $var(\alpha) \cap var(\beta) = \emptyset$, $var(\beta) \cap R(\Sigma) = \emptyset$, β not a tautology, then $\Sigma \subseteq M(\alpha \vee \beta) \Rightarrow \Sigma \subseteq M(\alpha)$.

Proof

Suppose not, so there is $\sigma \in \Sigma$ such that $\sigma \models \alpha \vee \beta$, $\sigma \not\models \alpha$. As β is not a tautology, there is an assignment to $var(\beta)$ which makes β wrong. Consider τ s.t. $\sigma = \tau$ except on $var(\beta)$, where τ makes β wrong, using this assignment. By $var(\alpha) \cap var(\beta) = \emptyset$, $\tau \models \neg\alpha$. By $var(\beta) \cap R(\Sigma) = \emptyset$, $\tau \in \Sigma$. So $\tau \not\models \alpha \vee \beta$ for some $\tau \in \Sigma$, contradiction.

$\square$

Fact 4.2

We use here normal forms (conjunctions of disjunctions).

Consider a finite language.

$\mid\!\sim$ has interpolation iff for all Σ, $\mu(\Sigma)$ $I(\Sigma) \subseteq I(\mu(\Sigma))$ holds.

In the infinite case, we need as additional prerequisite that $\mu(\Sigma)$ is definable if Σ is.

($\mu(\Sigma)$ are the minimal models of Σ, seen as sequences.)

Proof

Work with reformulations of Σ etc. which use only essential (= relevant) variables.

"$\Rightarrow$":

Suppose the condition is wrong. Then $X := I(\Sigma) - I(\mu(\Sigma)) = I(\Sigma) \cap R(\mu(\Sigma)) \neq \emptyset$. Then there is some $\sigma' \in \mu(\Sigma) \restriction R(\Sigma)$ which cannot be continued by some choice ρ in $X \cup (I(\Sigma) \cap I(\mu(\Sigma)))$ in $\mu(\Sigma)$, i.e. $\sigma' \circ \rho \notin \mu(\Sigma)$.

We first do the finite case: Consider the formula $\phi := \sigma' \rightarrow \neg\rho = \neg\sigma' \vee \neg\rho$. We have $Th(\Sigma) \mid\!\sim \phi$, as $\mu(\Sigma) \subseteq M(\phi)$. Suppose Σ'' is a semantical interpolant for Σ and ϕ. So $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\phi)$, and Σ'' does not contain any variables in ρ as essential variables. By Fact 4.1 (page 5), $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\neg\sigma')$, but $\sigma' \in \mu(\Sigma) \restriction R(\Sigma)$, contradiction.

We turn to the infinite case. Consider again $\sigma' \circ \rho$. As $\sigma' \circ \rho \notin \mu(\Sigma)$, and $\mu(\Sigma)$ is definable, there is some formula ϕ which holds in $\mu(\Sigma)$, but fails in $\sigma' \circ \rho$. Thus, $Th(\Sigma) \mid\!\sim \phi$. Write ϕ as a disjunction of conjunctions. Let Σ'' be an interpolant for Σ and $M(\phi)$. Thus $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\phi)$, and $\sigma' \circ \rho \notin M(\phi)$, so $\mu(\Sigma) \subseteq \Sigma'' \subseteq M(\phi) \subseteq M(\neg\sigma' \vee \neg\rho)$, so $\Sigma'' \subseteq M(\neg\sigma')$ by Fact 4.1 (page 5). So $\mu(\Sigma) \models \neg\sigma'$, contradiction, as $\sigma' \in \mu(\Sigma) \restriction R(\Sigma)$.

"$\Leftarrow$":

Let $I(\Sigma) \subseteq I(\mu(\Sigma))$. Let $\Sigma \mid\!\sim \Sigma'$, i.e. $\mu(\Sigma) \subseteq \Sigma'$. Write $\mu(\Sigma)$ as a (possibly infinite) conjunction of disjunctions, using only relevant variables. Form Σ'' from $\mu(\Sigma)$ by omitting all variables in this description which are not in $R(\Sigma')$. Note that all remaining variables are in $R(\mu(\Sigma)) \subseteq R(\Sigma)$, so Σ'' is a candidate for interpolation.

(1) $\mu(\Sigma) \subseteq \Sigma''$: Trivial.

(2) $\Sigma'' \subseteq \Sigma'$: Let $\sigma \in \Sigma''$. Then there is $\tau \in \mu(\Sigma) \subseteq \Sigma'$ s.t. $\sigma \restriction R(\Sigma') = \tau \restriction R(\Sigma')$, so $\sigma \in \Sigma'$.

