HOLONOMY OF TAME WEYL STRUCTURES

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Abstract. A Weyl structure on a compact conformal manifold is not complete in general. If, however, the life-time of incomplete geodesics can be controlled on compact subsets of the tangent bundle, the Weyl connection is called tame. We prove that every closed, non-exact, tame Weyl structure on a compact conformal manifold is either flat, or has irreducible holonomy, generalizing an analogous statement for Riemannian cone metrics [6].

2000 *Mathematics Subject Classification*: Primary 53A30, 53C05, 53C29.

Keywords: Weyl structures, reducible holonomy, cone metrics, tame connections.

1. Introduction

The holonomy of Riemannian manifolds is an extensively studied topic for a long time, and it is a basic fact that if the manifold is not locally symmetric, then the local holonomy group either acts reducibly (in which case the manifold locally splits as a Riemannian product and the holonomy group is the product of the holonomy groups of the factors), or it belongs to the Berger list [4].

On a conformal manifold (M, c), the rôle of the Levi-Civita connection is played by the family of compatible *Weyl structures*, which are conformal, torsion-free connections on the tangent bundle TM [15]. Weyl structures can be *closed* or *exact*, i.e. locally, resp. globally equal to the Levi-Civita connection of some Riemannian metric in the conformal class, or *non-closed*.

As a consequence of the Merkulov-Schwachhöfer classification of groups occurring as holonomy of torsion-free connections [12], the holonomy group of every non-closed *irreducible* Weyl structure is the full conformal group in dimensions other than 4. In [3] we show that the reducible case is very interesting and, so far, little understood: The holonomy reduction defines locally a *conformal product* structure, and the holonomy group, although included in a product group, is not necessarily a product itself. In short, the reduced holonomy of a non-closed Weyl structure is either trivial, the full conformal group, some special groups in dimension 4, or it is reducible (in which case no complete description exists yet).

In contrast to that, the restricted holonomy of a closed Weyl structure is always a Riemannian holonomy (see Remark 2.4 below). However, not every Riemannian holonomy group occurs as holonomy of a closed, non-exact Weyl structure. More precisely, we show in Section 5.2 that the locally symmetric case and the quaternion Kähler holonomy

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This work was partially supported by the French-German cooperation project Procope no. 17825PG.

$\mathrm{Sp}_k \cdot \mathrm{Sp}_1$ do not occur, while all other holonomy groups in the Berger list occur as restricted holonomy groups of closed, non-exact Weyl structures. Moreover, one can even realize them on compact manifolds, by means of the cone construction, cf. Proposition 5.8 for details.

On the other hand, no compact examples of closed, non-exact Weyl structures with reducible holonomy are known so far, besides the flat ones. In the main result of this paper (Theorem 4.1), we show that, under a certain restriction, called *tame* (see below), the holonomy of a closed, non-exact, Weyl structure on a compact conformal manifold is either discrete or irreducible.

It is remarkable that an analogous statement has no chance to hold for exact or non-closed Weyl structures. Simple counter-examples are Riemannian products for the first case and *conformal products* (see [3]) for the second one. In order to understand why this result is expectable for closed, non-exact Weyl structures, we consider it from the point of view of a classical result by S. Gallot [6], which states that the metric cone over a complete Riemannian manifold is either irreducible or flat.

Recall that a metric cone over a Riemannian manifold (N, g^N) is $\bar{N} := \mathbb{R}^*_+ \times N$, with the metric $g^{\bar{N}} := dt^2 + t^2 g^N$, and it carries a global flow of homotheties (the flow of the *radial* vector field $t(\partial/\partial t)$). The metric $g^{\bar{N}}$ and its Levi-Civita connection define by projection a conformal structure and a closed, non-exact *canonical* Weyl structure on the quotient of the cone by the discrete group Γ generated by one of these homotheties. Gallot's result can thus be restated as follows: If N is complete, then the canonical Weyl structure on $\bar{N}/\Gamma$ is irreducible or flat.

We generalize this construction in Section 2. To every closed, non-exact Weyl structure D on a conformal manifold (M, c), we associate its *minimal Riemannian cover* (M_0, g_0), with the property that the deck transformation group acts on M_0 by strict homotheties, and the pull-back of D to M_0 is the Levi-Civita connection of g_0. We obtain in this way a one-to-one correspondence between closed, non-exact, Weyl structures on compact conformal manifolds and incomplete Riemannian manifolds carrying a co-compact group Γ of strict homotheties acting freely and properly discontinuously, called *cone-like* manifolds (Remark 2.4).

In the first result of this paper, Theorem 2.6, we show that every cone-like manifold (M_0, g_0) can be completed as a metric space by adding exactly on point ω, called the *singularity* of (M_0, g_0), which emphasizes again the similarity with a cone. The crucial point of the proof is Lemma 2.8, which states that on a cone-like manifold, the distance from a fixed point to its image through any contracting homothety in Γ is bounded (a fact which does not necessarily hold on the universal covering of M_0).

If the restricted Riemannian holonomy of a cone-like manifold (M_0, g_0) is reducible, our main result (Theorem 4.1) states that, under the *tame* assumption, the metric g_0 is flat. The idea of the proof is to show that the sets of leaves of any of the two integrable foliations (corresponding to the parallel splitting of the tangent bundle) contain "large" families of complete (immersed) submanifolds, all isometric to each other. On the other hand, we also show that the homotheties of (M_0, g_0) preserve these families, and we end up with pairs of complete submanifolds which are at the same time isometric *and* homothetic to each other, thus flat.

Roughly, these ideas are inspired by the original proof of Gallot [6] of the irreducibility of the *cone* over a complete manifold. However, in our more general *cone-like* setting, the difficulty comes from the lack of information about the incomplete geodesics (which, for a *cone*, are just its *rays*, the orbits of the homothety flow). The notion of a *tame* connection, which we introduce in Section 3, is equivalent to the existence of uniform bounds for the life-times of the incomplete geodesics generated by vectors belonging to any compact subset of the tangent bundle, and allows us to construct the *large* families discussed above.

Theorem 4.1 applies to a wide class of Weyl structures: We show in Section 5 that the *tame* condition is fulfilled by any small deformations of a cone metric, more precisely by a C^1-neighbourhood of the canonical Weyl structure on the quotient of a cone by one of its homotheties. This is a consequence of a more general result (Theorem 5.4), which states that every Weyl structure is *tame*, provided that it satisfies an analytic (open) condition, called *analytically tame*, which is tautologically fulfilled by the canonical Weyl structure on cone quotients.

We classify, at last, all possible restricted holonomy groups of closed tame Weyl structures (Propositions 5.8 and 5.9).

2. The minimal Riemannian cover of a closed Weyl structure

In this section, (M, c) denotes a connected conformal manifold and D denotes a closed, non-exact, Weyl structure on (M, c) (see e.g. [3] for the basic definitions). Let $\tilde{\pi} : \tilde{M} \to M$ be the universal cover of M, endowed with the induced conformal structure $\tilde{c} := \tilde{\pi}^* c$, and Weyl derivative $\tilde{D} := \pi^* D$. Since $\tilde{M}$ is simply connected, $\tilde{D}$ is exact, so $\tilde{M}$ carries a Riemannian metric $\tilde{g}_0 \in \tilde{c}$, unique up to a multiplicative constant, whose Levi-Civita covariant derivative is just $\tilde{D}$.

Lemma 2.1. *The group $\mathcal{A} \simeq \pi_1(M)$ of deck transformations of the covering $\tilde{M} \to M$ consists of homotheties of $\tilde{g}_0$.*

Proof. Every element $\alpha \in \mathcal{A}$ is a conformal transformation of $(\tilde{M}, \tilde{c})$, so there exists a positive function ρ such that $\alpha^* \tilde{g}_0 = \rho^2 \tilde{g}_0$. On the other hand, α preserves $\tilde{D}$, so the Riemannian metric $\alpha^* \tilde{g}_0$ is $\tilde{D}$-parallel, therefore ρ is constant. ☐

For every $\alpha \in \mathcal{A}$ we denote by $\rho(\alpha)$ the constant of homothety. Consider the subgroup of isometric deck transformations of $(\tilde{M}, \tilde{g}_0)$:

$$\mathcal{I} := \{\alpha \in \mathcal{A} \mid \rho(\alpha) = 1\}.$$

Of course, ρ being a group homomorphism from $(\mathcal{A}, \circ)$ to $(\mathbb{R}_+^*, \times)$, $\mathcal{I}$ is a normal subgroup of $\mathcal{A}$. The quotient manifold $M_0 := \tilde{M}/\mathcal{I}$ is a Galois covering of M with Abelian deck transformation group $\Gamma := \mathcal{A}/\mathcal{I}$, isomorphic to the subgroup $\rho(\mathcal{A})$ of $(\mathbb{R}_+^*, \times)$. Moreover $\tilde{g}_0$ projects to a Riemannian metric g_0 on M_0. Clearly ρ descends to a group homomorphism, also denoted by $\rho : \Gamma \to \mathbb{R}^*_+$, such that $f^* g_0 = \rho(f)^2 g_0$ for every $f \in \Gamma$. The pull-back of D to M_0 (still denoted by D) is the Levi-Civita connection of g_0, and the deck transformation group Γ acts by *pure* homotheties on (M_0, g_0) (i.e. the only isometry in Γ is the identity). This motivates the following:

Definition 2.2. *Let D be a closed Weyl structure on a connected conformal manifold (M, c). The triple (M_0, g_0, Γ), together with the covering $\pi : M_0 \to M = M_0/\Gamma$ is called the* minimal Riemannian cover *of (M, c, D).*

Notice that there is no canonical way to choose g_0 in its homothety class, but all the properties we will consider in the sequel will not depend on such a choice.

If d denotes the geodesic distance on M_0 induced by the Riemannian metric g_0, every $f \in \Gamma$ is a homothety of the metric space (M_0, d), i.e. $d(f(x), f(y)) = \rho(f) d(x, y)$ for each $x,\ y \in M_0$.

Definition 2.3. *A* cone-like *space is a locally compact metric space (M_0, d) together with a finitely generated, non-trivial group Γ acting freely and properly discontinuously by homotheties on (M_0, d), such that Γ contains no isometry besides the identity, and such that the quotient M_0/Γ is a compact topological space.*

Remark 2.4. The above considerations show that the minimal Riemannian cover defines a one-to-one correspondence between the set of triples (M, c, D) consisting in a compact manifold M, a conformal structure c and a closed, non-exact Weyl structure D on it, and the set of cone-like Riemannian manifolds (M_0, g_0, Γ) (modulo constant rescalings of the metric g_0).

A fundamental example, which is the *Leitfaden* of our present study, is the following:

Example 2.5. Let (N, g^N) be a complete Riemannian manifold and let

$$(M_0, g_0) := (\mathbb{R}^*_+ \times N, dt^2 + t^2 g^N)$$

be the *metric cone* over N (note that g_0 and the product metric g on $M_0 \simeq \mathbb{R} \times N$ are conformally related by setting $t = e^s,\ t \in \mathbb{R}^*_+,\ s \in \mathbb{R}$). The multiplication by some $\lambda > 1$ on the $\mathbb{R}$-factor is a strict homothety of g_0 and an isometry of g. It generates a group Γ acting freely and properly discontinuously on M_0. The metric g projects to the product metric, also denoted by g, on the quotient manifold $M := M_0/\Gamma \simeq S^1 \times N$. The Levi-Civita connection D_0 of g_0 is Γ-invariant, inducing therefore a closed, non-exact Weyl structure D on $(M, [g])$. It is straightforward to check that (M_0, g_0) is the minimal Riemannian cover of $(M, [g], D)$. A slightly more general example of Weyl manifold having (M_0, g_0) as minimal Riemannian cover can be constructed by projecting the Levi-Civita connection of g_0 onto the *mapping torus* of an isometry of N and, more generally, on any compact quotient of $M_0 \simeq \mathbb{R} \times N$ by an Abelian group acting by isometries (and preserving the corresponding splitting) of the product metric.

