Embedded Systems Department, ESIEE Paris, University of Paris East

Trading Quantization Precision for Sampling Rates in Systems with Limited Communication in the Uplink Channel

Technical Report 1-06 / 2009

Mohamed El Mongi Ben Gaid and Arben Çela

Emails : mongi.ben-gaid@ifp.fr and a.cela@esiee.fr

Abstract

There are many situations where distributed control applications have to exchange information through limited bandwidth communication channels. Examples may be found in areas ranging from the underwater robotics to the control of satellites clusters. Bandwidth limitations affect the behavior of these systems. For that reason, there is a strong need for developing methods that maximize the relevancy of the exchanged control information. In general, increasing control inputs update frequency improves the disturbance rejection abilities whereas increasing their quantization precision improves the steady state performance (set point tracking precision for example). However, when the bandwidth is limited, increasing the update frequency necessitates the reduction of the quantization precision and vice versa. Motivated by these observations, and focusing on the uplink bandwidth limitations, an approach for the dynamical on-line state-feedback assignment of control inputs quantization precision and update rate is proposed. This approach, which is based on the model predictive control (MPC) technique, enables to choose the update rate and the quantization levels of control signals from a predefined set, in order to optimize the control performance. A heuristic approach allowing an efficient choice of the elements of this set is proposed. Practical stability properties of the approach are then studied. Finally, the effectiveness of the proposed method is illustrated on a simulation example.

Keywords: Quantized systems, networked control systems, model predictive control, practical stability.

Contents

1 Introduction

Recent years have seen a considerable development in microelectronics. Many small-size and low cost sensors and actuators, based on the micro-electro-mechanical systems (MEMS) technology have emerged. The price of the hardware components has continuously decreased to make them more affordable. This impacted the price of embedded processors, such as digital signal processors (DSPs), as well as wired and wireless communication networks hardware components. Consequently, designers have an increasing choice of inexpensive, ubiquitous and pervasive means of communication, computation, sensing and actuation, offering considerable opportunities for the development of new distributed control applications.

These developments make it more and more important to study the interplay between control theory and information theory, especially when the communications aspects influence the behavior of the controlled dynamical systems. These communication aspects that influence the control may be classified into four categories: bandwidth limitations [10, 13, 15, 18, 27, 30, 33], medium access constraints [6, 11, 22, 25], transmission delays [20, 26, 34] and information loss [28].

Bandwidth limitations affect many areas such as embedded systems [1], formations control [2], underwater robotics [29] or large arrays of MEMS [8]. However, the majority of the works studying the problem of the control over limited bandwidth channels have been focused on stability issues in presence of quantization and information limitations. Fewer works addressed the problems of the performance in presence of quantization [19, 24, 32].

The proposed approach is also motivated by practical issues. In fact, as shown in [17], improving the efficiency of a real-time communication protocol amounts to sending messages with a “maximized” data field. In fact, in all communication networks, protocol frames contain fields with fixed and incompressible length. Gathering the data in the same message leads to a more efficient use of the communication resources, since it helps avoiding the fragmentation of the data into several messages and prevents the resulting overhead. This may be done using a “smart” quantization of the control information.

This paper studies the problem of the control over limited bandwidth communication channels and focuses on practical stability and performance aspects in presence of information limitations. It extends the preliminary results exposed in [4]. The communication constrains are modeled at the bit level, in bits per second. Using this modeling, we have to determine the control inputs that have to be updated as well as their quantization precision. Naturally, handling dynamically the quantization precision requires some communication resources and some extra computational resources. Consequently, we have to jointly handle the computational complexity, the protocol bandwidth consumption and performance improvements. In order to limit the inherent complexity of the proposed protocol, we suppose that the quantization precision choices of the input control signals belong to a reduced finite set. At each sampling period, quantization possibilities may be chosen from this set. We propose a methodology allowing the construction of this set, in order to ensure the stability and to comply with the computational requirements. Our approach contrasts with the approach of [18], where the quantization precision of control signals is fixed. The proposed approach aims to capture the intuitive idea that high sampling rates improve the disturbance rejection and the transient behavior whereas the fine quantization improves the static precision near the origin [15]. At the same time, it allows to dynamically choose the pertinent control information to send, knowing the plant state and subject to the communication limitations [5, 6].

This paper is organized as follows. Section 2 describes the considered model of a control system with communication constraints affecting the transmission of the control inputs to the actuators. In Section 3, we describe the basic static control and communication scheme, and investigate its practical stability properties. Then, in Section 4, the MPC algorithm, allowing the dynamical assignment of control signals quantization precision and update rate is introduced and studied. Finally, in Section 5, simulation results are presented in order to illustrate the effectiveness of the proposed approach.

2 Problem setting

Consider the discrete-time LTI system described by the state equation

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where $x(k) \in \mathbb{R}^n$ and $u(k) \in \mathbb{R}^m$. We assume that the pair (A, B) is reachable and that the full state vector $x(k)$ is available to the controller at each sampling period.

The controller is connected to the actuators of the plant through a limited bandwidth communication channel. At each sampling period, at most R bits can be sent to the actuators through the communication channel. The considered communication scheme is described in Figure 1. The control inputs, which are computed by the controller, need to be properly en-

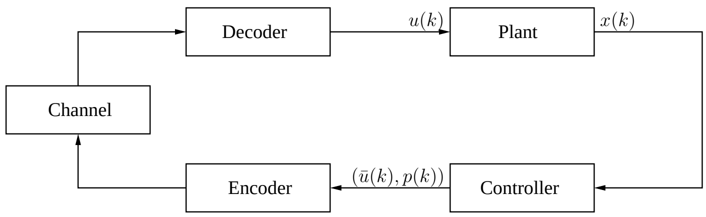

Figure 1: Information pattern

coded before their transmission over the network. This function is performed by the encoder, which converts these control inputs into a sequence of binary symbols. The transmitted information is then decoded by the decoder and applied to the inputs of the plant. We assume that the inputs of the plant are subject to saturation constraints at the actuators, which are defined by

$$-U_i \leq u_i(k) \leq U_i, \text{ where } U_i > 0 \text{ and } i = 1, \ldots, m. \tag{2}$$

In this paper, $\|.\|$ denotes a given matrix norm, $0_{n,m}$ represents the $n \times m$ matrix whose elements are equal to zero and I_n the $n \times n$ identity matrix.

Remark 1 (Down-link resources limitations). The literature addressing the problem of control under communication constraints mainly considered the down-link resources limitations, rather than the up-link. For that reason, our contribution may be seen as complementary approach. For its practical implementation on systems with both down-link and up-link limitations, the previously proposed methods in the literature may be used in order to obtain a state estimation at the controller, which could be subsequently used in our proposed approach.

2.1 Quantization aspects

Quantization is the process of approximating a continuous range of values into a finite set of discrete values, called *reconstruction levels* (or *quantization levels*). In this paper, the quantization is performed using *mid-tread uniform quantizers*, which are characterized by an odd number of reconstruction levels (which include the value of zero). Let u be a bounded continuous scalar signal verifying

$$|u| \leq U, U \in \mathbb{R}_{+}^{*}. \tag{3}$$

The set of reconstruction levels of the mid-tread uniform quantizer which may be encoded using M bits ($M > 1$), given the lower and upper bounds $-U$ and $+U$, is defined by the set-valued function

$$\mathbb{U}(M,U) = \left\{-U + lL(M,U),\ l = 0,\ldots,2^M - 2\right\}, \tag{4}$$

where l is the *quantization index* and $L(M,U)$ defined by

$$L(M,U) = \frac{U}{2^{M-1}-1} \tag{5}$$

is the *quantization step size*.

Remark 2 (Odd number of quantization levels). In the definition of the set valued function $\mathbb{U}(M,U)$, we have intentionally chosen to obtain an odd number of reconstruction levels $0,\ldots,2^M-2$ instead of having an even number of reconstruction levels $0,\ldots,2^M-1$. This choice was motivated by the need of integrating the zero reconstruction level, which have a particular signification, from the control point of view.