It remains to show in the infinite case that Σ'' is definable. This can be shown as in Fact 3.2 (page 4).

$\square$

4.3 $\phi \vdash \alpha \mid\!\sim \psi$

4.3.1 Product size

We work here with a notion of "big" and "small" subsets, which may be thought of as defined by a filter (ideal), though we usually will not need the full strength of a filter (ideal). But assume as usual that $A \subseteq B \subseteq C$ and $A \subseteq C$ is big imply $B \subseteq C$ is big, that $C \subseteq C$ is big, and define $A \subseteq B$ is small iff $(B - A) \subseteq B$ is big, call all subsets which are neither big nor small medium size. For an extensive discussion, see [GS09a].

Let $X' \cup X'' = X$ be a disjoint cover, so $\Pi X = \Pi X' \times \Pi X''$. We consider subsets Σ etc. of ΠX. If not said otherwise, Σ etc. need not be a product $\Sigma' \times \Sigma''$. As usual, $\Pi' := \Pi X'$, $\Sigma'' := \Sigma \restriction X''$ etc. The roles of X' and X'' are interchangeable, e.g., instead of $\Gamma \restriction X' \subseteq \Sigma \restriction X'$, we may also write $\Gamma \restriction X'' \subseteq \Sigma \restriction X''$.

We consider here the following three finite product rules about size and μ. $\mu(X) \subseteq X$ - read: the minimal elements of X - will generate the principal filter $\mathcal{F}(X)$, whose elements are big subsets of X.

Definition 4.1

$(S*1)$ $\Delta \subseteq \Sigma' \times \Sigma''$ is big iff there is $\Gamma = \Gamma' \times \Gamma'' \subseteq \Delta$ s.t. $\Gamma' \subseteq \Sigma'$ and $\Gamma'' \subseteq \Sigma''$ are big

$(S*2)$ $\Gamma \subseteq \Sigma$ is big $\Rightarrow \Gamma \upharpoonright X' \subseteq \Sigma \upharpoonright X'$ is big - where Σ is not necessarily a product.

$(S*3)$ $A \subseteq \Sigma$ is big $\Rightarrow$ there is $B \subseteq \Pi' \times \Sigma''$ big s.t. $B \upharpoonright X'' \subseteq A \upharpoonright X''$ - again, Σ is not necessarily a product.

$(\mu*1)$ $\mu(\Sigma' \times \Sigma'') = \mu(\Sigma') \times \mu(\Sigma'')$

$(\mu*2)$ $\mu(\Sigma) \subseteq \Gamma \Rightarrow \mu(\Sigma \upharpoonright X') \subseteq \Gamma \upharpoonright X'$

$(\mu*3)$ $\mu(\Pi' \times \Sigma'') \upharpoonright X'' \subseteq \mu(\Sigma) \upharpoonright X''$.

A generalization to more than two factors is obvious.

One can also consider weakenings, e.g.,

$(S*1')$ $\Gamma' \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is big iff $\Gamma' \subseteq \Sigma'$ is big.

Fact 4.3

Let $(S*1)$ hold. Then:

$\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small iff $\Gamma' \subseteq \Sigma'$ or $\Gamma'' \subseteq \Sigma''$ is small.

Proof

"$\Leftarrow$":

Suppose $\Gamma' \subseteq \Sigma'$ is small. Then $\Sigma' - \Gamma' \subseteq \Sigma'$ is big and $(\Sigma' - \Gamma') \times \Sigma'' \subseteq \Sigma' \times \Sigma''$ is big by $(S*1)$. But $(\Gamma' \times \Gamma'') \cap ((\Sigma' - \Gamma') \times \Sigma'') = \emptyset$, so $\Gamma' \times \Gamma'' \subseteq \Sigma' \times \Sigma''$ is small.

"$\Rightarrow$":

For the converse, suppose that neither $\Gamma' \subseteq \Sigma'$ nor $\Gamma'' \subseteq \Sigma''$ are small. Let $A \subseteq \Sigma' \times \Sigma''$ be big, we show that $A \cap (\Gamma' \times \Gamma'') \neq \emptyset$. By $(S*1)$ there are $B' \subseteq \Sigma'$ and $B'' \subseteq \Sigma''$ big, and $B' \times B'' \subseteq A$. Then $B' \cap \Gamma' \neq \emptyset$, $B'' \cap \Gamma'' \neq \emptyset$, so there is $\langle x', x'' \rangle \in (B' \times B'') \cap (\Gamma' \times \Gamma'') \subseteq A \cap (\Gamma' \times \Gamma'')$.