Metric cones can be equivalently characterized by the existence of a global *homothetic gradient flow*, i.e. a complete vector field which is locally a gradient (with respect to a local D-parallel metric g_0), and acts infinitesimally by homotheties of g_0. We will exhibit in this section some further properties which the class of cone-like spaces shares with the (much more restricted) class of metric cones.

The terminology in Definition 2.3 is justified by the following result which shows that (M_0, d) can be completed by adding one single point.

Theorem 2.6. *Let (M_0, d, Γ) be a cone-like space. Then the metric completion of (M_0, d) is a metric space $\widehat{M_0}$ such that $\widehat{M_0} \smallsetminus M_0$ is a single point ω, called the* singularity *of M_0.*

Proof. Since every commutator of Γ is an isometry of (M_0, d), the hypothesis ensures that Γ is Abelian. We need to show that (M_0, d) contains at least one non-convergent Cauchy sequence, and that any two such sequences are equivalent.

Let $f \in \Gamma$ be any element with $\rho(f) < 1$. For every $x \in M_0$ and $m < n \in \mathbb{N}$ we have

$$d(f^m(x), f^n(x)) \leq \sum_{k=m}^{n-1} d(f^k(x), f^{k+1}(x)) = d(x, f(x)) \sum_{k=m}^{n-1} \rho(f)^k < d(x, f(x)) \frac{\rho(f)^m}{1 - \rho(f)},$$

thus showing that $\{f^n(x)\}$ is a Cauchy sequence. If this sequence had a limit l in M_0, then l would be a fixed point of f, contradicting the fact that Γ acts freely. Thus (M_0, d) is non-complete.

Lemma 2.7. *Let $\{x_n\}$ be a non-convergent Cauchy sequence in (M_0, d). Then there exists $x \in M_0$ and a sequence $\{f_n\}$ of elements of Γ satisfying $\lim_{n\to\infty} \rho(f_n) = 0$ such that $\{x_n\}$ is equivalent to $\{f_n(x)\}$.*

Proof. Let π denote the projection of M_0 onto the compact space $M := M_0/\Gamma$. By choosing a subsequence if necessary, we may assume that $\pi(x_n)$ converges to some $y \in M$. Take $x \in \pi^{-1}(y)$. Since Γ acts properly discontinuously, there exists some open neighbourhood U_0 of x such that $h(U_0) \cap U_0 = \emptyset$ for every $h \in \Gamma$ different from the identity. We choose $r > 0$ such that the ball $B_x(2r)$ of radius $2r$ in x lies in U_0. Then $U := \pi(B_x(r))$ is a neighbourhood of y in the quotient topology, so there exists some n_0 such that $\pi(x_n) \in U$ for $n \geq n_0$. This shows that for $n \geq n_0$ there exist $z_n \in B_x(r)$ and $f_n \in \Gamma$ such that $x_n = f_n(z_n)$.

Suppose that $\rho(f_n)$ does not tend to zero. By taking a subsequence if necessary, we may assume that $\rho(f_n) > \delta$ for every n. For every m, n such that $f_n \neq f_m$, the open balls $f_n(B_x(2r))$ and $f_m(B_x(2r))$ are disjoint, being included in $f_n(U_0)$, and $f_m(U_0)$ respectively. As $f_n(z_n) \in f_n(B(x, r))$ and $f_m(z_m) \in f_m(B(x, r))$, we get $d(x_n, x_m) = d(f_n(z_n), f_m(z_m)) \geq 2r\delta$. The fact that $\{x_n\}$ is a Cauchy sequence ensures therefore the existence of an index N such that $f_n = f_N$ for every $n > N$. Since $B_x(r)$ is relatively compact, we may assume (passing to some subsequence, if necessary) that z_n tends to $z \in \overline{B_x(r)}$. Thus $\{x_n\}$ converges to $f_N(z)$, contradicting the fact that $\{x_n\}$ does not converge.

This shows that $\lim_{n\to\infty} \rho(f_n) = 0$. Since $d(f_n(z_n), f_n(x)) = \rho(f_n) d(z_n, x) < r\rho(f_n)$, the sequences $\{x_n\}$ and $\{f_n(x)\}$ are equivalent, thus proving the lemma. $\square$

In order to conclude the proof of the theorem we need one more technical result.

Lemma 2.8. *For every fixed point $x \in M_0$ there exists a constant K_x, depending on x, such that $d(x, f(x)) < K_x$ for every contracting $f \in \Gamma$ (i.e. with $\rho(f) < 1$).*

Proof. Let $\{h_1, \ldots, h_n\}$ be a system of generators of Γ with $\rho_i := \rho(h_i) > 1$. Let $D_x := \max_{\{i=1,\ldots,n\}} d(x, h_i(x))$. For every $(a_1, \ldots, a_k) \in \mathbb{N}^k$, we claim that

$$d\Big(x, \Big(\prod_{i=1}^{k} h_i^{a_i}\Big)(x)\Big) \leq D_x \prod_{i=1}^{k} \frac{\rho_i^{a_i+1} - 1}{\rho_i - 1}. \tag{1}$$

We prove the claim by induction on k. For $k = 1$ we have

$$d(x, h_1^{a_1}(x)) \leq \sum_{s=0}^{a_1-1} d(h_1^s(x), h_1^{s+1}(x)) = d(x, h_1(x)) \sum_{s=0}^{a_1-1} \rho_1^s \leq D_x \frac{\rho_1^{a_1} - 1}{\rho_1 - 1} < D_x \frac{\rho_1^{a_1+1} - 1}{\rho_1 - 1}.$$

Assume now that (1) holds for each $k \leq l$ and for every $(a_1, \ldots, a_k) \in \mathbb{N}^k$ and consider some element $(a_1, \ldots, a_{l+1}) \in \mathbb{N}^{l+1}$. We denote by

$$h := \prod_{i=1}^{l} h_i^{a_i} \quad \text{and by} \quad y_j := \left(h_{l+1}^j \circ h\right)(x), \ \forall\, j = 0, \ldots, a_{l+1}.$$

Using (1) for $k = l$ we have

$$d(x, y_0) \leq D_x \prod_{i=1}^{k} \frac{\rho_i^{a_i+1} - 1}{\rho_i - 1},$$

and $d(y_j, y_{j+1}) = \rho_{l+1}^j d(y_0, y_1) = \rho(h)\rho_{l+1}^j d(x, h_{l+1}(x))$, which further imply

$$\begin{aligned}
d\Big(x, \prod_{i=1}^{l+1} h_i^{a_i}(x)\Big) &= d(x, y_{a_{l+1}}) \leq d(x, y_0) + \sum_{j=0}^{a_{l+1}-1} d(y_j, y_{j+1}) \\
&\leq D_x \prod_{i=1}^{l} \frac{\rho_i^{a_i+1} - 1}{\rho_i - 1} + D_x \prod_{i=1}^{l} \rho_i^{a_i} \sum_{j=0}^{a_{l+1}-1} \rho_{l+1}^j \\
&\leq D_x \prod_{i=1}^{l} \frac{\rho_i^{a_i+1} - 1}{\rho_i - 1} \Big(1 + \sum_{j=0}^{a_{l+1}-1} \rho_{l+1}^j\Big) \leq D_x \prod_{i=1}^{l} \frac{\rho_i^{a_i+1} - 1}{\rho_i - 1} \Big(\sum_{j=0}^{a_{l+1}} \rho_{l+1}^j\Big) \\
&= D_x \prod_{i=1}^{l+1} \frac{\rho_i^{a_i+1} - 1}{\rho_i - 1},
\end{aligned}$$

thus proving our claim for $k = l + 1$. In order to finish the proof of the lemma, let $f \in \Gamma$ be an element with $\rho(f) < 1$. By reordering the system of generators if necessary, we can write

$$f = \prod_{i=1}^{n} h_i^{a_i}, \quad \text{with } a_i \geq 0 \text{ for } i \leq m \text{ and } a_i \leq 0 \text{ for } i \geq m + 1.$$

We denote $b_i := -a_i \geq 0$ for $i \geq m+1$. Using (1) we obtain

$$\begin{aligned} d(x, f(x)) &= \Big(\prod_{i=m+1}^{n} \rho_i^{a_i}\Big) d\Big(\prod_{i=1}^{m} h_i^{a_i}(x), \prod_{i=m+1}^{n} h_i^{b_i}(x)\Big) \\ &\leq \Big(\prod_{i=m+1}^{n} \rho_i^{a_i}\Big)\Big(d\big(x, \prod_{i=1}^{m} h_i^{a_i}(x)\big) + d\big(x, \prod_{i=m+1}^{n} h_i^{b_i}(x)\big)\Big) \\ &\leq \Big(\prod_{i=m+1}^{n} \rho_i^{a_i}\Big)\Big(D_x \prod_{i=1}^{m} \frac{\rho_i^{a_i+1}-1}{\rho_i - 1} + D_x \prod_{i=m+1}^{n} \frac{\rho_i^{b_i+1}-1}{\rho_i - 1}\Big). \end{aligned}$$

We neglect the -1 terms in the numerators above and multiply the brackets. Remembering that $\prod_{i=1}^{n} \rho_i^{a_i} = \rho(f) < 1$, we finally get

$$\begin{aligned} d(x, f(x)) &\leq D_x\Big(\prod_{i=1}^{m} \frac{\rho_i}{\rho_i - 1} \prod_{i=1}^{n} \rho_i^{a_i} + \prod_{i=m+1}^{n} \frac{\rho_i}{\rho_i - 1}\Big) \\ &\leq D_x\Big(\prod_{i=1}^{m} \frac{\rho_i}{\rho_i - 1} + \prod_{i=m+1}^{n} \frac{\rho_i}{\rho_i - 1}\Big) \\ &\leq D_x\Big(\prod_{i=1}^{n} \frac{\rho_i}{\rho_i - 1} + 1\Big) =: K_x, \end{aligned}$$

where the last inequality follows from the fact that $a + b \leq ab + 1$ for all $a, b \geq 1$. □

Let now $\{x_n\}$ be a non-convergent Cauchy sequence in M_0. Choose $y \in M_0$ and $f \in \Gamma$ such that $\rho := \rho(f) < 1$. We claim that $\{x_n\}$ is equivalent to $\{f^n(y)\}$.

By Lemma 2.7, there exists $x \in M_0$ and a sequence $\{f_n\}$ of elements of Γ satisfying $\lim_{n\to\infty} \rho(f_n) = 0$, such that $\{x_n\}$ is equivalent to $\{f_n(x)\}$. Since $\lim_{n\to\infty} \rho(f_n) = 0$, there exists an increasing sequence of integers $\{k_n\}$ such that $\rho(f_{k_n}) < \rho^n$. As $\rho(f^{-n} \circ f_{k_n}) < 1$, Lemma 2.8 yields

$$d(f^n(x), f_{k_n}(x)) = \rho^n d(x, (f^{-n} \circ f_{k_n})(x)) \leq K_x \rho^n.$$

The sequences $\{f_{k_n}(x)\}$ and $\{f^n(x)\}$ are thus equivalent, so the same holds for $\{x_n\}$ and $\{f^n(x)\}$. Finally, for any $y \neq x$, $\{f^n(x)\}$ is clearly equivalent to $\{f^n(y)\}$, thus finishing the proof of the theorem. □

Theorem 2.6 shows that cone-like spaces still have the one-point completion, although, in contrast to metric cones, they only carry a discrete group of homotheties. Note that the universal covering of a metric cone is a metric cone itself, therefore admits the one-point completion as well. It is unknown whether this fact holds for the universal covering of an arbitrary cone-like space (see Section 5.4).

Functions on a metric cone measuring geometric quantities like lengths, are equivariant with respect to the radial flow (acting by homotheties), and thus vary linearly on the rays.

In the more general case of cone-like spaces, we introduce, for further use, the following simple notion:

Definition 2.9. *Two positive functions* $f_1, f_2 : M_0 \to \mathbb{R}^*_+$ *are said to be* equivalent *if their ratio is bounded above and below by positive constants. A function which is equivalent to the distance to the singularity* ω *is called* quasi-linear*.*

Denote by $\delta : M_0 \to \mathbb{R}^*_+$ the distance to the singularity $\omega \in \widehat{M_0}$: $\delta(x) := d(x, \omega)$.