Given $M \in \mathbb{N}^*$ such that $M > 1$ and $U \in \mathbb{R}_{+}^{*}$, the quantizer $\mathcal{Q}_{(M,U)}$ is defined by

$$\begin{array}{rcl} \mathcal{Q}_{(M,U)} : \mathbb{R} & \longrightarrow & \mathbb{U}(M,U) \\ u & \longmapsto & c, \end{array}$$

such that

$$c = \mathcal{Q}_{(M,U)}(u) = \begin{cases} +U & \text{if } u > U, \\ -U & \text{if } u \leq -U, \\ -U + \left\lfloor \frac{u+U}{L(M,U)} + \frac{1}{2} \right\rfloor L(M,U) & \text{if } -U < u \leq U. \end{cases}$$

The quantizer $\mathcal{Q}_{(M,U)}$ uniquely associates to a real scalar $u \in \mathbb{R}$ a nearest neighbor $c \in \mathbb{U}(M,U)$.

Remark 3. (Quantization when $M = 1$) When $M = 1$, the expression (5) is not defined. For the sake of simplicity in the following discussions, we will assume that when $M = 1$, for all $u \in \mathbb{R}$, $\mathcal{Q}_{(1,U)}(u) = 0$.

2.2 Information pattern

The number of reconstruction levels of the components of a given control input may vary over the time according to an optimization policy (which will be developed in Section 0.4). Consequently, it is necessary for the decoder to identify how to reconstruct the different components of the control input of the plant from the received binary symbols. To this end, the *precision vector* $p(k) \in \mathbb{N}^m$ is introduced. The i^{th} component $p_i(k)$ of the precision vector $p(k)$ describes the number of bits that are required to encode all the reconstruction levels of the quantized control signal $u_i(k)$ (using the quantizer $\mathcal{Q}_{(p_i(k),U_i)}$). A compact representation of the precision vector has to be sent to the decoder, together with the control information. To minimize the necessary decoding information (which is described by $p(k)$), and to be able to produce compact representations of $p(k)$, the set $\mathcal{P}$ of possible values of $p(k)$ should contain a limited number of elements.

Let R_I and R_D be the number of bits which are used to encode respectively the control information and the decoding information. The communication constraints impose

$$R_I + R_D = R. \tag{6}$$

Then $\mathcal{P}$ must verify

$$\mathcal{P} \subseteq \left\{ p \in \mathbb{N}^m \text{ such that } \sum_{i=1}^{m} p_i = R_I \right\}, \tag{7}$$

and

$$|\mathcal{P}| \leq 2^{R_D}, \tag{8}$$

where $|\mathcal{P}|$ denotes the cardinality of $\mathcal{P}$.

2.3 Notion of quantization sequence

The notion of communication sequence was introduced by Brokett [11] and generalized by Hristu [21] in order to quantify the notion of *attention* [12]. A communication sequence describes, at each sampling period, which control inputs of the system are updated, assuming an infinite quantization precision, which represents an idealized situation.

Definition 1 (Periodic communication sequence [21])**.** A periodic communication δ^{T-1} sequence of period T and width m is an infinite sequence $(\delta(0), \ldots, \delta(T-1), \ldots)$ of elements of $\{0, 1\}^m$ verifying $\forall i \in \mathbb{N}$, $\delta(k + iT) = \delta(k)$. A periodic communication sequence is fully characterized by the sequence $\delta^{T-1} = (\delta(0), \ldots, \delta(T-1))$ corresponding to the first period.

The notion of communication sequence is well suited to model the problems of medium access arbitration, when quantization aspects are disregarded. In this paper, since quantization is a main concern, we propose a generalization of the notion of communication sequence, to take into account quantization aspects. This leads to the notion of quantization sequence.

Definition 2 (Periodic quantization sequence)**.** A periodic quantization sequence s^{T-1} of period T, width m and maximal precision M is an infinite sequence $(s(0), \ldots, s(T-1), \ldots)$ of elements of $\{0, \ldots, M\}^m$ verifying $\forall i \in \mathbb{N}$, $s(k + iT) = s(k)$. A periodic quantization sequence is fully characterized by the sequence $s^{T-1} = (s(0), \ldots, s(T-1))$ corresponding to the first period.

***Example* 1.** The sequence

$$s^2 = \left(\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \ldots \right)$$

is a periodic quantization sequence of period $T = 3$, width $m = 3$ and maximal precision $M = 4$.

In this paper, precision vectors are restricted to belong to the set $\mathcal{P}$. For that reason, we will only consider quantization sequences whose components are precision vectors in $\mathcal{P}$. These particular periodic quantization sequences are called *admissible periodic quantization sequences*. Since $\mathcal{P}$ contains only precision vectors that respect the communication constraints, an admissible periodic quantization sequence implicitly respects the required communication constraints.

Remark 4 (Communication delays and packet-dropouts issues). The proposed approach could be easily extended to take into account packet-dropouts which are known in advance (and which may be easily modeled using the notion of quantization sequence). However, the problem of the quantized control with unknown packet loss and communication delays is challenging and is out of the scope of this contribution.

2.4 Performance index definition

The performance of the controlled system (1) is evaluated using a quadratic cost function, which may be seen as the design specification of its ideal controller.

$$J(x, u, 0, N) = \sum_{k=0}^{N-1} \ell(x(k), u(k)). \tag{9}$$

where N is the final time, and

$$\ell(x(k), u(k)) = \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^T \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix},$$

and Q_1, Q_2 and Q_{12} are respectively $n \times n$, $m \times m$ and $n \times m$ matrices. Let

$$Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix}.$$

In the following, we assume that Q is positive definite.

3 Static strategy

3.1 Algorithm description

Let s^{T-1} be an admissible T-periodic quantization sequence. Let Λ be the function defined by

$$\begin{array}{rccl} \Lambda : & \{0, \ldots, M\}^m & \longrightarrow & \{0, 1\}^m \\ & s & \longmapsto & \delta, \end{array}$$

such that

$$\begin{cases} \delta_i = 1 & \text{if } s_i \neq 0 \\ \delta_i = 0 & \text{if } s_i = 0. \end{cases}$$

Let $L_s^{T-1} = (L_s(0), \dots, L_s(T-1))$ be the periodic sequence of control gains characterizing the optimal T-periodic controller of system (1) if the communication constraints are described by the periodic communication sequence $\delta^{T-1} = (\delta(0), \dots, \delta(T-1)) = (\Lambda(s(0)), \dots, \Lambda(s(T-1)))$. In this situation, the optimal controller, taking into account the medium access constraints modeled by δ^{T-1}, is the state feedback control law

$$u(k) = L_s(k)\tilde{x}(k), \tag{10}$$

where

$$\tilde{x}(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}.$$

The control gains sequence L_s^{T-1} may be derived using the approach described in [6].

Remark 5 (Control gains and communication constraints). The control gains $L_s(k)$ are designed to take into account the communication constraints $\delta_i(k) = 0 \Longrightarrow u_i(k) = u_i(k-1)$ (i.e. if $\delta_i(k) = 0$, then the i^{th} line of $L_s(k)$ verifies $L_{s_{i,n+i}}(k) = 1$ and $L_{s_{i,j}}(k) = 0$ for $j \neq n+i$).

To each admissible T-periodic quantization sequence s^{T-1}, a periodic control gains sequence L_s^{T-1}, derived as mentionned previously, may be associated. A simple approach for controlling the system (1) is to use the following control algorithm

$$\begin{aligned} &p(k) = s(k) \\ &v(k) = L_s(k)\tilde{x}(k) \\ &u_i(k) = \mathcal{Q}_{(p_i(k),U_i)}\left(v_i(k)\right) \text{ if } p_i(k) \neq 0 \\ &u_i(k) = u_i(k-1) \text{ if } p_i(k) = 0. \end{aligned} \tag{11}$$

This algorithm is called static strategy, because it is based on a fixed periodic quantization sequence, in opposition to the adaptive strategy, that will be described in the Section 0.4.

3.2 Practical stabilization using the static strategy

In this subsection, we are interested in the practical stability properties of the static strategy. Since the control inputs can only take a finite number of values, achieving the asymptotic stability is practically impossible. In fact, in its seminal paper [14], Delchamps has proved that when the input of the system passes through a quantizer having a finite number of quantization levels, then the set of trajectories that correspond to the asymptotic stability has Lebesgue measure zero. This fundamental result motivates the use of practical stability for the studied problem. In the following, the main definitions that are related to the stability notions that will be considered in this paper are reviewed.