□

Fact 4.4

If the filters over A are principal filters, generated by $\mu(A)$, i.e. $B \subseteq A$ is big iff $\mu(A) \subseteq B \subseteq A$ for some $\mu(A) \subseteq A$, then:

$(S*i)$ is equivalent to $(\mu*i)$, $i = 1, 2, 3$.

Proof

(1)

"$\Rightarrow$"

"$\subseteq$": $\mu(\Sigma') \subseteq \Sigma'$ and $\mu(\Sigma'') \subseteq \Sigma''$ are big, so by $(S*1)$ $\mu(\Sigma') \times \mu(\Sigma'') \subseteq \Sigma' \times \Sigma''$ is big, so $\mu(\Sigma' \times \Sigma'') \subseteq \mu(\Sigma') \times \mu(\Sigma'')$.

"$\supseteq$": $\mu(\Sigma' \times \Sigma'') \subseteq \Sigma' \times \Sigma''$ is big $\Rightarrow$ by $(S*1)$ there is $\Gamma' \times \Gamma'' \subseteq \mu(\Sigma' \times \Sigma'')$ and $\Gamma' \subseteq \Sigma'$, $\Gamma'' \subseteq \Sigma''$ big $\Rightarrow \mu(\Sigma') \subseteq \Gamma'$, $\mu(\Sigma'') \subseteq \Gamma'' \Rightarrow \mu(\Sigma') \times \mu(\Sigma'') \subseteq \mu(\Sigma' \times \Sigma'')$.

"$\Leftarrow$"

Let $\Gamma' \subseteq \Sigma'$ be big, $\Gamma'' \subseteq \Sigma''$ be big, $\Gamma' \times \Gamma'' \subseteq \Delta$, then $\mu(\Sigma') \subseteq \Gamma'$, $\mu(\Sigma'') \subseteq \Gamma''$, so by $(\mu*1)$ $\mu(\Sigma) = \mu(\Sigma') \times \mu(\Sigma'') \subseteq \Gamma' \times \Gamma'' \subseteq \Delta$, so Δ is big.

Let $\Delta \subseteq \Sigma$ be big, then by $(\mu*1)$ $\mu(\Sigma') \times \mu(\Sigma'') = \mu(\Sigma) \subseteq \Delta$.

(2)

"$\Rightarrow$"

$\mu(\Sigma) \subseteq \Gamma \Rightarrow \Gamma \subseteq \Sigma$ big $\Rightarrow$ by $(S*2)$ $\Gamma \upharpoonright X' \subseteq \Sigma \upharpoonright X'$ big $\Rightarrow \mu(\Sigma \upharpoonright X') \subseteq \Gamma \upharpoonright X'$.

"$\Leftarrow$"

$\Gamma \subseteq \Sigma$ big $\Rightarrow \mu(\Sigma) \subseteq \Gamma \Rightarrow$ by $(\mu*2)$ $\mu(\Sigma \upharpoonright X') \subseteq \Gamma \upharpoonright X' \Rightarrow \Gamma \upharpoonright X' \subseteq \Sigma \upharpoonright X'$ big.

(3)

"$\Rightarrow$"

$\mu(\Sigma) \subseteq \Sigma$ big $\Rightarrow \exists B \subseteq \Pi' \times \Sigma''$ big s.t. $B \upharpoonright X'' \subseteq \mu(\Sigma) \upharpoonright X''$ by $(S*3)$, thus in particular $\mu(\Pi' \times \Sigma'') \upharpoonright X'' \subseteq \mu(\Sigma) \upharpoonright X''$.

"$\Leftarrow$"

$A \subseteq \Sigma$ big $\Rightarrow \mu(\Sigma) \subseteq A$. $\mu(\Pi' \times \Sigma'') \subseteq \Pi' \times \Sigma''$ is big, and by $(\mu*3)$ $\mu(\Pi' \times \Sigma'') \upharpoonright X'' \subseteq \mu(\Sigma) \upharpoonright X'' \subseteq A \upharpoonright X''$.

□

Discussion

We compare these rules to probability defined size.

Let "big" be defined by "more than 50%". If $\Pi X'$ and $\Pi X''$ have 3 elements each, then subsets of $\Pi X'$ or $\Pi X''$ of $card \geq 2$ are big. But taking the product may give $4/9 < 1/2$. So the product rule "$big * big = big$" will not hold there. One direction will hold, of course.