Lemma 2.10. *Let* $\psi : M_0 \to \mathbb{R}^*_+$ *be any* Γ*-equivariant function* of weight 1 *on* M_0 *(i.e. satisfying* $\psi \circ f = \rho(f)\psi$ *for every element* $f \in \Gamma$*), such that* ψ *and* ψ^{-1} *are locally bounded (e.g.,* ψ *is continuous). Then* ψ *is quasi-linear.*

Proof. Consider a compact fundamental domain Ω of the action of Γ on M_0 and define

$$k_1 := \inf_{x \in \Omega} \frac{\psi(x)}{\delta(x)}, \qquad k_2 := \sup_{x \in \Omega} \frac{\psi(x)}{\delta(x)}.$$

Because δ is continuous and ψ and its inverse are locally bounded, their quotients δ/ψ and ψ/δ are bounded on the compact set Ω. It follows that k_1, k_2 are positive real numbers, so that

$$\frac{\psi(x)}{\delta(x)} \in [k_1, k_2]$$

holds tautologically on Ω. Let now y be an arbitrary point of M_0 and $f \in \Gamma$ such that $x := f^{-1}(y) \in \Omega$. From the equivariance property of ψ we get

$$\frac{\psi(y)}{\delta(y)} = \frac{\psi(f(x))}{\delta(f(x))} = \frac{\rho(f)\psi(x)}{\rho(f)\delta(x)} = \frac{\psi(x)}{\delta(x)} \in [k_1, k_2],$$

which finishes the proof. □

As a consequence of the previous lemma, we show for later use that if (M_0, g_0) is the minimal Riemannian cover of a closed non-exact Weyl structure D on a compact conformal manifold (M, c), then any conformal factor relating g_0 to the pull-back on M_0 of a metric in the conformal class c on M is equivalent to the distance function δ to the singularity $\omega \in M_0$:

Lemma 2.11. *Let* g *be the pull-back to* M_0 *of a metric in* c *on* M *and let* $\varphi : M_0 \to \mathbb{R}^*_+$ *be defined by* $g_0 = \varphi^2 g$*. The function* φ *is then quasi-linear on* M_0*.*

Proof. Every element $f \in \Gamma$ being an isometry of g, we obtain

$$\rho(f)^2 \varphi^2 g = \rho(f)^2 g_0 = f^* g_0 = (\varphi \circ f)^2 g,$$

showing that $\varphi \circ f = \rho(f)\varphi$. The assertion thus follows from Lemma 2.10. □

3. Tame Weyl structures and their geodesics

In contrast to the Riemannian situation, a Weyl structure on a compact conformal manifold is not necessarily geodesically complete.

Example 3.1. Let (M, c) be a compact conformal manifold and let D be a closed, non-exact, Weyl structure on M. Theorem 2.6 shows that the minimal Riemannian cover (M_0, g) of (M, c, D) is incomplete, so through every point of M_0 passes an incomplete geodesic. Its projection onto M is thus an incomplete geodesic of D.

In order to study the geometry of M_0 in the neighbourhood of its singularity ω, we need to understand the behaviour of the geodesics passing through or near ω. In principle, the dynamics of the geodesic flow of (M, g) can be rather wild near ω. Here is a list of phenomena which may occur:

1. The lengths of the geodesics starting at some given point P and passing through ω (i.e. the life-time of an incomplete geodesic) might not be bounded.
2. There might exist closed geodesics through ω (i.e. geodesics having finite life-time in both directions).
3. There might even exist a complete geodesic whose adherence contains ω.

3.1. **Tame connections.**

To begin with, let us recall some basic facts about the geodesic flow of an affine connection D on a manifold M, or, equivalently, the exponential map

$$\exp^D : \mathcal{U} \to M$$

defined on an open subset $\mathcal{U}$ of TM, and of regularity depending on the ones of M and D. For our purposes, we assume it is C^∞. For $X \in \mathcal{U}_x := \mathcal{U} \cap T_x M$, $\exp^D(X)$ is the point $\gamma(1)$ on the geodesic defined by

$$\gamma(0) = x \text{ and } \dot\gamma(0) = X.$$

We define the *life-time* $\mathcal{L}^D : TM \to (0, +\infty]$ of a half-geodesic generated by $X \in TM$, by

$$\mathcal{L}^D(X) := \sup\{t > 0 \mid tX \in \mathcal{U}\},$$

in other words, the supremum of the time for which the half-geodesic tangent to X is defined. Of course, if (M, D) is geodesically complete, all life-times are infinite.

We split the complement $TM \smallsetminus \{0\}$ of the zero section in the tangent bundle into two sets, the set $\mathcal{I}^D$ of vectors generating incomplete half-geodesics, and its complement $\mathcal{C}^D$. These subsets are both star-shaped, i.e. for a vector $X \in TM$

$$X \in \mathcal{I}^D \iff sX \in \mathcal{I}^D, \qquad \forall\, s > 0.$$

We exclude the zero section in TM because it generates complete geodesics, but admits a neighbourhood in $\mathcal{C}^D$ if and only if D is geodesically complete (and thus $\mathcal{C} = TM$ in this case). There is nothing to say, in general, about the topology of the two subsets of the partition, as one of them is obtained as an infinite intersection of open sets.

Definition 3.2. *An affine connection D on a manifold M is called* weakly tame *if the set $\mathcal{C}^D$ of non-zero vectors generating a complete half-geodesic is open in $TM \smallsetminus \{0\}$.*

In other words, the set of vectors generating incomplete geodesics $\mathcal{I}^D$ is closed in $TM \smallsetminus \{0\}$.

Example 3.3. Let (M,g) be the metric cone over a complete Riemannian manifold (N,h). The only incomplete geodesics on M are then the *rays* $\{x\} \times (0,t]$ (connecting the point (x,t) with the cone apex), and their length is t (see [6]). The Levi-Civita connection of (M,g) is thus weakly tame.

It is equally easy to construct examples of affine connections which are not weakly tame: Consider the flat torus T^2, choose a point ω on it, and pick a dense geodesic avoiding ω. This can be seen as a limit of geodesics passing through ω. The induced flat connection on $T^2 \smallsetminus \{\omega\}$ is thus not weakly tame.

Definition 3.4. *A weakly tame connection D on a manifold M is called* tame *if the life-time $\mathcal{L}^D : \mathcal{I}^D \to \mathbb{R}^*_+$ is locally bounded.*

Note that $\mathcal{L}^D$ is lower semi-continuous on $\mathcal{I}^D$: If a geodesic generated by $X \in TM$ is defined up to time T, all geodesics generated by vectors in some neighbourhood of X in TM are also defined up to time T.

We will see later on that the metric cone over a complete Riemannian manifold (Example 3.3) is tame.

Example 3.5. Not all weakly tame connections are tame, as the following example of a domain in $\mathbb{R}^2$ shows. Take

$$S := \{(x,y) \in \mathbb{R}^2 \mid x > 1,\ x^2y^2 < 1\} \smallsetminus \{(n,0) \mid n \in \mathbb{N},\ n > 3\}$$

and let D be the restriction to S of the canonical flat connection in $\mathbb{R}^2$.

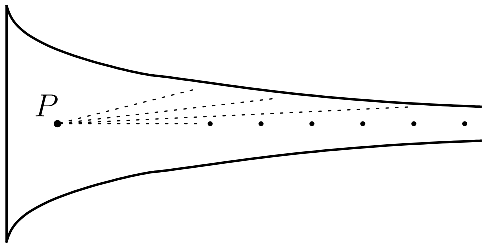

Figure 1. An example of weakly tame connection which is not tame.

The connection D on S is obviously weakly tame, since there are no complete half-geodesics. On the other hand, the life-time of a half-geodesic from any point $P \in S$ on the x-axis gets arbitrarily large as its defining vector approaches $(1,0) \in T_PS$ (for points $Q \in S$ away from the x-axis, the length of any geodesic through Q is bounded by a constant depending on Q). Therefore (S,D) is not tame.

Proposition 3.6. *An affine connection D on M is tame if and only if $\mathcal{L}^D$ is bounded on any intersection of $\mathcal{I}^D$ with a compact set in $TM \smallsetminus \{0\}$.*

Proof. This condition easily follows if D is *tame*, because such intersections are themselves compact (as D is weakly tame, $\mathcal{I}^D$ is closed).

Conversely, if $X \in \mathcal{C}^D$, then for any $t > 0$, there exists a compact neighbourhood $U(t)$ of X in $TM \smallsetminus \{0\}$, for which all the generated geodesics are defined at least up to time t: We choose, for example, a compact neighbourhood $V_t \subset \mathcal{U}$ of tX in TM (recall that the domain of definition $\mathcal{U}$ of exp is open in TM), then define $U(t) := (1/t)V_t$.

As $U(t)$ is compact, we get by hypothesis an upper bound $K(t)$ for the life-time of all incomplete geodesics generated by vectors in $U(t) \cap \mathcal{I}^D$. Let us fix $t_0 > 0$ and choose some T larger than $K(t_0)$. The set $U_0 := U(t_0) \cap U(T)$ is again a compact neighbourhood of X, but now, the upper bound $K(t_0)$ for $\mathcal{L}^D$ on $U_0 \cap \mathcal{I}^D \subset U(t) \cap \mathcal{I}^D$ and the lower bound T are contradictory, therefore $U_0 \cap \mathcal{I}^D$ is empty, which shows that $\mathcal{C}^D$ is open and D is weakly tame.

Moreover, the hypothesis implies that $\mathcal{L}^D$ is locally bounded, therefore D is tame as well. □

Remark 3.7. In the definition of a tame structure, we ask for $\mathcal{L}^D$ to be locally bounded *and* for D to be weakly tame. The following simple example, along with Example 3.5, shows that these two conditions are independent:

Example 3.8. Let $Z := S^1 \times (a, b)$ be a bounded, flat cylinder. The only complete geodesics on Z are the circles, hence $\mathcal{C}^D \subset TZ \smallsetminus \{0\}$ is the sub-bundle of tangent lines to the geodesic circles, in particular it is not open, therefore Z is not weakly tame.

On the other hand, the length of an incomplete geodesic depends *continuously* on its *slope* with respect to the geodesic circles, therefore $\mathcal{L}^D : \mathcal{I}^D \to (0, \infty)$ is a continuous function defined on an open subset of TZ, in particular it is locally bounded.

3.2. **Tame Riemannian metrics.**

We will now investigate the notions defined above in the particular case where D is the Levi-Civita connection of a Riemannian manifold, in particular on the minimal Riemannian cover (M_0, g_0) of a closed, non-exact Weyl manifold. Note that a connection on M is (weakly) tame if and only if its pull-back to any covering of M is (weakly) tame. In order to characterize tame closed Weyl structures on a compact manifold, it thus suffices to understand tame cone-like Riemannian manifolds. In the following lemmas (that hold in a general setting), we intentionally use notations from the previous sections, to emphasize where our main interest lies:

Lemma 3.9. *A Riemannian manifold (M_0, g_0) is weakly tame if and only if the set*

$$\mathcal{I}^{g_0} := \mathcal{I}^D \cap S(TM_0)$$

is closed. Here D is the Levi-Civita connection of g_0 and $S(TM_0)$ is the sphere bundle of unit vectors in TM_0.

The proof is obvious. Note the slight difference between $\mathcal{I}^D$ and $\mathcal{I}^{g_0}$. We also denote the restriction of $\mathcal{L}^D$ to $\mathcal{I}^{g_0}$ by $\mathcal{L}^{g_0}$ and $\mathcal{I}^{g_0} \cap T_xM_0 =: \mathcal{I}^{g_0}_x$.

For an incomplete Riemannian manifold (M_0, g_0), we set

$$\mu : M_0 \to (0, +\infty], \qquad \mu(x) := \sup_{X \in \mathcal{I}_x^{g_0}} \mathcal{L}^D(X).$$

It is the supremum of the lengths of all incomplete half-geodesics starting in x.