Definition 3 ((W, V)-stability [16]). Let V and W be two compact sets of $\mathbb{R}^{n+m}$ containing the origin in their interiors and such that $V \subseteq W$. System (1)(2)(11) is called (W, V)-*stable* if

- W is positively invariant for system (1)(2)(11),
- for all $\tilde{x}(0) \in W$, there exists k_0 (function of $\tilde{x}(0)$) such that any state trajectory of (1)(2)(11) with initial condition $\tilde{x}(0)$ satisfies $\tilde{x}(k) \in V$ for all $k \geq k_0$.

Furthermore, system (1)(2)(11) is called (W, V)-*stable in* $\mathcal{N}$-*steps* if $k_0 \leq \mathcal{N}$.

Assume that system (1), is uniformly exponentially stable using the control law (10). Let $\eta > 0$ and $\mathcal{B}_\eta$ the ball defined by

$$\mathcal{B}_\eta = \left\{\tilde{x} \in \mathbb{R}^{n+m}, \text{such that } \|\tilde{x}\| \leq \eta\right\}. \tag{12}$$

When system (1)(10) is uniformly exponentially stable, a question that arises is whether it is possible to guarantee the ultimate boundedness of the trajectories of system (1)(2)(11) to any desired final set $\mathcal{B}_\eta$, if its inputs are quantized with a sufficiently high precision. However, due to the possibility of saturation of the control inputs, this ultimate boundedness property may not be verified if the initial extended state is situated in some regions of $\mathbb{R}^{n+m}$, where even the non quantized system (1)(2)(10) subject to control inputs saturations may not be stabilizable. For that reason, and due to these saturation constraints that are considered in the proposed problem formulation, the latter question is reformulated in terms of (W,V)-stability. Consider the following perturbed system

$$x(k+1) = Ax(k) + Bu(k) + Bw(k). \tag{13}$$

Assume that the disturbance $w(k)$ belongs to a convex and compact set $\mathcal{W}$. The disturbance $w(k)$ may be seen as a model of the quantization error and its effect on the system.

Several methods have been proposed in the literature for computing robustly positively invariant sets, for example [9, 23, 31]. Let $\mathcal{F}$ be the set of states that do not lead to the saturation of the control commands using the control law (10). The set $\mathcal{F}$ is defined by

$$\mathcal{F} = \left\{\tilde{x} \in \mathbb{R}^{n+m} \text{ such that } \forall k \in \{0,\ldots,T-1\},\, L_s(k)\tilde{x} \in \prod_{i=1}^{m}[-U_i, U_i]\right\}. \tag{14}$$

Let $\mathcal{G}_\gamma$ the greatest convex and compact set in $\mathcal{F}$ that is robustly positively invariant for the perturbed system (13)(2)(10) and for bounded disturbances $w(k)$ belonging to $\mathcal{W}_\gamma$ defined by

$$\mathcal{W}_\gamma = \{w(k) \in \mathbb{R}^m \text{ such that for all } k \in \mathbb{N}, \|w(k)\| \leq \gamma\}. \tag{15}$$

Under these assumptions, the set $\mathcal{G}_\gamma$ is positively invariant for system (1)(2)(11) if the "quantization error" is less than or equal to γ. The $(\mathcal{G}_\gamma, \mathcal{B}_\eta)$-stability of system (1)(2)(11) is addressed in the following theorem.

Theorem 1 ((W,V)-stability of the static strategy)**.** *For all $\varepsilon > 0$, if system (1)(10) is uniformly exponentially stable, and the set $\mathcal{G}_{\frac{\varepsilon}{\varphi}}$ (as previously defined) is robustly positively invariant for the system(13)(2)(10), where φ is a constant that depends on the plant model (1), the controller (10) and ε, then there exists $p_0 \in \mathbb{N}$ such that system (1)(2), controlled by the control law (11) that is based on the periodic quantization sequence $p_0 \times s^{T-1}$ and the periodic control gains sequence L_s^{T-1} is $(\mathcal{G}_{\frac{\varepsilon}{\varphi}}, \mathcal{B}_\varepsilon)$-stable.*

Proof. Let $\gamma > 0$. Let $\tilde{x}(0) \in \mathcal{G}_\gamma$ be the initial condition of system(1)(2)(11) and $\tilde{x}(k)$ its extended state at instant k. Assume first that $\mathcal{G}_\gamma$ is robustly positively invariant for the system(13)(2)(10). Let $p \in \mathbb{N}$ and $v(k) = L_s(k)\tilde{x}(k)$. The quantization error $e^{(p)}(k)$ is defined for $i \in \{1,\ldots,m\}$ by

$$e_i^{(p)}(k) = \begin{cases} \mathcal{Q}_{(p\times s_i(k),U_i)}(v_i(k)) - v_i(k) & \text{if } s_i(k) \neq 0, \\ 0 & \text{if } s_i(k) = 0. \end{cases}$$

Let

$$\bar{A}(k) = \begin{bmatrix} A & 0_{n,m} \\ 0_{m,n} & 0_{m,m} \end{bmatrix} + \begin{bmatrix} BL_s(k) \\ L_s(k) \end{bmatrix}.$$

At a given discrete instant k, the extended state $\tilde{x}(k)$ of system(1)(11) verifies

$$\tilde{x}(k) = \Phi(k,0)\tilde{x}(0) + \sum_{i=0}^{k-1} \Gamma(k,i)e^{(p)}(i), \tag{16}$$

where

$$\Phi(k,i) = \begin{cases} \prod_{j=1}^{k-i} \bar{A}(k-j) & \text{if } k > i, \\ I_{n+m} & \text{if } k = i, \end{cases} \tag{17}$$

and

$$\Gamma(k,i) = \begin{cases} \left(\prod_{j=1}^{k-i-1} \bar{A}(k-j)\right) B & \text{if } i < k-1, \\ B & \text{if } i = k-1, \\ 0_{n,m} & \text{if } i = k. \end{cases} \tag{18}$$

The norm $\|.\|$ being a matrix norm, we may write

$$\|\tilde{x}(k)\| \leq \|\Phi(k,0)\| \, \|\tilde{x}(0)\| + \sum_{i=0}^{k-1} \|\Gamma(k,i)\| \left\| e^{(p)}(i) \right\|. \tag{19}$$

Since $\mathcal{G}_\gamma \subseteq \mathcal{F}$ and $\mathcal{G}_\gamma$ is positively invariant for system (1)(2)(11), then the control inputs corresponding to any state trajectory starting from $\tilde{x}(0) \in \mathcal{G}_\gamma$ never saturate. Consequently, the quantization error converges to zero as p tends to infinity. Hence, for any desired worst-case quantization error γ, there exists $p_0 \in \mathbb{N}$ such that

$$\forall p \geq p_0, \forall k \in \mathbb{N}, \left\| e^{(p)}(k) \right\| \leq \gamma.$$

For $p \geq p_0$,

$$\|\tilde{x}(k)\| \leq \|\Phi(k,0)\| \, \|\tilde{x}(0)\| + \sum_{i=0}^{k-1} \|\Gamma(k,i)\| \, \gamma. \tag{20}$$

Let ε' sufficiently small and verifying $0 < \varepsilon' < \varepsilon$. Let $\mathcal{M} = \max_{\tilde{x} \in \mathcal{F}} \|\tilde{x}\|$. Since system (1)(10) is uniformly exponentially stable, then $\Phi(k,0)$ converges exponentially to zero as k tends to infinity. Consequently, there exists $k_0^{'} \in \mathbb{N}$ such that $\forall k \geq k_0^{'}$, $\|\Phi(k,0)\| \leq \frac{\varepsilon'}{2\mathcal{M}}$. Furthermore, we have

$$\begin{aligned} \sum_{i=0}^{k} \|\Gamma(k,i)\| &= \|\Phi(k,1)B\| + \|\Phi(k,2)B\| + \ldots + \|\Phi(k,k)B\| + \|B\| \\ &\leq (\|\Phi(k,1)\| + \|\Phi(k,2)\| + \ldots + \|\Phi(k,k)\| + 1) \, \|B\|. \end{aligned} \tag{21}$$