Next, we discuss the prerequisite $\Sigma = \Sigma' \times \Sigma''$. Consider the following example:

Example 4.2

Take a language of 5 propositional variables, with $X' := \{a,b,c\}$, $X'' := \{d,e\}$. Consider the model set $\Sigma := \{\pm a \pm b \pm cde, -a-b-c-d\pm e\}$, i.e. of 8 models of de and 2 models of $-d$. The models of de are 8/10 of all elements of Σ, so it is reasonable to call them a big subset of Σ. But its projection on X'' is only 1/3 of Σ''.

So we have a potential *decrease* when going to the coordinates.

This shows that weakening the prerequisite about Σ as done in $(S*2)$ is not innocent.

Remark 4.5

When we set small sets to 0, big sets to 1, we have the following boolean rules for filters:

(1) $0+0=0$

(2) $1+x=1$

(3) $-0=1$, $-1=0$

(4) $0*x=0$

(5) $1*1=1$

There are no such rules for medium size sets, as the union of two medium size sets may be big, but also stay medium.

Such multiplication rules capture the behaviour of Reiter defaults and of defeasible inheritance.

4.3.2 Interpolation of the form $\phi \vdash \alpha \mid\sim \psi$

Corollary 4.6

We assume definability as shown in Fact 3.2 (page 4).

Interpolation of the form $\phi \vdash \alpha \mid\sim \psi$ exists, if

(1) both $(S*1)$ and $(S*2)$,

or

(2) $(S*3)$ hold,

when $\beta \mid\sim \gamma$ is defined by:

$\beta \mid\sim \gamma :\Leftrightarrow \mu(\beta) = \mu(M(\beta)) \subseteq M(\gamma)$, and

$\mu(X)$ is the generator of the principal filter over X.

Proof

Let $\Sigma := M(\phi)$, $\Gamma := M(\psi)$, X' the set of variables only in ϕ, so $\Gamma = \Pi' \times \Gamma \restriction X''$, where $\Pi' = \Pi X'$. Set $\alpha := Th(\Pi' \times \Sigma'')$, where $\Sigma'' = \Sigma \restriction X''$. Note that variables only in ψ are automatically taken care of, as Σ'' can be written as a product without mentioning them.

By prerequisite, $\mu(\Sigma) \subseteq \Gamma$, we have to show $\mu(\Pi' \times \Sigma'') \subseteq \Gamma$.

(1)

$\mu(\Pi' \times \Sigma'') = \mu(\Pi') \times \mu(\Sigma'')$ by $(S*1)$ and Fact 4.4 (page 6) (1). By $\mu(\Sigma) \subseteq \Gamma$, $(S*2)$, and Fact 4.4 (page 6) (2), $\mu(\Sigma'') = \mu(\Sigma \restriction X'') \subseteq \Gamma \restriction X''$, so $\mu(\Pi' \times \Sigma'') = \mu(\Pi') \times \mu(\Sigma'') \subseteq \mu(\Pi') \times \Gamma \restriction X'' \subseteq \Gamma$.

(2)

$\mu(\Pi' \times \Sigma'') \restriction X'' \subseteq \mu(\Sigma) \restriction X'' \subseteq \Gamma \restriction X''$ by $(S*3)$ and Fact 4.4 (page 6) (3). So $\mu(\Pi' \times \Sigma'') \subseteq \Pi' \times (\mu(\Pi' \times \Sigma'') \restriction X'') \subseteq \Pi' \times (\Gamma \restriction X'') = \Gamma$.

□

4.3.3 Hamming relations

Definition 4.2

Call a relation $\preceq \subseteq (\Pi X \times \Pi X) \cup (\Pi' \times \Pi') \cup (\Pi'' \times \Pi'')$ (where $\Pi' = \Pi X'$, $\Pi'' = \Pi X''$) a (generalized) Hamming relation iff

$\sigma \preceq \tau \Leftrightarrow \sigma' \preceq \tau'$ and $\sigma'' \preceq \tau''$,

where $\sigma = \sigma' \circ \sigma''$, $\tau = \tau' \circ \tau''$ ($\circ$ concatenation).

Define $x \prec y :\Leftrightarrow x \preceq y$ and $x \neq y$.

Thus, $\sigma \prec \tau$ iff $\sigma \preceq \tau$ and ($\sigma' \prec \tau'$ or $\sigma'' \prec \tau''$).

Again, generalization to more than two components is straightforward.