Lemma 3.10. *The Riemannian manifold (M_0, g_0) is tame if and only if μ is locally bounded.*

Proof. Note first that, since M_0 is incomplete, from every point starts an incomplete half-geodesic, and so the set $\mathcal{I}_x^{g_0}$ is non-empty for any $x \in M_0$, hence μ is well-defined.

If D is tame, the fact that μ is locally bounded follows immediately from Proposition 3.6.

Conversely, let $K \subset TM_0 \setminus \{0\}$ be a compact set. Then its projection K_0 on M_0 is compact as well, and there exists $q > 1$ such that

$$K \subset \{X \in T_x M_0 \mid x \in K_0, \ \frac{1}{q} \leq \|X\| \leq q\}.$$

The local boundedness of μ, together with the inequality above implies that

$$\mathcal{L}^D(X) \leq q \sup_{K_0} \mu, \qquad \forall \ X \in K \cap \mathcal{I}^D,$$

so D is tame by Proposition 3.6. $\square$

We give now a criterion characterizing closed tame Weyl structures on compact manifolds, or, equivalently, on cone-like Riemannian spaces:

Proposition 3.11. *Let (M, c) be a compact conformal manifold, and let D be a closed, non-exact Weyl structure on it. Then D is tame on M (or, equivalently, on M_0) if and only if $\mu : M_0 \to (0, +\infty]$ is (finite and) quasi-linear on M_0.*

Proof. The distance δ to the singularity $\omega \in M_0$ is always continuous on M_0, and $\mu \geq \delta$, therefore μ^{-1} is locally bounded. On the other hand, δ and μ are clearly Γ-equivariant of weight 1, because both denote geometrical lengths. If D is tame, Lemma 2.10 implies, together with Lemma 3.10, the quasi-linearity of μ.

Conversely, if μ is quasi-linear, then it is locally bounded, therefore, again by Lemma 3.10, D is tame. $\square$

In the next section we prove our main result, concerning the holonomy of a closed tame Weyl structure. Finally, in Section 5 we will show that the tame condition applies to an open set of Weyl structures (in the C^1-topology), in particular the class of tame closed Weyl structures is significantly large. We also give the complete classification of their possible restricted holonomy groups.

4. Closed tame Weyl structures with reducible holonomy

The goal of this section is to prove the following

Theorem 4.1. *If the restricted holonomy representation of a closed, non-exact, tame Weyl structure D on a compact conformal manifold (M,c) is reducible, then D is flat.*

Proof. We start by showing that if the restricted holonomy $\mathrm{Hol}_0(D)$ is reducible, then there exists a finite covering $\bar M$ of M on which the full holonomy of the pull-back of D has reducible holonomy. In order to keep the argument as simple as possible, we will not be very precise on the holonomy groups and consider them as abstract groups rather than as transformation groups of each tangent space.

Consider the metric $\tilde g_0$ on the universal cover $\tilde M$ of M (defined up to a multiplicative constant), whose Levi-Civita covariant derivative $\tilde\nabla$ is the pull-back of D to $\tilde M$. The holonomy of $\tilde\nabla$ is clearly equal to the restricted holonomy of D. By Theorem IV.5.4 in [10], the tangent bundle of $\tilde M$ splits in a direct sum $T\tilde M = T_0 \oplus \ldots \oplus T_m$ of $\tilde\nabla$-parallel sub-bundles and the holonomy group of $\tilde\nabla$ satisfies $\mathrm{Hol}(\tilde\nabla) = H_1 \times \ldots \times H_m$, where H_i acts irreducibly on T_i and trivially on T_j for $j \neq i$ (T_0 being the flat component). Moreover this decomposition is unique up to a permutation of the set $\{1,\ldots,m\}$ (such permutations may occur if some of the factors H_i coincide).

By Lemma 2.1, every element $f \in \mathcal{A}$ of the deck transformation group of the covering $\tilde M \to M$ is affine with respect to $\tilde\nabla$, so there exists a permutation σ_f of $\{1,\ldots,m\}$ such that $f_*(T_i) = T_{\sigma_f(i)}$. Let $\mathcal{B} \subset \mathcal{A}$ be the kernel of the group homomorphism $\mathcal{A} \to S_m$ given by $f \mapsto \sigma_f$. The metric $\tilde g_0$ and the connection $\tilde\nabla$ on $\tilde M$ descend to a conformal structure $\bar c$ and a Weyl structure $\bar D$ on the quotient $\bar M := M/\mathcal{B}$, which is a finite covering of M with group $\mathcal{A}/\mathcal{B} \subset S_m$. By construction, the holonomy group of $\bar D$ on $\bar M$ is reducible.

Replacing (M,c,D) by $(\bar M,\bar c,\bar D)$, we can from now on assume that the full holonomy of D is reducible. This implies that the tangent bundle of the minimal Riemannian cover (M_0,g_0) of (M,c,D) splits in a direct sum of orthogonal distributions $TM = V_1 \oplus V_2$, parallel with respect to the Levi-Civita connection $\nabla^0 = D$ of g_0. These distributions are integrable, hence define two orthogonal (and complementary) foliations on M_0.

We will use the notion *maximal leaf* M_i, $i = 1,2$, through $x \in M_0$ to denote the set of points that can be connected to x by means of a smooth curve tangent to V_i. It is a standard fact that M_i are immersed submanifolds of M_0.

We start with two preliminary results which hold on every (not necessarily complete) reducible Riemannian manifold (M_0,g_0).

Lemma 4.2. *Let U_1 be a leaf of V_1 (not necessarily complete) and let $X \in V_2$ be a parallel vector field along U_1. Assume that $\exp_x tX$ is defined for all $x \in U_1$ and $t \in [0,1]$. Then $x \mapsto \psi(x) := \exp_x X$ maps U_1 isometrically onto its image.*

Proof. The statement is classical but as we could not find a precise reference, we provide the proof for the reader's convenience.

We first show that the map ψ is a local isometry between U_1 and some other leaf U_1' of V_1. Consider the map $\varphi : U_1 \times [0,1] \to M_0$ defined by $\varphi(x,t) := \exp_x tX$. Define $X_{(x,t)} \in T_{\varphi(x,t)}M_0$ by

$$X_{(x,t)} := \frac{d}{ds}\Big|_{s=t} \varphi(x,s).$$

In other words, $X_{(x,t)}$ is the tangent vector to the geodesic $s \to \exp_x sX$ at $s = t$, so we clearly have the relation

$$\varphi(x, t+s) = \exp_{\varphi(x,t)} sX_{(x,t)}. \tag{2}$$

Let us fix $x \in U_1$ and denote $x_t := \varphi(x,t)$. The local de Rham decomposition theorem (Proposition IV.5.2 in [10]) states that each x_t has a neighbourhood $U(t)$ in M_0 isometric to a Riemannian product $U(t) \simeq U_1(t) \times U_2(t)$, where $U_1(t)$ and $U_2(t)$ are local leaves of V_1 and V_2 through x_t.

The geodesic segment $\varphi(\{x\} \times [0,1])$ is compact, so it can be covered by a finite number of neighbourhoods $U_2(s_1), \dots, U_2(s_n)$, with $0 = s_1 < \dots < s_n = 1$. Choose now $t_i \in (s_i, s_{i+1})\ \forall\ i = 1, \dots, n-1$, such that $\varphi(x, t_i) \in U_2(s_i) \cap U_2(s_{i+1})$, and set $t_n := 1$. For $k = 1, \dots, n$, let V_k be the open subset of U_1 defined by

$$V_k := \{y \in U_1 \mid \varphi(y, s_k) \in U(s_k)\}.$$

We denote by V the intersection of the V_k's and by W_k the subset of $U(t_k)$ given by $W_k := \varphi(V \times \{t_k\})$.

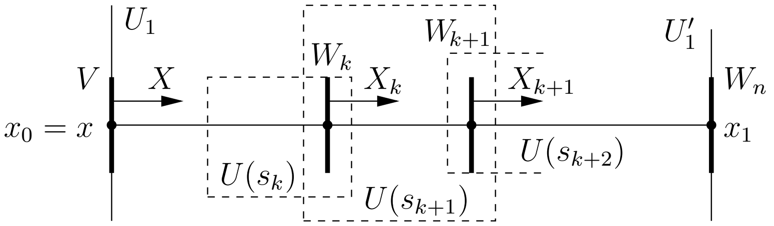

FIGURE 2. Stepwise exponentiation along the geodesic segment $\varphi(\{x\} \times [0,1])$.

Consider the vector field X_k along W_k whose value at $\varphi(y, t_k)$ is $X_{(y,t_k)}$. By construction, there exists a bijection $\varphi_k : W_k \to W_{k+1}$ defined by $\varphi_k(\varphi(y,t_k)) := \varphi(y, t_{k+1})$ for all $y \in V$.

We claim that for every $k = 1, \dots, n$,

(1) φ_{k-1} is an isometry;
(2) the vector field X_k is parallel along W_k;
(3) W_k is an open subset of the leaf $U_1(t_k)$.

For $k = 1$, the first statement is empty, and the other two hold by hypothesis.

Assume that the claim holds for some $k \geq 1$. Since X_k is parallel along W_k, it is constant in the product coordinates on $W_k \times U_2(s_k) \subset U(s_k)$ (i.e. there exists $Z \in T_{x_{t_k}} U_2(s_k)$ such that $(X_k)_{(z, x_{t_k})} = (0, Z)$, $\forall\ z \in W_k$). By (2) we have $\varphi_k(\varphi(y,t_k)) = \exp_{\varphi(y,t_k)}(t_{k+1} - t_k)X_{(y,t_k)}$ so in the product coordinates $\varphi_k(z, x_{t_k}) = (z, x_{t_{k+1}})$ for all

$z \in W_k$, showing that $W_k \subset U_1(t_k)$ and that φ_k is an isometry. Moreover X_{k+1} is constant along W_{k+1} in these coordinates, thus proving the induction step.

We have shown that in the neighbourhood V of x in U_1, the map $x \mapsto \psi(x) = \exp_x X$ is the composition of $n-1$ isometries $\varphi_{n-1} \circ \cdots \circ \varphi_1$ between local leaves of the distribution V_1. As this holds in the neighbourhood of every point x of U_1, ψ is a local isometry from U_1 to its image U_1'. In particular, this shows that U_1' is an open subset of the complete integral leaf of V_1 passing through $\varphi(x, 1)$.

Moreover, the claim above shows that $Y := \psi_*(X)$ is a well-defined parallel vector field along U_1'. Consider the map $\tilde{\psi} : U_1' \to M_0$ defined by $y \mapsto \exp_y(-Y)$. From the local considerations above, it is clear that $\tilde{\psi} \circ \psi$ is the identity of U_1. This shows that ψ is one-to-one. ☐

The next result, which is somewhat folkloric, like the previous one, shows that exponentiating a geodesic tangent to V_1 in the direction of a constant or affine Jacobi field tangent to V_2 yields another geodesic whenever it is defined.

Lemma 4.3. *Let $\gamma : [a, b] \to M_0$ be a geodesic tangent to V_1 parametrized by arc-length and let $X \in T_{\gamma(a)} M_0$ be a vector tangent to V_2. Extend X to a parallel vector field along γ.*

(i) *Assume that $\gamma_s(t) := \exp_{\gamma(t)}(sX)$ is well-defined for all $t \in [a, b]$ and $s \in [0, 1]$. Then $\gamma_1(t)$ is a geodesic in M_0 and its tangent vector at t is the parallel transport of $\dot{\gamma}(t)$ at $\exp_{\gamma(t)}(X)$ along the geodesic $s \mapsto \exp_{\gamma(t)}(sX)$.*

(ii) *Assume that $\gamma^X(t) := \exp_{\gamma(t)}(tX)$ is well-defined for all $t \in [a, b]$. Then $\gamma^X(t)$ is a geodesic in M_0 and the projections of $\dot{\gamma}^X(t)$ onto V_1 and V_2 are parallel vector fields along γ^X of length 1 and $|X|$ respectively.*

Proof. (i) The first statement follows immediately from Lemma 4.2. The second one is a consequence of the claim used to prove the same lemma.