The uniform exponential stability of system (1)(10) implies that there exists two positive constants c and $\beta < 1$ such that for all i

$$\|\Phi(k,i)\| \leq c\beta^{k-i} \text{, for } k \geq i.$$

Consequently

$$\sum_{i=0}^{k} \|\Gamma(k,i)\| \leq \|B\| (1 + c\sum_{i=0}^{k} \beta^{k-i}) = \|B\| (1 + c\sum_{i=0}^{k} \beta^{i}) = \|B\| (1 + c\frac{1-\beta^{k+1}}{1-\beta}).$$

Since the sequence $d_k = \sum_{i=0}^{k} \|\Gamma(k,i)\|$ is increasing and upper bounded by the convergent sequence $\|B\| (1 + c\frac{1-\beta^{k+1}}{1-\beta})$, then it is convergent too. Let

$$\varphi' = \lim_{k\to+\infty} \sum_{i=0}^{k} \|\Gamma(k,i)\| .$$

It follows that there exists $k_0^{''} \in \mathbb{N}$ such that $\forall k \geq k_0^{''}, \left|\sum_{i=0}^{k} \|\Gamma(k,i)\| - \varphi'\right| \leq \frac{\varepsilon'}{2\gamma}$. Let $k_0 = max(k_0^{'}, k_0^{''})$. Consequently,

$$\forall k \geq k_0, \|\tilde{x}(k)\| \leq \frac{\varepsilon'}{2\mathcal{M}} \|\tilde{x}(0)\| + (\frac{\varepsilon'}{2\gamma} + \varphi')\gamma \leq \varepsilon' + \varphi'\gamma. \tag{22}$$

Chossing $\varphi = \frac{\varphi'\varepsilon}{\varepsilon-\varepsilon'}$ and $\gamma = \frac{\varepsilon}{\varphi}$, the theorem is proved. □ □

Theorem 1 states conditions that guarantee the positive invariance of the extended state as well its the ultimate boundedness to a given ball including the origin, using the static strategy.

4 Model predictive control

4.1 Algorithm description

Assume that a set $\mathcal{C}$ of admissible periodic quantization sequences is defined (a method for the construction of this sequence will be presented in Section 4.2). Then, the problem of the integrated control and communication may be solved on-line using the model predictive control algorithm. The model predictive control is an elegant solution to tackle the hybrid aspect of the considered model, where both the control inputs and the quantization frequency decisions need to be determined at each sampling period. Model predictive control was successfully applied to the control of hybrid systems [3] and to the problems of integrated control and medium access allocation [6].

Using the MPC approach, an optimization problem is solved at each sampling period, in order to determine both the control inputs of the plant $u(k)$ and their quantization precision

$p(k)$. This problem is formulated as in the set of equations (23) below.

$$\begin{cases} \min_{s^{T-1}\in\mathcal{C}} \sum_{h=0}^{N-1} \ell\left(\hat{x}(k+h|k), \hat{u}(k+h|k)\right) \\ \text{subject to} \\ \hat{x}(k|k) = x(k) \\ \hat{u}(k-1|k) = u(k-1) \\ \text{and for all } h \in \{0,\dots,N-1\}, \text{ for all } i \in \{1,\dots,m\} \\ \hat{p}(k+h|k) = s(h) \\ \hat{v}(k+h|k) = L_{s^{T-1}}(h)\left[\hat{x}^T(k+h|k)\ \ \hat{u}^T(k+h-1|k)\right]^T \\ \hat{u}_i(k+h|k) = \mathcal{Q}_{(\hat{p}_i(k+h|k),U_i)}\left(\hat{v}_i(k+h|k)\right) \text{ if } \hat{p}_i(k+h|k) \neq 0 \\ \hat{u}_i(k+h|k) = \hat{u}_i(k+h-1|k) \text{ if } \hat{p}_i(k+h|k) = 0 \\ \hat{x}(k+h+1|k) = A\hat{x}(k+h|k) + B\hat{u}(k+h|k) \end{cases} \tag{23}$$

Using this approach, the control performance $J(\hat{x}, \hat{u}, k, N+k)$ over a horizon of N (which is assumed to be a multiple of T) is predicted, for the $|\mathcal{C}|$ communication sequences of $\mathcal{C}$. $\hat{x}(k+h|k)$ and $\hat{u}(k+h|k)$ constitute the predicted values of $x(k+h)$ and $u(k+h)$, for a given quantization sequence s^{T-1} and associated control gains sequence $L^{T-1}_{s^{T-1}}$. $\hat{p}(k+h|k)$ represents the quantization precision of the predicted inputs $\hat{u}(k+h|k)$. If $\hat{p}_i(k+h|k) = 0$, then the predicted control input $\hat{u}(k+h|k)$ cannot be updated. Consequently, its previous value $\hat{u}(k+h-1|k)$ will be maintained. Control inputs $u_i(k+h|k)$ which have to be updated (i.e. whose precision vectors satisfy $\hat{p}_i(k+h|k) \neq 0$) are computed by the application of the quantization map $\mathcal{Q}$ to the i^{th} element of the result of the state feedback operation $\hat{v}(k+h|k) = L_{s^{T-1}}(h)\left[\hat{x}^T(k+h|k)\ \ \hat{u}^T(k+h-1|k)\right]^T$. The solution of this optimization problem is the admissible periodic quantization sequence $s^{T-1^*} = (s^*(0), \dots, s^*(T-1))$ that minimizes the cost function $J(\hat{x}, \hat{u}, k, k+N)$ subject to the communication constraints. According to the MPC philosophy, the precision vector and control inputs at instant k are respectively given by

$$p(k) = \hat{p}^*(k|k) = s^*(0), \tag{24}$$

and

$$\begin{cases} u_i(k) = \hat{u}_i^*(k|k) = \mathcal{Q}_{(p_i(k),U_i)}\left(\hat{v}_i^*(k|k)\right) & \text{if } p_i(k) \neq 0 \\ u_i(k) = u_i(k-1) & \text{if } p_i(k) = 0, \end{cases} \tag{25}$$

where

$$\hat{v}^*(k|k) = L_{s^{T-1^*}}(0)\tilde{x}(k). \tag{26}$$

In the following, we will represent the model predictive control law (23)(24)(26)(25), when based on a given set of admissible periodic quantization sequences $\mathcal{C}$, by the function $\kappa_{\mathcal{C}}\left(\tilde{x}(k)\right)$ of the extended state $\tilde{x}(k)$.

The choice of the set $\mathcal{C}$ plays an important role in ensuring the practical stability and performance improvements. In the following, a heuristic method for choosing the elements of the set $\mathcal{C}$ is proposed. To simplify the notation, the cost function $J(\hat{x}, \hat{u}, k, k+N)$ will be simply denoted by $J(\tilde{x}, s)$, where s is the quantization sequence that is used (as well as its associated control gains sequence) to evaluate the predicted values $\hat{x}$ and $\hat{u}$. It may be also denoted as $J(\tilde{x}(k), \mathcal{U}(k))$, where $\mathcal{U}(k) = (\hat{u}(k|k), \dots, \hat{u}(k+N-1|k))$ is a sequence of control inputs that determine the evolution of the system from state $\tilde{x}(k)$ and the associated cost.

4.2 A heuristic approach for the choice of quantization sequences

Action domains

In order to simplify the presentation, and without loss of generality, we assume that

$$R_I = bM, \tag{27}$$

where M represents the maximal precision of the digital-to-analog converters, generally considered as the "infinite precision", and b is the maximal number of control inputs that may be sent with the maximal precision M, during the sampling period. Let $a^{1,Q} = (a_1, \ldots, a_Q)$ be a sequence of increasing integers such that $a_1 = 1$ and $\forall i \in \{1, \ldots, Q\}$, a_i divides M. Let $\mathcal{S}(m, M)$ be the set of periodic quantization sequences of width m and maximal precision M. Based on the sequence $a^{1,Q}$, it is possible to characterize Q remarkable sub-sets of $\mathcal{S}(m, M)$. These particular subsets are called *action domains*. Formally, the l^{th} action domain $\mathcal{A}_l, l \in \{1, \ldots, Q\}$ is the set of all quantization sequences $s^{T_l-1} = (s(o), \ldots, s(T_l - 1))$ such that $\forall i \in \{1, \ldots, m\}, \forall k \in \{0, \ldots, T_l - 1\}$, $s_i(k) \in \{\frac{M}{a_l}, 0\}$. This means that in the l^{th} action domain, $a_l b$ inputs may be updated with the precision $\frac{M}{a_l}$. The first action domain $\mathcal{A}_1$ represents the quantization sequences that update the minimal number of plant inputs with the maximal precision M. The last action domain $\mathcal{A}_Q$ represents the set of quantization sequences that update the maximal number of plant inputs with the minimal precision $\frac{M}{a_Q}$.