Example 4.3

The circumscription relation is a Hamming relation with $\neg p \leq p$ and $\bigwedge \pm q_i \leq \bigwedge \pm q'_i$ iff $\forall i(\pm q_i \leq \pm q'_i)$.

Remark 4.7

(1) The independence makes sense because the concept of models, and thus the usual interpolation for classical logic relie on the independence of the assignments.

(2) This corresponds to social choice for many independent dimensions.

(3) We can also consider such factorisation as an approximation: we can do part of the reasoning independently.

4.3.4 Hamming relations and product size

Definition 4.3

Given a relation $\preceq$, define as usual a principal filter $\mathcal{F}(X)$ generated by the $\preceq -$minimal elements:

$\mu(X) := \{x \in X : \neg\exists x' \prec x.x' \in X\}$,

$\mathcal{F}(X) := \{A \subseteq X : \mu(X) \subseteq A\}$.

Fact 4.8

Let $\preceq$ be a smooth Hamming relation. Then $(\mu * 2)$ holds, and thus $(S * 2)$ by Fact 4.4 (page 6), (2).

Proof

Suppose $\mu(\Sigma) \subseteq \Gamma$ and $s \in \Sigma \restriction X' - \Gamma \restriction X'$, we show $s \notin \mu(\Sigma \restriction X')$.

Let $\sigma = s \circ \sigma'' \in \Sigma$, then $\sigma \notin \Gamma$, so $\sigma \notin \mu(\Sigma)$. So here is $\rho \prec \sigma$, $\rho \in \mu(\Sigma) \subseteq \Gamma$ by smoothness. We have $\rho' \preceq s$. $\rho' = s$ cannot be, as $\rho' \in \Gamma \restriction X'$, and $s \notin \Gamma \restriction X'$. So $\rho' \prec s$, and $s \notin \mu(\Sigma \restriction X')$.

□

Example 4.4

Even for smooth Hamming relations, the converse of $(\mu * 2)$ is not necessarily true:

Let $\sigma' \prec \tau'$, $\tau'' \prec \sigma''$, $\sigma := \{\sigma, \tau\}$, then $\mu(\Sigma) = \Sigma$, but $\mu(\Sigma') = \{\sigma'\}$, $\mu(\Sigma'') = \{\tau''\}$, so $\mu(\Sigma) \neq \mu(\Sigma') \times \mu(\Sigma'')$.

We need the additional assumption that $\mu(\Sigma') \times \mu(\Sigma'') \subseteq \Sigma$, see Fact 4.9 (page 8) (1).

Fact 4.9

Let again $\Sigma' := \Sigma \restriction X'$, $\Sigma'' := \Sigma \restriction X''$.

(1) Let $\preceq$ be a smooth Hamming relation. Then:

$\mu(\Sigma') \times \mu(\Sigma'') \subseteq \Sigma \Rightarrow \mu(\Sigma) = \mu(\Sigma') \times \mu(\Sigma'')$.

(Here not necessarily $\Sigma = \Sigma' \times \Sigma''$.)

(2) Let $\preceq$ be a Hamming relation, and $\Sigma = \Sigma' \times \Sigma''$. Then $(\mu * 1)$ holds, and thus, by Fact 4.4 (page 6), (1) $(S * 1)$.

Proof

(1)

"$\supseteq$": Let $\sigma' \in \mu(\Sigma')$, $\sigma'' \in \mu(\Sigma'')$. By prerequisite, $\sigma' \circ \sigma'' \in \Sigma$. Suppose $\tau \prec \sigma' \circ \sigma''$, then $\tau' \prec \sigma'$ or $\tau'' \prec \sigma''$, contradiction.

"$\subseteq$": Let $\sigma \in \mu(\Sigma)$, suppose $\sigma' \notin \mu(\Sigma')$ or $\sigma'' \notin \mu(\Sigma'')$. So there are $\tau' \preceq \sigma'$, $\tau'' \preceq \sigma''$ with $\tau' \in \mu(\Sigma')$, $\tau'' \in \mu(\Sigma'')$ by smoothness. Moreoever, $\tau' \prec \sigma'$ or $\tau'' \prec \sigma''$. By prerequisite $\tau' \circ \tau'' \in \Sigma$, and $\tau' \circ \tau'' \prec \sigma$, so $\sigma \notin \mu(\Sigma)$.

(2)

"$\supseteq$": As in (1), the prerequisite holds trivially.

"$\subseteq$": As in (1), but we do not need $\tau' \in \mu(\Sigma')$, $\tau'' \in \mu(\Sigma'')$, as $\tau' \circ \tau''$ will be in Σ trivially. So smoothness is not needed.