(ii) Assume first that M_0 is a global Riemannian product $M_0 = M_1 \times M_2$. If $\gamma(a) = (m_1, m_2)$, then $\gamma(t) = (\gamma_1(t), m_2)$ for some geodesic γ_1 in M_1 parametrized by arc-length. The vector field X along γ can be written $X = (0, X_2)$, where X_2 is a constant vector tangent to M_2 at m_2. Denoting by $\gamma_2(t) = \exp_{m_2} tX_2$ the geodesic in M_2 starting at m_2 with initial speed X_2, then $\gamma^X(t) = (\gamma_1(t), \gamma_2(t))$, is a geodesic in M_0. The projections of $\dot{\gamma}^X(t)$ onto V_1 and V_2 are $(\dot{\gamma}_1, 0)$ and $(0, \dot{\gamma}_2)$, which are clearly parallel vector fields along γ^X of length 1 and $|X|$ respectively.

Back to the general case, it is of course enough to show that the statement holds in the neighbourhood of every point $\gamma^X(t_0)$. Since the domain of definition of the exponential on the normal bundle of a geodesic is open, the curve $c(t) := \exp_{\gamma(t)}(t_0 X)$ is well-defined for t near t_0. By Lemma 4.2, $c(t)$ is a geodesic through $x := \gamma^X(t_0)$, parametrized by arc-length. Moreover, if Y denotes the parallel vector field along $c(t)$ with $Y_x = d\exp_{\gamma(t_0)}(t_0 X)$, Lemma 4.2 also shows that $Y_{\gamma(t)} = d\exp_{\gamma(t)}(t_0 X)$, so by (2), $\gamma^X(t) = \exp_{\gamma(t)}((t - t_0)Y)$.

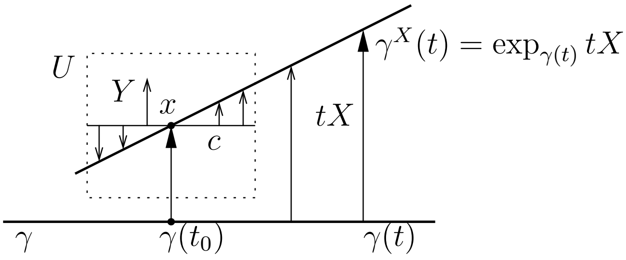

Figure 3. Idea of the proof of Lemma 4.3 (ii).

By the local de Rham theorem, the point x has a neighbourhood U isometric to $U_1 \times U_2$, where U_i is some local leaf of V_i through x. As $\gamma^X(t)$ lies in U for t near t_0, the statement follows from the first part of the proof. ☐

We assume from now on that D is a closed tame Weyl structure on a compact conformal manifold (M, c) which has reducible holonomy and that (M_0, g_0) is the minimal Riemannian cover of (M, c, D). We denote as before by d the distance induced by g_0 on M_0, by ω the singularity of M_0 and by δ the distance to the singularity: $\delta(x) := d(x, \omega)$. Since $D = \nabla^0$, g_0 has reducible holonomy, so the above results apply to the present setting. We will need the following quantitative version of the local de Rham decomposition theorem for (M_0, g_0).

Lemma 4.4. *There exists a quasi-linear function $\sigma : M_0 \to \mathbb{R}^*_+$ such that each point $x \in M_0$ has a neighbourhood U and an isometry $F : U \to B^1_x(\sigma(x)) \times B^2_x(\sigma(x))$, with $F(x) = (x, x)$, where $B^i_x(r)$ is the ball of radius r around x in the maximal leaf M_i though x, tangent to the distribution V_i.*

Proof. By Lemma 2.10, it is enough to define σ on a relatively compact fundamental domain $K \subset M_0$ of the covering $M_0 \to M$, and to extend it to M_0 in a Γ-equivariant way by $\sigma(f(x)) = \rho(f)\sigma(x)$ for all $f \in \Gamma$ and $x \in K$.

The local de Rham theorem ensures that for every $x \in M_0$ there exist neighbourhoods $U_i(x)$ of x in the maximal leaf $M_i(x)$ though x, tangent to the distribution V_i, such that $U_1(x) \times U_2(x)$ is isometric to a neighbourhood $U(x)$ of x in M_0. Take a finite number of points x_i such that $\overline{K} \subset \cup_i U(x_i)$. Each neighbourhood $U_1(x_i)$ and $U_2(x_i)$ contains a geodesic ball centered in x_i of radius $r_1(x_i)$ and $r_2(x_i)$ respectively. It is then enough to define σ on K to be the minimum of all these radii. ☐

We now come to a key point of the proof of Theorem 4.1, namely the existence of *complete* maximal leaves tangent to the distributions V_i.

Proposition 4.5. *If M_1 is a maximal leaf of V_1 which is incomplete, then every maximal leaf of V_2 which intersects M_1 is complete.*

Proof. Since M_1 is totally geodesic and incomplete, through every point $x \in M_1$ passes a geodesic $\gamma : (0, r] \to M_1$ parametrized by arc-length, such that $\gamma(r) = x$, which can not be defined at $t = 0$. Since M_i is totally geodesic in M_0, γ is also a geodesic in M_0. By Theorem 2.6, we must have $\lim_{t\to 0} \gamma(t) = \omega$ in $(\widehat{M_0}, d)$.

Let $X \in T_xM \cap V_2$ be any unit normal vector to M_1 at x, extended as before to a parallel vector field along γ. We claim that the geodesic generated by X in M_0 is complete.

The crucial point here is the fact that every point $\gamma(t)$ is far enough from the singularity ω, in order to ensure that the exponential function is well-defined in a suitable neighbourhood. More precisely, Proposition 3.11 shows that there exists a constant $\kappa > 0$ such that for every $t \in (0, r]$, the distance $\delta(\gamma(t))$ from $\gamma(t)$ to ω (in M_0) is bounded from below by κt. Consequently, $\exp_{\gamma(t)} sX$ is well-defined for $|s| \leq \kappa t$, so by Lemma 4.3 (ii), the curve $\gamma_1 : (0, r] \to M_0$ defined by $\gamma_1(t) := \exp_{\gamma(t)} \kappa tX$ is a geodesic in M_0 with $|\dot\gamma_1|^2 = 1 + \kappa^2$. Moreover, the limit in $\widehat{M_0}$ of $\gamma_1(t)$ as $t \to 0$ is clearly ω. Proposition 3.11 applied this time to the geodesic parametrized by arc-length $\tilde\gamma_1$ defined by

$$\tilde\gamma_1(t) := \gamma_1((1 + \kappa^2)^{-1/2}t)$$

yields $\delta(\tilde\gamma_1(t)) > \kappa t$, whence

$$\delta(\gamma_1(t)) > (1 + \kappa^2)^{1/2}\kappa t > \kappa t.$$

Consequently, for every $t \in (0, r]$, every geodesic defined by a unit vector $Y \in T_{\gamma_1(t)}M_0$ is defined at least up to the time κt. Taking Y to be the speed vector of the geodesic $s \to \exp_{\gamma(t)} sX$ at $s = \kappa t$, we obtain that this geodesic can actually be extended for $s \in [0, 2\kappa t]$, for any $t \in (0, r]$. By Lemma 4.3 (ii), the curve $\gamma_2 : (0, r] \to M_0$ defined by $\gamma_2(t) := \exp_{\gamma(t)} 2\kappa tX$ is thus a geodesic in M_0 with $|\dot\gamma_2|^2 = 1 + 4\kappa^2$. Again, we check that the distance from $\gamma_2(t)$ to the singularity is at least κt, showing that for every $t \in (0, r]$, $\exp_{\gamma(t)} sX$ is well-defined for $|s| \leq 3\kappa t$. Iterating the same argument shows that the geodesic $\exp_{\gamma(t)} sX$ is actually defined for every $t \in (0, r]$ and for every $s \in \mathbb{R}$.

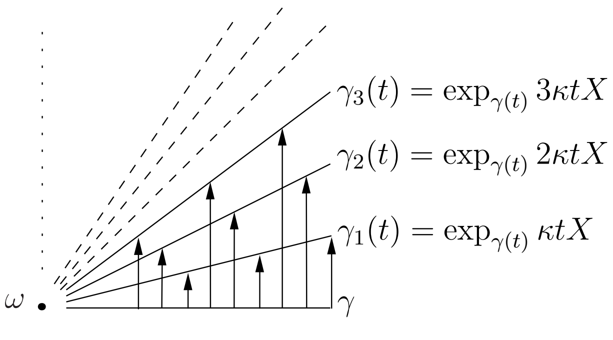

Figure 4. Idea of the proof of Proposition 4.5.

In particular, for $t = r$, $\gamma(t) = x$, we have proved that the geodesic through x tangent to $X \in V_2$ is complete. Since X was arbitrarily chosen, the whole integral leaf of V_2 through x is thus complete. □

In order to apply this result, we need to show that incomplete leaves actually exist.

Lemma 4.6. *There exist incomplete maximal leaves M_i of V_i or, equivalently, incomplete geodesics γ_i tangent to V_i for $i = 1$ and $i = 2$.*

Proof. Let $\gamma : (0, 1] \to M_0$ be an incomplete geodesic, such that $\lim_{t\to 0} \gamma(t) = \omega$ in $(\widehat{M_0}, d)$. We may assume that γ is not tangent to V_1 or V_2: If for instance γ were tangent to V_1, we replace it by γ^X given by Lemma 4.3 (ii), which is neither tangent to V_1 nor to V_2.

Let X_1 and X_2 denote the projections of $\dot\gamma$ on V_1 and V_2 respectively, which are clearly parallel along γ. We denote by $r_i := |X_i| \neq 0$ the norms of X_i and by $r := \sqrt{r_1^2 + r_2^2}$ the norm of $\dot\gamma$. Define the *slope* of γ to be the quotient $q(\gamma) := r_1/r_2$.

We claim that $\gamma_1(t) := \exp_{\gamma(t)}(-tX_1)$ is defined for every $t \in (0, 1]$, and is an incomplete geodesic tangent to V_2, such that $\lim_{t\to 0} \gamma_1(t) = \omega$ in $(\widehat{M_0}, d)$. The argument is similar to that used in the proof of Proposition 4.5: The exponential, denoted $\gamma^s(t)$ of $-tsX_1$ at $\gamma(t)$ is well-defined by Proposition 3.11 for $|s| \leq \frac{\kappa r}{r_1}$. For every fixed s in this interval, $\gamma^s(t)$ is an incomplete geodesic and its slope is (see Lemma 4.3 (ii)) $q(\gamma^s) = (1 - s)q(\gamma)$. If $\frac{\kappa r}{r_1} \geq 1$, which is equivalent to $q(c) \leq (\kappa^{-2} - 1)^{-1/2}$, the incomplete geodesic γ^s has zero slope for $s = 1$, i.e. it is tangent to V_2. Otherwise, we replace γ by γ^s with $s = \frac{\kappa r}{r_1}$ and repeat this procedure. The slope of the new geodesic is

$$q(\gamma^s) = \left(1 - \frac{\kappa r}{r_1}\right) q(\gamma) = q(\gamma) - \frac{\kappa r}{r_2} \leq q(c) - \kappa,$$

showing that the procedure stops after a finite number of iterations. Since V_1 and V_2 play symmetric rôles, this finishes the proof. ☐

Corollary 4.7. *If there exists an incomplete geodesic γ passing through a point $x \in M_0$ such that $\dot\gamma$ is neither tangent to V_1 nor to V_2, then x belongs to a complete leaf of V_1 which intersects an incomplete maximal leaf of V_2, and to a complete leaf of V_2 which intersects an incomplete maximal leaf of V_1.*

Proof. The result follows directly from the proof of Lemma 4.6 together with Proposition 4.5. ☐

Lemma 4.8. *If M_1 is a maximal leaf of V_1 which is incomplete, then all maximal leaves of V_2 which intersect M_1 are isometric.*