Basic sequences

Let $\Gamma^*(r)$ be an optimal offline periodic communication sequence that minimizes the $\mathcal{H}_2$ norm of the system, assuming that at most r control inputs may be updated during the sampling period with an infinite quantization precision. An efficient method for obtaining optimal and suboptimal solutions (with a guaranteed error bounds) to this optimization problem, using the branch and bound algorithm, was proposed in [7]. The generated optimal offline communication sequences, using this method, are only dependent on the intrinsic characteristics of the system. In order to further reduce the computational requirements of the algorithm, the search is restricted to the most relevant quantization sequences. To this end, to each action domain, a particular quantization sequence, called *basic sequence*, is assigned. More formally, to action domain A_l, a basic sequence s_B^l is associated according to the following relation

$$s_B^l = \frac{M}{a_l}\Gamma^*(ba_l). \tag{28}$$

Although this assignment is suboptimal since the optimization is performed assuming an infinite precision of the control inputs; it allows assigning the update rate of the different control signals according to the systems's dynamics. In fact, solving offline the problem of finding the optimal quantization sequence in the sense of the $\mathcal{H}_2$ performance index is very complex, since it suffers from the curse of dimensionality. The assumption of an infinite quantization precision considerably reduces this complexity, since convex optimization problems may be used inside the optimization algorithm (i.e. bouding phase of the branch and bound algorithm), instead of searching over all the possible discrete values of the control inputs, which increase exponentially with the quantization precision. In the following, we will denote by $\mathcal{C}_l$ the set of admissible periodic quantization sequences that are obtained by the circular permutation of the basic sequence s_B^l. Unless stated otherwise, the set $\mathcal{C}$ that will be used by the proposed MPC algorithm is defined by

$$\mathcal{C} = \bigcup_{l=1}^{Q} \mathcal{C}_l.$$

4.3 Attraction properties of the MPC

Theorem 1 stated the conditions allowing to guarantee the ultimate boundedness to a given ball including the origin, using the static strategy. Corollary 1 showed that it is possible to refine this neighborhood of the origin if the quantization precision is increased. When the MPC algorithm is used, an interesting question is to determine the conditions under which the MPC will allow to perform the tradeoff between quantization precision and update rates and to "attract" the system to the smallest ball around the origin in steady state, while improving the convergence rate by changing action domains in transient states. The notion of attraction is formalized in the following definition.

Definition 4 ((W,V)-attraction)**.** Let W a compact set and V a set containing the origin in its interior and verifying $W \cap V = \emptyset$. System (1)(2) controlled by a control law $\kappa(\tilde{x}(k))$ is called (W,V)-attractive if for all trajectory starting in W (i.e. verifying $\tilde{x}(0) \in W$), there exists a time instant k_0 such that $\tilde{x}(k_0) \in V$.

A system is called $(W, V)-$attractive when any trajectory starting in W is driven in a finite time to V. A $(W, V)-$stable system is $(W, V)-$attractive, but the reciprocal is not necessarily true. In the following, sufficient conditions under which these attraction properties are verified, are stated. But before stating these results, the following technical lemma is needed.

Lemma 1 (Characterization of the (W,V)-attraction)**.** *Assume that there exists a control law $\kappa(\tilde{x}(k))$, a positive definite function $P(\tilde{x})$ of the extended state of the closed loop system (1)(2) (controlled by κ), and a compact set W that does not contain the origin and such that for all $\tilde{x}(k) \in W$, $\Delta P(\tilde{x}(k)) = P(\tilde{x}(k+1)) - P(\tilde{x}(k)) < 0$. Let $V = \{\tilde{x} \in \mathbb{R}^{n+m} - W$ for which there exists $\tilde{y} \in W$ such that $P(\tilde{x}) < P(\tilde{y})\}$. If $\tilde{x}(0) \in W$, then there exists an instant k_0 such that $\tilde{x}(k_0) \in V$.*

Proof. The proof is performed by contradiction. Let $\tilde{x}(0)$ a given initial state in W. Assume that for all $k \in \mathbb{N}$, $\tilde{x}(k) \in W$. Since W is compact (and consequently closed) and does not contain the origin, then there exists $\delta < 0$ such that $\max_{\tilde{x} \in W} \Delta P(\tilde{x}) = \delta$. Therefore, at any instant k,

$$P(\tilde{x}(k)) = P(\tilde{x}(0)) + \sum_{i=1}^{k} (P(\tilde{x}(i)) - P(\tilde{x}(i-1))) \le P(\tilde{x}(0)) + \delta k.$$

Consequently, when $k \longrightarrow +\infty$, $P(\tilde{x}(k)) \longrightarrow -\infty$, which contradicts the fact that $P(\tilde{x}(k))$ is a positive definite. For that reason, we conclude that there exists a time instant k_0 such that $\tilde{x}(k_0) \notin W$. Since $P(\tilde{x}(k))$ is strictly decreasing for $k \le k_0$, then necessarily $P(\tilde{x}(k_0)) < P(\tilde{x}(0))$. Thus, $\tilde{x}(k_0) \in V$. □ □

Lemma 1 shows that when a positive definite function of the state is strictly decreasing along the trajectories of the system in a region W that does not contain the origin; the state cannot stay indefinitely in this region and must approach the origin. This Lemma will be the basis for proving the attraction properties of the MPC algorithm.

Let $P(\tilde{x}(k)) = J(\tilde{x}(k),\mathcal{U}^*(k))$. The function $P(\tilde{x}(k))$ represents the cost function that is minimized by the MPC algorithm and will play the role of a "Lyapunov function". By construction, P is a positive definite function. We have the following result.

Theorem 2 ((W,V)-attraction of the MPC law $\kappa_{\mathcal{C}_l}$)**.** *Let $\xi > 0$. Let*

$$W = \{\tilde{x} \in \mathbb{R}^{n+m} \text{ such that } P(\tilde{x}) \leq \xi\}$$

and V two compact sets of $\mathbb{R}^{n+m}$ containing the origin in their interiors and such that $V \subseteq W$. Let ϵ be a small positive number, $\bar{W} = \{\tilde{x} \in W$ such that $\ell(x, \kappa_{\mathcal{C}_l}(\tilde{x})) \geq \max_{\tilde{x}_v \in V} \ell(x_v, \kappa_{\mathcal{C}_l}(\tilde{x}_v)) + \epsilon\}$ and $\bar{V} = W - \bar{W}$. If the $(W, \bar{V})$-stability in $\mathcal{N}$-steps of system (1)(2) is ensured by the static strategy based on any quantization sequence $s^l \in \mathcal{C}_l$, and if the horizon N of the MPC algorithm is chosen such that $N > \mathcal{N}$, then the MPC algorithm $\kappa_{\mathcal{C}_l}$ is $(\bar{W}, \bar{V})$-attractive.