□

We now show the main property for $\vdash \circ \mid\!\sim$ interpolation in a direct proof. Note that interpolation treats the two components of the product differently, which also shows that our requirements are sufficient, but not necessary - see Section 4.4 (page 9).

Fact 4.10

Let $\preceq$ be a smooth Hamming relation, then $(\mu * 3)$ holds, and thus by Fact 4.4 (page 6) (3) $(S * 3)$.

Proof

Let again $\Pi' = \Pi X'$, $\Sigma'' = \Sigma \restriction X''$.

Let $\Delta := \Pi' \times \Sigma''$, $\sigma'' \in \mu(\Delta) \restriction X''$, $\sigma = \sigma' \circ \sigma'' \in \mu(\Delta)$ for some $\sigma' \in \Pi'$. Suppose $\sigma'' \notin \mu(\Sigma) \restriction X''$. There cannot be any $\tau \prec \sigma$, $\tau \in \Sigma$, by $\Sigma \subseteq \Delta$. So $\sigma \notin \Sigma$, but $\sigma'' \in \Sigma''$, so there is $\tau \in \Sigma$ $\tau'' = \sigma''$. As τ is not minimal, there must be minimal $\rho = \rho' \circ \rho'' \prec \tau$, $\rho \in \Sigma$ by smoothness. As ρ is minimal, $\rho'' \neq \sigma''$, and as $\rho \prec \tau$, $\rho'' \prec \sigma''$. By prerequisite $\sigma' \circ \rho'' \in \Delta$, and $\sigma' \circ \rho'' \prec \sigma$, contradiction.

Note that smoothness is essential. Otherwise, there might be an infinite descending chain τ_i below τ, all with $\tau_i'' = \sigma''$, but none below σ.

□

Corollary 4.11

Interpolation in the form $\phi \vdash \alpha \mid\!\sim \psi$ exists, when $\mid\!\sim$ is defined by a smooth Hamming relation.

Proof

We give two proofs:

(1)

By Fact 4.8 (page 8) and Fact 4.9 (page 8) (2) $(S*1)$ and $(S*2)$ hold. Thus, by Corollary 4.6 (page 7) (1), interpolation exists.

(2)

By Fact 4.10 (page 8), $(S*3)$ holds, so by Corollary 4.6 (page 7) (2), interpolation exists.

□

Fact 4.12

$(\mu*1)$ and $(\mu*2)$ and the usual axioms for smooth relations characterize smooth Hamming relations.

Proof

Define as usual $\sigma \prec \tau :\Leftrightarrow \tau \notin \mu(\{\sigma,\tau\})$.

We have to show:

$\sigma \prec \tau$ iff $\sigma' \preceq \tau'$ and $\sigma'' \preceq \tau''$ and $(\sigma' \prec \tau'$ or $\sigma'' \prec \tau'')$.

"$\Leftarrow$":

Suppose $\sigma' \prec \tau'$ and $\sigma'' \preceq \tau''$. Then $\mu(\{\sigma',\tau'\}) = \{\sigma'\}$, and $\mu(\{\sigma'',\tau''\}) = \{\sigma''\}$ (either $\sigma'' \prec \tau''$ or $\sigma'' = \tau''$, so in both cases $\mu(\{\sigma'',\tau''\}) = \{\sigma''\}$). As $\tau' \notin \mu(\{\sigma',\tau'\})$, $\tau \notin \mu(\{\sigma',\tau'\} \times \{\sigma'',\tau''\}) = \mu(\{\sigma',\tau'\}) \times \mu(\{\sigma'',\tau''\})$ (by $(\mu*1)$) $= \{\sigma'\} \times \{\sigma''\} = \{\sigma\}$, by $(\mu*1)$, so by smoothness $\sigma \prec \tau$.

"$\Rightarrow$":

Conversely, if $\sigma \prec \tau$, so $\Gamma := \{\sigma\} = \mu(\Sigma)$ for $\Sigma := \{\sigma,\tau\}$, so by $(\mu*2)$ $\mu(\Sigma \restriction X') = \mu(\{\sigma',\tau'\}) \subseteq \Gamma \restriction X' = \{\sigma'\}$, so $\sigma' \preceq \tau'$, analogously $\mu(\Sigma \restriction X'') = \mu(\{\sigma'',\tau''\}) \subseteq \Gamma \restriction X'' = \{\sigma''\}$, so $\sigma'' \preceq \tau''$, but both cannot be equal.