Proof. Let $M_2(x)$ denote the maximal leaf of V_2 through x. Since every two points of M_1 can be joined by a broken geodesic, it is enough to show that $M_2(\gamma(0))$ and $M_2(\gamma(1))$ are isometric for every geodesic $\gamma : [0, 1] \to M_1$. Consider the normal vector field $X(t)$ on $M_2(\gamma(t))$ obtained by parallel transport of $\dot\gamma(t)$. Since $M_2(\gamma(0))$ is complete, every point $y \in M_2(\gamma(0))$ can be expressed as $y = \exp_{\gamma(0)}(Y)$ for some $Y \in T_{\gamma(0)}M_2(\gamma(0))$. We extend Y along γ by parallel transport. By Proposition 4.5, the leaves $M_2(\gamma(t))$ are complete, hence $\exp_{\gamma(t)}(sY)$ is well-defined for every $s \in \mathbb{R}$ and $t \in [0, 1]$. By Lemma 4.3 (i) we get $\exp_{\gamma(t)}(Y) = \exp_y(tX(0))$. The exponential of $X(0)$ is thus defined for all $y \in M_2(\gamma(0))$ so we conclude by Lemma 4.2. ☐

Consider now a metric g on M_0 obtained as the pull-back of a metric in the conformal class c on M. Let φ be the conformal factor relating g to g_0 by $g_0 = \varphi^2 g$ and let $\tilde{d}$ the geodesic distance induced on M_0 by g. Denote by $B_x(r)$ and $\tilde{B}_x(r)$ the set of points at distance less than r from x with respect to d and $\tilde{d}$ respectively. Recall that by Lemma 2.11 φ is quasi-linear, so there exist positive constants $k_1,\ k_2$ such that

$$k_1\delta(x) \le \varphi(x) \le k_2\delta(x), \qquad \forall\ x \in M_0. \tag{3}$$

Lemma 4.9. *For every $x \in M_0$ and positive real number r, the open ball $B_x(r)$ contains the open ball $\tilde{B}_x(\tilde{r})$, where $\tilde{r} = \frac{r}{k_2(\delta(x)+r)}$.*

Proof. For every $y \in \overline{B_x(r)}$ we have $\delta(y) \le \delta(x) + r$, so by (3) $\varphi(y) \le k_2(\delta(x) + r)$. Consequently, the g^0-length $l^0(c)$ and g-length $l(c)$ of every path contained in $\overline{B_x(r)}$ are related by

$$l^0(c) \le k_2(\delta(x) + r)l(c). \tag{4}$$

Assume there exists $z \in \tilde{B}_x(\tilde{r}) \smallsetminus B_x(r)$ and let $c : [0, 1] \to M_0$ be any path joining x and z. Define $s_0 = \inf\{s \mid c(s) \notin B_x(r)\}$ and consider the path $c' = c|_{[0,s_0]}$ which is clearly contained in $\overline{B_x(r)}$. Then $l^0(c') \ge r$, so by (4), $l(c) \ge l(c') \ge \tilde{r}$. Since this holds for every path c, we must have $\tilde{d}(x, z) \ge \tilde{r}$, contradicting the fact that $z \in \tilde{B}_x(\tilde{r})$. ☐

Lemma 4.10. *Let $\gamma : (0, a] \to M_0$ be an incomplete g_0-geodesic parametrized by arc-length, such that $\lim_{t\to 0} \delta(\gamma(t)) = 0$. There exist positive real numbers $\rho,\ q \in (0, 1)$ such that the open balls $B_n := B_{\gamma(q^n)}(\rho q^n)$, $n \in \mathbb{N}$, are all pairwise disjoint.*

Proof. Recall that by Proposition 3.11 we have control on the distance from $\gamma(t)$ to the singularity ω, i.e. there exists a constant $\kappa \in (0, 1)$, independent of γ, such that:

$$\kappa t \le \delta(\gamma(t)) \le t, \qquad \forall\ t \in (0, a]. \tag{5}$$

We start with arbitrary ρ and q in $(0, 1)$. For every $y \in B_n$, Equation (5) yields

$$(\kappa - \rho)q^n \le \delta(\gamma(q^n)) - \rho q^n \le \delta(y) \le \delta(\gamma(q^n)) + \rho q^n \le (\rho + 1)q^n,$$

so by (3) we get

$$k_1(\kappa - \rho)q^n \le \varphi(y) \le k_2(\rho + 1)q^n.$$

It is thus enough to choose ρ and q such that $k_2(\rho + 1)q^{n+1} < k_1(\kappa - \rho)q^n$ for every n, which is equivalent to $\rho < \kappa$ and $q < \frac{k_1}{k_2(\rho+1)}(\kappa - \rho)$. ☐

Corollary 4.11. *Consider the open subset $B := \cup_{n\ge 1} B_n$ in M_0. There exists $f \in \Gamma$ different from the identity such that $f(B) \cap B \ne \emptyset$.*

Proof. Lemma 4.9 applied to $x = \gamma(q^n)$ and $r = \rho q^n$ shows that B_n contains the open ball $\tilde{B}_x(\tilde{r})$ where

$$\tilde{r} = \frac{r}{k_2(\delta(x) + r)} = \frac{\rho q^n}{k_2(\delta(x) + \rho q^n)} \ge \frac{\rho q^n}{k_2(q^n + \rho q^n)} = \frac{\rho}{k_2(1 + \rho)}.$$

Recall that Γ acts by isometries on (M_0, g) and that $(M_0, g)/\Gamma = (M, g)$. If $f(B_n)\cap B_m = \emptyset$ for every $f \in \Gamma$ and $m \ne n$, the projections $\pi(B_n)$ of B_n onto M would be pairwise disjoint sets, each of them containing a ball of g-radius $\frac{\rho}{k_2(1+\rho)}$ in M. This is impossible since M is compact, thus proving our assertion. ☐

The last step in the proof of Theorem 4.1 is the following:

Lemma 4.12. *Let M_1 be a maximal leaf of V_1 which is incomplete. Then for every $x \in M_1$, the maximal leaf $M_2(x)$ of V_2 through x is flat.*

Proof. Let $\gamma : (0,1] \to M_1$ be an incomplete geodesic with respect to g_0 parametrized by arc-length, such that $\gamma(1) = x$ and $\gamma(t)$ converges to ω (with respect to d) as t tends to 0. By Lemma 4.10 one can find $\rho,\ q \in (0,1)$ such that the open balls $B_n := B_{\gamma(q^n)}(\rho q^n)$ are pairwise disjoint. Moreover, one can choose ρ such that each maximal leaf of V_2 through a point of $B = \cup_{n\geq 1} B_n$ intersects M_1. Indeed, this follows from Lemma 4.4 provided that ρq^n is smaller than $\sigma(\gamma(q^n))$ for every n. Since σ is quasi-linear, there exists some σ_0 such that $\sigma(x) \geq \sigma_0 \delta(x)$, so from (5) it suffices to take $\rho < \kappa\sigma_0$.

Corollary 4.11 now shows that there exists $f \in \Gamma$ different from the identity and $y, z \in B$ such that $y = f(z)$. Then f maps the integral leaf $M_2(z)$ of V_2 through z to the integral leaf of V_2 through y. Since both leaves intersect M_1, Lemma 4.8 shows that they are isometric. Composing f with this isometry we obtain a strict homothety of $M_2(z)$. Since $M_2(z)$ is complete, Lemma 2, page 242 in [10] shows that it must be flat. By Lemma 4.8 again, all the other leaves tangent to V_2 must be flat as well. ☐

We are now in position to complete the proof of Theorem 4.1. From Corollary 4.7, and Lemma 4.12, the sectional curvature of g_0 vanishes at each point x which belongs to an incomplete geodesic which is neither tangent to V_1 nor to V_2. The proof of Lemma 4.3 (ii) shows that the set of such points is dense in M_0. Thus (M_0, g_0) is a flat Riemannian manifold, so the holonomy group of $D = \nabla^0$ is discrete. ☐

Remark 4.13. The only place where the compactness assumption on M is needed in Theorem 4.1, is to ensure, by Theorem 2.6, that the minimal Riemannian cover of (M, c, D) has exactly one singularity. Theorem 4.1 thus holds in a slightly more general setting, and applies in particular to all metric cones over complete Riemannian manifolds.

5. Examples and applications

5.1. **Analytically tame Weyl structures.** In order to show that the hypotheses in Theorem 4.1 are satisfied by a large variety of Weyl structures, we introduce the following:

Definition 5.1. *A Weyl structure D on a conformal manifold (M, c) is called* analytically tame *if there exists a complete Riemannian metric $g \in c$ and a positive real number $\varepsilon > 0$ such that*

$$|\theta|^2 g(X,X) + (\nabla_X \theta)(X) \geq 2\varepsilon g(X,X), \qquad \forall\ X \in TM, \tag{6}$$

where ∇ denotes the Levi-Civita covariant derivative of g and θ denotes the Lee form of D with respect to g.

Recall that the Lee form θ of a Weyl structure D with respect to a metric $g \in c$ measures the difference between D and the Levi-Civita connection $\nabla = \nabla^g$ of g:

$$D_X Y - \nabla_X Y = \tilde{\theta}_X(Y) := \theta(X)Y + \theta(Y)X + \theta^\sharp g(X,Y),\ \forall\ X, Y \in TM, \tag{7}$$

where $\theta = g(\theta^\sharp, \cdot)$.

Example 5.2. With the notations form Example 2.5, it is easy to see that the standard Weyl structure D_0 on (a compact quotient of) a metric cone is analytically tame. Indeed, the Lee form of D_0 with respect to the complete metric g is $\theta_0 := ds$ and it is parallel for $\nabla = \nabla^g$. Therefore $|\theta_0|^2 g + \nabla\theta = g$ thus (6) is even an equality for $\varepsilon = 1/2$.

Remark 5.3. An exact Weyl structure on a compact conformal manifold M can not be analytically tame. Indeed, its Lee form θ with respect to any metric is exact, $\theta = d\varphi$ so (6) cannot hold at points where φ reaches its maximum on M.

The remaining part of this section is devoted to the proof of the following result, which also justifies the terminology in Definition 5.1:

Theorem 5.4. *If D is an analytically tame Weyl structure with respect to some complete metric $g \in c$ on a conformal manifold (M, c), then D is tame.*

Proof. Let θ be the Lee form of D with respect to g. If ∇ denotes the Levi-Civita covariant derivative of g, then we get from (7), see also [8]:

$$D_X - \nabla_X = (\theta \wedge X)_* - p\theta(X)\mathrm{Id} \tag{8}$$

on $T^*M^{\otimes p}$, where $(\theta \wedge X)_*$ is the usual extension of the endomorphism $\theta \wedge X$ as a derivation (in fact, the right hand side of (8) is just $\tilde{\theta}_X$, acting as a derivation on $T^*M^{\otimes p}$). It is easy to check that $(\theta \wedge X)_* g = 0$, so (8) yields

$$D_X g = -2\theta(X)g. \tag{9}$$

On the other hand, applying (8) to the Lee form θ itself yields

$$D_X\theta = \nabla_X\theta + g(\theta,\theta)g(X,.) - 2\theta(X)\theta,$$

thus showing that (6) is equivalent to

$$(D_X\theta)(X) \geq 2\varepsilon g(X,X) - 2\theta(X)^2, \qquad \forall\ X \in TM. \tag{10}$$

Let $\gamma(t)$ be a geodesic with respect to D on M and let $I = (a,b)$ denote its maximal domain of definition, with $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. We introduce the functions $F(t) := g(\dot\gamma(t), \dot\gamma(t))^{-\frac{1}{2}}$ and $H(t) := \theta(\dot\gamma(t))$, defined on I. Using (9) we get

$$F'(t) = \dot\gamma(t).F(t) = -\frac{1}{2}\left[(D_{\dot\gamma(t)}g)(\dot\gamma(t),\dot\gamma(t))\right]\left[g(\dot\gamma(t),\dot\gamma(t))^{-\frac{3}{2}}\right] = F(t)H(t), \tag{11}$$

and from (10),

$$H'(t) = \dot\gamma(t).H(t) = (D_{\dot\gamma(t)}\theta)(\dot\gamma(t)) \geq 2\varepsilon F(t)^{-2} - 2H(t)^2. \tag{12}$$

Lemma 5.5. *If $b < \infty$ (i.e. γ is incomplete toward the future) then $\underline{\lim}_{t\to b}F(t) = 0$. Similarly, if $a > -\infty$, then $\underline{\lim}_{t\to a}F(t) = 0$.*

Proof. We have to show that the g-norm of the speed vector of an incomplete D-geodesic cannot be bounded on M. Consider the geodesic flow of D, viewed as a vector field on the tangent bundle TM. Let γ be the maximal half-geodesic with respect to D, issued from some $X \in T_xM$. There exists $T > 0$ such that the maximal integral curve through X of the geodesic flow is defined only for $t < T$. Assume that the g-norm of $\dot\gamma$ is bounded: $g(\dot\gamma(t), \dot\gamma(t)) < k^2$ for all $t \in [0, T)$. Then the corresponding integral curve is contained in the subset $K(k, T)$ of TM defined by

$$K(k,T) := \{Y_y \in TM \mid d(x,y) \le kT \text{ and } g(Y,Y) \le k^2\},$$

where d denotes the geodesic distance with respect to g. Since g is complete, the closed geodesic balls are compact, so $K(k,T)$ is a compact subset of TM. On the other hand, it is well-known that an incomplete integral curve of a vector field cannot be contained in any compact subset, thus proving the lemma. ☐

In order to fix the ideas, we will assume from now on that

$$0 \in I \qquad \text{and} \qquad H(0) \le 0 \tag{13}$$

(this can always be achieved by making a translation in time and replacing $\gamma(t)$ with $\gamma(-t)$ if necessary).