Proof. Let $\bar{\eta} = \max_{\tilde{x}_v \in V} \ell(x_v, \kappa_{\mathcal{C}_l}(\tilde{x}_v))$. At time step $k = 0$, the solutions of the open-loop optimal control and communication problem are $\mathcal{U}^*(0) = (\hat{u}^*(0|0), \ldots, \hat{u}^*(N-1|0))$ and s^*. The control input that will be applied to the plant, according to the receding horizon philosophy will be $u(0) = \hat{u}^*(0|0)$. At time step $k = 1$, the sequence $\check{\mathcal{U}}(1) = (\hat{u}^*(1|0), \ldots, \hat{u}^*(N|0))$ is a feasible sequence, where $\hat{u}^*(N|0)$ is defined for $i \in \{1, \ldots, m\}$ by

$$\hat{u}_i^*(N|0) = \begin{cases} \mathcal{Q}_{(s_i^*(N), U_i)}\left(\hat{v}_i^*(N|0)\right) & \text{if } s_i^*(N) \neq 0, \\ \hat{u}_i^*(N-1|0) & \text{if } s_i^*(N) = 0, \end{cases}$$

and

$$\hat{v}(N|0) = L_{s^*}(N)[\hat{x}^*(N|0) \;\; \hat{u}^*(N-1|0)]^T.$$

The associated cost is

$$J(\tilde{x}(1),\check{\mathcal{U}}(1)) = J(\tilde{x}(0),\mathcal{U}^*(0)) - \ell(x(0), u(0)) + \ell(\hat{x}^*(N|0), \hat{u}^*(N|0)). \tag{29}$$

Since the static strategy based on the quantization sequence $s^* \in \mathcal{C}_l$ ensures $(W, \bar{V})$-stability in $\mathcal{N}$-steps of system (1)(2), and knowing that $N > \mathcal{N}$, then $\hat{x}^*(N|0) \in \bar{V}$. Consequently, by construction of $\bar{V}$, $\ell(\hat{x}^*(N|0), \hat{u}^*(N|0)) < \bar{\eta}$. We may write

$$\begin{aligned} P(\tilde{x}(1)) - P(\tilde{x}(0)) &= J(\tilde{x}(1),\mathcal{U}^*(1)) - J(\tilde{x}(0),\mathcal{U}^*(0)) \\ &\leq J(\tilde{x}(1),\check{\mathcal{U}}(1)) - J(\tilde{x}(0),\mathcal{U}^*(0)) \\ &< -\ell(x(0), \kappa_{\mathcal{C}_l}(\tilde{x}(0))) + \bar{\eta}. \end{aligned} \tag{30}$$

This reasoning remains true for all other instants $k > 0$. The positive definite function P associated to the MPC control law is then strictly decreasing when $\tilde{x}(0) \in \bar{W} \subseteq \{\tilde{x} \in \mathbb{R}^{n+m}$ such that $\ell(x, \kappa_{\mathcal{C}_l}(\tilde{x})) > \bar{\eta}\}$. By construction, $\bar{W}$ is compact, does not contain the origin and verifies $\bar{W} \cap \bar{V} = \emptyset$. Consequently, applying the previous lemma, there exists an instant k_0 such that $\tilde{x}(k_0) \notin \bar{W}$ and $P(\tilde{x}(k_0)) < P(\tilde{x}(0)) \leq \xi$, thus, $\tilde{x}(k_0) \in \bar{V}$. □ □

Theorem 2 states the conditions under which any trajectory starting in $\bar{W}$ is driven in finite time to $\bar{V}$, using the MPC law $\kappa_{\mathcal{C}_l}$. Let $\mathcal{G}$ the set of extended states that can be pratically stabilized by at least one static strategy (as defined by the previous heuristic approach). $\mathcal{G}$ is formally defined by

$$\mathcal{G} = \bigcup_{i=1}^{Q} \mathcal{G}_{\frac{M}{a_i}}.$$

In the following, to simplify the notion, let $J(\tilde{x}, l_i) = \min_{s \in \mathcal{C}_i} J(\tilde{x}, s)$. Let $\mathcal{R}_i$ be the region defined by

$$\mathcal{R}_i = \{\tilde{x} \in \mathcal{G} \text{ such that for all } j > i,\ J(\tilde{x}, l_i) < J(\tilde{x}, l_j)\} .$$

It is easy to see that regions $\mathcal{R}_i$ ($i \in \{1, \ldots, Q\}$), form a partition of $\mathcal{G}$. An interesting question is whether the MPC algorithm $\kappa_{\mathcal{C}}$ is able to ensure the attraction of any trajectory starting in $\mathcal{G}$ to a desired ball X_f. Theorem 3 provides sufficient conditions to ensure this property. To simplify the notation, let $\mathcal{R}_0 = X_f$.

Theorem 3 (Convergent switching between action domains in the MPC law $\kappa_{\mathcal{C}}$)**.** *If $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_Q$ and if the MPC algorithm $\kappa_{\mathcal{C}_l}$ ensures $(\mathcal{R}_l, \mathcal{R}_{l-1})$-attraction of system (1)(2) for $1 \le l \le Q$, then system (1)(2), controlled by the MPC control law $\kappa_{\mathcal{C}}$ is $(\mathcal{G}, X_f)$-attractive.*

Proof. $(\mathcal{R}_1, \ldots, \mathcal{R}_Q)$ form a partition of $\mathcal{G}$. For that reason, there exists $l \in \{1, \ldots, Q\}$ such that the extended state $\tilde{x}$ belongs to region $\mathcal{R}_l$. Consequently, by construction of the MPC algorithm $\kappa_{\mathcal{C}}$, as long as the extended state $\tilde{x}$ belongs to $\mathcal{R}_l$, the MPC algorithm $\kappa_{\mathcal{C}_l}$ will be applied, until the extended state reaches a region $\mathcal{R}_j$ with $j \neq l$. The $(\mathcal{R}_l, \mathcal{R}_{l-1})$-attraction implies that this region is necessarily included in $\mathcal{R}_{l-1}$. Consequently, $j < l$. Applying recursively the same reasoning to the region $\mathcal{R}_j$, the theorem is proved. □ □

In simulation, it is observed that when the prediction horizon is sufficiently large, adding more elements to $\mathcal{C}$ leads to a better control performance. However, if the set $\mathcal{C}$ is reduced to $\mathcal{C} = \{s_B^1, \ldots, s_B^Q\}$, proving the (W,V)-stability becomes easy, as stated in the following result.

Corollary 1 (Convergent switching between action domains in $\kappa_{\mathcal{C}}$ for $\mathcal{C} = \{s_B^1, \ldots, s_B^Q\}$)**.** *If $\mathcal{R}_1 \subset \mathcal{R}_2 \subset \ldots \subset \mathcal{R}_Q$ and if the static strategy s_B^l ensures $(\mathcal{R}_l, \mathcal{R}_{l-1})$-stability of system (1)(2) for $1 \le l \le Q$, then system (1)(2), controlled by the MPC control law $\kappa_{\mathcal{C}}$ for $\mathcal{C} = \{s_B^1, \ldots, s_B^Q\}$ is $(\mathcal{G}, X_f)$-stable.*

The proof may be easily established following the same lines as the previous one.

5 Simulation results

Consider the continuous-time LTI system defined by the state and input matrices

$$A_c = \begin{bmatrix} A_{c_s} & 0 & 0 & 0 \\ 0 & A_{c_s} & 0 & 0 \\ 0 & 0 & A_{c_s} & 0 \\ 0 & 0 & 0 & A_{c_s} \end{bmatrix} \text{ and } B_c = \begin{bmatrix} B_{c_s} & 0 & 0 & 0 \\ 0 & B_{c_s} & 0 & 0 \\ 0 & 0 & B_{c_s} & 0 \\ 0 & 0 & 0 & B_{c_s} \end{bmatrix},$$

where

$$A_{c_s} = \begin{bmatrix} 0 & 110 \\ -900 & 10 \end{bmatrix} \text{ and } B_{c_s} = \begin{bmatrix} 0 \\ 210 \end{bmatrix}.$$

The system is composed of four independent and identical second order open-loop unstable subsystems, with eigenvalues $5 \pm 314.6j$. The design criteria of the ideal controller for the closed-loop of each subsystem are defined by the matrices $Q_{c_s} = \mathrm{Diag}(30, 10)$ and $R_{c_s} = 0.01$. The communication channel linking the controller to the four distant actuators has a bandwidth of 10 kbps, which means that every 2 ms, at most 20 bits of information can be sent to the actuators.