□

4.4 Context and structure

We take the importance of condition $(\mu*3)$ (or $(S*3)$) as occasion for a broader remark.

(1) This condition points to a weakening of the Hamming condition:

Adding new "branches" in X' will not give new minimal elements in X'', but may destroy other minimal elements in X''. This can be achieved by a sort of semi-rankedness: If ρ and σ are different only in the $X'-$part, then $\tau \prec \rho$ iff $\tau \prec \sigma$, but not necessarily $\rho \prec \tau$ iff $\sigma \prec \tau$.

(2) In more abstract terms:

When we separate support from attack (support: a branch σ' in X' supports a continuation σ'' in X'' iff $\sigma \circ \sigma''$ is minimal, i.e. not attacked, attack: a branch τ in X' attacks a continuation σ'' in X'' iff it prevents all $\sigma \circ \sigma''$ to be minimal), we see that new branches will not support any new continuations, but may well attack continuations.

More radically, we can consider paths σ'' as positive information, σ' as potentialy negative information. Thus, Π' gives maximal negative information, and thus smallest set of accepted models.

(3) We can interpret this as follows: X'' determines the base set. X' is the context. This determines the choice (subset of the base set). We compare to preferential structures: In preferential structures, $\prec$ is not part of the language either, it is context. And we have the same behaviour as shown in the fundamental property of preferential structures: the bigger the set, the more attacks are possible.

(4) The concept of size looks only at the result of support and attack, so it is necessarily somewhat coarse. Future research should also investigate both concepts separately.

We broaden this.

Following a tradition begun by Kripke, one has added structure to the set of classical models, reachability, preference, etc. Perhaps one should emphasize a more abstract approach, as done by one the authors e.g. in [Sch92], and elaborated in [Sch04], see in particular the distinction between structural and algebraic semantics in the latter. Our suggestion is to separate structure from logic in the semantics, and to treat what we called context above by a separate "machinery". So, given a set X of models, we have some abstract function f, which chooses the models where the consequences hold, $f(X)$.

Now, we can put into this "machinery" whatever we want.

The abstract properties of preferential or modal structures are well known.

But we can also investigate non-static f, where f changes in function of what we already did - "reacting" to the past.

We can look at usual properties of f, complexity, generation by some simple structure like a special machine, etc.

So we advocate the separation of usual, classical semantics, from the additional properties, which are treated "outside". It might be interesting to forget altogether about logic, classify those functions or more complicated devices which correspond to some logical property, and investigate them and their properties.

5 Hamming distances and revision

Definition 5.1

Given $x, y \in \Sigma$, a set of sequences over an index set I, the Hamming distance comes in two flavours:

$d_s(x,y) := \{i \in I : x(i) \neq y(i)\}$, the set variant,

$d_c(x,y) := card(d_s(x,y))$, the counting variant.

We define $d_s(x,y) \leq d_s(x',y')$ iff $d_s(x,y) \subseteq d_s(x',y')$,

thus, $s-$distances are not always comparabel.

There are straightforward generalizations of the counting variant:

We can also give different importance to different i in the counting variant, so e.g., $d_c(\langle x,x'\rangle, \langle y,y'\rangle)$ might be 1 if $x \neq y$ and $x' = y'$, but 2 if $x = y$ and $x' \neq y'$.

If the $x \in \Sigma$ may have more than 2 different values, then a varying individual distance may also reflect to the distances in Σ. So, if $d(x(i), x'(i)) < d(x(i), x''(i))$, then (the rest being equal), we may have $d(x,x') < d(x,x'')$.

The (for us) essential property of the set variant Hamming distance is that we cannot compensate, i.e. a difference in x cannot be compensated by equality in x' and x'', as in the counting variant.

Definition 5.2

Call a function $d : (\Pi X \times \Pi X) \cup (\Pi' \times \Pi') \cup (\Pi'' \times \Pi'') \rightarrow Z$

(where $\Pi' = \Pi X'$, $\Pi'' = \Pi X''$)

a (generalized) Hamming distance iff

(1) there is a total order $\leq$ on Z

(2) $d(\sigma_1, \tau_1) \leq d(\sigma_2, \tau_2)$ iff $d(\sigma_1', \tau_1') \leq d(\sigma_2', \tau_2')$ and $d(\sigma_1'', \tau_1'') \leq d(\sigma_2'', \tau_2'')$.

(In the strict part, at least one has to be strict on the right hand side.)