Lemma 5.6. *The function F has at most one critical point. If this happens, then the critical point is an absolute minimum of F, and γ is complete. Conversely, if γ is complete, then F has a critical point.*

Proof. Let t_0 be a critical point of F. From (11), $H(t_0) = 0$. Moreover, (12) shows that H' is strictly positive at each point where H vanishes, so actually H cannot vanish more than once. Thus $H(t)$ is positive for $t \ge t_0$ and negative for $t \le t_0$, so t_0 is a global minimum of F. Lemma 5.5 then shows that γ cannot be incomplete.

Conversely, assume that γ is complete, i.e. $I = \mathbb{R}$. If F has no critical point, H does not vanish, so by our assumption (13), $H < 0$ on $\mathbb{R}$. F is thus a decreasing positive function, so $\lim_{t\to\infty} H(t)F(t) = \lim_{t\to\infty} F'(t) = 0$. Dividing by $H(t)^2$ in (12) yields

$$\frac{H'(t)}{H(t)^2} > \frac{2\varepsilon}{F(t)^2H(t)^2} - 2.$$

Since the right hand side tends to infinity as $t \to \infty$, an integration shows that $\lim_{t\to\infty} H(t) = 0$. Using (12) again, we then see that there exist some $t_0 \in \mathbb{R}$ and $\delta > 0$ such that $H'(t) > \delta$ for $t > t_0$. This of course contradicts the fact that H is negative on the whole real line $\mathbb{R}$, thus proving the lemma. ☐

The convention (13) together with Lemma 5.6 ensures that if γ is an incomplete geodesic, H is negative on I, so F is decreasing. By Lemma 5.5, γ is complete toward $-\infty$, i.e. $I = (-\infty, b)$, with $b \in \mathbb{R}_+$.

Lemma 5.7. *If $\gamma : I \to M_0$ is a geodesic with respect to D which is incomplete in the positive direction, then*

$$H(t) \le -\frac{\sqrt{\varepsilon}}{F(t)}, \qquad \forall\, t \in I. \tag{14}$$

Proof. If (14) does not hold, there exists $t_0 \in I$ such that

$$-\frac{\sqrt{\varepsilon}}{F(t_0)} < H(t_0) < 0.$$

We define the open set

$$I' := \{t \in I \mid -\frac{\sqrt{\varepsilon}}{F(t)} < H(t)\},$$

and let (a', b') be the connected component of I' containing t_0. By (12), H is strictly increasing on I'. In the other hand, we have seen that F is decreasing on I. If $b' \in I$, we would have

$$H(b') = -\frac{\sqrt{\varepsilon}}{F(b')} < -\frac{\sqrt{\varepsilon}}{F(t_0)} < H(t_0),$$

a contradiction. The only possibility left is thus $b' = b$. But this is impossible as well, since by (11),

$$\lim_{t \to b'} \log(F(t)) = \log(F(t_0)) + \int_{t_0}^{b'} H(t)dt > -\infty,$$

contradicting Lemma 5.5. $\square$

Back to the proof of Theorem 5.4, using (14) and (11) we get $F'(t) \leq -\sqrt{\varepsilon}$, and thus

$$\sqrt{\varepsilon}b = \int_0^b \sqrt{\varepsilon}dt \leq -\int_0^b F'(t)dt \leq F(0) = g(\dot{\gamma}(0), \dot{\gamma}(0))^{-\frac{1}{2}}.$$

In other words, the life-time of every geodesic γ, incomplete in the positive direction, is bounded from above by $(\varepsilon g(\dot{\gamma}(0), \dot{\gamma}(0)))^{-\frac{1}{2}}$. Let K be any compact subset of $TM \smallsetminus \{0\}$ and let $l(K)$ denote

$$l(K) := \inf_{X \in K} \{g(X, X)\}.$$

With the notations from Section 3, for every $X \in \mathcal{I}^D \cap K$ we have $\mathcal{L}^D(X) \leq (\varepsilon l(K))^{-\frac{1}{2}}$, so D is tame by Proposition 3.6. $\square$

The analytically tame condition (6) is clearly open in the C^1 topology defined by the metric g on the space of Weyl structures. Therefore, Theorem 4.1 applies to open subsets of the space of closed Weyl structures.

5.2. **Holonomy issues.** An exact Weyl structure on a conformal manifold is just the Levi-Civita connection of some metric in the conformal class. The possible restricted holonomy groups of exact Weyl structures are thus given by the Berger-Simons theorem ([4], p. 300). The analogous question for non-closed Weyl structures can be answered from [12] in the irreducible case and was studied in [3] in the reducible case. It thus remains to understand the case of closed, non-exact Weyl structures. The next result gives a complete list in the compact case, under the assumption that the connection is tame.

Proposition 5.8. *The restricted holonomy group of a closed, non-exact, tame Weyl structure D on a compact n-dimensional conformal manifold (M, c) is one of the following:*

$$\mathrm{SO}(n),\ \mathrm{U}(n/2),\ \mathrm{SU}(n/2),\ \mathrm{Sp}(n/4),\ G_2\ \textit{(for } n = 7\textit{)},\ \mathrm{Spin}(7)\ \textit{(for } n = 8\textit{)},\ 0.$$

Conversely, each of the groups listed above can be realized as the restricted holonomy of a closed, non-exact Weyl structure on a compact conformal manifold.

Proof. Since D is locally the Levi-Civita connection of metrics in the conformal class c, the Berger-Simons theorem applies. Assume first that D is locally symmetric. The metric g_0 on the minimal Riemannian cover M_0 of (M, c, D) is then locally symmetric. Every nontrivial homothety f satisfies $f^*g_0 = \rho(f)^2 g_0$ and preserves the Riemannian curvature tensor R_0. In particular $f^*(|R_0|^2) = \rho(f)^{-4}|R_0|^2$. On the other hand, R_0 being parallel with respect to the Levi-Civita connection of g_0, $|R_0|^2$ is constant on M_0. Since $\rho(f) \neq 1$, this shows that (M_0, g_0) is flat, so $\mathrm{Hol}_0(D) = 0$.

Assuming from now on that $\mathrm{Hol}_0(D) \neq 0$, D is irreducible by Theorem 4.1, so $\mathrm{Hol}_0(D)$ is in the Berger list [4], p. 301. It remains to show that $\mathrm{Sp}(k)\cdot\mathrm{Sp}(1)$ can not be realized as the restricted holonomy group of a closed, non-exact Weyl structure. The argument is similar to the one used above. If $\mathrm{Hol}_0(D) = \mathrm{Sp}(k) \cdot \mathrm{Sp}(1)$ then the minimal cover (M_0, g_0) is *quaternion-Kähler*, therefore Einstein with non-zero Ricci tensor $\mathrm{Ric} = \lambda g_0$ [4]. Since the homotheties that act on M_0 preserve the Levi-Civita connection of g_0, they also preserve the Ricci tensor. We infer that every homothety has to be an isometry, which contradicts the fact that D is not exact.

Conversely, we will show that every group in the above list can be realized as the holonomy of a closed, non-exact Weyl structure on a compact manifold. In fact, the examples that we provide are all quotients of cone constructions by a non-trivial homothety [1].

Indeed, let $n > 2$ (the case $n = 2$ is trivial) and (N^{n-1}, h) be a compact Riemannian manifold of a certain *weak holonomy* type (that we precise in each case), and let $M := S^1 \times N$ with the product metric g. The conformal manifold (M, c) (where c is the conformal class of g) can equally be seen as the quotient of the cone $M_0 := \mathbb{R}^*_+ \times N$, $g_0 = dt^2 + t^2 h$ by the infinite cyclic group generated by $\eta(t, x) := (kt, x)$, for $0 < k < 1$. The cone-like manifold (M_0, g_0) is the minimal Riemannian cover of the Weyl manifold (M, c, D), where D is the projection to M of the Levi-Civita connection of the cone (M_0, g_0).

The remaining part of the proof is based on well-known results in Riemannian geometry (see [1] for details):

(1) If (N^{2m-1}, h) is *Sasakian* and not Einstein, then $\mathrm{Hol}_0(M, D) = \mathrm{U}(m)$ [14];
(2) If (N^{2m-1}, h) is *Sasaki-Einstein* and neither *3-Sasakian*, nor locally isometric to the unit sphere, then $\mathrm{Hol}_0(M, D) = \mathrm{SU}(m)$;
(3) If (N^{4k-1}, h) is *3-Sasakian* but not locally isometric to the unit sphere, then $\mathrm{Hol}_0(M, D) = \mathrm{Sp}(k)$;

(4) If (N^6, h) is strictly *nearly Kähler* [9], has scalar curvature equal to 30, and is not locally isometric to S^6, then $\mathrm{Hol}_0(M, D) = G_2$;
(5) If (N^7, h) has a *proper nearly parallel G_2-structure* (i.e. the space of Killing spinors with Killing constant 1/2 is one-dimensional, [5]), then $\mathrm{Hol}_0(M, D) = \mathrm{Spin}(7)$.
(6) If (N^{n-1}, h) is the unit sphere S^{n-1}, then $\mathrm{Hol}_0(M, D) = 0$. □

Note that the case $\mathrm{Hol}_0(M, D) = \mathrm{U}(m)$ is well-known in the literature and corresponds to *locally conformally Kähler* (l.c.K.) manifolds. The l.c.K. structure constructed above on $S^1 \times N$ for every Sasakian manifold N has the following special property: There exists a metric g in the conformal class such that the Lee form of D with respect to g is ∇^g-parallel [14]. This special kind of l.c.K. metric is called *Vaisman* metric and it is known that not every l.c.K. structure contains such a metric in the conformal class, not even for a deformation of the l.c.K. conformal class (see [11], [13] for examples of l.c.K. manifolds which can not be conformally Vaisman for topological reasons, having non-zero Euler characteristic, and also [2] for a classification of Vaisman structures on compact 4-manifolds).