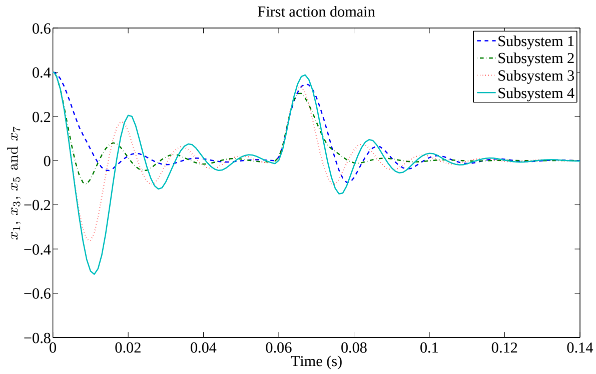

Figure 2: Global system responses obtained for the basic sequence of action domain $\mathcal{A}_1$

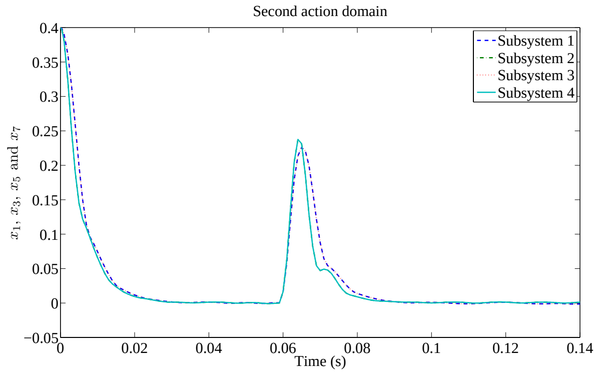

Figure 3: Global system responses obtained for the basic sequence of action domain $\mathcal{A}_2$

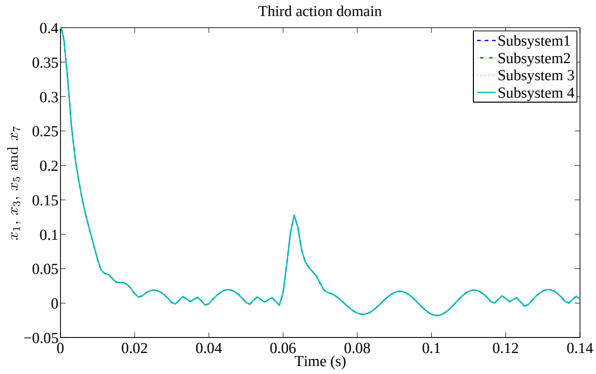

Figure 4: Global system responses obtained for the basic sequence of action domain $\mathcal{A}_3$

Based on these bandwidth constraints, choosing $M = 20$, and using the heuristic approach of the previous section, three action domains $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are associated to the global system, and corresponding to the sequence $a^{1,3} = (a_1, a_2, a_3) = (1, 2, 4)$. Since the four subsystems are identical, the basic sequences that are associated to each action domain are defined by

$$s_B^1 = \left(\begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 20 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 20 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 20 \end{bmatrix}, \ldots \right),$$

$$s_B^2 = \left(\begin{bmatrix} 0 \\ 10 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \\ 10 \end{bmatrix}, \begin{bmatrix} 10 \\ 0 \\ 10 \\ 0 \end{bmatrix}, \ldots \right),$$

and

$$s_B^3 = \left(\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}, \ldots \right).$$

The static strategy is first evaluated. In the simulation results (depicted in Figures 2, 3 and 4), the control performance corresponding to the application of the basic sequences s_B^1, s_B^2 and s_B^3 (of action domains $\mathcal{A}_1$ $\mathcal{A}_2$ and $\mathcal{A}_3$) is evaluated and compared. In these simulations, the global system is started from the initial condition $[0.4\ \ 0\ \ 0.4\ \ 0\ \ 0.4\ \ 0\ \ 0.4\ \ 0]^T$ and disturbed at $t = 59$ ms. It may be observed that the response corresponding to the basic sequence s_B^1 presents an oscillatory behavior, resulting from the effect of the long effective update period (which is equal to 8 ms). These oscillations disappear when the basic sequence s_B^2 is used. Using the basic sequence s_B^3 further improves the disturbance rejection capabilities as well as the response time of the systems, but lead to a degradation of the steady state performance (chaotic behavior near the origin). The main observation that we learn from this example is that using a high action domain improves the transient behavior whereas using low action domain improves the steady state precision. Performing a switching between these actions domains in an automatic and clairvoyant way will allow to take advantage of the benefits of each action domain and to improve the control performance by trading quantization precision for update rates.

The MPC algorithm is next evaluated. The used MPC algorithm was designed using the heuristic approach of the previous section. According to this approach, the set $\mathcal{C}$ shall contain the basic sequences $s_B^1 \times \frac{R_I}{M}$, $s_B^2 \times \frac{R_I}{M}$ and $s_B^3 \times \frac{R_I}{M}$ as well as their circular permutations. The cardinality of $\mathcal{C}$ (i.e. the number of used quantization sequences) is equal to 7. For that reason, 3 bits must be reserved to the encoding of the decoding key. Consequently, the parameters R_I and R_D were chosen such that

$$R_I = 16 \text{ and } R_D = 3.$$

The set $\mathcal{P}$ containing the possible precision vectors (corresponding to this choice of $\mathcal{C}$) is defined by

$$\mathcal{P} = \left\{ \begin{bmatrix} 16 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 16 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 16 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 16 \end{bmatrix}, \begin{bmatrix} 0 \\ 8 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 8 \\ 0 \\ 8 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 4 \\ 4 \end{bmatrix} \right\}$$

The control gains that are associated to each quantization sequence were derived in the same way as for the static strategy.

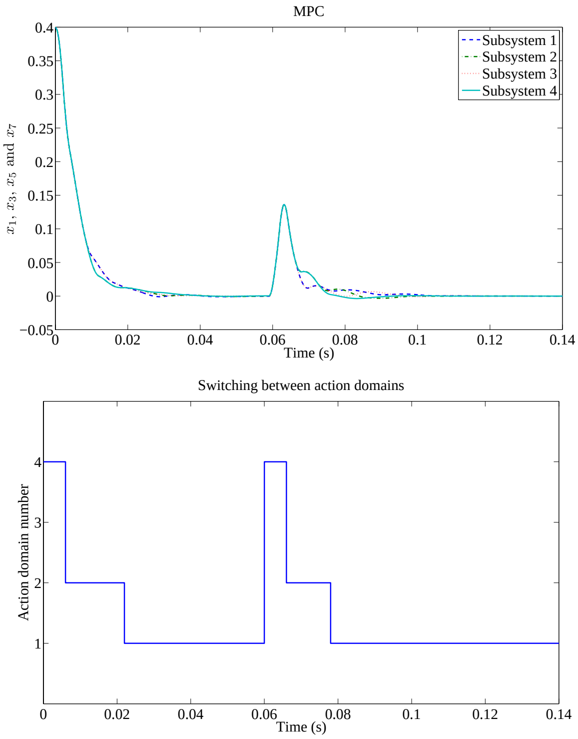

Figure 5: Global system responses obtained for the MPC algorithm (left) and used action domains (right)

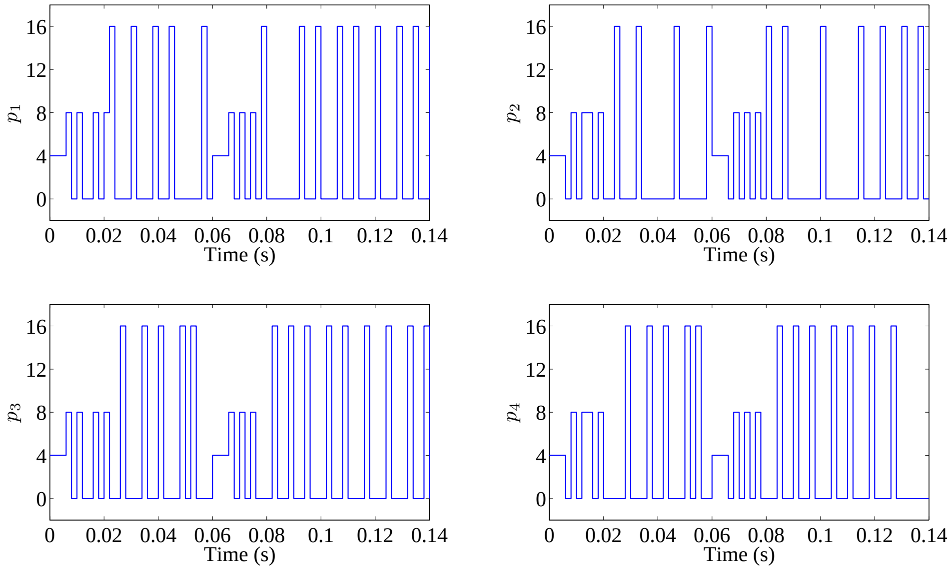

Figure 6: Quantization precision of control signals

Global system responses corresponding to the use of the MPC algorithm are depicted in Figure 5 (left). It may be observed that using the MPC approach, the tradeoff between precision and rapidity is performed. The advantages resulting from the usage of each basic sequence are achieved. The oscillations caused by the long update period as well as the chaotic behavior near the practical stability region are eliminated and the response to the unpredictable disturbances improved. These improvements are due to a more intelligent choice of the number of quantization levels of the control signals, which are allocated according to the state value of the system. The time evolution of the used action domains is depicted in Figure 5 (right). It may be observed that when the four systems are disturbed at the same time, the MPC algorithm chooses the send at the same time the control inputs to the four systems, using a precision vector from the first action domain. As long as the systems are stabilized, precision vectors from the second and finally from the third action domain are chosen. The corresponding quantization precision of the different control signals is depicted in Figure 6.