Definition 5.3

Given a distance d, define for two sets X, Y

$X \mid Y := \{y \in Y : \exists x \in X(\neg\exists x' \in X, y' \in Y. d(x',y') < d(x,y))\}$.

We assume that $X \mid Y \neq \emptyset$ if $X, Y \neq \emptyset$.

We have results analogous to the relation case.

Fact 5.1

Let $\mid$ be defined by a Hamming distance, then:

(1) $(\Sigma_1' \times \Sigma_1'') \mid (\Sigma_2' \times \Sigma_2'') = (\Sigma_1' \mid \Sigma_2') \times (\Sigma_1'' \mid \Sigma_2'')$

(2) $(\Sigma_1' \mid \Sigma_2') \times (\Sigma_1'' \mid \Sigma_2'') \subseteq \Sigma_2$ and $(\Sigma_2' \mid \Sigma_1') \times (\Sigma_2'' \mid \Sigma_1'') \subseteq \Sigma_1 \Rightarrow (\Sigma_1) \mid (\Sigma_2) = (\Sigma_1' \mid \Sigma_2') \times (\Sigma_1'' \mid \Sigma_2'')$, if the distance is symmetric

(where Σ_i here is not necessarily $\Sigma_i' \times \Sigma_i''$, etc.).

Proof

(1) and (2)

"$\subseteq$":

Suppose $d(\sigma, \tau)$ is minimal for $\sigma \in \Sigma_1$, $\tau \in \Sigma_2$. If there is $\alpha' \in \Sigma_1'$, $\beta' \in \Sigma_2'$ s.t. $d(\alpha', \beta') < d(\sigma', \tau')$, then $d(\alpha' \circ \sigma'', \beta' \circ \tau'') < d(\sigma, \tau)$, so $d(\sigma', \tau')$ and $d(\sigma'', \tau'')$ have to be minimal.

For the argument to go through, we need $\alpha' \circ \sigma''$, $\beta' \circ \tau''$ to be in the sets considered. If $\Sigma_i = \Sigma_i' \times \Sigma_i''$, this is trivially satisfied. Otherwise, we use the usual minimality assumption of revision, this replaces smoothness, and add the supplementary conditions.

"$\supseteq$":

For the converse, suppose $d(\sigma',\tau')$ and $d(\sigma'',\tau'')$ are minimal, but $d(\sigma,\tau)$ is not, so $d(\alpha,\beta) < d(\sigma,\tau)$ for some α,β, then $d(\alpha',\beta') < d(\sigma',\tau')$ or $d(\alpha'',\beta'') < d(\sigma'',\tau'')$, contradiction.

□

Corollary 5.2

By Fact 5.1 (page 10), Hamming distances generate decomposable revision operators a la Parikh, see [Par96], also in the generalized form of variable K and ϕ.

Fact 5.3

Let $\mid$ be defined by a Hamming distance, then:

$\Pi \mid \Sigma \subseteq \Gamma \Rightarrow \Pi' \mid (\Sigma \restriction X') \subseteq \Gamma \restriction X'$.

Proof

Let $t \in \Sigma \restriction X' - \Gamma \restriction X'$, we show $t \notin \Pi' \mid (\Sigma \restriction X')$. Let $\tau \in \Sigma$ be s.t. $\tau' = t$, then $\tau \notin \Gamma$ (otherwise $t \in \Gamma \restriction X'$), so $\tau \notin \Pi \mid \Sigma$, so there is $\alpha \in \Pi$, $\beta \in \Sigma$, $d(\alpha,\beta) < d(\sigma,\tau)$ for all $\sigma \in \Pi$. If $d(\sigma',\tau')$ were minimal for some σ, then we would consider $\sigma' \circ \alpha''$, $\tau' \circ \beta''$, and have $\tau' \circ \beta'' \in \Pi \mid \Sigma$, so $\tau' \circ \beta'' \in \Gamma$, and $t \in \Gamma \restriction X'$, contradiction.

□

5.1 Discussion of representation

It would be nice to have a representation result like the one for Hamming relations. But this is impossible, for the following reason:

In constructing the representing distance from revision results, we made arbitrary choices (see the proofs in [LMS01] or [Sch04]). I.e., we choose sometimes arbitrarily $d(x,y) \leq d(x',y')$, when we do not have enough information to decide. (This is an example of the fact that the problem of "losing ignorance" should not be underestimated, see e.g. [GS08f].) As we do not follow the same procedure for all cases, there is no guarantee that the different representations will fit together.

Of course, it might be possible to come to a uniform choice, and one could then attempt a representation result. This is left as an open problem.

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