For the other holonomy groups in the above list we have the following structure result (note that the tame assumption is no longer required):

Proposition 5.9. *Let (M, c, D) be a compact Weyl manifold of dimension $n > 2$, such that D is a closed Weyl structure whose restricted holonomy is one of the following subgroups of $\mathrm{SO}(n)$: $\mathrm{SU}(n/4)$, $\mathrm{Sp}(n/4), G_2 \subset \mathrm{SO}(7)$, $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$ or $0 \subset \mathrm{SO}(n)$ (cases (2)–(6) in the above list). Then the following hold:*

(1) *There exists a compact Riemannian manifold (N, g_N) satisfying one of the conditions (2)–(6) above, such that the universal cover $\tilde{M}$ of M, together with the metric g_0 whose Levi-Civita connection is the pull-back of D, is a metric cone over (N, g_N), i.e. $\tilde{M} = \mathbb{R}^*_+ \times N$, and $g_0 = dt^2 + t^2 g_N$.*
(2) *The minimal Riemannian cover of (M, c, D) is the metric cone over a finite quotient of (N, g_N).*
(3) *The manifold M, endowed with its Gauduchon metric, is the mapping torus of an isometry of this finite quotient of (N, g_N).*

Proof. Let $g \in c$ denote the *Gauduchon metric* of D on M (which is determined up to a multiplicative constant by the fact that the Lee form of D with respect to g is δ^g–co-closed, see [7]), as well as its pull-back to the universal cover $\tilde{M}$ of M. We denote by g_0 the metric on $\tilde{M}$ having D as Levi-Civita covariant derivative. In all five cases (2)–(6), the metric g_0 is Ricci-flat, so D is an *Einstein-Weyl* structure. This also holds on the compact manifold M, therefore Theorem 3 in [8] implies that the Lee form of D with respect to g is parallel. The same is true on the complete, simply connected manifold $(\tilde{M}, g)$, which is therefore a Riemannian product $(\mathbb{R}, ds^2) \times (N, g_N)$.

The Lee form of D with respect to g on $\tilde{M}$ is just ds, so $g_0 = e^{2s} g$, i.e. $g_0 = dt^2 + t^2 g_N$ after a coordinate change $t := e^s$. This means that $(\tilde{M}, g_0)$ is the metric cone over (N, g_N). It is well-known, see for example [1], that if the holonomy of the metric cone of (N, g_N) is one of the five groups above, then (N, g_N) is Einstein with positive scalar curvature. This, together with the fact that N is closed in $\tilde{M}$, (and thus complete), implies that N has to be compact.

Let $f \in \pi_1(M)$ be any deck transformation, thus acting isometrically on $(\tilde{M}, g)$. Since f is affine with respect to D, it has to preserve the Lee form of D with respect to g, i.e. $f^*(ds) = ds$, and therefore it preserves its g-dual $\partial/\partial s$. This means that f commutes with the flow of $\partial/\partial s$ on $\tilde{M}$, so it is induced by an isometry, also denoted by f, of (N, g_N): $f(s,x) = (s + \ln(\rho(f)), f(x))$ (recall that $\rho(f)$ is the homothety constant of f with respect to g_0: $f^*g_0 = \rho(f)^2 g_0$). It follows that the group $\mathcal{I} \subset \pi_1(M)$ of deck transformations preserving g_0 induces a group of isometries $\mathcal{I}_N$ acting freely on (N, g_N), so the minimal Riemannian cover (M_0, g_0) of (M, c, D) is the metric cone over $(N, g_N)/\mathcal{I}_N$.

Finally, the compactness of N implies that the deck transformation group $\Gamma = \pi_1(M)/\mathcal{I}$ of the covering $M_0 \to M$ is discrete, hence isomorphic to $\mathbb{Z}$, showing that (M, g) is the mapping torus of an isometry of $(N, g_N)/\mathcal{I}_N$. ☐

As a consequence, $\chi(M) = 0$ and the fundamental group of M is a finite extension of $\mathbb{Z}$. Note that if $\dim M = 2$ and D is flat, its minimal covering may be $\mathbb{C}^*$ or $\mathbb{C}$. In both cases $\pi_1(M)$ is (a finite extension of) $\mathbb{Z}^2$.

Remark 5.10. The Berger-Simons theorem, along with the de Rham decomposition theorem, completely classify the restricted holonomy groups of torsion-free connections with *bounded* full holonomy group (as a subset of $\mathrm{GL}(n, \mathbb{R}) \subset \mathbb{R}^{n^2}$). On the other hand, a closed, non-exact Weyl structure is just a torsion-free connection whose restricted holonomy group is *compact*, but its full holonomy group is *not bounded*. Theorem 4.1 and the results in this section can thus be interpreted as an holonomy classification for this kind of connections (under the tame assumption).

5.3. **An example of cone-like manifold which is not tame.** Let $\widehat{C_0}$ be the following rotation cone in $\mathbb{R}^3$:

$$\widehat{C_0} := \{(x, y, z) \mid z = \sqrt{x^2 + y^2}\}.$$

The set $C_0 := \widehat{C_0} \smallsetminus \{0\}$ is a smooth Riemannian submanifold of $\mathbb{R}^3$ and its metric completion is $\widehat{C_0}$. The homothety $X \mapsto 2X$ in $\mathbb{R}^3$ defines by restriction a homothety f of C_0, which generates a group of homotheties $\Gamma := \{f^n \mid n \in \mathbb{Z}\}$ acting freely and properly discontinuously on C_0. The quotient space is a topological torus T^2. The Riemannian metric on C_0 defines by projection a conformal structure on T^2, and its Levi-Civita connection projects to a closed, non-exact Weyl structure on T^2.

We are going to apply some surgery and smoothening to get by similar methods a closed, non-exact Weyl structure on a surface of genus 2.

To do that, consider the domain $B_0 := C_0 \cap \{1 < z < 2\}$, remove the two topological discs obtained as intersection of B_0 with the full cylinder

$$Z_0 := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + (z - 3/2)^2 \leq 1/16\},$$

connect the borders of the two removed discs by the part of the boundary of Z_0 that lies *inside* the cone C_0, then smoothen it up to get a new surface $B \subset \mathbb{R}^3$ such that:

(1) Only the part of B_0 inside the (larger) cylinder

$$Z := \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + (z - 3/2)^2 \leq 1/8\}$$

has been changed (in particular there are neighbourhoods of the two boundary circles of B_0 that are unchanged, so the gluing with the remaining part of C_0 can be done smoothly);

(2) The symmetries

$$S^x, S^y : \mathbb{R}^3 \to \mathbb{R}^3,\ S^x(x, y, z) := (-x, y, z),\ S^y(x, y, z) := (x, -y, z)$$

still act as isometries of B.

The union

$$N_0 := \bigcup_{n\in\mathbb{Z}} f^n \left(\overline{B}\right)$$

is then a non-closed (hence incomplete) smooth submanifold in $\mathbb{R}^3$ which can be completed as a metric space by adding the origin to it. Let g_0 denote the induced Riemannian metric from $\mathbb{R}^3$. The group Γ acts on (N_0, g_0) by homotheties and the quotient space $N := N_0/\Gamma$ is a genus 2 surface (obtained by gluing together the two circles that constitute the boundary of B). The Riemannian metric g_0 and its Levi-Civita connection define, by projection, a conformal structure c, and a closed, non-exact Weyl structure D on N.

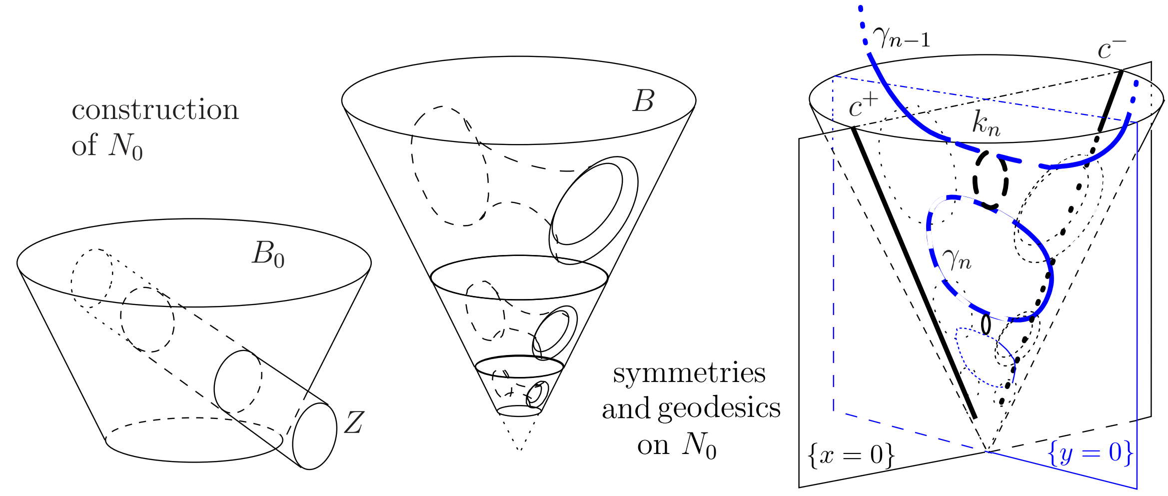

FIGURE 5. Construction and properties of N_0.

We are going to study the geodesics of N_0 and prove the following

Proposition 5.11. *The Weyl structure D is not tame on (N, c).*

Proof. Consider the isometries S^x and S^y acting on (N_0, g_0), whose fixed point sets consist of unions of geodesics in N_0:

(1) $\mathrm{Fix}(S^x) = N_0 \cap \{x = 0\}$, which is a union of two half lines $c^+, c^- : (0, \infty) \to N_0$, $c^\pm(t) := (0, \pm t, t)$ and an infinity of circles $k_n := \{(0, y, z) \mid y^2 + (z - 3 \cdot 2^{n-1})^2 = 4^{n-2}\}$, $n \in \mathbb{Z}$;
(2) $\mathrm{Fix}(S^y) = N_0 \cap \{y = 0\}$, which is a union of closed curves γ_n connecting $f^n(B)$ with $f^{n+1}(B)$ and intersecting k_n in $P_n := (0, 0, 7 \cdot 2^{n-2})$, and k_{n+1} in $Q_{n+1} := (0, 0, 3 \cdot 2^{n-1})$.

We denote $P := P_n$, for some positive integer n. The point P is a fixed point for both isometries S^x and S^y, and hence for their composition $S := S^x \circ S^y$. The latter induces the map $X \mapsto -X$ on $T_P N_0$ and associates to a point $Q \in N_0$ the *geodesic reflection through* P, i.e. the point $\bar{Q}$ such that, for any geodesic $c^Q : (-\varepsilon, a] \to N_0$ with $c^Q(a) = Q$ and $c^Q(0) = P$, c^Q can be defined on a symmetric interval $[-a, a]$, and $\bar{Q} = c^Q(-a)$.

On the other hand, there exists a geodesic $\gamma : (-\varepsilon, T) \to N_0$ such that $\gamma(0) = P$ and $\gamma(t)$ tends to $\omega = (0, 0, 0)$ as t tends to T. The remark above implies that the geodesic is actually defined on $(-T, T)$ (and this is its maximal domain of definition), and

$$\lim_{t\to T} \gamma(t) = \omega = \lim_{t\to -T} \gamma(t),$$

so both ends of the incomplete geodesic γ tend to the singularity.

For any $\varepsilon > 0$, the point $\gamma(T - \varepsilon)$ can thus be connected by at least two half-geodesics with ω, namely the two branches of γ, of lengths ε and $2T - \varepsilon$ respectively. As ε can be chosen arbitrarily small, we see that there is no bound for the ratios of those lengths, therefore N_0 is not tame by the converse statement in Proposition 3.11. □

5.4. **Open problems.** Several natural questions emerge from the considerations in the present paper. We list here some of the most interesting ones:

(1) Theorem 2.6 shows that if D is a closed Weyl structure on a compact conformal manifold (M, c), then the minimal Riemannian cover (M_0, g_0) can be metrically completed by adding exactly one point. The metric completion of its universal covering $(\tilde{M}, \tilde{g})$ is, however, not well understood: The *boundary* of $\tilde{M}$ in its metric completion may be more complicated in general, possibly depending on the growth of the fundamental group of M.
(2) One can check that the analytically tame condition forces the Lee form θ to be non-vanishing, therefore restricting the topology of M_0 to products $\mathbb{R} \times N$, in particular $\chi(M) = 0$. Does the tame condition also imply a topological restriction? And if this restriction is satisfied, is the structure automatically tame?
(3) Ultimately, is the tame condition necessary in the proof of Theorem 4.1?

The answer to this last question seems to be the most challenging problem in the holonomy theory of Weyl structures.

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