6 Conclusion

In this paper, the problem of the control over limited bandwidth communication channels was studied. A finely grained model was adopted, ensuring the respect of the bandwidth constraints and allowing the characterization of the influence of the update frequency and quantization precision on the control performance. A simple static strategy was first proposed, and its (W,V)-stability properties studied. An efficient approach for the improvement of disturbance rejection capabilities and steady state precision was then proposed. This approach dynamically assigns the quantization precision of the control signals in order to improve the control performance, taking into account the communication and computation requirements of the introduced dynamic protocol. It naturally allows handling LTI systems with multiple inputs. Sufficient conditions for ensuring the practical stability of this approach were stated. Finally, the proposed approach was evaluated and illustrated through a numerical example.

Bibliography

[1] K.-E. Årzén and A. Cervin. Control and embedded computing: Survey of research directions. In *Proceedings of the 16th IFAC World Congress on Automatic Control*, pages 342–353, Prague, Czech Republic, July 2005.

[2] G. Belanger, J. Speyer, S. Ananyev, D. Chichka, and R. Carpenter. Decentralized control of satellite clusters under limited communication. In *Proceedings of the AIAA Guidance, Navigation, and Control Conference and Exhibit*, pages 134–145, Providence, Rhode Island, USA, August 2004.

[3] A. Bemporad and M. Morari. Control of systems integrating logic, dynamics, and constraints. *Automatica*, 35(3):407–427, 1999.

[4] M-M. Ben Gaid and A. Çela. Trading quantization precision for sampling rates in networked systems with limited communication. In *Proceedings of the 45th IEEE Conference on Decision an Control*, pages 1135–1140, San Diego, CA, USA, December 2006.

[5] M-M. Ben Gaid, A. Çela, and Y. Hamam. Optimal integrated control and scheduling of systems with communication constraints. In *Proceedings of the Joint 44th IEEE Conference on Decision and Control and European Control Conference*, Seville, Spain, December 2005.

[6] M-M. Ben Gaid, A. Çela, and Y. Hamam. Optimal integrated control and scheduling of networked control systems with communication constraints: Application to a car suspension system. *IEEE Transactions on Control Systems Technology*, 14(4):776–787, July 2006.

[7] M-M. Ben Gaid, A. Çela, Y. Hamam, and C. Ionete. Optimal scheduling of control tasks with state feedback resource allocation. In *Proceedings of the 2006 American Control Conference*, Minneapolis, Minnesota, USA, June 2006.

[8] A. Berlin and K. Gabriel. Distributed MEMS: New challenges for computation. *IEEE Computational Science and Engineering Magazine*, 4(1):12–16, January 1997.

[9] F. Blanchini. Set invariance in control. *Automatica*, 35:1747–1767, 1999.

[10] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Transactions on Automatic Control*, 45(7):1279–1289, July 2000.

[11] R.W. Brockett. Stabilization of motor networks. In *Proceedings of the 34th IEEE Conference on Decision and Control*, pages 1484–1488, New Orleans, Los Angeles, USA, December 1995.

[12] R.W. Brockett. Minimum attention control. In *Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, California, USA, December 1997.

[13] C. Canudas-de-Wit, F. Rubio, J. Fornes, and F. Gomez-Estern. Differential coding in networked controlled linear systems. In *Proceedings of the 2006 American Control Conference*, Minneapolis, Minnesota, June 2006.

[14] D. F. Delchamps. Stabilizing a linear system with quantized state feedback. *IEEE Transactions on Automatic Control*, 35(8):916–924, 1990.

[15] N. Elia and S.K. Mitter. Stabilization of linear systems with limited information. *IEEE Transactions on Automatic Control*, 46(9):1384–1400, September 2001.

[16] F. Fagnani and S. Zampieri. Quantized stabilization of linear systems: complexity versus performance. *IEEE Transactions on Automatic Control*, 49(9):1534–1548, 2004.

[17] D. Georgiev and D. Tilbury. Packet-based control: The $\mathcal{H}_2$-optimal solution. *Automatica*, 42(1):137–144, 2006.

[18] G. C. Goodwin, H. Haimovich, D. E. Quevedo, and J. S. Welsh. A moving horizon approach to networked control systems design. *IEEE Transactions on Automatic Control*, 49(9):1562–1572, September 2004.

[19] V. Gupta, A. F. Dana, R. M. Murray, and B. Hassibi. On the effect of quantization on performance. In *Proceedings of the 2006 American Control Conference*, pages 1364–1369, Minneapolis, Minnesota, USA, June 2006.

[20] J.P. Hespanha, P. Naghshtabrizi, and Yonggang X. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1), January 2007.

[21] D. Hristu. *Optimal control with limited communication*. PhD thesis, Division of Engineering and Applied Sciences, Harvard University, June 1999.

[22] D. Hristu and K. Morgansen. Limited communication control. *System and Control Letters*, 37(4):193–205, July 1999.

[23] I. Kolmanovsky and E. G. Gilbert. Theory and computation of disturbance invariant sets for discrete-time linear systems. *Mathematical Problems in Engineering*, 4(4):317–367, 1998.

[24] M. D. Lemmon and Q. Ling. Control system performance under dynamic quantization : The scalar case. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, pages 1884 – 1888, Paradise Island, Bahamas, December 2004.

[25] B. Lincoln and B. Bernhardsson. LQR optimization of linear system switching. *IEEE Transactions on Automatic Control*, 47(10):1701–1705, October 2002.

[26] L.A. Montestruque and P.J. Antsaklis. On the model-based control of networked systems. *IEEE Transactions on Automatic Control*, 49(9):1562–1572, September 2004.

[27] G. N. Nair and R. J. Evans. Stabilization with data-rate-limited feedback: tightest attainable bounds. *Systems and Control Letters*, 41(1):49–56, September 2000.

[28] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S.S. Sastry. Foundations of control and estimation over lossy networks. *Proceedings of the IEEE*, 95(1), January 2007.

[29] A. Speranzon, J. Silva, J. de Sousa, and K. H. Johansson. On collaborative optimization and communication for a team of autonomous underwater vehicles. In *Proceedings of Reglermöte*, Göteborg , Sweden, May 2004.

[30] S. Tatikonda and S. Mitter. Control under communication constraints. *IEEE Transactions on Automatic Control*, 49(7):1056–1068, July 2004.

[31] Rakovic S. V., Kerrigan E. C., Kouramas K. I., and Mayne D. Q. Invariant approximations of the minimal robust positively invariant set. *IEEE Transactions on Automatic Control*, 50(3):406–410, 2005.

[32] E. Verriest and M. Egerstedt. Control with delayed and limited information: A first look. In *Proceedings of the 42nd IEEE Conference on Decision and Control*, pages 1231 – 1236, Hawaii, USA, December 2003.

[33] W. S. Wong and R. W. Brockett. Systems with finite communication bandwidth constraints – Part II : Stabilization with limited information feedback. *IEEE Transactions on Automatic Control*, 44(5):1049–1053, May 1999.

[34] W. Zhang, M. S. Branicky, and S. M. Phillips. Stability of networked control systems. *IEEE control systems magazine*, 21(1):84–99, February 2001.